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ENDOMORPHISMS OF THE CUNTZ ALGEBRAS

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Abstract. This mainly expository article is devoted to recent advances in the study of dynamical aspects of the Cuntz algebras $\mathcal{O}_n$, $n < \infty$, via their automorphisms and, more generally, endomorphisms. A combinatorial description of permutative automorphisms of $\mathcal{O}_n$ in terms of labelled, rooted trees is presented. This in turn gives rise to an algebraic characterization of the restricted Weyl group of $\mathcal{O}_n$. It is shown how this group is related to certain classical dynamical systems on the Cantor set. An identification of the image in $\mathrm{Out}(\mathcal{O}_n)$ of the restricted Weyl group with the group of automorphisms of the full two-sided n-shift is given, for prime n, providing an answer to a question raised by Cuntz in 1980. Furthermore, we discuss proper endomorphisms of $\mathcal{O}_n$ which preserve either the canonical UHF-subalgebra or the diagonal MASA, and present methods for constructing exotic examples of such endomorphisms.

1. Introduction. The C^*-algebras $\mathcal{O}_n$, $n \in \{2, 3, 4, \dots\} \cup \{\infty\}$ were first defined and investigated by Cuntz in his seminal paper [23], and they bear his name ever since. It is difficult to overestimate the importance of the Cuntz algebras in theory of operator

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algebras and many other areas. It suffices to mention that Cuntz's original article, [23], is probably the most cited ever paper in the area of operator algebras (MSC class 46L). Indeed, as C^*-algebras naturally generated by Hilbert spaces, the Cuntz algebras continue to provide a convenient framework for several different areas of investigations. In order to illustrate the variety of applications, without pretending in any way to be exhaustive, we only mention a very small sample of papers dealing with Fredholm theory, classification of C^*-algebras, self-similar sets, coding theory, continuous fractions, spectral flow and index theory for twisted cyclic cocycles, see e.g. [4, 40, 49, 46, 39, 11].

This mainly expository article is devoted to recent advances in the study of dynamical aspects of the Cuntz algebras $\mathcal{O}_n$ with $n < \infty$ via their automorphisms and, more generally, endomorphisms. It is not a comprehensive review but a selective one, biased towards the contributions made by the three authors. Some original results are also presented in this article, as will be explained later.

Systematic investigations of endomorphisms of $\mathcal{O}_n$, $n < \infty$ were initiated by Cuntz in [24]. A fundamental bijective correspondence between unital $*$-endomorphisms and unitaries in $\mathcal{O}_n$ was established therein (see equation (2), below). Using this correspondence Cuntz proved a number of interesting results, in particular with regard to those endomorphisms which globally preserve either the core UHF-subalgebra $\mathcal{F}_n$ or the diagonal MASA $\mathcal{D}_n$.

Investigations of automorphisms of $\mathcal{O}_n$ began almost immediately after the birth of the algebras in question, see [2, 24, 28, 27, 10, 47, 56]. Classification of group actions on $\mathcal{O}_n$ came to the fore somewhat later, see [35, 48]. In the present article, we review more recent results on automorphisms of $\mathcal{O}_n$ contained in [55, 22, 17, 18, 16].

Proper endomorphisms of the Cuntz algebras have also attracted a lot of attention. In particular, they played a role in certain aspects of index theory, both from the C^*-algebraic and von Neumann algebraic point of view. The problem of computing the Jones(-Kosaki-Longo) index of (the normal extensions) of localized endomorphisms of $\mathcal{O}_n$ was posed in [36]. Progress on this and other related problems was then achieved in a number of papers. Of particular note in this regard are contributions made by Longo, [43, 44, 45], and Izumi, [32, 33, 34], but see also [25, 19, 1, 15, 29, 37, 38, 21, 20, 30]. There is also a parallel line of research dealing with various entropy computations, e.g. see [12, 53, 54]. Recently, one of the most interesting applications of endomorphisms of $\mathcal{O}_n$, found by Bratteli and Jørgensen in [8, 9], is in the area of wavelets. Before that, shift endomorphisms of Cuntz algebras have been systematically employed in the analysis of structural aspects of quantum field theory, see e.g. the discussion in [26, Section 2] and references therein.

The present article is organized as follows. After setting the stage with some preliminaries in Section 2, we discuss localized automorphisms in Section 3. Localization refers to the fact that the corresponding unitary lies in one of the matrix algebras constituting a building block of the UHF-subalgebra $\mathcal{F}_n$. In Section 3.2, we review fundamental results about permutative automorphisms of $\mathcal{O}_n$, mainly contained in [22]. The key breakthrough obtained therein was a clear-cut correspondence between such automorphisms and certain combinatorial structure related to labelled trees. This in turn served as a platform for further theoretical analysis, classification results, and construction of non-trivial examples.

In Section 3.3, we present a more direct approach to finding automorphisms, based on solving certain polynomial matrix equations. Even though these equations are relatively easy to derive, finding a complete set of solutions is a highly non-trivial task.

Section 3.4 contains a complete classification of those permutative endomorphisms of $\mathcal{O}_3$ in level $k = 3$ which are either automorphisms of $\mathcal{O}_3$ or restrict to automorphisms of the diagonal $\mathcal{D}_3$. These results were obtained in [17] and in the subsequent unpublished work [18], with aid of massive computer calculations. We also give tables summarizing the results of our automorphism search for all values of parameters n and k with $n + k \leq 6$.

In Section 4, we review very recently obtained description of the so-called restricted Weyl group of $\mathcal{O}_n$ in terms of automorphisms of the full two-sided n-shift, [16]. On one hand, this result provides an answer to a question raised by Cuntz in [24]. On the other hand, it establishes a very interesting correspondence between an important class of automorphisms of a purely infinite, simple C^*-algebra $\mathcal{O}_n$ and much studied automorphism group of a classical system of paramount importance in symbolic dynamics, [41, 42]. Some aspects of this correspondence are related to the problem of extension of an automorphism from $\mathcal{D}_n$ to the entire $\mathcal{O}_n$. A similar question for the UHF-subalgebra $\mathcal{F}_n$ rather than the diagonal $\mathcal{D}_n$ has been studied recently in [14].

Some recent results related to proper endomorphisms of $\mathcal{O}_n$ are reviewed in Section 5. Subsection 5.1 deals with those endomorphisms which globally preserve the UHF-subalgebra $\mathcal{F}_n$, while Subsection 5.2 with those which globally preserve the diagonal $\mathcal{D}_n$. The main theme in here is construction of endomorphisms which globally preserve one of these subalgebras but whose corresponding unitary does not belong to the relevant normalizer. The problem of existence of such exotic endomorphisms was left open in [24] and remained unresolved until the recent works of [20] and [30].

2. Preliminaries. If n is an integer greater than 1, the *Cuntz algebra* $\mathcal{O}_n$ is the unital C^*-algebra generated by n isometries $S_1, \ldots, S_n$, satisfying $\sum_{i=1}^{n} S_i S_i^* = I$, [23]. Then it turns out that $\mathcal{O}_n$ is separable, simple, nuclear and purely infinite. We denote by W_n^k the set of k-tuples $\mu = (\mu_1, \ldots, \mu_k)$ with $\mu_m \in \{1, \ldots, n\}$, and by W_n the union $\bigcup_{k=0}^{\infty} W_n^k$, where $W_n^0 = \{0\}$. We call elements of W_n *multi-indices.* If $\mu \in W_n^k$ then $|\mu| = k$ is the *length* of μ. If $\mu = (\mu_1, \ldots, \mu_k) \in W_n$, then $S_\mu = S_{\mu_1} \ldots S_{\mu_k}$ ($S_0 = 1$ by convention) is an isometry with range projection $P_\mu = S_\mu S_\mu^*$. Every word in $\{S_i, S_i^* : i = 1, \ldots, n\}$ can be uniquely expressed as $S_\mu S_\nu^*$, for $\mu, \nu \in W_n$ [23, Lemma 1.3].

