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INVARIANT MEASURE FOR SOME DIFFERENTIAL OPERATORS AND UNITARIZING MEASURE FOR THE REPRESENTATION OF A LIE GROUP. EXAMPLES IN FINITE DIMENSION

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Abstract. Consider a Lie group with a unitary representation into a space of holomorphic functions defined on a domain $\mathcal{D}$ of $\mathbb{C}$ and in $L^2(\mu)$, the measure μ being the unitarizing measure of the representation. On finite-dimensional examples, we show that this unitarizing measure is also the invariant measure for some differential operators on $\mathcal{D}$. We calculate these operators and we develop the concepts of unitarizing measure and invariant measure for an *OU operator* (*differential operator associated to the representation*) in the following elementary cases:
A) The commutative groups $(\mathbb{R},+)$ and $(\mathbb{R}^* = \mathbb{R}-0,\times)$.
B) The multiplicative group M of 2×2 complex invertible matrices and some subgroups of M.
C) The three-dimensional Heisenberg group.

Contents

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0. Introduction. First we define the important concepts: 1) unitarizing measure for the representation of a Lie group G in a space of holomorphic functions (the unitarizing measure is a real measure), 2) infinitesimal representation of the Lie algebra, 3) from the vector fields of the infinitesimal representation, construction of a second order differential operator Δ^{OU} with a drift term and real valued coefficients, and finally we ask 4) whether the measure μ of the representation of the group G is also the invariant measure for this differential operator Δ^{OU}. Such a study is motivated by the work and ideas in [6], [3], [17], [18] and [9]. In particular, according to [3], [18], in certain cases, the measure for the representation of an infinite-dimensional Lie group should be obtained as the invariant measure for an Ornstein–Uhlenbeck operator Δ^{OU}. In [9], the space of holomorphic functions is identified with a complex line bundle, each section of this bundle being a complex valued function, the operator Δ^{OU} is constructed in this abstract setting by the stochastic calculus of variation and the example of the group $SU(1,1)$ with the Poincaré disk $SU(1,1)/S^1$ is given. It is interesting to have more examples and explicit formulas for the operator Δ^{OU}. In Section 4, we consider discrete series representations for $Gl(2, \mathbb{R})$ and for elementary finite-dimensional complex linear groups. The realization of discrete series representation in concrete functions spaces has been done by several authors, see the historical notes, Chapter V in [24]. In the present work, G is a finite-dimensional group. In the first part (Section 1), we explain the relations between Δ^{OU} operators and representations of the form $T_g f(z) = h_g(z) f(k_g(z))$ where h_g and k_g are holomorphic functions on a domain $\mathcal{D}$. We show in Theorems 1.10 and 1.18 how to find, in a systematic way, the differential operator Δ^{OU} in terms of the infinitesimal representation of the group. Theorems 1.10 and 1.18 can be applied to all the examples (Sections 2, 3, 4) and also to the 3-dimensional Heisenberg group (Section 5). We illustrate Section 1 and along the lines of the ideas in [18], by direct calculation on the examples, we establish the expressions for the differential operator Δ^{OU}, the measure of the representation and the vector fields in the infinitesimal representation.

1. Unitary representations and Δ^{OU} differential operators

1.1. Unitarizing measure and unitary representation. Given a Lie group G and μ a *real measure* on a domain $\mathcal{D}$ in $\mathbb{C}^n = \mathbb{R}^{2n}$, we denote by $L^2_{\mathrm{Hol}}(\mu)$ the set of holomorphic functions $f : \mathcal{D} \to \mathcal{D}$ such that $|f|^2$ is μ-integrable, i.e.

$$L^2_{\mathrm{Hol}}(\mathcal{D}; \mu) = \mathrm{Hol}(\mathcal{D}) \cap L^2(\mu).$$

We put $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$ and $z_j = x_j + \sqrt{-1}\, y_j$, $x_j, y_j \in \mathbb{R}$. For $g \in G$, we consider the operators

$$T_g : L^2_{\mathrm{Hol}}(\mathcal{D}; \mu) \to L^2_{\mathrm{Hol}}(\mathcal{D}; \mu)$$

such that T_g has the following properties:

(a) For any holomorphic function f on $\mathcal{D}$,

$$T_{g_1 g_2} f = T_{g_1}(T_{g_2} f)$$

where $g_1 g_2$ is the product of g_1 and g_2 in the group G. If e is the neutral element of G,

$$(T_e f)(z) = f(z),$$

and therefore, for any $g \in G$, we have

$$(T_g)^{-1} = T_{g^{-1}}.$$

In fact, T_g is a semi-group of operators indexed by a group G.

(b) T_g is unitary or, equivalently, μ is *unitarizing* for T_g, that is

$$\int |T_g(f)(z)|^2 \, d\mu(z) = \int |f(z)|^2 \, d\mu(z)$$

(c) T_g is of the form

$$(T_g f)(z) = h_g(z) f(k_g(z))$$

where $h_g : \mathcal{D} \to \mathbb{C}$ and $k_g : \mathcal{D} \to \mathcal{D}$ are *holomorphic* functions on $\mathcal{D}$. The conditions (a) and (c) together give the following system on h_g and k_g

$$h_{g_1}(z) h_{g_2}(k_{g_1}(z)) = h_{g_1 g_2}(z) \tag{1.1}$$

and

$$k_{g_2}(k_{g_1}(z)) = k_{g_1 g_2}(z). \tag{1.2}$$

Moreover, since $(T_e f)(z) = z$, we deduce that

$$k_e(z) = z \quad \text{and} \quad h_e(z) = 1. \tag{1.3}$$

When (a)–(b)–(c) are satisfied, T_g is a unitary representation of G into $L^2_{\mathrm{Hol}}(\mathcal{D}; \mu)$. We denote this representation by (T_g, μ). We assume that $k_g(z)$ depends effectively upon g; thus the trivial cases where $k_g(z) = z$ for any g will be eliminated from our considerations.

Remark 1.1. Assume that k_g is determined and that $\widehat{h_g}$ and $\widetilde{h}_g$ are both solutions of (1.1), then the product $h_g = \widehat{h_g} \times \widetilde{h}_g$ is also a solution of (1.1). If $k_g(z)$ is a solution of (1.2), then for any holomorphic function $\psi : \mathcal{D} \to \mathbb{C}$,

$$h_g(z) = \frac{\psi(k_g(z))}{\psi(z)}$$

as well as the determinant of the complex Jacobian matrix of $k_g(z)$ are solutions of (1.1). If $k_g(z) = (k_1(z), k_2(z), \dots, k_n(z))$, we define the complex Jacobian matrix of $k_g(z)$ as the matrix $\left(\frac{\partial k_m}{\partial z_j}\right)$. In the following, for simplicity, $\mathcal{D}$ is a subset of $\mathbb{C}$. Then in these cases we have $h_g(z) = \left(\frac{k_g(z)}{z}\right)^q$ for any positive integer q, $h_g^{\varphi}(z) = \exp(\varphi(k_g(z)) - \varphi(z))$ where φ is holomorphic and

$$h_g(z) = k_g'(z).$$

On the other hand, if $k_g(z)$ satisfies (1.2) and $h_g(z)$ satisfies (1.1), then for any integer q,

$$\widehat{k_g}(z) = [k_g(z^q)]^{(1/q)}$$

satisfies also (1.2) and $\widehat{h}_g(z) = h_g(z^q)$ satisfies $\widehat{h}_{g_1}(z)\widehat{h}_{g_2}(\widehat{k}_{g_1}(z)) = \widehat{h}_{g_1 g_2}(z)$.

REMARK 1.2. Assume that the group law is not known, but that two functions $k_g(z)$ and $h_g(z)$ are given. If $k_g(z)$ and $h_g(z)$ satisfy (1.1)–(1.2), from the following example, we see that it is possible to determine the group law. Assume $G = \mathbb{C}^3$, $g = (a, b, c) \in \mathbb{C}^3$. Assume that $h_g(z) = e^{bz+c}$ and $k_g(z) = z + a$. We determine the group law $*$

$$(a_1, b_1, c_1) * (a_2, b_2, c_2) = (a_3, b_3, c_3)$$

as follows. The condition (1.2) implies $a_3 = a_1 + a_2$ and the condition (1.1) implies that $b_3 = b_1 + b_2$, $c_3 = c_1 + c_2 + a_1 b_2$. This defines the group structure $G = (\mathbb{C}^3, *)$ associated to the multiplication of 3×3 Heisenberg matrices $\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$. We can find a subgroup G_1 of G such that $\mu = e^{-(x^2+y^2)}\,dx\,dy$ is a *unitarizing measure* for the representation $T_g f(z) = e^{bz+c} f(z + a)$, $g \in G_1$. Writing the condition (b), we obtain

$$a + \overline{b} = 0, \quad c + \overline{c} + a\overline{a} = 0.$$

The set of $g_1 = (a, -\overline{a}, c)$ with $c+\overline{c}+a\overline{a} = 0$ is a subgroup G_1 of G with the representation

$$T_{g_1} f(z) = e^{-\overline{a}z+c} f(z + a) \quad \text{in} \quad L^2_{\mathrm{Hol}}(e^{-(x^2+y^2)}\,dx\,dy).$$

1.2. Unitarizing measure and the infinitesimal representation of the Lie algebra $\mathcal{G}$. In [3], [1], [4], the unitarizing measure and its relation to Δ^{OU} have been studied in the case of representations of the Lie algebra $\mathcal{G}$ of G. *The infinitesimal representation* of the Lie algebra $\mathcal{G}$ into $L^2_\mu(\mathcal{D})$ is obtained as follows: Let $\epsilon \mapsto g_\epsilon$ be a curve on G such that $g_0 =$ identity of G, we put

$$v = \frac{d}{d\epsilon}\Big|_{\epsilon=0} g_\epsilon. \tag{1.4}$$

For f holomorphic on $\mathcal{D}$ and $f \in L^2_\mu(\mathcal{D})$, we put

$$\rho(v)f = \frac{d}{d\epsilon}\Big|_{\epsilon=0} T_{g_\epsilon} f, \tag{1.5}$$

the unitarizing condition gives

$$\int [\rho(v)f]\,\overline{\phi}\,d\mu + \int f\,\overline{\rho(v)\phi}\,d\mu = 0 \tag{1.6}$$

for any f and ϕ holomorphic. Let $(e_j)_{(j=1,\dots,n)}$ be a basis of the Lie algebra $\mathcal{G}$, and let $(\rho(e_j))$ be the corresponding operators in the infinitesimal representation,

$$\rho(e_j)f = \frac{d}{d\epsilon}\Big|_{\epsilon=0} T_{g^j_\epsilon} f, \quad e_j = \frac{d}{d\epsilon}\Big|_{\epsilon=0} g^j_\epsilon$$

where g^j_ϵ are curves on G.

LEMMA 1.3. *If we assume that* (T_g, μ) *is of the form* (c)*, we have*

$$\rho(e_j) = \alpha_j(z)\,\frac{\partial}{\partial z} + \beta_j(z),$$

$$\alpha_j(z) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} k_{g^j_\epsilon}(z), \quad \beta_j(z) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} h_{g^j_\epsilon}(z),$$

where g^j_ϵ *are curves on* G *such that* $e_j = \frac{d}{d\epsilon}\big|_{\epsilon=0} g^j_\epsilon$. *We see that* $\alpha_j(z)$ *and* $\beta_j(z)$ *are holomorphic functions of* z. *If*

$$T_g f(z) = [k'_g(z)]^\gamma \, \frac{\psi(k_g(z))}{\psi(z)} \, f(k_g(z)) \quad \textit{where } \psi \textit{ is holomorphic,} \tag{1.7}$$

we have

$$\rho(e_j) = \alpha_j(z) \, \frac{\partial}{\partial z} + \gamma \alpha'_j(z) + \alpha_j(z) \, \frac{\psi'(z)}{\psi(z)} \, . \tag{1.8}$$

In this work, we shall assume that γ is an integer though other real values of γ are admissible, see [6]. Assume that the measure μ has the density $R(z, \overline{z})$ with respect to the Lebesgue measure $dz \, d\overline{z}$ on $\mathcal{D}$, let

$$d\mu = 2R(z, \overline{z}) \, dx \, dy = R(z, \overline{z}) \, dz \, d\overline{z},$$

and put

$$\Gamma_j(z, \overline{z}) = \frac{1}{R(z, \overline{z})} \, \frac{\partial}{\partial z} (\alpha_j(z) R(z, \overline{z})) = \alpha'_j(z) + \alpha_j(z) \, \frac{\partial}{\partial z} \log R(z, \overline{z}). \tag{1.9}$$

Equivalently

$$\Gamma_j(z, \overline{z}) = \alpha_j(z) \, \frac{\partial}{\partial z} \log(\alpha_j(z) R(z, \overline{z})) \quad \text{for} \quad j = 1, \ldots, n.$$

We obtain by writing (1.6) for $v = e_j$, $j = 1, \ldots, n$, that the functions $\beta_j(z)$ and $\Gamma_j(z, \overline{z})$ have the same real part,

$$\Re \beta_j(z) = \Re \Gamma_j(z, \overline{z}) \quad \text{for} \quad j = 1, \ldots, n. \tag{1.10}$$

