

## CLUSTERS IN MIDDLE-PHASE PERCOLATION ON HYPERBOLIC PLANE

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**Abstract.** I consider  $p$ -Bernoulli bond percolation on transitive, nonamenable, planar graphs with one end and on their duals. It is known from [BS01] that in such a graph  $G$  we have three essential phases of percolation, i.e.

$$0 < p_c(G) < p_u(G) < 1,$$

where  $p_c$  is the critical probability and  $p_u$ —the unification probability. I prove that in the middle phase a.s. all the ends of all the infinite clusters have one-point boundaries in  $\partial\mathbb{H}^2$ . This result is similar to some results in [Lal].

**1. Introduction.** For any graph  $G$ , let  $V(G)$  denote its set of vertices and  $E(G)$ —its set of edges. A *percolation* on  $G$  is a random subgraph of  $G$  or, one can say, a probability measure on the space of subgraphs of  $G$ . For any infinite connected graph  $G$  and  $p \in [0; 1]$  let  $\omega^{(p)}(G)$  denote the process of  *$p$ -Bernoulli bond percolation* on  $G$ , which is a random subgraph of  $G$  formed by taking stochastically independently each edge of  $G$  with probability  $p$  to the random graph (we call these edges *open*) and taking all the vertices of  $G$  to it. The components of  $\omega^{(p)}(G)$  are often called *clusters*. One can say that in some sense the clusters of  $\omega^{(p)}$  increase with the value of parameter  $p$ .<sup>1</sup> When  $p$  increases from 0 to 1, first we have a.s. no infinite clusters and, suddenly, above some threshold we have a.s. some infinite cluster in  $\omega^{(p)}$ . Then, it turns out that in the case of transitive, nonamenable, planar graphs with one end (which are exactly vertex-transitive tiling graphs in the hyperbolic plane  $\mathbb{H}^2$ ) we have a.s. infinitely many infinite clusters in  $\omega^{(p)}$  for some period of time, and then, after another threshold, we get exactly one infinite cluster till

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<sup>1</sup>This is formalized in [Grim], Chapter 2.1.

the value of 1 (these infinitely many clusters “merge” into one). Therefore we say about three phases of Bernoulli bond percolation in such graphs.<sup>2</sup> This was established in [BS01] for transitive, nonamenable, planar graphs with one end and in [Lal] for Cayley graphs of a wide class of Fuchsian groups (see Remark 7). Earlier examples of graphs on which we have three phases (in a generalized sense) were products of regular tree and  $\mathbb{Z}^d$  given in [GN]. Let us define precisely the thresholds described above. The *critical probability* (or *critical parameter*)  $p_c(G)$  of any graph  $G$  is defined to be the infimum of  $p \in [0; 1]$  such that a.s. there is some infinite cluster in  $\omega^{(p)}(G)$ . Similarly, the *unification probability*  $p_u(G)$  is the infimum of  $p \in [0; 1]$  such that  $\omega^{(p)}(G)$  has exactly one infinite cluster a.s.

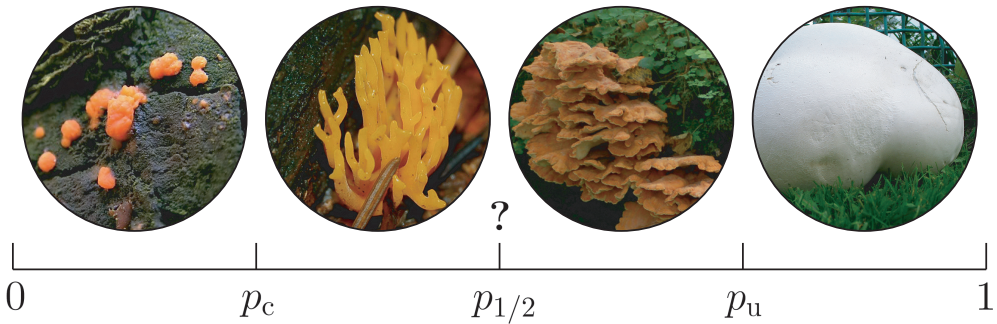


Fig. 1. The idea of four phases of percolation in  $\mathbb{H}^3$ . Authors of the photos (from the left): 1. Dorota R. (radzido); 2, 3. Agata Piestrzyńska-Kajtoch; 4. Kazimiera Stelmach

A couple of important book on percolation, including the basics of percolation, are [Grim] and [LP]. Also the papers [BS01] and [Lal] considering Bernoulli percolation on planar graphs in  $\mathbb{H}^2$  have references on percolation on other planar graphs (e.g. trees and lattice  $\mathbb{Z}^2$ ); I base on the paper [BS01] in this work.