We denote by $\mathcal{F}_n^k$ the C^*-subalgebra of $\mathcal{O}_n$ spanned by all words of the form $S_\mu S_\nu^*$, $\mu, \nu \in W_n^k$, which is isomorphic to the matrix algebra $M_{n^k}(\mathbb{C})$. The norm closure $\mathcal{F}_n$ of $\bigcup_{k=0}^{\infty} \mathcal{F}_n^k$, is the UHF-algebra of type n^∞, called the *core UHF-subalgebra* of $\mathcal{O}_n$, [23]. It is the fixed point algebra for the gauge action of the circle group $\gamma : U(1) \to \mathrm{Aut}(\mathcal{O}_n)$ defined on generators as $\gamma_t(S_i) = tS_i$. For $k \in \mathbb{Z}$, we denote by $\mathcal{O}_n^{(k)} := \{x \in \mathcal{O}_n : \gamma_t(x) = t^k x\}$, the spectral subspace for this action. In particular, $\mathcal{F}_n = \mathcal{O}_n^{(0)}$. The C^*-subalgebra of $\mathcal{F}_n$ generated by projections P_μ, $\mu \in W_n$, is a *MASA* (maximal abelian subalgebra) both in $\mathcal{F}_n$ and in $\mathcal{O}_n$. We call it the *diagonal* and denote by $\mathcal{D}_n$. The spectrum of $\mathcal{D}_n$ is naturally identified with X_n—the full one-sided n-shift space. We also set $\mathcal{D}_n^k := \mathcal{D}_n \cap \mathcal{F}_n^k$. Throughout this paper we are interested in the inclusions

$$\mathcal{D}_n \subseteq \mathcal{F}_n \subseteq \mathcal{O}_n.$$

The UHF-subalgebra $\mathcal{F}_n$ possesses a unique normalized trace, denoted by τ. We will refer to the restriction of τ to $\mathcal{D}_n$ as to the canonical trace on $\mathcal{D}_n$.

We denote by $\mathcal{S}_n$ the group of those unitaries in $\mathcal{O}_n$ which can be written as finite sums of words, i.e., in the form $u = \sum_{j=1}^m S_{\mu_j} S_{\nu_j}^*$ for some $\mu_j, \nu_j \in W_n$. It turns out that $\mathcal{S}_n$ is isomorphic to the Higman–Thompson group $G_{n,1}$ [50]. One can also identify a copy of Thompson's group F sitting in canonical fashion inside $\mathcal{S}_2$. We also define $\mathcal{P}_n = \mathcal{S}_n \cap \mathcal{U}(\mathcal{F}_n)$. Then $\mathcal{P}_n = \bigcup_k \mathcal{P}_n^k$, where $\mathcal{P}_n^k$ are permutation unitaries in $\mathcal{U}(\mathcal{F}_n^k)$. That is, for each $u \in \mathcal{P}_n^k$ there is a unique permutation σ of multi-indices W_n^k such that

$$u = \sum_{\mu \in W_n^k} S_{\sigma(\mu)} S_\mu^*. \tag{1}$$

As shown by Cuntz in [24], there exists the following bijective correspondence between unitaries in $\mathcal{O}_n$ and unital $*$-endomorphisms of $\mathcal{O}_n$ (whose collection we denote by $\mathrm{End}(\mathcal{O}_n)$). A unitary u in $\mathcal{O}_n$ determines an endomorphism λ_u by[1]

$$\lambda_u(S_i) = uS_i, \quad i = 1, \dots, n. \tag{2}$$

Conversely, if $\rho : \mathcal{O}_n \to \mathcal{O}_n$ is an endomorphism, then $\sum_{i=1}^n \rho(S_i) S_i^* = u$ gives a unitary $u \in \mathcal{O}_n$ such that $\rho = \lambda_u$. If the unitary u arises from a permutation σ via formula (1), the corresponding endomorphism will be sometimes denoted by λ_σ. Composition of endomorphisms corresponds to a 'convolution' multiplication of unitaries as follows:

$$\lambda_u \circ \lambda_w = \lambda_{u*w}, \quad \text{where} \quad u * w = \lambda_u(w)u. \tag{3}$$

We denote by φ the canonical shift:

$$\varphi(x) = \sum_{i=1}^n S_i x S_i^*, \quad x \in \mathcal{O}_n.$$

If we take $u = \sum_{i,j} S_i S_j S_i^* S_j^*$ then $\varphi = \lambda_u$. It is well-known that φ leaves invariant both $\mathcal{F}_n$ and $\mathcal{D}_n$, and that φ commutes with the gauge action γ. We denote by ϕ the standard left inverse of φ, defined as $\phi(a) = \frac{1}{n} \sum_{i=1}^n S_i^* a S_i$.

If $u \in \mathcal{U}(\mathcal{O}_n)$ then for each positive integer k we define

$$u_k := u\varphi(u) \cdots \varphi^{k-1}(u). \tag{4}$$

We agree that u_k^* stands for $(u_k)^*$. If α and β are multi-indices of length k and m, respectively, then $\lambda_u(S_\alpha S_\beta^*) = u_k S_\alpha S_\beta^* u_m^*$. This is established through a repeated application of the identity $S_i a = \varphi(a) S_i$, valid for all $i = 1, \dots, n$ and $a \in \mathcal{O}_n$. If $u \in \mathcal{F}_n^k$ for some k then, following [19], we call endomorphism λ_u *localized.* Even though systematic investigations of such endomorphisms were initiated in [24], it should be noted that automorphisms constructed this way appeared already in the work of Connes in the context of the hyperfinite type II_1 factor, [13].

For algebras $A \subseteq B$ we denote by $\mathcal{N}_B(A) = \{u \in \mathcal{U}(B) : uAu^* = A\}$ the normalizer of A in B, and by $A' \cap B = \{b \in B : (\forall a \in A)\, ab = ba\}$ the relative commutant of A in B. We also denote by $\mathrm{Aut}(B, A)$ the collection of all those automorphisms α of B such that $\alpha(A) = A$, and by $\mathrm{Aut}_A(B)$ those automorphisms of B which fix A point-wise.

[1]In some papers, e.g. [24], [55] and [22], a different convention $\lambda_u(S_i) = u^* S_i$ is used.

3. Localized endomorphisms and automorphisms. In this section, we mostly deal with automorphisms of $\mathcal{O}_n$. However, it may be useful to broaden our horizon for a little while and consider more general endomorphisms of Cuntz algebras from the point of view of subfactor/sector theory.

3.1. One example. Since dealing simultaneously with all unitaries in matrix algebras is very difficult, in order to discuss interesting cases it is convenient to focus on some selected classes of unitaries which arise in specific situations like in the study of integrable systems. Let H be a Hilbert space with $\dim(H) = n$. Cuntz already noticed that unitary solutions $Y \in \mathcal{U}(H \otimes H)$ of the quantum YBE (without spectral parameter)

$$Y_{12}Y_{23}Y_{12} = Y_{23}Y_{12}Y_{23} \tag{5}$$

can be characterized in Cuntz algebra terms as those unitaries Y in $\mathcal{F}_n^2$ satisfying

$$\lambda_Y(Y) = \varphi(Y). \tag{6}$$

As a simple exercise, it is instructive to observe that no nontrivial unitary solution of the YBE induces an automorphism of $\mathcal{O}_n$. Indeed, we claim that if $Y \in \mathcal{F}_n^2$ then Y satisfies equation (5) if and only if[2] $Y \in (\lambda_Y^2, \lambda_Y^2) := \lambda_Y^2(\mathcal{O}_n)' \cap \mathcal{O}_n$. Here one needs the composition rule of endomorphisms, namely $\lambda_Y^2 = \lambda_{\lambda_Y(Y)Y} = \lambda_{Y\varphi(Y)Y\varphi(Y^*)}$, along with the characterization of self-intertwiners recalled in Section 5.1 below. That is, thanks to equation (13) one has that $Y \in (\lambda_Y^2, \lambda_Y^2)$ if and only if

$$\big(Y\varphi(Y)Y\varphi(Y^*)\big)^*YY\varphi(Y)Y\varphi(Y^*) = \varphi(Y). \tag{7}$$

Now, the left hand side of (7) is precisely $\varphi(Y)Y^*\varphi(Y^*)Y\varphi(Y)Y\varphi(Y^*)$ and the claim is now clear. If Y is not a multiple of the identity, this shows already that $\lambda_Y^2(\mathcal{O}_n)' \cap \mathcal{O}_n$ contains non-scalar elements and therefore, $\mathcal{O}_n$ being simple, λ_Y^2 is not an automorphism, as well as λ_Y. The computation of the Jones index for subfactors associated to Yang–Baxter unitaries has been discussed in more detail in [19, 15].