Since the condition (1.10) is concerned only with the real parts, a representation may have several unitarizing measures and this is the case in Lemma 3.2, formula (3.4). On the other hand, when $n > 1$, the system of equations (1.9)–(1.10) or equivalently the system

$$\frac{1}{2} \left(\alpha_j(z) \frac{\partial}{\partial z} + \overline{\alpha_j(z)} \, \frac{\partial}{\partial \overline{z}} \right) \log R = \Re(\beta_j(z) - \alpha'_j(z)), \quad j = 1, \ldots, n, \tag{1.11}$$

must have a common solution $R(z, \overline{z})$. This fact is verified on our examples. In particular, in the case of (1.8) with $\psi(z) = 1$, we have $\beta_j(z) = \gamma \alpha'_j(z)$,

$$\Gamma_j - \beta_j = (1 - \gamma) \alpha'_j + \alpha_j \, \frac{\partial}{\partial z} \log R \tag{1.12}$$

and the system (1.11) reduces to

$$\frac{1}{2} \Big(\alpha_j(z) \, \frac{\partial}{\partial z} + \overline{\alpha_j(z)} \, \frac{\partial}{\partial \overline{z}} \Big) \log R = (\gamma - 1) \Re(\alpha'_j(z)), \quad j = 1, \ldots, n.$$

If $R(z, \overline{z})$ has been determined, we have the following

LEMMA 1.4. *Assume that* $d\mu = R(z, \overline{z}) \, dz \, d\overline{z}$, *then*

(i) $$\Re \Big[\alpha_j(z) \, \frac{\partial}{\partial z} \log R(z, \overline{z}) \Big] \quad \textit{is harmonic on } \mathcal{D} \quad \textit{for } j = 1, \ldots, n.$$

The function $\beta_j(z)$ *is completely determined by* $\alpha_j(z)$ *and* $R(z, \overline{z})$. *We have*

(ii) $$\Re \frac{\partial}{\partial z} \Big[\alpha_j(z) \, \frac{\partial^2}{\partial z \, \partial \overline{z}} \log R \Big] = 0, \quad j = 1, \ldots, n.$$

Proof. Since $\beta_j(z)$ and $\alpha'_j(z)$ are holomorphic, $\Re(\beta_j)$ is harmonic. To prove (ii), we calculate the Laplacian of the expression in (i). ∎

For the bracket in $\mathcal{G}$,

$$[e_i, e_j] = \sum_k c_{ij}^k e_k, \tag{1.13}$$

the condition $\rho([e_i, e_j]) = [\rho(e_i), \rho(e_j)]$ together with (1.13) implies that

$$\alpha_i \alpha'_j - \alpha_j \alpha'_i = \sum_k c_{ij}^k \alpha_k, \tag{1.14}$$

$$\alpha_i \beta'_j - \alpha_j \beta'_i = \sum_k c_{ij}^k \beta_k. \tag{1.15}$$

The relation (1.15) is a consequence of (1.14) when $\rho(e_j)$ is given by (1.8). Moreover (1.14) and (1.9) imply

$$\alpha_i \frac{\partial}{\partial z}\Gamma_j - \alpha_j \frac{\partial}{\partial z}\Gamma_i = \sum_k c_{ij}^k \Gamma_k. \tag{1.16}$$

Definition 1.5. If V is a vector field on $\mathcal{D}$, we define the *divergence function* $\mathrm{div}_\mu(V)$ by the condition

$$\int \mathrm{div}_\mu(V)(z,\overline{z})\Phi(z,\overline{z})\, d\mu = \int (V\Phi)\, d\mu \quad \text{for any differentiable function } \Phi(z,\overline{z}).$$

If $d\mu = R(z,\overline{z})\, dz\, d\overline{z}$, we have

$$\mathrm{div}_\mu\Big(u(z,\overline{z})\frac{\partial}{\partial z}\Big) = -\frac{1}{R}\frac{\partial}{\partial z}(u(z,\overline{z})R(z,\overline{z})). \tag{1.17}$$

We consider the vector fields

$$H_j = \alpha_j(z)\frac{\partial}{\partial z} \quad \text{and} \quad \overline{H_j} = \overline{\alpha_j(z)}\frac{\partial}{\partial \overline{z}}, \tag{1.18}$$

then (1.9) implies $\mathrm{div}_\mu(H_j) = -\Gamma_j(z,\overline{z})$ and $\mathrm{div}_\mu(\overline{H_j}) = -\overline{\Gamma_j(z,\overline{z})}$.

The vector field V has divergence zero with respect to μ if and only if $\int (V\Phi)\, d\mu = 0$ for any differentiable $\Phi(z,\overline{z})$. For example, for a differentiable function $h : \mathbb{R} \to \mathbb{R}$,

$$V = \overline{z}\frac{\partial}{\partial \overline{z}} - z\frac{\partial}{\partial z} = i\frac{\partial}{\partial \theta} \quad \text{with} \quad z = re^{i\theta} \tag{1.19}$$

has divergence zero with respect to $\mu = h(z\overline{z})\, dz\, d\overline{z}$.

Definition 1.6. We call a vector field V such that $\mathrm{div}_\mu(V) = 0$ a *divergence-free vector field* associated to the representation (T_g, μ).

1.3. Invariant measure with respect to $u(z,\overline{z})\frac{\partial^2}{\partial z \partial \overline{z}} + v(z,\overline{z})\frac{\partial}{\partial \overline{z}}$

Definition 1.7. We say that the second order differential operator A *has the real measure μ as invariant measure* if for any differentiable function $\Phi(z,\overline{z})$ we have

$$\int A\Phi\, d\mu = 0. \tag{1.20}$$

Remark 1.8. Since the measure μ is real, we have $\int (C_1 A + C_2\overline{A})\Phi\, d\mu = 0$ for any constants C_1, C_2. If $C_1 A + C_2\overline{A}$ reduces to a first order operator, then $V = C_1 A + C_2\overline{A}$ is a vector field and $\mathrm{div}_\mu(V) = 0$.

Lemma 1.9. *An operator of the form*

$$u(z,\overline{z})\,\frac{\partial^2}{\partial z\partial\overline{z}}+v(z,\overline{z})\,\frac{\partial}{\partial\overline{z}} \tag{1.21}$$

has $R(z,\overline{z})\,dz\,d\overline{z}$ as invariant measure if and only if

$$v(z,\overline{z})=\frac{1}{R}\,\frac{\partial}{\partial z}(u(z,\overline{z})R(z,\overline{z})). \tag{1.22}$$

Comparing (1.22) and (1.9), we see that the concept of unitarizing measure and that of invariant measure are closely related.

Theorem 1.10. *Assume that the Lie algebra $\mathcal{G}$ has dimension n, $n>1$. Consider the unitarizing measure $\mu=R(z,\overline{z})\,dz\,d\overline{z}$ for the representation T_g. Assume that there exist constants A_{jk}, $j,k=1,\dots,n$, such that*

$$\sum_{j,k}A_{jk}\alpha_j(z)\alpha_k(z)=0. \tag{1.23}$$

Let

$$\Delta=\sum_{j,k}A_{jk}(\rho(e_j)+\overline{\rho(e_j)})\overline{H_k}, \tag{1.24}$$

then

(i) *Δ has μ as invariant measure,*
(ii) *Δ is an operator of the form* (1.21).

The condition (1.23) *means that $\sum_{j,k}A_{jk}H_jH_k$ reduces to a* (*holomorphic*) *vector field.*

Proof. Let $\Gamma_j(z,\overline{z})$ be as in (1.9). The conditions (1.9)–(1.10) imply that for any j,k

$$\Delta_{jk}=(\rho(e_j)+\overline{\rho(e_j)})\overline{H_k} \tag{1.25}$$

has μ as invariant measure, thus Δ has μ as invariant measure. The condition (1.23) implies that Δ is of the form (1.21), it has no term in $\frac{\partial^2}{\partial\overline{z}^2}$. ∎

Independently of Theorem 1.10, we have

Theorem 1.11. *The functions $\Gamma_j(z,\overline{z})$ are given by* (1.9). *If there exist constants A_{jk} such that*

$$\sum_{j,k}A_{jk}(\beta_j(z)-\Gamma_j(z,\overline{z}))\,\overline{\alpha_k(z)}=0, \tag{1.26}$$

then $\Delta=\sum_{j,k}A_{jk}\rho(e_j)\overline{H_k}$ has $R(z,\overline{z})\,dz\,d\overline{z}$ as invariant measure.

Proof. By (1.9). ∎

Corollary 1.12. *Assume that $\beta_j=\gamma\alpha_j'$ and that $\sum_{j,k}A_{jk}\alpha_j(z)\overline{\alpha_k(z)}$ is real. If* (1.26) *is satisfied, we have*

$$R(z,\overline{z})=\text{constant}\times\left[\sum_{j,k}A_{jk}\alpha_j(z)\overline{\alpha_k(z)}\right]^{\gamma-1}.$$

Proof. Substituting (1.12) into (1.26), we obtain

$$\sum_{j,k}A_{jk}(1-\gamma)\alpha_j'\overline{\alpha_k}+\sum_{j,k}A_{jk}\alpha_j\overline{\alpha_k}\,\frac{\partial}{\partial z}\log R=0.$$

This condition implies that $R = \Phi(\overline{z}) \times [\sum_{j,k} A_{jk}\alpha_j\overline{\alpha_k}]^{\gamma-1}$. Since $\sum_{j,k} A_{jk}\alpha_j(z)\overline{\alpha_k(z)}$ and $R(z,\overline{z})$ are real, we deduce that $\Phi(\overline{z})$ is constant. ■

The conditions of Theorem 1.11 are satisfied for the one-dimensional cases: in Sections 2 and 3, we calculate the density R in such a way that $\beta = \Gamma$, see (2.4) and Remark 3.3. They are also satisfied for some of the examples in Section 4 but we cannot apply Theorem 1.11 to the Heisenberg group, Section 5, where we use Theorem 1.10 instead. The conditions (1.23) are satisfied both for the Heisenberg group and for the examples in Section 4. We relate Theorems 1.10 and 1.11.

Lemma 1.13. *Assume that both conditions* (1.23) *and* (1.26) *are satisfied and that the constants A_{jk} are real numbers. Then $\sum_{j,k} A_{jk}\rho(e_j)H_k = 0$.*

Proof. We have to verify that

$$\sum_{j,k} A_{jk}(\alpha_j\alpha_k' + \beta_j\alpha_k) = 0. \tag{1.27}$$

The condition (1.23) implies that $\sum_{j,k} A_{jk}\alpha_j\alpha_k' = -\sum_{j,k} A_{jk}\alpha_j'\alpha_k$. Using (1.9), we replace α_j'. Thus for proving (1.27), we have to verify that

$$\sum_{j,k} A_{jk}\Big(-\Gamma_j + \alpha_j\frac{\partial}{\partial z}\log R + \beta_j\Big)\alpha_k = 0. \tag{1.28}$$

The condition (A_{jk} are real) as well as (1.10) and (1.26) imply that

$$\sum_{j,k} A_{jk}(\beta_j - \Gamma_j)\alpha_k = 0. \tag{1.29}$$

Then (1.29) and (1.23) imply (1.28). ■

1.4. Ornstein–Uhlenbeck operator. For the classical Ornstein–Uhlenbeck operator,

$$D = \frac{d^2}{dx^2} - \frac{x}{t}\frac{d}{dx} = \delta\,\frac{d}{dx} \quad \text{where} \quad \delta = \frac{d}{dx} - \frac{x}{t}\,I,$$

the measure $\mu = e^{-x^2/2t}\,dx$ is invariant. For the first order term, we have the divergence condition

$$\frac{x}{t} = \mathrm{div}_\mu\Big(\frac{d}{dx}\Big).$$

The divergence operator δ satisfies

$$\delta\,\frac{d}{dx} - \frac{d}{dx}\,\delta = \frac{1}{t}\,I.$$

The n-th Hermite polynomial is $H_n(x) = \delta^n 1$.

On the complex plane $\mathbb{C}$, we consider only Hermitian metrics, they are of the form $ds^2 = g(z,\overline{z})\,dz\,d\overline{z}$ where $g(z,\overline{z})$ is a real valued function (see for example p. 289 in [14]). In fact, these metrics are also Kählerian, [11]. This will give restrictions on the choice of the unitarizing measure as we can see in Remark 3.3, and on the operator Δ which has μ as invariant measure, see Remark 4.8.

By analogy with the classical Ornstein–Uhlenbeck operator, we introduce the following definition.

DEFINITION 1.14. We call an operator a *real Ornstein–Uhlenbeck operator* (*OU-operator*) associated to the holomorphic representation (T_g, μ) when it is of the form

$$u(x,y)\Big[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial Q}{\partial x}\frac{\partial}{\partial x}+\frac{\partial Q}{\partial y}\frac{\partial}{\partial y}\Big] \tag{1.30}$$

such that

(i) it has the measure μ of the representation as invariant measure,
(ii) $u(x,y)$ is real positive.