The motivation for investigating the boundaries of ends of infinite clusters comes from considering percolation phases in case of regular tilings of  $\mathbb{H}^2$  and the 3-dimensional hyperbolic space  $\mathbb{H}^3$ . Let us visualize  $\mathbb{H}^2$  and  $\mathbb{H}^3$  as the Poincaré disc models.

On graphs of regular tilings of  $\mathbb{H}^3$  we conjecturally have three phases of percolation. (It is due to Conjecture 6 and Question 3 in [BS96], see also Theorem 10 of [BB]. The inequality  $p_c(G) < p_u(G)$  has been also established for some Cayley graphs of all nonamenable finitely generated groups in [PSN] and for some kind of continuous percolation in  $\mathbb{H}^n$  in [Tyk].) So in the first phase (for  $0 \leq p \leq p_c$ ) we have a.s. only finite clusters, which roughly look like points (in large scale), so 0-dimensional objects. In the last phase (for  $p_u \leq p \leq 1$ ) there is only one big one-ended infinite cluster (one-ended means: after throwing out a bounded set it still has only one infinite component), so it looks like the whole Poincaré ball, which is of dimension 3. The conjecture of my advisor is that in the middle phase we have a.s. “1-dimensional” (fibrous) infinite clusters with  $p$  below some threshold  $p_{1/2} \in (p_c; p_u)$  and “2-dimensional” (fan-shaped) with  $p$  above  $p_{1/2}$  (see Fig. 1). So we should have four phases—one more than the dimension of the space ( $\mathbb{H}^3$ ).

<sup>2</sup>See also Remark 10.

Following this idea, in the percolation on a graph of tiling of  $\mathbb{H}^2$  we should have only three such phases. We already know three phases of it by [BS01]—see Theorem 8 and Corollary 9, so we want the clusters to be 0-dimensional in the first phase, 1-dimensional in the second and 2-dimensional in the third.

My formalization of 1-dimensional is the following: all the ends of each infinite cluster have one-point boundaries (which is to be explained further). The main result in this paper (Theorem 20 and Corollary 21) says that in the middle phase (for a transitive, nonamenable, planar graph with one end and also for its dual) the infinite clusters are a.s. 1-dimensional.

**Acknowledgements.** I wish to express gratitude to my advisor, Jan Dymara, who once proposed me percolation as master thesis topic and led me through doing it. My master thesis has developed to this article.

**2. Boundaries of ends.** Now I am going to define the boundary of an end of an infinite cluster in  $\mathbb{H}^2$ , but the definition is formulated in case of any “nice” topological space.

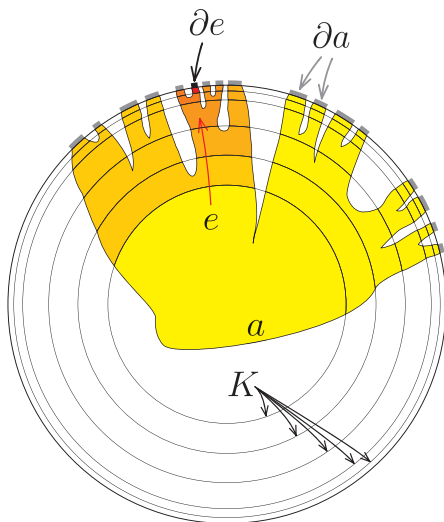


Fig. 2. An end  $e$  of a set  $a$ , its boundary  $\partial e$  and the boundary  $\partial a$  of the whole set in case of Poincaré disc. ( $K$  in the picture is shown as getting bigger and bigger.)

**DEFINITION 1.** Let  $X$  be a completely regular Hausdorff  $(T_{3\frac{1}{2}})$ , locally compact topological space. Then:

- An *end of a subset*  $a \subseteq X$  is a function  $e$  defined on the family of all compact subsets of  $X$ , satisfying the following:
  - for any compact  $K \subseteq X$  the set  $e(K)$  is one of the components of  $a \setminus K$ ;
  - for  $K \subseteq K' \subseteq X$ —both compact—we have

$$e(K) \supseteq e(K').$$

Now let  $\hat{X}$  be an arbitrary compactification of  $X$ . Then

- The *boundary* of  $a \subseteq X$  is

$$\partial a = \bar{a}^{\hat{X}} \setminus X$$

(by  $\bar{a}^Y$  I mean the closure taken in the space  $Y$ ).