It is well-known that finding all solutions of the YBE in dimension n is a difficult problem that has been dealt with only for very small values of n. This is closely related with the classification problem for braiding in categories of representations of quantum groups and/or conformal nets. It is expected that attaching to these solutions invariants from subfactor theory will lead to a much better understanding.

Other families of unitaries related to the study of spin/vertex models might also provide a useful playground:

Problem 3.1.

(a) Examine Cuntz algebra endomorphisms associated to normalized Hadamard matrices, cf. [36];
(b) Discuss from the Cuntz algebra point of view the tetrahedron equation and/or its several variations (see e.g. [3]).

[2] At first sight this condition might look a bit strange, however one should then remember that in algebraic quantum field theory the canonical braiding ϵ_ρ of a localized morphism ρ of the observable net $\mathfrak{A}$ indeed satisfies $\epsilon_\rho \in \rho^2(\mathfrak{A})' \cap \mathfrak{A}$.

Finally, it is worth to recall that a throughout discussion of localized endomorphisms associated to (finite-dimensional) unitaries satisfying the so-called pentagon equation (which is a basic ingredient of quantum group theory) has been provided in [45, 19].

3.2. Permutative automorphisms and labelled trees. We begin by recalling[3] some results from [22]. Let u be a unitary in $\mathcal{F}_n^k$. For $i, j \in \{1, \dots, n\}$, one defines linear maps $a_{ij}^u : \mathcal{F}_n^{k-1} \to \mathcal{F}_n^{k-1}$ by $a_{ij}^u(x) = S_i^* u^* x u S_j$, $x \in \mathcal{F}_n^{k-1}$. We put $V_u = \mathcal{F}_n^{k-1}/\mathbb{C}1$. Since $a_{ij}^u(\mathbb{C}1) \subseteq \mathbb{C}1$, there are induced maps $\tilde{a}_{ij}^u : V_u \to V_u$. We define A_u as the subring of $\mathcal{L}(V_u)$ generated by $\{\tilde{a}_{ij}^u : i, j = 1, \dots, n\}$. We denote by H the linear span of the S_i's. Following [19], we define inductively $\Xi_0 = \mathcal{F}_n^{k-1}$ and $\Xi_r = \lambda_u(H)^* \Xi_{r-1} \lambda_u(H)$, $r \geq 1$. It follows that $\{\Xi_r\}$ is a nonincreasing sequence of subspaces of $\mathcal{F}_n^{k-1}$ and thus it eventually stabilizes. If p is the smallest integer for which $\Xi_p = \Xi_{p+1}$, then $\Xi_u := \bigcap_r \Xi_r = \Xi_p$. The following result is contained in [22].

Theorem 3.2. *Let u be a unitary in $\mathcal{F}_n^k$. Then the following conditions are equivalent*:

(1) λ_u *is invertible with localized inverse*;
(2) A_u *is nilpotent*;
(3) $\Xi_u = \mathbb{C}1$.

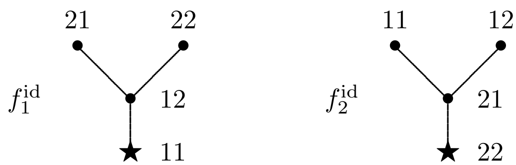
In the case of a permutation unitary $u \in \mathcal{P}_n^k$, Theorem 3.2 may be strengthened and very conveniently reformulated in combinatorial terms, as follows. As shown in [22], the corresponding λ_u is an automorphism of $\mathcal{O}_n$ if and only if u satisfies two conditions, called (b) and (d) therein. Condition (b) by itself guarantees that endomorphism λ_u restricts to an automorphism of the diagonal $\mathcal{D}_n$.[4] To describe these two conditions we will identify unitary $u \in \mathcal{P}_n^k$ with the corresponding permutation of W_n^k.

For $i = 1, \dots, n$, one defines a mapping $f_i^u : W_n^{k-1} \to W_n^{k-1}$ so that $f_i^u(\alpha) = \beta$ if and only if there exists $m \in W_n^1$ such that $(\beta, m) = u(i, \alpha)$. Then u satisfies condition (b) if and only if there exists a partial order $\leq$ on $W_n^{k-1} \times W_n^{k-1}$ such that:

(i) Each element of the diagonal (α, α) is minimal.
(ii) Each (α, β) is bounded below by some diagonal element.
(iii) For every i and all (α, β) such that $\alpha \neq \beta$, we have

$$(f_i^u(\alpha), f_i^u(\beta)) \leq (\alpha, \beta). \tag{8}$$

For this condition (b) to hold it is necessary that the diagram of each mapping f_i^u is a rooted tree, with the root its unique fixed point and with an edge going down from α to β if $f_i^u(\alpha) = \beta$. By convention, we do not include in the diagram the loop from the root to itself. For example, if $u = \text{id}$ is viewed as an element of $\mathcal{P}_2^3$, then the corresponding pair of labelled trees is:



[3]Note the difference in convention regarding the definition of λ_u.

[4]Since $\mathcal{P}_n^k \subset \mathcal{N}_{\mathcal{O}_n}(\mathcal{D}_n)$ for all k, every permutative endomorphism of $\mathcal{O}_n$ maps $\mathcal{D}_n$ into itself.

To describe condition (d) we define $\mathcal{W}_n^{k-1}$ as the union of all off-diagonal elements of $W_n^{k-1} \times W_n^{k-1}$ and one additional element $\dagger$. For $i, j \in W_n^1$, we also define mappings $f_{ij}^u : \mathcal{W}_n^{k-1} \to \mathcal{W}_n^{k-1}$ so that $f_{ij}^u(\alpha, \beta) = (\gamma, \delta)$ if there exists an $m \in W_n^1$ such that $(\gamma, m) = u(i, \alpha)$ and $(\delta, m) = u(j, \beta)$. Otherwise, we set $f_{ij}^u(\alpha, \beta) = \dagger$. We also put $f_{ij}^u(\dagger) = \dagger$ for all i, j. Then u satisfies condition (d) if and only if there exists a partial order $\leq$ on $\mathcal{W}_n^{k-1}$ such that:

(i) The only minimal element with respect to $\leq$ is $\dagger$.
(ii) For every $(\alpha, \beta) \in \mathcal{W}_n^{k-1}$ and all $i, j = 1, \ldots, n$, we have

$$f_{ij}^u(\alpha, \beta) \leq (\alpha, \beta). \tag{9}$$

With help of this combinatorial approach, a complete classification has been achieved in [22], [17] and [18] of permutations in $\mathcal{P}_n^k$ with $n + k \leq 6$ such that the corresponding endomorphism λ_u either is automorphism of $\mathcal{O}_n$ or restricts to an automorphism of the diagonal $\mathcal{D}_n$. Considering the image of $\lambda(\mathcal{P}_n^k)^{-1}$ in the outer automorphism group of $\mathcal{O}_n$, it was shown in [22] with respect to the case of $\mathcal{O}_2$, that no outer automorphisms apart from the flip-flop arise in this way for $k = 3$ (a much simpler case $k = 2$ being already known). For $k = 4$, twelve new classes in $\mathrm{Out}(\mathcal{O}_2)$ were found.

3.3. Inverse pairs of localized automorphisms. In this section, we gather together a few facts about pairs of unitaries in some finite matrix algebras giving rise to automorphisms of $\mathcal{O}_n$ that are inverses of each other. We also briefly discuss interesting algebraic equations such unitaries must satisfy. These equations provide a useful background for the considerations in Section 3 of [22] (e.g. Theorem 3.2, Corollary 3.3 therein), which are reviewed in the present article in Section 3.2 above. They have also been useful for several other concrete computations in [22], e.g. in computing explicitly the inverse of λ_A, introduced and analyzed in Section 5, filling the tables of Section 6, and in the search of square-free automorphisms, [22]. Although these equations are not difficult to derive, we think that highlighting them may be of benefit, especially to the readers who do not use the machinery of Cuntz algebras on the daily basis.