Since $d\mu = R\,dz\,d\overline{z}$ is real, this implies that Q is a real valued function. If $z = z + iy$, then

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial Q}{\partial x}\frac{\partial}{\partial x}+\frac{\partial Q}{\partial y}\frac{\partial}{\partial y} = 4\frac{\partial^2}{\partial z\partial\overline{z}}+2\Big[\frac{\partial Q}{\partial\overline{z}}\frac{\partial}{\partial z}+\frac{\partial Q}{\partial z}\frac{\partial}{\partial\overline{z}}\Big]. \tag{1.31}$$

We have

$$d\mu = R\,dz\,d\overline{z} = e^{Q(z,\overline{z})}\,dv \quad \text{with} \quad dv = \frac{dz\,d\overline{z}}{u(z,\overline{z})} \tag{1.32}$$

and

$$Q = \log(Ru). \tag{1.33}$$

If the product Ru is real positive, then Q is real valued. The volume element $\frac{1}{2}\,dv$ is associated to the metric ds^2 on $\mathcal{D} = \{u(z,\overline{z}) > 0\}$,

$$ds^2 = \frac{1}{u}\,(dx^2+dy^2).$$

DEFINITION 1.15. We call an operator a *complex OU-operator associated to* (T_g,μ), when it is of the form

$$u(z,\overline{z})\Big[\frac{\partial^2}{\partial z\partial\overline{z}}+\frac{\partial Q}{\partial z}\frac{\partial}{\partial\overline{z}}\Big] \tag{1.34}$$

with the measure μ of the representation as invariant measure and such that Q is real valued.

Consider the symplectic form

$$\omega = \frac{dz\wedge d\overline{z}}{u(z,\overline{z})}\,. \tag{1.35}$$

We define the complex gradient of Q with respect to ω as the vector field

$$\operatorname{grad}_\omega Q = u\,\frac{\partial Q}{\partial z}\,\frac{\partial}{\partial\overline{z}}\,. \tag{1.36}$$

With this notation, the complex OU-operator associated to (T_g,μ) is

$$\Delta_{\text{laplacian}} + \operatorname{grad}_\omega Q, \tag{1.37}$$

where $\Delta_{\text{laplacian}}$ is the *Riemannian Laplacian* on $\mathcal{D}$. The classical Ornstein–Uhlenbeck has been extended to an analogue on the Wiener space and in the infinite-dimensional setting, commutations and integration by parts identities have been obtained with the divergence operator, see [23]. In our context, we obtained the following factorization.

LEMMA 1.16. *Let $h(z,\overline{z})$ be a function of $(z,\overline{z})$ with real or complex values. Consider the divergence operator δ_h*

$$\delta_h = h\Big[\frac{\partial}{\partial z} + \Big(\frac{\partial}{\partial z}\log(Rh)\Big)I\Big]. \tag{1.38}$$

If $Q = \log(Ru)$, we have

$$u(z,\overline{z})\Big[\frac{\partial^2}{\partial z\partial\overline{z}} + \frac{\partial Q}{\partial z}\frac{\partial}{\partial\overline{z}}\Big] = \delta_h\Big(\frac{u}{h}\frac{\partial}{\partial\overline{z}}\Big). \tag{1.39}$$

In particular, if we put $\Gamma_\alpha = \alpha\frac{\partial}{\partial z}\log(\alpha R)$ as in (1.9), we have $\delta_\alpha = \alpha\,\frac{\partial}{\partial z} + \Gamma_\alpha$.

In the following, our problem is to express (1.37) or equivalently (1.39) in terms of the infinitesimal representation as in Theorems 1.10 or 1.11. For example, Theorem 1.11 comes down to writing the factorization (1.39) as a sum of factors $\rho(e_j)\overline{\alpha_k(z)}\frac{\partial}{\partial\overline{z}}$ where the $\rho(e_j)$ have the shape of a divergence operator.

LEMMA 1.17. *Assume that $\sum_{j,k} A_{jk}\alpha_j(z)\alpha_k(z) = 0$. Let Δ be as in Theorem 1.10, then Δ is of the form (1.39) with $Q = \log(Ru)$,*

$$u(z,\overline{z}) = \sum_{j,k} A_{jk}\alpha_j(z)\overline{\alpha_k(z)},$$

$$\Delta = \sum_{j,k} A_{jk}(\rho(e_j) + \overline{\rho(e_j)})\overline{H_k} = u(z,\overline{z})\Big[\frac{\partial^2}{\partial z\partial\overline{z}} + \frac{\partial Q}{\partial z}\frac{\partial}{\partial\overline{z}}\Big].$$

Proof. We verify that the coefficient of $\frac{\partial}{\partial\overline{z}}$ in Δ is equal to $u\,\frac{\partial Q}{\partial z}$. ∎

We deduce

THEOREM 1.18. *Assume that the conditions of Theorem 1.10 are satisfied, i.e. there exist constants A_{jk} such that $\sum_{j,k} A_{jk}\alpha_j(z)\alpha_k(z) = 0$. Assume that up to a multiplicative constant, $u(z,\overline{z}) = \sum_{j,k} A_{jk}\alpha_j(z)\overline{\alpha_k(z)}$ is real and positive. Let $\mathcal{D}$ be the subset of $\mathbb{C}$ defined by*

$$\mathcal{D} = \Big\{z \;\Big|\; \sum_{j,k} A_{jk}\alpha_j(z)\overline{\alpha_k(z)} \neq 0\Big\}. \tag{1.40}$$

Assume that $\mathcal{D} \neq \emptyset$. Then there is an Ornstein–Uhlenbeck operator Δ^{OU} associated to the representation (T_g,μ), and we have

$$\Delta^{OU} = \Delta + \overline{\Delta} \quad \textit{where} \quad \Delta = \sum_{j,k} A_{jk}(\rho(e_j) + \overline{\rho(e_j)})\overline{H_k}. \tag{1.41}$$

Moreover, $\mathrm{div}_\mu(V) = 0$ for the vector field

$$V = \Delta - \overline{\Delta} = \sum_{j,k} A_{jk}(\beta_j + \overline{\beta_j})\Big(\overline{\alpha_k}\,\frac{\partial}{\partial\overline{z}} - \alpha_k\,\frac{\partial}{\partial z}\Big). \tag{1.42}$$

Proof. According to Theorem 1.10, Δ is of the form (1.21) and

$$v(z,\overline{z}) = u(z,\overline{z})\,\frac{\partial}{\partial z}\log(Ru).$$

By Lemma 1.17, $\Delta + \overline{\Delta}$ is an OU-operator. On the other hand since $u(z,\overline{z})$ is real, we deduce that $V = \Delta - \overline{\Delta}$ is a vector field. ∎

We have the identification

$$\begin{aligned}&\text{Unitarizing measure for a representation of a group } G \text{ on } L^2_{\mathrm{Hol}}(\mathcal{D};\mu)\\ &\qquad\qquad\qquad\qquad = \text{invariant measure for an OU-process on } \mathcal{D}.\end{aligned}$$

The so-called OU-process is the diffusion on $\mathcal{D}$ with an infinitesimal generator Δ^{OU}. This identification would permit the construction of the invariant measure from the OU-process, see [18], [3] and [8] and a study of unitary representations of the group G. Given the infinitesimal representation $(\rho(e_j))$, let Δ be as in Theorem 1.10 with appropriate constants A_{jk}, a problem is to construct an invariant real measure μ for Δ, and find the Hilbert space $L^2_{\mathrm{Hol}}(\mu)$ of the representation. From our examples, the infinitesimal representation and Δ^{OU} determine the support of μ.

1.5. Equivalent representations. Since our purpose is to relate the unitarizing measure of the representation to a second order differential operator which has this measure as invariant measure, it is worthwhile to consider equivalent representations and to compare the differential operators and invariant measures for these representations. Moreover, the unitarizing measures for two equivalent representations do not have in general the same class of integrable functions. See Subsection 4.1.

1.5.1. *The representation T^ψ_g.* Assume that $k_g(z)$ satisfies (1.2). For any holomorphic function ψ, let

$$h^\psi_g(z) = \frac{\psi(k_g(z))}{\psi(z)}\,.$$

It satisfies (1.1). Consider the representations

$$(T_g f)(z) = h_g(z)(f(k_g(z))) \quad \text{and} \quad (T^\psi_g f)(z) = h^\psi_g(z)(T_g f)(z). \tag{1.43}$$

If T_g is unitary for μ, then T^ψ_g is unitary for the measure $d\mu^\psi(z) = |\psi(z)|^2\, d\mu(z)$. If $\psi(z) \neq 0$, we define the linear operator A^ψ

$$(A^\psi f)(z) = \frac{f(z)}{\psi(z)}\,.$$

Since we consider mainly the cases $\psi(z) = \exp(\varphi(z))$ and $\psi(z) = z^n$ where n is a positive integer, we do not investigate the zeroes of $\psi(z)$. The identities $\int |A^\psi f|^2\, d\mu^\psi = \int |f|^2\, d\mu$ and $(T^\psi_g)(A^\psi f)(z) = A^\psi(T_g f)(z)$ show that (T_g, μ) and (T^ψ_g, μ^ψ) are equivalent. For the concept of equivalent representations, see for example page 7 in [24]. In Section 3.1, solving the functional equation $\alpha(t_1) + t_1\alpha(t_2) = \alpha(t_1 t_2)$ yields representations like T_g and $T^{\exp\varphi}_g = \exp(\varphi(k_g(z)) - \varphi(z))(T_g f)(z)$. The representation $T^{\exp\varphi}_g$ is unitary for $d\mu^\varphi(z) = \exp(\varphi(z) + \overline{\varphi(z)})\, d\mu(z)$ and $(A^{\exp\varphi} f)(z) = \exp(-\varphi(z)) f(z)$. Define

$$\rho^\psi(v) f = \frac{d}{d\epsilon}\Big|_{\epsilon=0} T^\psi_{g_\epsilon} f(z),$$

then

$$[\rho^\psi(v) f](z) = [\rho(v) f](z) + \frac{\psi'(z)}{\psi(z)}\Big[\frac{d}{d\epsilon}\Big|_{\epsilon=0} k_{g_\epsilon}(z)\Big] f(z).$$

Since $k_g(z)$ is the same for T_g and T_g^{ψ}, we have

$$[\rho^{\psi}(e_j)f](z) = [\rho(e_j)f](z) + \frac{\psi'(z)}{\psi(z)}\,\alpha_j(z)f(z).$$

Let Δ be as in (1.24), we have $\Delta = u(z,\overline{z})\left[\frac{\partial^2}{\partial z\partial\overline{z}} + \frac{\partial Q}{\partial z}\frac{\partial}{\partial\overline{z}}\right]$ where

$$u(z,\overline{z}) = \sum_{j,k} A_{jk}\alpha_j(z)\overline{\alpha_k(z)}.$$

According to Theorem 1.10, with the infinitesimal representation ρ^{ψ}, we obtain

$$\Delta^{\psi} = \Delta + \frac{\psi'(z)}{\psi(z)}\,u(z,\overline{z})\,\frac{\partial}{\partial\overline{z}} = u(z,\overline{z})\Big[\frac{\partial^2}{\partial z\,\partial\overline{z}} + \frac{\partial Q_{\psi}}{\partial z}\frac{\partial}{\partial\overline{z}}\Big] \quad \text{with} \quad e^{Q_{\psi}} = e^{Q}\psi. \tag{1.44}$$

The measure $d\mu^{\psi} = |\psi|^2\,d\mu$ of the equivalent representation is invariant for Δ^{ψ} if and only if $d\mu$ is invariant for Δ.

1.5.2. *The representation* $\widehat{T}_g$. Assume that q is an integer. Let

$$(T_g f)(z) = h_g(z)(f(k_g(z))) \quad \text{and} \quad (\widehat{T}_g f)(z) = h_g(z^q)f((k_g(z^q))^{1/q}). \tag{1.45}$$

We put $(A^q f)(z) = f(z^q)$. Since $\widehat{T}_g\,A^q = A^q\,T_g$, the two representations (1.45) are equivalent. If T_g is unitary for μ, let μ^q be the measure such that $\int |A^q f|^2\,d\mu^q = \int |f|^2\,d\mu$, then $\widehat{T}_g$ is unitary for μ^q. Let $\rho(v)f(z) = \frac{d}{d\epsilon}\big|_{\epsilon=0}T_g f(z) = \alpha(z)f'(z) + \beta(z)$, we obtain

$$\widehat{\rho}(v)f(z) = \frac{d}{d\epsilon}\Big|_{\epsilon=0}\widehat{T}_{g_\epsilon}f = \frac{\alpha(z^q)}{qz^{q-1}}\,f'(z) + \beta(z^q)f(z).$$

1.6. Classical examples. In the next sections, for holomorphic representations of finite-dimensional elementary groups, we start from the system (1.1)–(1.2). We determine T_g in the form (c) with (1.1)–(1.2), then we construct the *real* measure μ such that (b) is satisfied, next we find a second order differential operator denoted by Δ (not necessarily real) which has μ as invariant measure. We require Δ to be of the form (1.21). We express this operator in terms of the infinitesimal representation of the Lie algebra as in (1.24). This explicitly relates the first order terms in Δ to the *infinitesimal representation of the Lie algebra* $\mathcal{G}$ of G. In the one-dimensional example in Section 3, one can associate several unitarizing measures to a representation T_g. However, there is only one density $R(z,\overline{z})$ such that $\Gamma(z,\overline{z}) = \beta(z)$ (see (1.9)–(1.10) and Lemma 1.3). For the 3-dimensional Heisenberg group in Section 5, $\mathcal{D} = \mathbb{C}$, see ([5], [20], [1]), the Δ^{OU} operator is completely determined by the infinitesimal representation. In Section 4, we examine

1) the group of 2×2 matrices $\begin{pmatrix} a & b \\ \overline{b} & \overline{a}\end{pmatrix}$ such that $|a|^2 - |b|^2 = 1$, $\mathcal{D}$ is the unit disk in $\mathbb{C}$, see [4].
2) the group of 2×2 matrices $\begin{pmatrix} a & b \\ -\overline{b} & \overline{a}\end{pmatrix}$ such that $|a|^2 + |b|^2 = 1$, $\mathcal{D} = \mathbb{C}$.