- Finally the *boundary of an end*  $e$  of  $a \subseteq X$  is

$$\partial e = \bigcap_{\substack{K \subseteq X \\ K - \text{compact}}} \partial e(K).$$

In this paper I always take  $X = \mathbb{H}^2$  and  $\hat{X} = \hat{\mathbb{H}}^2$  (the closed Poincaré ball, i.e.  $\hat{\mathbb{H}}^2 = \mathbb{H}^2 \cup \partial\mathbb{H}^2$ ). The role of  $a$  will be played by clusters of percolation in  $\mathbb{H}^2$ .

**3. The graph.** Now I introduce some notions needed to explain what class of graphs I am considering. Then I provide two equivalent definitions of this class.

**DEFINITION 2.** By a *polygonal tiling* of  $\mathbb{H}^2$ , or *tiling* of  $\mathbb{H}^2$  for short, I mean a locally finite family of hyperbolic polygons (in this paper by a *polygon* I mean only a bounded finite-sided polygon whose perimeter is a simple closed polygonal chain) which covers the hyperbolic plane in such a way that they have pairwise disjoint interiors and any two different of them either are disjoint, or intersect exactly at a sum of some of their common sides and vertices. The *graph of such a tiling* as above is just the graph obtained from all the vertices and edges of the tiling. Obviously such a graph is always a planar graph. A *regular tiling* is a polygonal tiling by congruent regular polygons (regular polygons means: equilateral and equiangular).

A *plane graph* is a geometric realization of a planar graph in the plane (in this definition only the topology plays a role, so it does not matter if the plane is hyperbolic). In this paper *faces* of a plane graph are the components of its complement in the plane. Here I overload the notation, denoting both the abstract planar graph and its plane realization by  $G$  (although it does matter, when taking the dual graph, see Definition 22).

**REMARK 3.** I declare all the graphs mentioned in this paper to be *locally finite*, i.e. having every vertex of finite degree.

Further in this paper I consider graphs of polygonal tilings of  $\mathbb{H}^2$  which are vertex-transitive in the sense that some groups of isometries of  $\mathbb{H}^2$  preserving them act on their vertices transitively. I call such graphs *vertex-transitive tiling graphs*. I consider also their duals as well (see Definition 22).

The class of vertex-transitive tiling graphs can be completely characterized as follows:

**PROPOSITION 4.** *Every vertex-transitive tiling graph  $G$  is a transitive, nonamenable, planar graph with one end.*

Below I define the graph properties mentioned above:

**DEFINITION 5.** Let  $G$  be any locally finite graph. We define it to:

- *have one end*, if for any finite set  $V_0 \subseteq V(G)$  the subgraph induced on its complement  $V(G) \setminus V_0$  has exactly one unbounded component;

- be *nonamenable*, if there is a constant  $\varepsilon > 0$  such that for every finite nonempty  $V_0 \subseteq V(G)$  we have  $|\partial V_0|/|V_0| \geq \varepsilon$ , where  $\partial V_0$  is the set of edges of  $G$  with exactly one vertex in  $V_0$  (a kind of boundary of  $V_0$ ). Otherwise we call  $G$  *amenable*.

*Proof of Proposition 4.* Planarity and transitivity are obviously satisfied, so the remaining properties of  $G$  to show are having only one end and nonamenability:

*One end:* Let  $V_0$  be finite subset of  $V(G)$ . Remove  $V_0$  from  $V(G)$  and take the induced subgraph  $G'$  (here I mean the plane graph). Take a hyperbolic ball  $B$  which covers  $V_0$  together with all the tiles meeting  $V_0$ . Now, for every two vertices not lying in  $B$  there is a polygonal path  $P_0$  in  $\mathbb{H}^2$  joining them and not intersecting  $B$ . We can replace this path by a path  $P$  in graph  $G$  chosen to go along perimeters of consecutive tiles visited by  $P_0$ . This path may meet  $B$ , but is still disjoint with  $V_0$ . Hence all vertices in  $V(G) \setminus B$  lie in one component of  $G'$ . But the rest of vertices of  $G$  lie in  $B$ , so there are finitely many of them, whence  $G'$  has exactly one unbounded component.

*Nonamenability:* The tiling related to  $G$  is vertex-transitive, so all its tiles belong to finitely many classes of congruence. Therefore this case is just a particular case of the following lemma:

LEMMA 6. *Let  $T$  be a tiling of  $\mathbb{H}^2$  by polygons of finitely many classes of congruence and let  $G$  be the graph of  $T$ . Then  $G$  is nonamenable.*

*Proof.* Let us build a tiling dual (in some sense) to  $T$  (for the notion of duality see Definition 22).