So let us suppose that $u \in \mathcal{F}_n^k$ and $w \in \mathcal{F}_n^h$ are unitaries such that

$$\lambda_u \lambda_w = \mathrm{id} = \lambda_w \lambda_u ,$$

i.e. $\lambda_u(w)u = 1 = \lambda_w(u)w$.[5] This readily leads to a system of coupled matrix equations

$$u_h w u_h^* = u^*, \quad w_k u w_k^* = w^*, \tag{10}$$

where both u_h and w_k are in $\mathcal{F}_n^{h+k-1}$. In passing, observe that the second equation is independent of the level h for which $w \in \mathcal{F}_n^h$.

[5]Since λ_u and λ_w are injective, one identity implies the other. Also, up to replacing k and h with their maximum, there would be no loss of generality in assuming that $k = h$. However as the inverse of an automorphism induced by a unitary in a matrix algebra might very well be induced by a unitary in a larger matrix algebra, it seems convenient to allow this more flexible asymmetric formulation. It is worth stressing that, given k, the subset of unitaries u's in $\mathcal{F}_n^k$ such that λ_u^{-1} (exists and) is still induced by a unitary in $\mathcal{F}_n^k$ is strictly smaller than the set of unitaries such that λ_u^{-1} is induced by a unitary in some $\mathcal{F}_n^h$. An a priori bound for h as a function of n, k is provided in [22, Corollary 3.3].

In practical situations, one is faced with the converse problem. Starting with some $u \in \mathcal{F}_n^k$, one might not know the precise value of h, or even if the corresponding w exists at all. It turns out that the existence of solutions (for w) of equations (10) imply invertibility of λ_u. The following proposition combined with [22, Corollary 3.3] gives an algorithmic procedure for finding these solutions. We omit its elementary proof.

PROPOSITION 3.3. *Let u be a unitary in $\mathcal{F}_n^k$ and suppose that $u_h^* u^* u_h \in \mathcal{F}_n^h$ for some h. Then λ_u is invertible and $\lambda_u^{-1} = \lambda_w$ with $w := u_h^* u^* u_h$.*

In particular, given a unitary $u \in \mathcal{F}_n^k$, one has $\lambda_u^2 = \mathrm{id}$ (i.e., $u = w$) if and only if $\lambda_u(u)u = 1$, if and only if $u_k u u_k^* = u^*$.

Finally, we present yet another computational strategy for determining invertibility of endomorphism λ_u and finding its inverse. Again, we omit an elementary proof of the following proposition.

PROPOSITION 3.4. *Let u and w be unitaries in $\mathcal{F}_n^k$ and $\mathcal{F}_n^h$, respectively, satisfying equations* (10). *Then u is a solution of the following polynomial matrix equation*

$$(u_r^* u^* u_r)_r u (u_r^* u^* u_r)_r^* = u_r^* u u_r \,, \tag{11}$$

where r can be taken as maximum of k and h.

Conversely, given r, every solution $u \in \mathcal{F}_n^r$ of equation (11) *gives rise to an automorphism λ_u of $\mathcal{O}_n$, with inverse induced by $w := u_r^* u^* u_r$.*

After some simplification, taking into account that $u \in \mathcal{F}_n^r$, it is straightforward to check that the first nontrivial equation in the family (11), for $r = 2$, is

$$\varphi(u)\varphi^2(u^*)\varphi(u^*)u = u\varphi(u)\varphi^2(u^*)\varphi(u^*), \tag{12}$$

i.e. u commutes with $\varphi(u\varphi(u^*)u^*)$. Similarly, for $r = 3$, one obtains

$$\begin{aligned} &u\varphi^2(u\varphi(u)\varphi^2(u))\varphi\big(\varphi^2(u^*)\varphi(u^*)u\varphi(u)\varphi^2(u)\big)\varphi^2(u^*)\varphi(u^*) \\ &= \varphi^2(u\varphi(u)\varphi^2(u))\varphi\big(\varphi^2(u^*)\varphi(u^*)u\varphi(u)\varphi^2(u)\big)\varphi^2(u^*)\varphi(u^*)u \end{aligned}$$

i.e., u commutes with $\varphi^2(u\varphi(u)\varphi^2(u))\varphi\big(\varphi^2(u^*)\varphi(u^*)u\varphi(u)\varphi^2(u)\big)\varphi^2(u^*)\varphi(u^*)$.

REMARK 3.5. The strategy of applying Proposition 3.4 is to find all pairs satisfying (10) by solving equations of the form (11) for all values of r. Implicitly, by solving such an equation, we predict w to take a particular form, namely $w = u_r^* u^* u_r$. However, we do not assume $w \in \mathcal{F}_n^r$. In fact, w automatically belongs to $\mathcal{F}_n^{2r-1}$. Combining this with equations (10) we obtain an additional relation u must satisfy, namely $u_r^* u^* u_r = u_{2r-1}^* u^* u_{2r-1}$.

We find it rather intriguing that in the case of permutation unitaries the polynomial matrix equations (11) turn out to be equivalent to the tree related conditions of [22, Corollary 4.12].

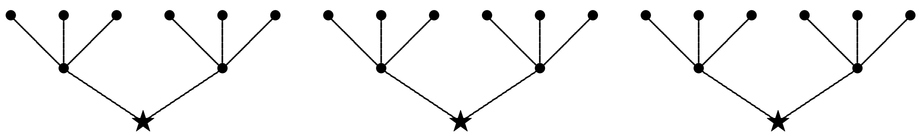
Of course, the above polynomial matrix equations apply to arbitrary unitaries in the algebraic part of $\mathcal{F}_n$ and not only to permutation matrices. Therefore, they can be used for finding other families of automorphisms of $\mathcal{O}_n$ with localized inverses. It is to be expected that new interesting classes of automorphisms different from the much studied quasi-free ones will be found this way. It seems also worth while to investigate the algebraic varieties in $\mathbb{R}^{2k^2}$ defined by these equations. At present, we are not aware of occurrences of these equations outside the realm of Cuntz algebras but we would not be surprised if such instances were found.

3.4. The classification of permutative automorphisms. The classification of automorphisms of $\mathcal{O}_n$ associated to unitaries in $\mathcal{P}_n^k$ for $n = k = 2$ goes back to [37]. Beyond that, the program of classifying permutative automorphisms corresponding to unitaries in $\mathcal{P}_n^k$ for small values of n and k was initiated in [22] and continued in [17]. In this section, we present a complete classification in the case $n = k = 3$. These results come from the unpublished manuscript, [18], and were obtained with aid of a massive scale computer calculations involving Magma software, [6].

As discussed in Section 3.2 above, determination of invertibility of a permutative endomorphism hinges upon verification of two combinatorial conditions, called (b) and (d), [22]. In short, condition (b) allows to determine when the corresponding endomorphism λ_σ of $\mathcal{O}_n$ restricts to an automorphism of $\mathcal{D}_n$, while condition (d), together with (b), determines the more stringent situation that $\lambda_\sigma \in \mathrm{Aut}(\mathcal{O}_n)$.[6] Detailed analysis of conditions (b) and (d) in terms of labelled, rooted trees, was then accomplished for $n = 2$ in [22] up to level $k = 4$, and in [17] for $n = 3$ up to level $k = 3$ (for $n = 3 = k$ only condition (b) was examined) and $n = 4$ up to level 2.