These two examples enter in the same framework of the group of 2×2 matrices $\begin{pmatrix} a & b \\ p\overline{b} & \overline{a}\end{pmatrix}$ such that $|a|^2 - p|b|^2 > 0$ and p is a real number.

Our future objective is to extend the results to infinite-dimensional Lie groups, see [18], [21]. We have in mind: 1) the infinite-dimensional Siegel disk, see [2], [22], 2) the group of diffeomorphisms of the circle and the Virasoro algebra, see [3], [16], [21], [18],

[19], [12], 3) the infinite-dimensional Heisenberg group and its representation with unitarizing measure being the Wiener measure on the Wiener space of continuous maps from $[0,1]$ to $\mathbb{C}$ (see [10], [13]) or its representation in a Gaussian space $(\mathcal{S}'(R),\gamma)$ of tempered Schwartz distributions where γ is the Gaussian measure given by its Fourier transform, see [7].

2. The additive group $(R,+)$. The system (1.1)–(1.2) becomes

$$\text{(i)}:\ h_{t_1}(z)h_{t_2}(k_{t_1}(z)) = h_{t_1+t_2}(z) \quad\text{and}\quad \text{(ii)}:\ k_{t_2}(k_{t_1}(z)) = k_{t_1+t_2}(z). \tag{2.1}$$

Looking for solutions $k_t(z) = az + tb$ where a, b are constants, implies $k_t(z) = z + tb$. We substitute in (2.1)(i)

$$h_{t_1}(z)h_{t_2}(z + t_1 b) = h_{t_1+t_2}(z). \tag{2.2}$$

Looking for solutions of (2.2) in the form $h_t(z) = \exp(\alpha t z + \beta)$ where α, β are constants, we find $\beta = 0$, $b = 0$, thus $(T_t f)(z) = e^{\alpha t z} f(z)$. Since $k_g(z)$ does not depend upon g, we do not consider these solutions. Now take for solutions of (2.1), $k_t(z) = e^{\lambda t} z$ and $h_t(z) = e^{\alpha t}$ where α and λ are constant, we obtain $(T_t f)(z) = e^{\alpha t} f(e^{\lambda t} z)$. The change of parameter $t \mapsto e^t$ leads to the multiplicative group $(\mathbb{R}^* = \mathbb{R} - 0, \times)$ studied in the next section. We also have for solutions of (1.2)

$$k_t(z) = \Big(\frac{z^k}{1 - t z^k}\Big)^{(1/k)} \quad \text{where } k \neq 0 \text{ is an integer}, \tag{2.3}$$

or equivalently $k_t(z) = (z^p - t)^{(1/p)}$ if $k = -p$. Compare with the flows (18)–(19) in [15]. By Remark 1.1, $h_t(z) = (1 - t z^k)^\gamma$ is solution of (1.1) (see Subsection 1.5.2). Let

$$\alpha(z) = \frac{d}{dt}\Big|_{t=0} k_t(z) = \frac{1}{k}\, z^{k+1}, \qquad \beta(z) = \frac{d}{dt}\Big|_{t=0} h_t(z) = -\gamma z^k.$$

As in (1.9), solving for real $R(z,\overline{z})$,

$$\beta(z) = \alpha'(z) + \alpha(z)\,\frac{\partial}{\partial z}\log R(z,\overline{z}),$$

we find $R(z,\overline{z}) = H(\theta)(z\overline{z})^{-(1+k(\gamma+1))}$ where $z = re^{i\theta}$ and

$$d\mu = (z\overline{z})^{-k\gamma}\,\frac{dz\,d\overline{z}}{(z\overline{z})^{k+1}} \quad \text{is unitarizing for} \quad T_t f(z) = (1 - t z^k)^\gamma f(k_t(z)).$$

We have $\operatorname{div}_\mu(z^{k+1}\,\frac{\partial}{\partial z}) = -k\gamma z^k$ and $d\mu$ is invariant with respect to

$$k^2\Delta = (z\overline{z})^{k+1}\Big[\frac{\partial^2}{\partial z\partial\overline{z}} - \frac{k\gamma}{z}\,\frac{\partial}{\partial\overline{z}}\Big]. \tag{2.4}$$

3. The multiplicative group $(\mathbb{R}^* = \mathbb{R} - 0, \times)$. Assume $t \neq 0$. The system (1.1)(1.2) becomes

$$h_{t_1}(z)h_{t_2}(k_{t_1}(z)) = h_{t_1 t_2}(z) \quad\text{and}\quad k_{t_2}(k_{t_1}(z)) = k_{t_1 t_2}(z). \tag{3.1}$$

We take $k_t(z) = tz$, then

$$h_{t_1}(z)h_{t_2}(t_1 z) = h_{t_1 t_2}(z). \tag{3.2}$$

Lemma 3.1. *The function* $h_t(z) = e^{a(t-1)z}$, *where* $a \in \mathbb{C}$, *is a solution of* (3.2). *Solving* (3.1), *we have from* (c)

$$(T_t f)(z) = e^{a(t-1)z} f(tz)$$

with $t \neq 0$ *and* a *constant. For the neutral* $t = 1$, $T_1 f(z) = f(z)$. *The operator* T_t *is invertible* $(T_t)^{-1} f(z) = T_{1/t} f(z)$.

Proof. Putting $t_1 = 0$ in (3.2) and assuming that $h_0(z) \neq 0$, we find $h_t(0) = 1$ for any t. We look for a solution of (3.2) in the form $h_t(z) = e^{\alpha(t)z}$ where $\alpha(t)$ is a function of t. The equation for $\alpha(t)$ is

$$\alpha(t_1) + t_1 \alpha(t_2) = \alpha(t_1 t_2) \qquad \forall\, t_1, t_2 \neq 0. \tag{3.3}$$

If $t_1 = t_2$, it becomes $(1+t)\alpha(t) = \alpha(t^2)$. A solution is $\alpha(t) = a(t-1)$ where a is a constant and this is also a solution of (3.3). This gives $h_t(z) = e^{a(t-1)z}$ where $a \in \mathbb{C}$. ∎

Lemma 3.2. *Let* $\alpha \in \mathbb{C}$, $\lambda, \beta, \delta \in \mathbb{R}$. *On* $\mathcal{D} = \{(x,y) \mid \lambda x^2 + \beta xy + \delta y^2 \neq 0\}$, *the real measures*

$$d\mu_{\alpha,A,B,\gamma}(z) = \frac{e^{az+\overline{az}}\, dx\, dy}{(\lambda x^2 + \beta xy + \delta y^2)^{1-\gamma}} = \frac{e^{az+\overline{az}}\, dz\, d\overline{z}}{(Az^2 + 2Bz\overline{z} + \overline{A}\overline{z}^2)^{1-\gamma}}, \qquad z = x+iy, \tag{3.4}$$

are unitarizing for T_t *with* $t \neq 0$,

$$(T_t f)(z) = t^\gamma\, e^{\alpha(t-1)z} f(tz). \tag{3.5}$$

The constant B *is real. The infinitesimal representation associated to* (T_t) *is*

$$\rho(1) = z\, \frac{\partial}{\partial z} + (az + \gamma) I \quad \textit{and} \quad (If)(z) = f(z). \tag{3.6}$$

Proof. We find the measure μ by writing (b), i.e. $\int |T_t f(z)|^2\, d\mu = \int |f(z)|^2\, d\mu$. Assume that $\mu = g(x,y)\, dx\, dy$. For unitarity, we must have

$$t^{2\gamma} \int e^{(t-1)(\alpha z + \overline{\alpha z})} |f(tz)|^2 g(x,y)\, dx\, dy = \int |f(z)|^2 g(x,y)\, dx\, dy.$$

With the change of variables $x' = tx$, $y' = ty$, $z' = tz$, the integral is equal to

$$t^{2\gamma} \int e^{(1-1/t)(\alpha z + \overline{\alpha z})} |f(z)|^2 g\Big(\frac{x}{t}, \frac{y}{t}\Big)\, \frac{1}{t^2}\, dx\, dy.$$

We deduce that

$$e^{-(\alpha z + \overline{\alpha z})/t} g\Big(\frac{x}{t}, \frac{y}{t}\Big)\, \frac{1}{t^{2-2\gamma}} = e^{-(\alpha z + \overline{\alpha z})} g(x,y).$$

This identity shows that the function $\psi(x,y) = e^{-(\alpha z + \overline{\alpha z})} g(x,y)$ is homogeneous of degree $-2+2\gamma$ and we can take $\psi(x,y) = \text{constant}\, [\lambda x^2 + \beta xy + \delta y^2]^{\gamma-1}$. The density is $g(x,y) = \text{constant} \times e^{(\alpha z + \overline{\alpha z})} \times [\lambda x^2 + \beta xy + \delta y^2]^{\gamma-1}$ with $z = x+iy$. Then we calculate $(\rho(1)f)(z) = \frac{d}{dt}\big|_{t=1} (T_t f)(z) = z f'(z) + (az+\gamma) f(z)$. ∎

Remark 3.3. With the notation from Lemma 1.3, we put $\alpha(z) = z$,

$$R(z, \overline{z}) = \frac{e^{az+\overline{az}}}{(Az^2 + 2Bz\overline{z} + \overline{A}\overline{z}^2)^{1-\gamma}},$$

$$\Gamma(z,\overline{z}) = \alpha'(z) + \alpha(z)\, \frac{\partial}{\partial z} \log R(z,\overline{z}) = az + \gamma + (\gamma - 1) z\, \frac{\partial}{\partial z} \log\Big(A\, \frac{z}{\overline{z}} + 2B + \overline{A}\, \frac{\overline{z}}{z}\Big).$$

In polar coordinates $z = re^{i\theta}$, $z\,\frac{\partial}{\partial z} + \overline{z}\,\frac{\partial}{\partial \overline{z}} = r\,\frac{\partial}{\partial r}$, thus the term with log cancels in $\Gamma + \overline{\Gamma}$. We obtain $\Re\Gamma(z,\overline{z}) = \Re(az+\gamma)$ and we deduce $\beta(z) = az + \gamma$.

The case $\beta(z) = \Gamma(z,\overline{z})$ occurs when $A = 0$, then we consider $d\mu_{\alpha,\gamma} = d\mu_{\alpha,0,1,\gamma}$,

$$d\mu_{\alpha,\gamma} = \frac{e^{\alpha z + \overline{\alpha z}}}{(x^2+y^2)^{1-\gamma}}\,dx\,dy.$$

We define $H_1 = z\,\frac{\partial}{\partial z}$ as the first order term in $\rho(1)$. The measure $d\mu_{\alpha,\gamma}$ is invariant with respect to

$$\Delta = \rho(1)\overline{H_1} = z\overline{z}\Big[\frac{\partial^2}{\partial z \partial \overline{z}} + a\,\frac{\partial}{\partial \overline{z}}\Big] + \gamma\,\overline{z}\,\frac{\partial}{\partial \overline{z}}\,.$$

Since $k_t(z) = tz$, the group $(\mathbb{R}^*, \times)$ acts on $\mathbb{R}^2$ by the homothetic transformations $z \mapsto tz$. The metric $ds^2 = \frac{dx^2+dy^2}{x^2+y^2}$ on $\mathbb{R}^2$ is invariant under the transformations $k_t(z) = tz$.

Combining with Section 1.5, we have

THEOREM 3.4. *Let φ be a holomorphic function, and define*

$$T_t f(z) = t^{\gamma}\, e^{\varphi(tz)-\varphi(z)} f(tz) \quad \textit{for} \quad t \in \mathbb{R}^*. \tag{3.7}$$

We have $T_1 f(z) = f(z)$, $T_t^{-1} f(z) = t^{-\gamma}\, e^{\varphi(z/t)-\varphi(z)} f(z/t)$ and $T_{t_1}T_{t_2} = T_{t_1 t_2}$. The infinitesimal representation is

$$\rho(1) = z\,\frac{\partial}{\partial z} + [z\varphi'(z) + \gamma]I.$$

The measure

$$d\mu(z) = e^{\varphi(z)+\overline{\varphi(z)}}\,(x^2+y^2)^{\gamma}\,\frac{dx\,dy}{(x^2+y^2)}\,, \quad z = x+iy, \tag{3.8}$$

is unitarizing for T_t, i.e. $\int |T_t f(z)|^2\,d\mu = \int |f(z)|^2\,d\mu$ and $d\mu$ is an invariant measure with respect to the complex operator

$$\Delta = \rho(1)\overline{H_1} = z\overline{z}\Big[\frac{\partial^2}{\partial z \partial \overline{z}} + \varphi'(z)\,\frac{\partial}{\partial \overline{z}} + \frac{\gamma}{z}\,\frac{\partial}{\partial \overline{z}}\Big]. \tag{3.9}$$

The real measure $d\mu$ is also invariant with respect to $\Delta^{OU} = \Delta + \overline{\Delta}$. The vector field $V = \Delta - \overline{\Delta}$ satisfies $\operatorname{div}_{\mu}(V) = 0$,

$$V = z\overline{z}\Big[\varphi'(z)\,\frac{\partial}{\partial \overline{z}} - \overline{\varphi'(z)}\,\frac{\partial}{\partial z}\Big] + \gamma\Big[\overline{z}\,\frac{\partial}{\partial \overline{z}} - z\,\frac{\partial}{\partial z}\Big]. \tag{3.10}$$

4. The group of 2×2 invertible matrices. The results of Sections 2 and 3 are consequences of the following. Consider the group G of 2×2 complex invertible matrices

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad g^{-1} = \frac{1}{\det g}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{4.1}$$

with the usual multiplication

$$g_1 g_2 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{pmatrix}. \tag{4.2}$$

If we put

$$\psi_g(z) = \frac{az+b}{cz+d}\,, \tag{4.3}$$

then

$$\psi_{g_1g_2}(z) = \frac{(a_1a_2+b_1c_2)z+(a_1b_2+b_1d_2)}{(c_1a_2+d_1c_2)z+(c_1b_2+d_1d_2)} = \psi_{g_1}(\psi_{g_2}(z)). \tag{4.4}$$

Let

$$k_g(z) = \psi_{g^{-1}}(z) = \frac{dz-b}{-cz+a}, \quad k'_g(z) = \frac{d}{dz}k_g(z) = \frac{\det g}{(-cz+a)^2}. \tag{4.5}$$

We verify that k_g satisfies (1.2) and that $\widehat{h}_g(z) = -cz+a$ satisfies (1.1) (see (4.5)). Another solution of (1.1) is $h_g(z) = \det g$. We express $\widehat{h}_g(z) = -cz+a$ in terms of $k'_g(z)$ and $\det g$. Consider

$$(T_gf)(z) = (\det g)^\nu [k'_g(z)]^\gamma \, \frac{k_g(z)^n}{z^n} \, e^{\varphi(k_g(z))-\varphi(z)} f(k_g(z)) \tag{4.6}$$

with $k_g(z) = \frac{dz-b}{-cz+a}$. As in Lemma 1.3, let

$$\alpha(z) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} k_{g_\epsilon}(z) \quad \text{where} \quad \frac{d}{d\epsilon}\Big|_{\epsilon=0} g_\epsilon = V.$$

The infinitesimal representation is $\rho(V)f(z) = \alpha(z)f'(z) + \beta(z)f(z)$ with

$$\beta(z) = \nu\, \frac{d}{d\epsilon}\Big|_{\epsilon=0} (\det g_\epsilon) + \gamma\alpha'(z) + \Big[\varphi'(z) + \frac{n}{z}\Big]\alpha(z). \tag{4.7}$$

Assume that the representation (4.6) is unitary with respect to $R(z,\overline{z})\,dz\,d\overline{z}$. As in (1.9), we put

$$\Gamma(z,\overline{z}) = \alpha'(z) + \alpha(z)\,\frac{\partial}{\partial z}\log R(z,\overline{z}). \tag{4.8}$$

4.1. The domain $\mathcal{D}$ is the complex plane or the complex half-plane. In this section, we also consider equivalent representations, see Subsection 1.5.1.

Lemma 4.1. *For $g \in G$, let $(T_gf)(z)$ be as in (4.6), then we have $T_{g_1}T_{g_2} = T_{g_1g_2}$. The infinitesimal representation is given by*

$$\begin{aligned}
\rho_1 &= \rho(a) = -z\,\frac{\partial}{\partial z} + [\nu - \gamma - n - z\varphi'(z)],\\
\rho_2 &= \rho(b)f(z) = -\frac{\partial}{\partial z} - \frac{n}{z} - \varphi'(z),\\
\rho_3 &= \rho(c) = z^2\,\frac{\partial}{\partial z} + [(2\gamma+n)z + z^2\varphi'(z)],\\
\rho_4 &= \rho(d) = z\,\frac{\partial}{\partial z} + [\nu + \gamma + n + z\varphi'(z)].
\end{aligned} \tag{4.9}$$

Proof. To obtain (4.9), we take small variations of the coefficients $a_\epsilon = a+\epsilon, \ldots$ and we calculate the partial derivatives $(\partial_a)\big|_{g=e} T_gf(z)$ at $g = e = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$. ■

Corollary 4.2. *Let ρ_j, $j = 1,2,3,4$, be as in Lemma 4.1, and α_j, β_j be defined by $\rho_j = \alpha_j(z)\,\frac{\partial}{\partial z} + \beta_j(z)$. Then*

(i) $\alpha_3\alpha_2 + \alpha_2\alpha_3 + 2\alpha_1^2 = 0$ *and* $\alpha_3\overline{\alpha_2} + \alpha_2\overline{\alpha_3} + 2\alpha_1\overline{\alpha_1} = -(z-\overline{z})^2$,
(ii) $\alpha_4\alpha_2 - \alpha_2\alpha_4 = 0$ *and* $\alpha_4\overline{\alpha_2} - \alpha_2\overline{\alpha_4} = \overline{z} - z$,
(iii) $\alpha_1\alpha_3 - \alpha_3\alpha_1 = 0$ *and* $\alpha_1\overline{\alpha_3} - \alpha_3\overline{\alpha_1} = z\overline{z}(z-\overline{z})$.

The conditions (1.23) *in Theorem* 1.10 *are satisfied for* (i), (ii) *or* (iii).

In order to write the system (1.10) to find the density $R(z,\overline{z})$, we put $\Gamma_j = \alpha'_j + \alpha_j \frac{\partial}{\partial z} \log R$ as in (1.9). We assume that $\varphi(z)=0$. We deduce from Lemma 4.1

$$\begin{aligned} &\beta_1-\Gamma_1 = \nu-(\gamma+n-1)+z\,\frac{\partial}{\partial z}\log R, \quad \beta_2-\Gamma_2=-\frac{n}{z}+\frac{\partial}{\partial z}\log R,\\ &\beta_3-\Gamma_3=(2\gamma+n-2)z-z^2\,\frac{\partial}{\partial z}\log R, \quad \beta_4-\Gamma_4=\nu+\gamma+n-1-z\,\frac{\partial}{\partial z}\log R. \end{aligned} \tag{4.10}$$

Lemma 4.3. *Assume $\nu=0$. The system $\Re(\beta_j-\Gamma_j)=0$ for $j=1,\dots,4$ (see (1.10)) has the solution*

$$R = \text{constant}\,(z-\overline{z})^{2(\gamma-1)}(z\overline{z})^n. \tag{4.11}$$

For this solution $R(z,\overline{z})$, letting $z=re^{i\theta}$, we have

$$\begin{aligned} &\beta_1-\Gamma_1=-i(\gamma-1)\,\frac{\cos\theta}{\sin\theta}=\frac{(\gamma-1)(z+\overline{z})}{z-\overline{z}}, \qquad \beta_2-\Gamma_2=-i\,\frac{\gamma-1}{r\sin\theta}=\frac{2(\gamma-1)}{z-\overline{z}},\\ &\beta_3-\Gamma_3=i\,\frac{(\gamma-1)r}{\sin\theta}=-\frac{2(\gamma-1)z\overline{z}}{z-\overline{z}}, \qquad \beta_4-\Gamma_4=i(\gamma-1)\,\frac{\cos\theta}{\sin\theta}=-(\beta_1-\Gamma_1). \end{aligned}$$

Proof. With $\Re(\beta_j-\Gamma_j)=0$ for $j=1$ and $j=4$, we obtain $\Re(\nu)=0$ and

$$\Big(z\,\frac{\partial}{\partial z}+\overline{z}\,\frac{\partial}{\partial\overline{z}}\Big)\log R = 2(\gamma+n-1).$$

Let $z=re^{i\theta}$. It gives $R=H(\theta)r^{2(\gamma+n-1)}$. We replace in $\Re(\beta_j-\Gamma_j)=0$, $j=2,3$,

$$\begin{aligned} &\Big[\cos\theta\,\frac{\partial}{\partial r}-\frac{1}{r}\sin\theta\,\frac{\partial}{\partial\theta}\Big]\log R = 2n\,\frac{\cos\theta}{r}\,,\\ &\Big[\cos\theta\,\frac{\partial}{\partial r}+\frac{1}{r}\sin\theta\,\frac{\partial}{\partial\theta}\Big]\log R = 2(2\gamma+n-2)\,\frac{\cos\theta}{r}\,. \end{aligned}$$

We deduce (4.11). ■

4.1.1. *Holomorphic representation of $Gl(2,\mathbb{R})$.* When R is given by (4.11), we have

$$(\beta_3-\Gamma_3)\overline{\alpha_2}+(\beta_2-\Gamma_2)\overline{\alpha_3}+2(\beta_1-\Gamma_1)\overline{\alpha_1}=0. \tag{4.12}$$

The conditions (1.26) of Theorem 1.11 are satisfied. With (i) in Corollary 4.2, Theorem 1.11 and Lemma 4.3, we obtain

Theorem 4.4. *Assume that a, b, c, d are real numbers (and that 2γ is an integer). Let*

$$\begin{aligned} &(T_g f)(z) = [k'_g(z)]^\gamma\,\frac{k_g(z)^n}{z^n}\,f(k_g(z)) \quad \textit{with} \quad k_g(z)=\frac{dz-b}{-cz+a}\,,\\ &d\mu=(z-\overline{z})^{2\gamma}\,(z\overline{z})^n\,dv \quad \textit{with} \quad dv=\frac{dz\,d\overline{z}}{(z-\overline{z})^2}\,1_{\{z\neq\overline{z}\}}\,. \end{aligned} \tag{4.13}$$

The measure μ is unitarizing for T_g. Let $H_a=-z\,\frac{\partial}{\partial z}$, $H_b=-\frac{\partial}{\partial z}$, $H_c=z^2\,\frac{\partial}{\partial z}$, $H_d=z\,\frac{\partial}{\partial z}$ and let ρ be the infinitesimal representation as in Lemma 4.1 (with $\nu=0$, $\varphi=0$). Then μ is an invariant measure for

$$\Delta=\rho(c)\overline{H_b}+\rho(b)\overline{H_c}+2\rho(a)\overline{H_a}=-(z-\overline{z})^2\Big[\frac{\partial^2}{\partial z\partial\overline{z}}+\Big(\frac{2\gamma}{z-\overline{z}}+\frac{n}{z}\Big)\frac{\partial}{\partial\overline{z}}\Big], \tag{4.14}$$

or, equivalently,

$$\Delta=-(z-\overline{z})^2\Big[\frac{\partial^2}{\partial z\partial\overline{z}}+\frac{\partial Q}{\partial z}\,\frac{\partial}{\partial\overline{z}}\Big] \quad \textit{with} \quad Q=\log[(z-\overline{z})^{2\gamma}(z\overline{z})^n]. \tag{4.15}$$

Proof. With (4.11), we obtain the unitarizing measure $d\mu = \text{constant} \times (z-\overline{z})^{2\gamma}(z\overline{z})^n\, dv$. In that case, the domain of integration is $\mathcal{D} = \{z \in \mathbb{C} \,|\, \Im z \neq 0\}$. If a, b, c, d are real numbers and the imaginary part $\Im z \neq 0$, then $k_g(z)$ in (4.6) is well defined and k_g is a map from $\mathcal{D}$ to $\mathcal{D}$. The expression (4.14) is a consequence of (4.12) and Theorem 1.10. Compare (4.14) with Definition 1.14. ∎

Corollary 4.5. *Let Δ be as in Theorem 4.4, we have*

$$\Delta = -\Big(\frac{\partial}{\partial z} + \frac{2(\gamma-1)}{z-\overline{z}} + \frac{n}{z}\Big)\Big((z-\overline{z})^2 \frac{\partial}{\partial \overline{z}}\Big).$$

Moreover,

$$\Delta + \overline{\Delta} = 2y^2\Big[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 4\,\frac{\gamma}{y}\frac{\partial}{\partial y} + \frac{2n}{x^2+y^2}\Big(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\Big)\Big]$$

has μ as invariant measure and we have $\operatorname{div}_\mu(V) = 0$ *for the vector field*

$$V = \Delta - \overline{\Delta} = -4i\gamma y\,\frac{\partial}{\partial x} - \frac{4iny^2}{x^2+y^2}\Big(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\Big). \tag{4.16}$$

Remark 4.6. If we consider the subgroup of matrices g such that $\det g > 0$ and the domain $\mathcal{D}_1 = \{z \in \mathbb{C} \,|\, \Im z > 0\}$, then k_g is a map from $\mathcal{D}_1$ to $\mathcal{D}_1$. As in Subsection 1.5.1, consider the holomorphic function $\psi(z) = e^{(i-1)z}$. We have $|\psi(z)|^2 = e^{-2(x+y)}$ with $z = x+iy$. Assume that $d\mu = dx\,dy$, then $d\mu^\psi = e^{-2(x+y)}\,dx\,dy$. The functions $1_{z+\overline{z}>0}(z-\overline{z})^n$ are integrable for $d\mu^\psi$ but not for $d\mu$.