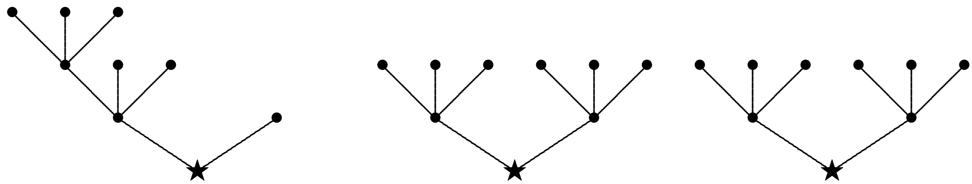
In the case $n = k = 3$, the involved rooted trees have nine vertices. By in-degree type of a rooted tree we mean the multiset of the in-degrees of its vertices; in [17, Figure 1], we have divided a relevant subset of 171 rooted trees with 9 vertices into 11 distinct in-degree types called $A, B, \ldots, K$ and described in Table 1 therein. For instance, the in-degree type A spots only trees with six vertices with no incoming edge (leaves) and three vertices with three incoming edges (also recall that there is always an invisible loop at the root), while the in-degree type B singles out trees with five leaves, one vertex with one incoming edge, one vertex with two incoming edges and two vertices with three incoming edges. It turns out that condition (b) is satisfied for a set $\mathcal{F}$ of 7390 3-tuples of labelled rooted trees, up to permutation of tree position (action of the symmetric group S_3) and consistent relabelling of all trees (action of S_9), as described in [17, Section 2.2]. The set $\mathcal{F}$ is then partitioned into 6 distinct three-element multisets of in-degree types, as listed in Table 1 below (based on Table 2 in [17], to which we refer for more details). Examples of triples of rooted trees with labels belonging to the in-degree types $A\,A\,A$ and $A\,F\,G$ are shown in Figure 2 of [17].

For instance, the six permutative Bogolubov automorphisms associated to permutations $u \in \mathcal{P}_3^1$, viewed as elements in $\mathcal{P}_3^3$, give raise to the 3-tuple of trees with in-degree type $A\,A\,A$



[6]It is also useful to observe that, the diagonal $\mathcal{D}_n$ being a MASA in $\mathcal{O}_n$, an automorphism of $\mathcal{O}_n$ mapping $\mathcal{D}_n$ into itself automatically restricts to an automorphism of $\mathcal{D}_n$.

but also other 3-tuples of trees still of in-degree type $A\,A\,A$ may correspond to permutative automorphisms of $\mathcal{O}_3$, e.g.



In fact, type A is comprised of the two kinds of trees entering the above two 3-tuples.

As deduced in [18] after very long and tedious computer-assisted computations, we can report that, among the permutations already selected on the basis of condition (b), the total number of permutations for $n = k = 3$ satisfying condition (d) is

$$907\,044 \cdot 9! = 329\,148\,126\,720.$$

This result relies very much on the extensive set of data already collected in [17]. For each of the 7 390 representatives in the set $\mathcal{F}$ we found the induced permutations that satisfy condition (d); it took about 7 processor years to compute.

Table 1. In-degree types of permutations satisfying condition (d)

ID types			some	none	total
A	A	A	290	1 878	2 168
A	B	B	611	2 171	2 782
A	C	D	86	864	950
A	E	E	290	782	1 072
A	F	G	35	357	392
A	H	H	12	14	26
total			1 324	6 066	7 390

In Table 1, we indicate how many instances of each in-degree type have some permutations satisfying condition (d) and how many instances have none. In Table 2, the 7 390 representative tree tuples are counted (second column headed $\#f$) according to exactly how many induced permutations satisfy condition (d) (first column headed $\#\sigma$), and according to the combined relabelling and repositioning orbit size (third column headed $\#o$). The fourth entry in each row is the product of the first three entries; so the sum of the fourth column is the given figure.

All in all, taking into account the results in [22, 17], Table 3 summarizes the up-to-date enumeration of permutations providing automorphisms of $\mathcal{O}_n$ and (in brackets, in the second line) of $\mathcal{D}_n$:

PROBLEM 3.6. Extend the results summarized in Tables 1, 2, and 3 to include a wider range of parameters, possibly developing new computational techniques to this end.

4. The restricted Weyl group of $\mathcal{O}_n$. We recall from [24] that $\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n)$ is the normalizer of $\mathrm{Aut}_{\mathcal{D}_n}(\mathcal{O}_n)$ in $\mathrm{Aut}(\mathcal{O}_n)$ and it can be also described as the group $\lambda(\mathcal{N}_{\mathcal{O}_n}(\mathcal{D}_n))^{-1}$ of automorphisms of $\mathcal{O}_n$ induced by elements in the normalizer $\mathcal{N}_{\mathcal{O}_n}(\mathcal{D}_n)$.

Table 2. Number of permutations σ satisfying condition (d) per f.

$\#\sigma$	$\#f$	$\#o$	$\#\sigma \cdot \#f \cdot \#o$
0	6 066	$6 \cdot 9!$	$0 \cdot 9!$
24	22	$6 \cdot 9!$	$3\,168 \cdot 9!$
48	288	$6 \cdot 9!$	$82\,944 \cdot 9!$
60	9	$6 \cdot 9!$	$3\,240 \cdot 9!$
72	10	$6 \cdot 9!$	$4\,320 \cdot 9!$
84	47	$6 \cdot 9!$	$23\,688 \cdot 9!$
96	213	$6 \cdot 9!$	$122\,688 \cdot 9!$
96	6	$3 \cdot 9!$	$1\,728 \cdot 9!$
108	103	$6 \cdot 9!$	$66\,744 \cdot 9!$
120	74	$6 \cdot 9!$	$53\,280 \cdot 9!$
132	107	$6 \cdot 9!$	$84\,744 \cdot 9!$
144	111	$6 \cdot 9!$	$95\,904 \cdot 9!$
156	121	$6 \cdot 9!$	$113\,256 \cdot 9!$
168	23	$6 \cdot 9!$	$23\,184 \cdot 9!$
180	3	$3 \cdot 9!$	$1\,620 \cdot 9!$
192	57	$6 \cdot 9!$	$65\,664 \cdot 9!$
192	8	$3 \cdot 9!$	$4\,608 \cdot 9!$
204	26	$6 \cdot 9!$	$31\,824 \cdot 9!$
204	4	$3 \cdot 9!$	$2\,448 \cdot 9!$
216	11	$6 \cdot 9!$	$14\,256 \cdot 9!$
216	7	$3 \cdot 9!$	$4\,536 \cdot 9!$
228	27	$6 \cdot 9!$	$36\,936 \cdot 9!$
240	38	$6 \cdot 9!$	$54\,720 \cdot 9!$
312	4	$6 \cdot 9!$	$7\,488 \cdot 9!$
312	4	$3 \cdot 9!$	$3\,744 \cdot 9!$
312	1	$1 \cdot 9!$	$312 \cdot 9!$
	7 390		$907\,044 \cdot 9!$

Table 3. Number of permutative automorphisms of $\mathcal{O}_n$ (and of $\mathcal{D}_n$) at level k

$n \setminus k$	1	2	3	4
2	2 (2)	4 (8)	48 (324)	564,480 (175,472,640)
3	6 (6)	576 (5184)	329,148,126,720 (161,536,753,300,930,560)	
4	24 (24)	5,771,520 (1,791,590,400)		

Furthermore, using [51], one can show that $\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n)$ has the structure of a semidirect product $\mathrm{Aut}_{\mathcal{D}_n}(\mathcal{O}_n) \rtimes \lambda(\mathcal{S}_n)^{-1}$ [22]. In particular, the group $\lambda(\mathcal{P}_n)^{-1}$ is isomorphic with the quotient of the group $\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n) \cap \mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n)$ by its normal subgroup $\mathrm{Aut}_{\mathcal{D}_n}(\mathcal{O}_n)$. We call it the *restricted Weyl group* of $\mathcal{O}_n$, cf. [24, 22]. We also note that every unital endomorphism of $\mathcal{O}_n$ which fixes the diagonal $\mathcal{D}_n$ point-wise is automatically surjective, i.e. it is an element of $\mathrm{Aut}_{\mathcal{D}_n}(\mathcal{O}_n)$ [14, Proposition 3.2] and that it is easy to construct product-type automorphisms of $\mathcal{D}_n$ that do not extend to (possibly proper) endomorphisms of $\mathcal{O}_n$ [14, Proposition 3.1]. A simple example of such an automorphism of $\mathcal{D}_2$ is given by $\bigotimes_{i=1}^{\infty} \mathrm{Ad}(u_i)$, where $u_i = 1$ for i even and $u_i = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for i odd and we have realized $\mathcal{D}_2$ as an infinite tensor product over $\mathbb{N}$ of diagonal matrices of size 2. In particular, it becomes important to characterize those automorphisms of $\mathcal{D}_n$ that can be obtained by restricting automorphisms (or even endomorphisms) of $\mathcal{O}_n$. As a variation on the theme, we mention the following

Problem 4.1. Find necessary and sufficient conditions for an automorphism of $\mathcal{D}_n$ to extend to an automorphism or a proper endomorphism of $\mathcal{F}_n$, respectively.