Remark 4.7. If $\gamma = 1$, then $d\mu = (z\overline{z})^n\,dz\,d\overline{z}$ is unitarizing for T_g, the domain $\mathcal{D}$ is the complex plane, we do not need the restriction of the assumption that a, b, c, d are real. Since $\beta_j - \Gamma_j = 0$, by Theorem 1.11, μ is invariant for any $\Delta = \sum_{j,k} A_{jk}\rho_j\overline{H_k}$ where A_{jk} are arbitrary constants, $j,k = 1,\dots,4$. For example, for $\gamma = 1$, $n = 0$, it is immediate that

$$\rho(a)\overline{H_a} + \rho(b)\overline{H_b} + \rho(c)\overline{H_c} + \rho(d)\overline{H_d} = (1+z\overline{z})^2\frac{\partial^2}{\partial z\partial\overline{z}} + 2(1+z\overline{z})\overline{z}\frac{\partial}{\partial\overline{z}} \tag{4.17}$$

has $dz\,d\overline{z}$ as invariant measure.

Remark 4.8. The case where the metric on $\mathcal{D}$ is not Hermitian. If we do not require Δ to be of the form (1.21) and allow

$$\Delta = w(z,\overline{z})\,\frac{\partial^2}{\partial \overline{z}^2} + u(z,\overline{z})\,\frac{\partial^2}{\partial z\partial\overline{z}} + v(z,\overline{z})\,\frac{\partial}{\partial\overline{z}}$$

with $w(z,\overline{z}) \neq 0$, there are many differential operators Δ with $\mu = (z-\overline{z})^{2(\gamma-1)}(z\overline{z})^n\,dz\,d\overline{z}$ as invariant measure. However, the principal symbol in such operators does not correspond to a Hermitian metric on $\mathbb{C}$. Examples of such Δ are given by 1.25,

$$[\rho(a) + \overline{\rho(a)}]\overline{H_a} = z\overline{z}\frac{\partial^2}{\partial z\partial\overline{z}} + \overline{z}^2\frac{\partial^2}{\partial\overline{z}^2} + (2\gamma+2n+1)\overline{z}\frac{\partial}{\partial\overline{z}}, \quad \dots$$

4.1.2. *The subgroup of transformations $k_g(z) = tz - b$ where t, b are complex numbers.* We consider the relation (ii) in Corollary 4.2. If R is given by (4.11), we verify that

$$(\beta_4 - \Gamma_4)\overline{\alpha_2} - (\beta_2 - \Gamma_2)\overline{\alpha_4} = -(\gamma - 1). \tag{4.18}$$

We cannot apply Theorem 1.11, we find an OU-operator with Theorem 1.10.

THEOREM 4.9. *Consider the group of transformations of the complex plane defined by* $k_g(z) = tz - b$ *where* t *is real,* $t \neq 0$ *and* b *is a complex number. Assume that* $2\gamma - 1 \geq 0$. *Let*

$$T_g f(z) = t^\gamma f(tz - b).$$

Consider the Hilbert space $\mathcal{H}$ *of entire functions* f *such that*

$$\|f\|^2 = \int |f(x+iy)|^2 y^s \, dx \, dy < +\infty \quad \text{for} \quad 0 \leq s \leq 2\gamma - 1.$$

Then the measure

$$d\mu = (z - \overline{z})^{2(\gamma-1)} \, dz \, d\overline{z}$$

satisfies the unitarity condition $\int |T_g f(z)|^2 \, d\mu = \int |f|^2 \, d\mu$ *for all* $f \in \mathcal{H}$ *and it is an invariant measure for*

$$\Delta = (\rho(t) + \overline{\rho(t)})\overline{H_b} - (\rho(b) + \overline{\rho(b)})\overline{H_t} = (\overline{z} - z)\frac{\partial^2}{\partial z \partial \overline{z}} - (2\gamma - 1)\frac{\partial}{\partial \overline{z}}$$

where $\rho(t) = \rho_4 = z\frac{\partial}{\partial z} + \gamma$ *and* $\rho(b) = \rho_2 = -\frac{\partial}{\partial z}$ *as in Lemma* 1.4.

Proof. If $f \in \mathcal{H}$, then for any real t and complex number b, the function $|f(tz - b)|^2$ is integrable for μ. ∎

4.1.3. *The subgroup of transformations* $k_g(z) = \frac{z}{-cz+a}$ *where* c, a *are complex numbers and* $a \neq 0$. We apply Theorem 1.10 with Corollary 4.2(iii). We find

$$(\beta_1 - \Gamma_1)\overline{\alpha_3} - (\beta_3 - \Gamma_3)\overline{\alpha_1} = -(\gamma - 1)\overline{z}^2.$$

THEOREM 4.10. *Consider the group of transformations*

$$k_g(z) = \frac{z}{-cz + a}$$

and let

$$T_g f(z) = \frac{a^\gamma}{(-cz + a)^{2\gamma}} f\Big(\frac{z}{-cz + a}\Big).$$

The measure $d\mu = (z - \overline{z})^{2(\gamma-1)} \, dz \, d\overline{z}$ *is unitarizing for* T_g, *i.e. for any holomorphic function such that* $|f(z)|^2$ *and* $T_g f(z)|^2$ *are integrable for* μ, *we have* $\int |T_g f(z)|^2 \, d\mu = \int |f(z)|^2 \, d\mu$. *The measure* μ *is invariant for*

$$\Delta = z\overline{z}(z - \overline{z})\frac{\partial^2}{\partial z \partial \overline{z}} - \overline{z}^2\frac{\partial}{\partial \overline{z}} + 2\gamma z\overline{z}\frac{\partial}{\partial \overline{z}} \, .$$

Proof. We apply Theorem 1.10. As in Subsection 4.1.2, the difficulty is to determine a class of holomorphic functions f such that $|f(k_g(z))|^2$ is integrable for μ for any c and a, $a \neq 0$. ∎

4.2. The subgroups A_p and G_p. If we require conditions on the parameters a, b, c, d, in order to have a subgroup of G, this gives restrictions on T_g; thus for a subgroup of G, we have more unitarizing measures since there are fewer equations in the system (1.10). For example, for the commutative subgroup of diagonal matrices $g = \left(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right)$ with $d \neq 0$, we obtain $k_g(z) = dz$ and we are in the case of Section 3, if d is a real number.

NOTATION 4.11. Given a fixed real number p, we denote by A_p the group constituted with matrices

$$g = \begin{pmatrix} a & b \\ p\,\overline{b} & \overline{a} \end{pmatrix} \quad \textit{with} \quad \det g = a\overline{a} - pb\overline{b} > 0.$$

We denote by G_p the subgroup of matrices in A_p with determinant equal to one.

Since we do not fix the value of the determinant for elements in A_p, the Lie algebra $\mathcal{A}_p$ of A_p is a four-dimensional real vector space. Fixing the determinant equal to one for elements in G_p takes off one degree of freedom, then the Lie algebra $\mathcal{G}_p$ of G_p is a three-dimensional real vector space. In the following, we consider representations of G_p. Because of 1.5.1, we assume $n = 0$ and $\varphi = 0$ in (4.6). Since $k'_g(z) = (a\overline{a} - pb\overline{b})[-p\overline{b}\,z + a]^{-2}$, the representation (4.6) for G_p is written as

$$(T_g f)(z) = \frac{1}{(-p\overline{b}\,z + a)^{2\gamma}}\, f\Big(\frac{\overline{a}z - b}{-p\overline{b}z + a}\Big). \tag{4.19}$$

To calculate the infinitesimal representation of $\mathcal{G}_p$, for $t > 0$, t small, we consider the following three curves $g_t^{(j)}$, $j = 1, 2, 3$, which are in G_p and $g_0^{(j)}$ = Identity of G_p. The vectors $e_j = \frac{d}{dt}\big|_{t=0} g_t^{(j)}$, $j = 1, 2, 3$, form a basis of the Lie algebra $\mathcal{G}_p$.

$$\begin{aligned}
g_t^{(1)} &= \begin{pmatrix} e^{it/2} & 0 \\ 0 & e^{-it/2} \end{pmatrix}, & e_1 &= \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \\
g_t^{(2)} &= \frac{1}{\sqrt{\cos^2(t/2) - p\sin^2(t/2)}} \begin{pmatrix} \cos(t/2) & \sin(t/2) \\ p\sin(t/2) & \cos(t/2) \end{pmatrix}, & e_2 &= \frac{1}{2}\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}, \\
g_t^{(3)} &= \frac{1}{\sqrt{\cos^2(t/2) - p\sin^2(t/2)}} \begin{pmatrix} \cos(t/2) & i\sin(t/2) \\ -ip\sin(t/2) & \cos(t/2) \end{pmatrix}, & e_3 &= \frac{1}{2}\begin{pmatrix} 0 & i \\ -ip & 0 \end{pmatrix}.
\end{aligned} \tag{4.20}$$

Then $[e_1, e_2] = e_3$, $[e_1, e_3] = -e_2$, $[e_2, e_3] = pe_1$ and $\{e, e_1, e_2, e_3\}$ is a basis of $\mathcal{A}_p$.

LEMMA 4.12. *The infinitesimal representation for $\mathcal{G}_p$ associated to* (4.19) *is given by*

$$\begin{aligned}
\rho(e_1)f(z) &= -i[zf'(z) + \gamma f(z)], \\
\rho(e_2)f(z) &= -\frac{1}{2}(1 - pz^2)f'(z) + \gamma pzf(z), \\
\rho(e_3)f(z) &= -\frac{i}{2}(1 + pz^2)f'(z) - i\gamma pzf(z).
\end{aligned}$$

In the following, we put $\rho_j = \rho(e_j) = \alpha_j(z)\frac{\partial}{\partial z} + \beta_j(z)$. We solve the system (1.9)–(1.10) with

$$\begin{aligned}
\beta_1 - \Gamma_1 &= -i(\gamma - 1) + iz\,\frac{\partial}{\partial z}\log R, \\
\beta_2 - \Gamma_2 &= (\gamma - 1)pz + \frac{1}{2}(1 - pz^2)\,\frac{\partial}{\partial z}\log R, \\
\beta_3 - \Gamma_3 &= -i(\gamma - 1)pz + \frac{i}{2}(1 + pz^2)\,\frac{\partial}{\partial z}\log R.
\end{aligned}$$

We obtain $R(z, \overline{z}) = \text{constant}\,(1 - pz\overline{z})^{2\gamma - 2}$. With the expression of R,

$$\beta_1 - \Gamma_1 = -i(\gamma - 1)\,\frac{1 + pz\overline{z}}{1 - pz\overline{z}}, \quad \beta_2 - \Gamma_2 = (\gamma - 1)\,\frac{p(z - \overline{z})}{1 - pz\overline{z}}, \quad \beta_3 - \Gamma_3 = -i(\gamma - 1)\,\frac{p(z + \overline{z})}{1 - pz\overline{z}}.$$

4.2.1. *Domains* $\mathcal{D}_p = \{1 - pz\overline{z} \geq 0\}$. We have

$$\alpha_2(z) = \overline{\alpha_2(z)} + \alpha_3(z)\overline{\alpha_3(z)} - p\alpha_1(z)\overline{\alpha_1(z)} = \frac{1}{2}(1 - pz\overline{z})^2$$

$$\text{and} \quad \alpha_2^2 + \alpha_3^2 - p\alpha_1^2 = 0, \tag{4.21}$$

$$(\beta_2 - \Gamma_2)\alpha_2 + (\beta_3 - \Gamma_3)\alpha_3 - p(\beta_1 - \Gamma_1)\alpha_1 = 0,$$

and $\Delta_1 = \rho(e_2)H_2 + \rho(e_3)H_3 - p\rho(e_1)H_1 = 0$. The conditions (1.26) of Theorem 1.11 are satisfied. In the following lemma, we write the unitarizing measure in terms of the volume measure. We deduce

Lemma 4.13. *Let $H_j = \alpha_j(z)\frac{\partial}{\partial z}$, $j = 1, 2, 3$, and $\overline{H_j} = \overline{\alpha_j(z)}\frac{\partial}{\partial \overline{z}}$.*

$$H_1 f(z) = -izf'(z), \quad H_2 f(z) = -\frac{1}{2}(1 - pz^2)f'(z), \quad H_3 f(z) = -\frac{i}{2}(1 + pz^2)f'(z),$$

then $\Delta = \rho(e_2)\overline{H_2} + \rho(e_3)\overline{H_3} - p\rho(e_1)\overline{H_1}$ has the unitarizing measure

$$d\mu_p = (1 - pz\overline{z})^{2\gamma}\, \frac{dz\, d\overline{z}}{(1 - pz\overline{z})^2} \tag{4.22}$$

as invariant measure.

Theorem 4.14. *The domain $\mathcal{D}_p = \{z \in \mathbb{C} \,|\, 1 - pz\overline{z} > 0\}$ is invariant under the transformations $z \mapsto (az + b)(p\overline{b}z + \overline{a})^{-1}$. Let*

$$(T_g f)(z) = \frac{1}{(-p\overline{b}\, z + a)^{2\gamma}} \times f\Big(\frac{\overline{a}z - b}{-p\overline{b}z + a}\Big) \quad \text{for} \quad g \in G_p, \quad z \in \mathcal{D}_p. \tag{4.23}$$

Given g_1 and $g_2 \in G_p$, we have $T_{g_1}T_{g_2} = T_{g_1 g_2}$. Here, we assume that 2γ is an integer. If $g \in G_p$, the operator T_g is unitary in $L^2_{\mathrm{Hol}}(\mathcal{D}_p; \mu_p)$ where μ_p is given by (4.22). Let $\rho(e_j)_{j=1,2,3}$, H_j and Δ as in Lemma 4.13, we have

$$\Delta = \rho(e_2)\overline{H_2} + \rho(e_3)\overline{H_3} - p\rho(e_1)\overline{H_1} = \frac{1}{2}(1 - pz\overline{z})^2\Big[\frac{\partial^2}{\partial z \partial \overline{z}} - \frac{2\gamma p\, \overline{z}}{(1 - pz\overline{z})}\frac{\partial}{\partial \overline{z}}\Big]. \tag{4.24}$$

The measure μ_p is invariant for Δ and for $\Delta^{OU} = \Delta + \overline{\Delta}$. We have

$$\begin{aligned}\Delta^{OU} &= (1 - pz\overline{z})^2\Big[\frac{\partial^2}{\partial z\partial\overline{z}} + \gamma\frac{\partial}{\partial z}\log(1 - pz\overline{z})\frac{\partial}{\partial\overline{z}} + \gamma\frac{\partial}{\partial\overline{z}}\log(1 - pz\overline{z})\frac{\partial}{\partial z}\Big] \\ &= (1 - pz\overline{z})^2\Big[\frac{\partial^2}{\partial z\partial\overline{z}} + \gamma h'(z\overline{z})\Big(\overline{z}\frac{\partial}{\partial\overline{z}} + z\frac{\partial}{\partial z}\Big)\Big]\end{aligned} \tag{4.25}$$

with $h(z\overline{z}) = \log(1 - pz\overline{z})$. The vector field $V = \Delta - \overline{\Delta}$ is a free-divergence vector field ($\mathrm{div}_\mu(V) = 0$),

$$V = \Delta - \overline{\Delta} = -\gamma p(1 - pz\overline{z})\Big(\overline{z}\frac{\partial}{\partial\overline{z}} - z\frac{\partial}{\partial z}\Big). \tag{4.26}$$

Proof. This is a consequence of Theorem 1.10 and Lemma 4.13. Below, we show how to calculate R from the condition of unitarity (b). Let $\mu = R(z, \overline{z})\, dz\, d\overline{z}$ be a real measure on $\mathcal{D}_p$ such that (b) is realized. With a change of variable in (b), we obtain

$$(\det g)^{2\nu}\, |k_g'(k_g^{-1}(z))|^{2\gamma} \times R(k_g^{-1}(z), \overline{k_g^{-1}(z)}) \times |(k_g^{-1})'(z)|^2 = R(z, \overline{z}).$$

Since $(k_g^{-1})'(z) = 1/k_g'(k_g^{-1}(z))$, it gives

$$(\det g)^{2\nu}\, |(k_g^{-1})'(z)|^{2(1-\gamma)} \times R(k_g^{-1}(z), \overline{k_g^{-1}(z)}) = R(z, \overline{z}) \quad \forall\, g \in G_p.$$

If $\det g = 1$, then $R(z, \overline{z}) = (1 - pz\overline{z})^{2(\gamma-1)}$ is a solution. ■

5. The 3-dimensional Heisenberg group and the system (1.1)–(1.2)

5.1. Commutators in non-commutative groups. If the group G is not commutative, we compare $T_{g_1}T_{g_2}$ and $T_{g_2}T_{g_1}$ for $g_1,\, g_2 \in G$. We say that $g \in G$ is a commutator in G if g is of the form $g = c(g_1, g_2) = g_2^{-1}g_1g_2g_1^{-1}$ with $g_1,\, g_2 \in G$. In that case, we have $g_1g_2 = g_2c(g_1, g_2)g_1$. If this property is satisfied, according to (a), we obtain

$$T_{g_1g_2}T_{(g_2g_1)^{-1}} = T_{g_1g_2g_1^{-1}g_2^{-1}} = T_{g_1g_1^{-1}c(g_2,g_1^{-1})g_2g_2^{-1}} = T_{c(g_2,g_1^{-1})}$$

This gives $T_{g_1g_2} = T_{c(g_2,g_1^{-1})}T_{g_2g_1}$ and $T_{g_1g_2} = T_{g_2g_1}T_{c(g_2^{-1},g_1)}$. In the case of the Heisenberg group, see Remark 1.2, commutators are $g = (0, 0, ik)$ where k is a real number.

5.2. Representations where $k_g(z) = z + u(g)$ and $h_g(z) = \exp(l(g)z + m(g))$. Let γ and δ be two fixed constants such that $\gamma - \delta \neq 0$. We assume that γ and δ are real numbers. Consider on the 3-dimensional real space the group law

$$(a_1, b_1, c_1) * (a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, c_3)$$

with

$$c_3 = c_1 + c_2 + \begin{pmatrix} a_1 & b_1 \end{pmatrix} \begin{pmatrix} 0 & \gamma \\ \delta & 0 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = c_1 + c_2 + \gamma a_1b_2 + \delta a_2b_1. \tag{5.1}$$

The commutators (see Subsection 5.1) are given by $g_1g_2 = g_2c(g_1, g_2)g_1$ with

$$c(g_1, g_2) = (0, 0, (\gamma - \delta)(a_1b_2 - a_2b_1)).$$

Let $u = (\alpha, \beta)$ and $l = (\lambda, \epsilon)$ be two complex vectors in $\mathbb{C}^2$, we have

$$\det(u, l) = \alpha\epsilon - \lambda\beta.$$

For $g = (a, b, c)$, we put

$$u(g) = \alpha a + \beta b \quad \text{and} \quad l(g) = \lambda a + \epsilon b,$$
$$\det(u, l) = \alpha\epsilon - \lambda\beta.$$

We have $u(g_1)l(g_2) - u(g_2)l(g_1) = \det(u, l)(a_1b_2 - a_2b_1)$. We put

$$m(g) = m(a, b, c) = \frac{1}{2}l(g)u(g) + \det(u, l)\Big[\frac{1}{\gamma - \delta}\, c - \frac{\gamma + \delta}{2(\gamma - \delta)}ab\Big]. \tag{5.2}$$

Let

$$k_g(z) = z + u(g) \quad \text{and} \quad h_g(z) = \exp(l(g)z + m(g)), \tag{5.3}$$

then the conditions (1.1)–(1.3) are satisfied.

LEMMA 5.1. *We define*

$$T_gf(z) = (T_{a,b,t}f)(z) = \exp(l(g)z + m(g)) \times f(z + u(g)). \tag{5.4}$$

Then $T_{g_1}T_{g_2} = T_{g_1g_2}$. The infinitesimal representation for (5.4) *is*

$$\begin{aligned} \rho(a)f(z) &= \alpha f'(z) + \lambda z f(z), \\ \rho(b)f(z) &= \beta f'(z) + \epsilon z f(z), \\ \rho(c)f(z) &= \frac{\det(u, l)}{\gamma - \delta}\, f(z). \end{aligned} \tag{5.5}$$

In the next lemma, we determine the density $R(z, \overline{z})$ of the unitarizing measure μ.

Lemma 5.2. *Let* $\rho(j) = \alpha(j)\frac{\partial}{\partial z} + \beta(j)$ *with* $j = a, b, c$ *and* $\Gamma(j) = \frac{1}{R}\frac{\partial}{\partial z}(\alpha(j)R)$*, then*

$$\begin{aligned}\beta(a) - \Gamma(a) &= \lambda z - \alpha\frac{\partial}{\partial z}\log R,\\ \beta(b) - \Gamma(b) &= \epsilon z - \beta\frac{\partial}{\partial z}\log R,\\ \beta(c) - \Gamma(c) &= \frac{\det(u,l)}{\gamma-\delta}.\end{aligned} \tag{5.6}$$

Solving the system $\Re(\beta(j) - \Gamma(j)) = 0$*,* $j = a, b, c$*, we obtain*

(i) $\Re[\det(u,l)] = 0$ *where* $\det(u,l) = \alpha\epsilon - \lambda\beta$,

and letting $\alpha = \alpha_1 + i\alpha_2$*,* $\beta = \beta_1 + i\beta_2$*,*

(ii) $\det(u,\overline{u}) = \alpha\overline{\beta} - \overline{\alpha}\beta = -2i(\alpha_1\beta_2 - \alpha_2\beta_1) \neq 0$.

When (i) *and* (ii) *are satisfied, we have, up to an additive constant*

$$\log R = \frac{1}{2}\,\frac{\lambda\overline{\beta} - \epsilon\overline{\alpha}}{\alpha\overline{\beta} - \beta\overline{\alpha}}\,z^2 + \frac{\overline{\lambda}\,\overline{\beta} - \overline{\epsilon}\,\overline{\alpha}}{\alpha\overline{\beta} - \beta\overline{\alpha}}\,z\,\overline{z} + \frac{1}{2}\,\frac{\overline{\epsilon}\alpha - \overline{\lambda}\beta}{\alpha\overline{\beta} - \beta\overline{\alpha}}\,\overline{z}^2. \tag{5.7}$$

We put $\log R = Az^2 + 2Bz\overline{z} + \overline{A}\overline{z}^2 + 2Bz\overline{z} + \overline{A}\overline{z}^2$*. The constant* B *is real.*

Proof. Since $\beta(c) - \Gamma(c) = 0$, we have (ii). The system (1.10) becomes

$$\alpha\frac{\partial}{\partial z}\log R + \overline{\alpha}\frac{\partial}{\partial \overline{z}}\log R = \lambda z + \overline{\lambda}\overline{z}, \quad \beta\frac{\partial}{\partial z}\log R + \overline{\beta}\frac{\partial}{\partial \overline{z}}\log R = \epsilon z + \overline{\epsilon z}$$

and has a unique solution if $\alpha\overline{\beta} - \overline{\alpha}\beta \neq 0$. ∎

Corollary 5.3. *If the density* $E(z,\overline{z})$ *is given by* (5.7)*, we have*

$$\beta(a) - \Gamma(a) = 2B(\overline{\alpha}z - \alpha\overline{z}) \quad \beta(b) - \Gamma(b) = 2B(\overline{\beta}z - \beta\overline{z}), \quad \beta(c) - \Gamma(c) = \frac{\det(u,l)}{\gamma-\delta}$$

and $[\beta(a) - \Gamma(a)]\overline{\alpha(b)} - [\beta(b) - \Gamma(b)]\overline{\alpha(a)} = (\lambda\beta - \epsilon\alpha)\overline{z}$*.*

5.2.1. *Example.* Let $u(g) = ha + 2\tau b$, $l(g) = ib$, $\gamma = 1$ and $\delta = 0$, then with the notation of (5.2), $m(g) = i\tau b^2 + ihc$, $\lambda = 0$, $\epsilon = i$, $\alpha = h$, $\beta = 2\tau$. A group element (a, b, c) acts on holomorphic functions as

$$U_{h,\tau}(a,b,c)f(z) = \exp(ihc)(S_a \circ T_b f)(z) = \exp(ihc)\exp(i\tau b^2 + ibz)f(z + ah + 2\tau b),$$

where $(S_a f)(z) = f(z + ha)$ and $(T_b f)(z) = \exp(i\tau b^2 + ibz)f(z + 2\tau b)$, see [20, pages 6–7],

$$U_{h,\tau}(a_1,b_1,c_1) \circ U_{h,\tau}(a_2,b_2,c_2) = U_{h,\tau}(a_1 + a_2, b_1 + b_2, c_1 + c_2 + a_1 b_2).$$

$U_{h,\tau}(a,b,c)$ is unitary on the Hilbert space of entire functions on the complex plane such that $\|f\|_\tau^2 = \int_C \exp(-y^2/2t)|f(x+iy)|^2\,dx\,dy$ is finite, see [20, pages 6–7]. The infinitesimal representation is

$$\rho(a)f(z) = hf'(z), \quad \rho(b)f(z) = 2\tau f'(z) + izf(z), \quad \rho(c)f(z) = ihf(z).$$

We assume that a, b, c are real numbers and $\tau = it$ where t is a real number. We obtain that $R(z,\overline{z}) = \text{constant} \times \exp(-y^2/2t)$ and $\log R = (z - \overline{z})^2/(8t)$. The operator

$$\Delta^{OU} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{y}{t}\frac{\partial}{\partial y}$$

has $\exp(-y^2/2t)\,dz\,d\overline{z}$ as invariant measure. To give an expression of Δ^{OU} in terms of $\rho(a)$, $\rho(b)$ and $H_a = h\,\frac{\partial}{\partial z}$, $H_b = 2it\,\frac{\partial}{\partial z}$ is not immediate. Δ^{OU} is not equal to $\rho(a)\overline{H_a}+\rho(b)\overline{H_b} = (h^2+4t^2)\,\frac{\partial^2}{\partial z\partial\overline{z}} + 2zt\,\frac{\partial}{\partial\overline{z}}$. But we verify that

$$(\rho(b)+\overline{\rho(b)})(\rho(a)-\overline{\rho(a)}) - \frac{2it}{h}(\rho(a)+\overline{\rho(a)})^2 = -2ith\Big[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\frac{y}{t}\frac{\partial}{\partial y}\Big].$$