In [16], a subgroup $\mathfrak{G}_n$ of $\mathrm{Aut}(\mathcal{D}_n)$ was defined. It consists of those automorphisms α for which there exists an m such that both $\alpha\varphi^m$ and $\alpha^{-1}\varphi^m$ commute with the shift φ.

Theorem 4.2 ([16]). *The restriction $r : \lambda(\mathcal{P}_n)^{-1} \to \mathfrak{G}_n$ is a group isomorphism.*

Recall that the spectrum of $\mathcal{D}_n$ may be naturally identified with the full one-sided n-shift space X_n. The above theorem identifies the restricted Weyl group of $\mathcal{O}_n$ with the group of those homeomorphisms of X_n which together with their inverses eventually commute with the shift. In a sense, this provides an answer to a question raised by Cuntz in [24].

Let $\mathfrak{IG}_n = \{\mathrm{Ad}(u)|_{\mathcal{D}_n} : u \in \mathcal{P}_n\}$. This is a normal subgroup of $\mathfrak{G}_n$, since for $u \in \mathcal{P}_n^k$ we have $\mathrm{Ad}(u)\varphi^k = \varphi^k$. We also denote by $\mathrm{Inn}\,\lambda(\mathcal{P}_n)^{-1}$ the normal subgroup of $\lambda(\mathcal{P}_n)^{-1}$ consisting of all inner permutative automorphisms $\{\mathrm{Ad}(u) : u \in \mathcal{P}_n\}$. We call the quotient $\lambda(\mathcal{P}_n)^{-1}/\,\mathrm{Inn}\,\lambda(\mathcal{P}_n)^{-1}$ the *restricted outer Weyl group* of $\mathcal{O}_n$. It follows from Theorem 4.2 that the restricted outer Weyl group of $\mathcal{O}_n$ is naturally isomorphic to the quotient $\mathfrak{G}_n/\mathfrak{IG}_n$. Further analysis reveals that this group in turn is related to automorphisms of the two-sided shift. Indeed, let $\mathrm{Aut}(\Sigma_n)$ denote the group of automorphisms of the full two-sided n-shift (that is, the group of homeomorphisms of the full two-sided n-shift space Σ_n that commute with the two-sided shift σ) and let $\langle\sigma\rangle$ be its subgroup generated by the two-sided shift σ. It is known that $\langle\sigma\rangle$ coincides with the center of $\mathrm{Aut}(\Sigma_n)$.

Theorem 4.3 ([16]). *There is a natural embedding of the group $\lambda(\mathcal{P}_n)^{-1}/\,\mathrm{Inn}\,\lambda(\mathcal{P}_n)^{-1}$ into $\mathrm{Aut}(\Sigma_n)/\langle\sigma\rangle$. If n is prime then this embedding is surjective and thus the two groups are isomorphic.*

The above theorem establishes a useful correspondences between permutative automorphisms of the Cuntz algebra $\mathcal{O}_n$ and automorphisms of a classical dynamical system. It opens up very attractive possibilities for two-fold applications: of topological dynamics to the study of automorphisms of a simple, purely infinite C^*-algebra, and of algebraic methods available for $\mathcal{O}_n$ to the study of symbolic dynamical systems. Thanks to a combined effort of a number of researchers (see [41] and [42]) several interesting properties of the group $\mathrm{Aut}(\Sigma_n)/\langle\sigma\rangle$ are known: it is countable, residually finite, contains all finite groups, and contains all free products of finitely many cyclic groups. However, a number of questions remain to date unsolved (see [7]). For example, is it generated by elements of finite order? And most importantly, is $\mathrm{Aut}(\Sigma_n)/\langle\sigma\rangle$ isomorphic to $\mathrm{Aut}(\Sigma_m)/\langle\sigma\rangle$ (as an abstract group) when $n \neq m$ are prime?

Along with automorphisms of the two-sided shift, automorphisms of the full one-sided shift X_n have been extensively studied (see [41] and [42]). As shown in [16], each element of $\mathrm{Aut}(X_n)$ (viewed as an element of $\mathrm{Aut}(\mathcal{D}_n)$) admits an extension to an outer permutative automorphism of $\mathcal{O}_n$. This leads to the following.

THEOREM 4.4. *There exists a natural* (*given by extensions*) *embedding of* $\mathrm{Aut}(X_n)$ *into* $\lambda(\mathcal{P}_n)^{-1}/\operatorname{Inn}\lambda(\mathcal{P}_n)^{-1}$.

5. Proper endomorphisms.

In this section, we mainly deal with proper endomorphisms of $\mathcal{O}_n$ which globally preserve either the core UHF-subalgebra $\mathcal{F}_n$ or the diagonal MASA $\mathcal{D}_n$. Two main references for the results reviewed below are [20] and [30].

5.1. Endomorphisms preserving $\mathcal{F}_n$.

Cuntz showed in [24] that if a unitary w belongs to $\mathcal{F}_n$ then the corresponding endomorphism λ_w globally preserves $\mathcal{F}_n$. The reversed implication was left open in [24]. This question was finally answered to the negative in [30], where a number of counterexamples were produced. The main method for finding such counterexamples is the following.

Let u be a unitary in $\mathcal{O}_n$ and let v be a unitary in the relative commutant $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$. Then the three endomorphisms λ_u, λ_{vu}, and $\lambda_{u\varphi(v)}$ coincide on $\mathcal{F}_n$. Assume further that $u \in \mathcal{F}_n$, and let w equal either vu or $u\varphi(v)$. Then $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ and thus $\lambda_w(\mathcal{F}_n) \subseteq \mathcal{F}_n$. However, w belongs to $\mathcal{F}_n$ if and only if v does.

The above observation shows how to construct examples of unitaries w outside $\mathcal{F}_n$ for which nevertheless $\lambda_w(\mathcal{F}_n) \subseteq \mathcal{F}_n$. To this end, it suffices to find a unitary $u \in \mathcal{F}_n$ such that the relative commutant $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ is not contained in $\mathcal{F}_n$. This is possible. In fact, one can even find unitaries in a matrix algebra $\mathcal{F}_n^k$ such that $\lambda_u(\mathcal{O}_n)' \cap \mathcal{O}_n$ is not contained in $\mathcal{F}_n$. The existence of such unitaries was demonstrated in [19]. The relative commutant $\lambda_u(\mathcal{O}_n)' \cap \mathcal{O}_n$ coincides with the space (λ_u, λ_u) of self-intertwiners of the endomorphism λ_u, which can be computed as

$$(\lambda_u, \lambda_u) = \{x \in \mathcal{O}_n : x = (\operatorname{Ad} u \circ \varphi)(x)\}. \tag{13}$$

In [20], an explicit example was given of a permutation unitary $u \in \mathcal{P}_2^4$ and a unitary v in $\mathcal{S}_2 \setminus \mathcal{P}_2$ such that $v \in (\lambda_u, \lambda_u)$. Notice that $\lambda_{u\varphi(v)}(\mathcal{F}_n) = \lambda_u(\mathcal{F}_n)$ naturally gives rise to a subfactor of the A.F.D. II_1 factor with finite Jones index.

PROBLEM 5.1. Provide a combinatorial algorithm to construct and possibly "classify" pairs (u, v) with $u \in \mathcal{P}_n^k$ and $v \in (\lambda_u, \lambda_u) \cap (\mathcal{S}_n \setminus \mathcal{P}_n)$.