We can find Δ^{OU} in a systematic way with Theorem 1.10: As in Lemma 5.1, $\alpha(a) = h$, $\alpha(b) = 2it$, $\alpha(c) = 0$, then $\alpha(a)\alpha(b)-\alpha(b)\alpha(a) = 0$ and $\alpha(a)\overline{\alpha(b)}-\alpha(b)\overline{\alpha(a)} = -2ith$. By Theorem 1.10,

$$\Delta = (\rho(a)+\overline{\rho(a)})\overline{H_b} - (\rho(b)+\overline{\rho(b)})\overline{H_a} = -4ith\Big[\frac{\partial^2}{\partial z\partial\overline{z}}+\frac{1}{4t}(z-\overline{z})\frac{\partial}{\partial\overline{z}}\Big]$$

as well as $\Delta-\overline{\Delta} = -2ith\Delta^{OU}$ have $R\,dz\,d\overline{z}$ as invariant measure and the vector field

$$V = \Delta+\overline{\Delta} = -4ith(z-\overline{z})\Big(\frac{\partial}{\partial z}+\frac{\partial}{\partial\overline{z}}\Big)$$

satisfies $\operatorname{div}_\mu(V) = 0$. We also have

$$\beta(a)-\Gamma(a) = -h\frac{\partial}{\partial z}\log R = \frac{h(z-\overline{z})}{4t},\quad \beta(b)-\Gamma(b) = iz-2it\frac{\partial}{\partial z}\log R = i\Big(\frac{z+\overline{z}}{2}\Big)$$

and $\beta(c)-\Gamma(c) = ih$. Since $\overline{\alpha(a)}(\beta(b)-\Gamma(b)) - \overline{\alpha(b)}(\beta(a)-\Gamma(a)) = ih\overline{z} \neq 0$, we cannot apply Theorem 1.11.

5.2.2. *The representation* (5.4). Assume that γ, δ are real constants. The constants α, β, λ, ϵ are complex numbers. Let $\alpha = \alpha_1+i\alpha_2$, $\beta = \beta_1+i\beta_2$, $\lambda = \lambda_1+i\lambda_2$, $\epsilon = \epsilon_1+i\epsilon_2$ and α_1, β_1, λ_1, ϵ_1 are the real parts, α_2, β_2, λ_2, ϵ_2 are the imaginary parts. We have

THEOREM 5.4. *Consider the* 3*-dimensional* real *Heisenberg group G_1 with the group law* (5.1)*. Let T_g as in* (5.4)*.*

$$T_gf(z) = (T_{a,b,t}f)(z) = \exp(l(g)z+m(g))\times f(z+u(g)).$$

Assume that $\Re[\det(u,l)] = 0$ where $\det(u,l) = \alpha\epsilon-\lambda\beta$ and assume that $\det(u,\overline{u}) = \alpha\overline{\beta}-\overline{\alpha}\beta = -2i(\alpha_1\beta_2-\alpha_2\beta_1)\neq 0$. Then T_g is a holomorphic unitary representation for the group G_1 on $L^2(\mu)$,

$$\int |T_gf(z)|^2\,d\mu(x,y) = \int |f(z)|^2\,d\mu(x,y), \qquad z = x+iy,$$

where $d\mu(x,y) = \exp(Q(x,y))\,dx\,dy$ and the quadratic form $Q(x,y)$ is

$$Q(x,y) = \frac{\lambda_1\beta_2-\alpha_2\epsilon_1}{\alpha_1\beta_2-\alpha_2\beta_1}x^2 + \frac{(\alpha_1\epsilon_1+\alpha_2\epsilon_2)-(\lambda_1\beta_1+\lambda_2\beta_2)}{\alpha_1\beta_2-\alpha_2\beta_1}xy + \frac{\lambda_2\beta_1-\alpha_1\epsilon_2}{\alpha_1\beta_2-\alpha_2\beta_1}y^2. \quad (5.8)$$

The real measure $d\mu$ is an invariant measure for

$$\Delta^{OU} = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\Big(\frac{\partial Q}{\partial x}\Big)\frac{\partial}{\partial x}+\Big(\frac{\partial Q}{\partial y}\Big)\frac{\partial}{\partial y}. \quad (5.9)$$

We have

$$\Delta^{OU} = \frac{2}{\overline{\alpha}\beta-\alpha\overline{\beta}}\times\big[(\rho(a)+\overline{\rho(a)})(H_b-\overline{H_b})+(\rho(b)+\overline{\rho(b)})(\overline{H_a}-H_a)\big] \quad (5.10)$$

where $H_a = \alpha\frac{\partial}{\partial z}$, $H_b = \beta\frac{\partial}{\partial z}$. *In particular,* Δ^{OU} *does not depend on the constants* δ *and* γ *in* (5.1). *On the complex plane, consider the metric* $ds^2 = dx^2 + dy^2$, *the Laplacian is* $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ *and* Δ^{OU} *is the usual two-dimensional Ornstein–Uhlenbeck operator.*

Proof. With the notation of (5.5)–(5.6), $\alpha(a) = \alpha$, $\alpha(b) = \beta$. As in example 5.2.1, $\alpha(a)\alpha(b) - \alpha(b)\alpha(a) = 0$ and $\alpha(a)\overline{\alpha(b)} - \alpha(b)\overline{\alpha(a)} = \alpha\overline{\beta} - \beta\overline{\alpha} \neq 0$. By Theorem 1.10,

$$\begin{aligned}\Delta &= (\rho(a) + \overline{\rho(a)})\overline{H_b} - (\rho(b) + \overline{\rho(b)})\overline{H_a} \\ &= (\alpha\overline{\beta} - \beta\overline{\alpha})\Big[\frac{\partial^2}{\partial z\,\partial\overline{z}} + \Big(\frac{\lambda\overline{\beta} - \epsilon\overline{\alpha}}{\alpha\overline{\beta} - \beta\overline{\alpha}}\, z + \frac{\overline{\lambda\beta - \epsilon\alpha}}{\alpha\overline{\beta} - \beta\overline{\alpha}}\,\overline{z}\Big)\frac{\partial}{\partial\overline{z}}\Big]\end{aligned}$$

has the measure $R\,dz\,d\overline{z}$ as invariant measure ($R = e^Q$ is given by (5.7) and $Q = Az^2 + 2Bz\overline{z} + \overline{A}\overline{z}^2$). Since $\alpha\overline{\beta} - \beta\overline{\alpha} \neq 0$ and $\Re(\alpha\overline{\beta} - \beta\overline{\alpha}) = 0$, we have

$$\Delta - \overline{\Delta} = (\alpha\overline{\beta} - \beta\overline{\alpha})\Big[2\,\frac{\partial^2}{\partial z\partial\overline{z}} + \frac{\partial}{\partial z}\log R\,\frac{\partial}{\partial\overline{z}} + \frac{\partial}{\partial\overline{z}}\log R\,\frac{\partial}{\partial z}\Big]$$

where R is given by (5.7) and

$$V = \Delta + \overline{\Delta} = \det\Big(lz + \overline{lz}, u\,\frac{\partial}{\partial z} + \overline{u}\,\frac{\partial}{\partial\overline{z}}\Big)$$

where

$$\det\Big(lz + \overline{lz}, u\,\frac{\partial}{\partial z} + \overline{u}\,\frac{\partial}{\partial\overline{z}}\Big) = (\det(l,\overline{u})z + \det(\overline{l},\overline{u})\overline{z})\,\frac{\partial}{\partial\overline{z}} + (\det(l,u)z + \det(\overline{l},u)\overline{z})\,\frac{\partial}{\partial z}.$$

We obtain (5.9)–(5.10) from (1.34). ∎

REMARK 5.5. Let $\psi(z) = z^n$, the unitarizing measure of the equivalent representation T_g^{ψ} of Subsection 1.5.1 is $d\mu^{\psi} = (x^2 + y^2)^n e^{Q(x,y)}\,dx\,dy$.

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References

[1] H. Airault, *Mesure unitarisante: algèbre de Heisenberg, algèbre de Virasoro*, C. R. Math. Acad. Sci. Paris 334 (2002), 787–792.

[2] H. Airault, P. Malliavin, *Regularized Brownian motion on the Siegel disk of infinite dimension*, Ukraïn. Mat. Zh. 52 (2000), 1158–1165; Ukrainian Math. J. 52 (2000), 1324–1333.

[3] H. Airault, P. Malliavin, *Unitarizing probability measures for representations of Virasoro algebra*, J. Math. Pures Appl. (9) 80 (2001), 627–667.

[4] H. Airault, H. Ouerdiane, *Ornstein–Uhlenbeck operators and unitarizing measures in the Poincaré disk*, Bull. Sci. Math. 133 (2009), 671–692.

[5] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform*, Comm. Pure Appl. Math. 14 (1961), 187–214.

[6] F. A. Berezin, *Quantization in complex symmetric spaces*, Izv. Akad. Nauk SSSR Ser. Mat. 39 (1975), 363–402; English transl.: Math. USSR–Izv. 9 (1975), 341–379.

[7] S. Chaari, F. Cipriano, H. H. Kuo, H. Ouerdiane, *Surface measure on the dual space of the Schwartz space*, Commun. Stoch. Anal. 4 (2010), 467–480.

[8] A. B. Cruzeiro, P. Malliavin, *Non perturbative construction of invariant measure through confinement by curvature*, J. Math. Pures Appl. (9) 77 (1998), 527–537.

[9] A. B. Cruzeiro, P. Malliavin, *Stochastic calculus of variations on complex line bundle and construction of unitarizing measures for the Poincaré disk*, J. Funct. Anal. 256 (2009), 385–408.

[10] B. K. Driver, B. C. Hall, *The energy representation has no non-zero fixed vectors*, in: Stochastic Processes, Physics and Geometry: New Interplays, II (Leipzig, 1999), CMS Conf. Proc. 29, Amer. Math. Soc., Providence, RI, 2000, 143–155.

[11] D. S. Freed, *Special Kähler Manifolds*, Comm. Math. Phys. 203 (1999), 31–52.

[12] M. Gordina, M. Wu, *Diffeomorphisms of the circle and Brownian motions on an infinite-dimensional symplectic group*, Commun. Stoch. Anal. 2 (2008), 71–95.

[13] B. C. Hall, A. N. Sengupta, *The Segal-Bargmann transform for path-groups*, J. Funct. Anal. 152 (1998), 220–254.

[14] S. Helgason, *Differential Geometry and Symmetric Spaces*, Pure Appl. Math. 12, Academic Press, New York, 1962.

[15] A. A. Kirillov, *Geometric approach to discrete series of unirreps for Vir*, J. Math. Pures Appl. (9) 77 (1998), 735–746.

[16] M. Kontsevich, Y. Suhov, *On Malliavin measures, SLE, and CFT*, Tr. Mat. Inst. Steklova 258 (2007), Anal. i Osob. Ch. 1, 107–153; English transl.: Proc. Steklov Inst. Math. 258 (2007), 100–146.

[17] P. Malliavin, *Heat measures and unitarizing measures for Berezinian representations on the space of univalent functions in the unit disk*, in: Perspectives in Analysis, Math. Phys. Stud. 27, Springer, Berlin, 2005, 253–268.

[18] P. Malliavin, *Invariant or quasi-invariant probability measures for infinite dimensional groups. I. Non-ergodicity of Euler hydrodynamic*, Jpn. J. Math. 3 (2008), 1–17; *II. Unitarizing measures or Berezinian measures*, Jpn. J. Math. 3 (2008), 19–47.

[19] I. Markina, D. Prokhorov, A. Vasil'ev, *Sub-Riemannian geometry of the coefficients of univalent functions*, J. Funct. Anal. 245 (2007), 475–492.

[20] D. Mumford, *Tata Lectures on Theta I*, with collaboration of C. Musili, M. Nori, E. Previato, M. Stillman, Modern Birkhäuser Classics, Birkhäuser, Boston, 2007.

[21] Y. A. Neretin, *Some remarks on quasi-invariant actions of loop groups and the group of diffeomorphisms of the circle*, Comm. Math. Phys. 154 (1994), 599–626.

[22] A. G. Sergeev, *The group of quasisymmetric homeomorphisms of the circle and quantization of the universal Teichmüller space*, SIGMA Symmetry Integrability Geom. Methods Appl. 5 (2009), Paper 015.

[23] I. Shigekawa, *Derivatives of Wiener functionals and absolute continuity of induced measures*, J. Math. Kyoto Univ. 20 (1980), 263–289.

[24] M. Sugiura, *Unitary Representations and Harmonic Analysis. An Introduction*, second ed., North-Holland Math. Library 44, North-Holland, Amsterdam, 1990.