An alert reader could spot intriguing resemblance of this problem with the classification of the so-called modular invariants (see [5]), although it is possible that this is nothing more than a formal analogy.

Furthermore, in [20] a striking example was found of a unitary element $u \in \mathcal{F}_2$ for which the relative commutant $\lambda_u(\mathcal{O}_2)' \cap \mathcal{O}_2$ contains a unital copy of $\mathcal{O}_2$. In this case the proof is non-constructive and involves a modification of Rørdam's proof of the isomorphism $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$, [52]. As a corollary, one obtains existence of a unital $*$-homomorphism $\sigma : \mathcal{O}_2 \otimes \mathcal{O}_2 \to \mathcal{O}_2$ such that $\sigma(\mathcal{F}_2 \otimes \mathcal{F}_2) \subseteq \mathcal{F}_2$. It is not clear though whether such a σ can be an isomorphism.

PROBLEM 5.2. Does there exist an isomorphism $\sigma\colon \mathcal{O}_2\otimes\mathcal{O}_2 \to \mathcal{O}_2$ such that $\sigma(\mathcal{F}_2\otimes\mathcal{F}_2) \subseteq \mathcal{F}_2$ or, better yet, $\sigma(\mathcal{F}_2 \otimes \mathcal{F}_2) = \mathcal{F}_2$?

At present, we still do not know whether the method described above captures all possible cases or not, and thus we would like to pose the following problem.

PROBLEM 5.3. Does there exist a unitary $w \in \mathcal{O}_n$ such that $\lambda_w(\mathcal{F}_n) \subseteq \mathcal{F}_n$ but there is no unitary $u \in \mathcal{F}_n$ such that $\lambda_w|_{\mathcal{F}_n} = \lambda_u|_{\mathcal{F}_n}$?

Under certain additional assumptions, condition $\lambda_w(\mathcal{F}_n) \subseteq \mathcal{F}_n$ implies $w \in \mathcal{F}_n$, [20]. In particular, this happens when:

(i) $\lambda_w(\mathcal{F}_n) = \mathcal{F}_n$. If moreover $\lambda_w|_{\mathcal{F}_n} = \mathrm{id}$ then $w = t1$, $t \in U(1)$, and thus λ_w is a gauge automorphism of $\mathcal{O}_n$;
(ii) $\lambda_w \in \mathrm{Aut}(\mathcal{O}_n)$;
(iii) $\lambda_w(\mathcal{F}_n)' \cap \mathcal{O}_n = \mathbb{C}1$;
(iv) $w \in \mathcal{S}_n$ and $\mathcal{D}_n \subseteq \lambda_w(\mathcal{F}_n)$.

5.2. Endomorphisms preserving $\mathcal{D}_n$. Cuntz showed in [24] that if a unitary w belongs to the normalizer $\mathcal{N}_{\mathcal{O}_n}(\mathcal{D}_n)$ of the diagonal $\mathcal{D}_n$ in $\mathcal{O}_n$ then the corresponding endomorphism λ_w globally preserves $\mathcal{D}_n$. The reversed implication was left open in [24]. This problem was investigated in depth in [30]. In particular, examples of unitaries $w \notin \mathcal{N}_{\mathcal{O}_n}(\mathcal{D}_n)$ such that $\lambda_w(\mathcal{D}_n) \subseteq \mathcal{D}_n$ were found therein, and the following convenient criterion of global preservation of $\mathcal{D}_n$ was given.

THEOREM 5.4. *Let $k \in \mathbb{N}$ and let $w \in \mathcal{U}(\mathcal{F}_n^k)$. For $i, j = 1, \dots, n$ let $E_{ij} : \mathcal{F}_n^k \to \mathcal{F}_n^{k-1}$ be linear maps determined by the condition that $a = \sum_{i,j=1}^n E_{ij}(a)\varphi^{k-1}(S_iS_j^*)$ for all $a \in \mathcal{F}_n^k$. Define by induction an increasing sequence of unital selfadjoint subspaces $\mathfrak{W}_r$ of $\mathcal{F}_n^{k-1}$ so that*

$$\begin{aligned} \mathfrak{S}_1 &= \mathrm{span}\{E_{jj}(wxw^*) : x \in \mathcal{D}_n^1,\ j = 1, \dots, n\}, \\ \widetilde{\mathfrak{S}}_{r+1} &= \mathrm{span}\{E_{jj}((\mathrm{Ad}\, w \circ \varphi)(x)) : x \in \mathfrak{S}_r,\ j = 1, \dots, n\}, \\ \mathfrak{S}_{r+1} &= \mathfrak{S}_r + \widetilde{\mathfrak{S}}_{r+1}. \end{aligned}$$

We agree that $\mathfrak{S}_0 = \mathbb{C}1$. Let R be the smallest integer such that $\mathfrak{S}_R = \mathfrak{S}_{R-1}$. Then $\lambda_w(\mathcal{D}_n) \subseteq \mathcal{D}_n$ if and only if $\lambda_w(\mathcal{D}_n^R) \subseteq \mathcal{D}_n$.

The above theorem leads to the following corollary, [30].

COROLLARY 5.5. *Let w be a unitary in $\mathcal{F}_n^k$. If $w\mathcal{D}_n^1 w^* = \varphi^{k-1}(\mathcal{D}_n^1)$ then $\lambda_w(\mathcal{D}_n) \subseteq \mathcal{D}_n$. Thus if $u \in \mathcal{F}_n^k$, $z \in \mathcal{U}(\mathcal{F}_n^1)$ and $u(z\mathcal{D}_n^1 z^*)u^* = \varphi^{k-1}(z\mathcal{D}_n^1 z^*)$ then $\mathcal{A} = \lambda_z(\mathcal{D}_n)$ is λ_u-invariant.*

The second part of the above corollary deals with one of the motivations for investigations of the question when λ_w preserves $\mathcal{D}_n$. Namely, this information can be useful when searching for MASAs of $\mathcal{O}_n$ globally invariant under an endomorphism. The simplest examples involve product type standard MASAs, arising as $\lambda_z(\mathcal{D}_n)$ for some Bogolubov automorphism λ_z of $\mathcal{O}_n$, $z \in \mathcal{U}(\mathcal{F}_n^1)$. Existence of invariant MASAs is in turn helpful in determining entropy of an endomorphism, as demonstrated in [54, 53].

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References

[1] P. T. Akemann, *On a class of endomorphisms of the hyperfinite II_1 factor*, Doctoral Dissertation, UC Berkeley, 1997.

[2] R. J. Archbold, *On the "flip-flop" automorphism of $C^*(S_1, S_2)$*, Quart. J. Math. Oxford Ser. (2) 30 (1979), 129–132.

[3] V. V. Bazhanov, V. V. Mangazeev, S. M. Sergeev, *Quantum geometry of 3-dimensional lattices and tetrahedron equation*, in: XVIth International Congress on Mathematical Physics, World Sci. Publ., Hackensack, NJ, 2010, 23–44; arXiv:0911.3693.

[4] C. Binnenhei, *Charged quantum fields associated with endomorphisms of CAR and CCR algebras*, Ph.D. Thesis, arXiv:math/9809035.

[5] J. Böckenhauer, D. E. Evans, *Modular invariants and subfactors*, in: Mathematical Physics in Mathematics and Physics (Siena, 2000), Fields Inst. Commun. 30, Amer. Math. Soc., Providence, RI, 2001, 11–37.

[6] W. Bosma, J. Cannon, C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. 24 (1997), 235–265.

[7] M. Boyle, *Open problems in symbolic dynamics*, in: Geometric and Probabilistic Structures in Dynamics, Contemp. Math. 469, Amer. Math. Soc., Providence, RI, 2008, 69–118.

[8] O. Bratteli, P. E. T. Jørgensen, *Isometries, shifts, Cuntz algebras and multiresolution wavelet analysis of scale N*, Integral Equations Operator Theory 28 (1997), 382–443.

[9] O. Bratteli, P. E. T. Jørgensen, *Iterated function systems and permutation representations of the Cuntz algebra*, Mem. Amer. Math. Soc. 139 (1999), no. 663.

[10] A. L. Carey, D. E. Evans, *On an automorphic action of $U(n, 1)$ on $\mathcal{O}_n$*, J. Funct. Anal. 70 (1987), 90–110.

[11] A. L. Carey, J. Phillips, A. Rennie, *Twisted cyclic theory and an index theory for the gauge invariant KMS state on the Cuntz algebra $\mathcal{O}_n$*, J. K-Theory 6 (2010), 339–380.

[12] M. Choda, *Entropy of Cuntz's canonical endomorphism*, Pacific J. Math. 190 (1999), 235–245.

[13] A. Connes, *Periodic automorphisms of the hyperfinite factor of type II_1*, Acta Sci. Math. (Szeged) 39 (1977), 39–66.

[14] R. Conti, *Automorphisms of the UHF algebra that do not extend to the Cuntz algebra*, J. Aust. Math. Soc. 89 (2010), 309–315.

[15] R. Conti, F. Fidaleo, *Braided endomorphisms of Cuntz algebras*, Math. Scand. 87 (2000), 93–114.

[16] R. Conti, J. H. Hong, W. Szymański, *The restricted Weyl group of the Cuntz algebra and shift endomorphisms*, J. Reine Angew. Math. (to appear), arXiv:1006.4791.

[17] R. Conti, J. Kimberley, W. Szymański, *More localized automorphisms of the Cuntz algebras*, Proc. Edinb. Math. Soc. (2) 53 (2010), 619–631.

[18] R. Conti, J. Kimberley, W. Szymański, unpublished notes.

[19] R. Conti, C. Pinzari, *Remarks on the index of endomorphisms of Cuntz algebras*, J. Funct. Anal. 142 (1996), 369–405.

[20] R. Conti, M. Rørdam, W. Szymański, *Endomorphisms of $\mathcal{O}_n$ which preserve the canonical UHF-subalgebra*, J. Funct. Anal. 259 (2010), 602–617.

[21] R. Conti, W. Szymański, *Computing the Jones index of quadratic permutation endomorphisms of $\mathcal{O}_2$*, J. Math. Phys. 50 (2009), 012705.

[22] R. Conti, W. Szymański, *Labeled trees and localized automorphisms of the Cuntz algebras*, Trans. Amer. Math. Soc. 363 (2011), 5847–5870.

[23] J. Cuntz, *Simple C^*-algebras generated by isometries*, Comm. Math. Phys. 57 (1977), 173–185.

[24] J. Cuntz, *Automorphisms of certain simple C^*-algebras*, in: Quantum Fields—Algebras, Processes (Bielefeld, 1978), Springer, Vienna, 1980, 187–196.

[25] J. Cuntz, *Regular actions of Hopf algebras on the C^*-algebra generated by a Hilbert space*, in: Operator Algebras, Mathematical Physics, and Low-Dimensional Topology (Istanbul, 1991), Res. Notes Math. 5, A K Peters, Wellesley, MA, 1993, 87–100.

[26] S. Doplicher, J. E. Roberts, *Why there is a field algebra with a compact gauge group describing the superselection structure in particle physics*, Comm. Math. Phys. 131 (1990), 51–107.

[27] M. Enomoto, M. Fujii, H. Takehana, Y. Watatani, *Automorphisms on Cuntz algebras. II*, Math. Japon. 24 (1979/80), 463–468.

[28] M. Enomoto, H. Takehana, Y. Watatani, *Automorphisms on Cuntz algebras*, Math. Japon. 24 (1979/80), 231–234.

[29] R. Gohm, *A probabilistic index for completely positive maps and an application*, J. Operator Theory 54 (2005), 339–361.

[30] J. H. Hong, A. Skalski, W. Szymański, *On invariant MASAs for endomorphisms of the Cuntz algebras*, Indiana Univ. Math. J. 59 (2010), 1873–1892.

[31] A. Hopenwasser, J. R. Peters, S. C. Power, *Subalgebras of graph C^*-algebras*, New York J. Math. 11 (2005), 351–386.

[32] M. Izumi, *Application of fusion rules to classification of subfactors*, Publ. Res. Inst. Math. Sci. 27 (1991), 953–994.

[33] M. Izumi, *Subalgebras of infinite C^*-algebras with finite Watatani indices I. Cuntz algebras*, Comm. Math. Phys. 155 (1993), 157–182.

[34] M. Izumi, *The structure of sectors associated with Longo-Rehren inclusions. II. Examples*, Rev. Math. Phys. 13 (2001), 603–674.

[35] M. Izumi, *Finite group actions on C^*-algebras with the Rohlin property. I*, Duke Math. J. 122 (2004), 233–280.

[36] V. F. R. Jones, *On a family of almost commuting endomorphisms*, J. Funct. Anal. 122 (1994), 84–90.

[37] K. Kawamura, *Polynomial endomorphisms of the Cuntz algebras arising from permutations. I. General theory*, Lett. Math. Phys. 71 (2005), 149–158.

[38] K. Kawamura, *Branching laws for polynomial endomorphisms of Cuntz algebras arising from permutations*, Lett. Math. Phys. 77 (2006), 111–126.

[39] K. Kawamura, Y. Hayashi, D. Lascu, *Continued fraction expansions and permutative representations of the Cuntz algebra $\mathcal{O}_\infty$*, J. Number Theory 129 (2009), 3069–3080.

[40] E. Kirchberg, N. C. Phillips, *Embedding of exact C^*-algebras in the Cuntz algebra $\mathcal{O}_2$*, J. Reine Angew. Math. 525 (2000), 17–53.

[41] B. P. Kitchens, *Symbolic Dynamics. One-Sided, Two-Sided and Countable State Markov Shifts*, Universitext, Springer, Berlin, 1998.

[42] D. Lind, B. Marcus, *An Introduction to Symbolic Dynamics and Coding*, Cambridge Univ. Press, Cambridge, 1995.

[43] R. Longo, *Index of subfactors and statistics of quantum fields. I*, Comm. Math. Phys. 126 (1989), 217–247.

[44] R. Longo, *Index of subfactors and statistics of quantum fields. II. Correspondences, braid group statistics and Jones polynomial*, Comm. Math. Phys. 130 (1990), 285–309.

[45] R. Longo, *A duality for Hopf algebras and for subfactors. I*, Comm. Math. Phys. 159 (1994), 133–150.

[46] Y. I. Manin, M. Marcolli, *Error-correcting codes and phase transitions*, arXiv:0910.5135.

[47] K. Matsumoto, J. Tomiyama, *Outer automorphisms of Cuntz algebras*, Bull. London Math. Soc. 25 (1993), 64–66.

[48] H. Matui, *Classification of outer actions of* $\mathbb{Z}^N$ *on* $\mathcal{O}_2$, Adv. Math. 217 (2008), 2872–2896.

[49] M. Mori, O. Suzuki, Y. Watatani, *Representations of Cuntz algebras on fractal sets*, Kyushu J. Math. 61 (2007), 443–456.

[50] V. Nekrashevych, *Cuntz-Pimsner algebras of group actions*, J. Operator Theory 52 (2004), 223–249.

[51] S. C. Power, *Homology for operator algebras, III. Partial isometry homotopy and triangular algebras*, New York J. Math. 4 (1998), 35–56.

[52] M. Rørdam, *A short proof of Elliott's theorem:* $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$, C. R. Math. Rep. Acad. Sci. Canada 16 (1994), 31–36.

[53] A. Skalski, *Noncommutative topological entropy of endomorphisms of Cuntz algebras II*, arXiv:1002.2276.

[54] A. Skalski, J. Zacharias, *Noncommutative topological entropy of endomorphisms of Cuntz algebras*, Lett. Math. Phys. 86 (2008), 115–134.

[55] W. Szymański, *On localized automorphisms of the Cuntz algebras which preserve the diagonal subalgebra*, in: New Development of Operator Algebras, RIMS Kôkyûroku 1587, Kyoto Univ., Kyoto, 2008, 109–115.

[56] S.-K. Tsui, *Some weakly inner automorphisms of the Cuntz algebras*, Proc. Amer. Math. Soc. 123 (1995), 1719–1725.