First, we construct the dual tiling on some finite set  $F \subset T$  which includes all types of congruence of tiles of  $T$  (more precisely, we construct traces of the dual tiling on the tiles from  $F$ ). Let  $A \in F$ . Let us choose a point  $A^\dagger \in \text{int } A$  (considered dual to  $A$ ). Join it to the midpoints of all the edges of  $A$  by polygonal paths which meet pairwise only in  $A^\dagger$  and meet boundary of  $A$  only in their ends. Call these paths *rays*.

Then we move the picture to any other tile of  $T$  by an isometry moving some  $A \in F$  to it. Let us denote the obtained dual tiling by  $T^\dagger$ .

Now, let  $K \subset V(G)$  be finite, nonempty. We are going to bound  $|\partial K|/|K|$  from below by a positive constant independent of  $K$ .

Let us estimate  $|\partial K|$  from below. Each edge of  $\partial K$  is hit by two rays, one from one side and second—from the other side. Let  $L$  be the maximal number of segments of rays (for all tiles in  $F$ ; obviously,  $L$  exists). Then the number of segments of sum of the two rays hitting our edge, which we think of as path dual to the edge, is not greater than  $2L$ . Let  $K^\dagger$  be the family of tiles of  $T^\dagger$  dual to vertices in  $K$ . Note that  $\partial(\bigcup K^\dagger)$  is the set of sides of  $\bigcup K^\dagger$ . Then

$$2L|\partial K| \geq |\partial(\bigcup K^\dagger)|.$$

Let  $a$  be the minimal area of a polygon in  $T^\dagger$  ( $a$  exists, because every polygon in  $T^\dagger$  contains one of the components of some polygon of  $T$  divided by its rays and there are finitely many classes of congruence of such components). Then

$$\text{Area}(\bigcup K^\dagger) \geq a|K^\dagger| = a|K|.$$

Now, we show that

$$\pi|\partial(\bigcup K^\dagger)| \geq \text{Area}(\bigcup K^\dagger).$$

Let us assume that  $\bigcup K^\dagger$  is simply connected (and therefore also connected). (It suffices to do the proof in this case, because then making “holes” in  $\bigcup K^\dagger$  increases  $|\partial(\bigcup K^\dagger)|$  and decreases  $\text{Area}(\bigcup K^\dagger)$ , hence preserves the inequality, and taking disjoint union of such “holed” polygons acts additively on both  $|\partial(\bigcup K^\dagger)|$  and  $\text{Area}(\bigcup K^\dagger)$ , hence also preserves the inequality.) In this case the Euler characteristic  $\chi(\bigcup K^\dagger) = 1$  and by the Gauss–Bonnet formula for polygons we have

$$-\text{Area}(\bigcup K^\dagger) + \sum_{\substack{v - \text{vertex} \\ \text{of } \partial(\bigcup K^\dagger)}} (\pi - \angle v) = 2\pi,$$

because  $\mathbb{H}^2$  has constant curvature  $-1$  (here  $\angle v$  denotes the angle between the consecutive edges of  $\partial(\bigcup K^\dagger)$ ). In the sum each vertex is counted each time the boundary path goes through it.) Hence

$$\text{Area}(\bigcup K^\dagger) = \sum_{\substack{v - \text{vertex} \\ \text{of } \partial(\bigcup K^\dagger)}} (\pi - \angle v) - 2\pi \leq \sum_{\substack{v - \text{vertex} \\ \text{of } \partial(\bigcup K^\dagger)}} \pi = \pi|\partial(\bigcup K^\dagger)|,$$

which we needed.

So for finite, nonempty  $K \subset V(G)$

$$2L|\partial K| \geq |\partial(\bigcup K^\dagger)| \geq \frac{\text{Area}(\bigcup K^\dagger)}{\pi} \geq \frac{a|K|}{\pi},$$

hence

$$\frac{|\partial K|}{|K|} \geq \frac{a}{2\pi L} > 0. \blacksquare$$

This completes the proof of Proposition 4.  $\blacksquare$

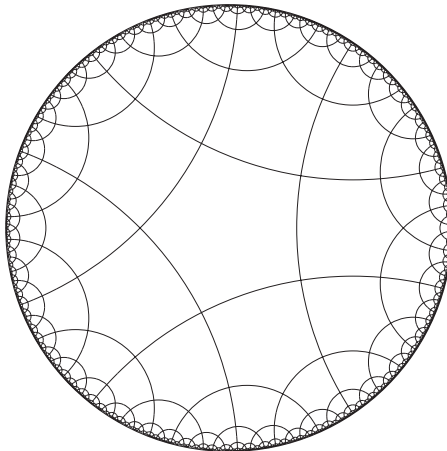


Fig. 3. An example of tiling of  $\mathbb{H}^2$  by regular right-angled pentagons

REMARK 7. The main theorem is proven for all vertex-transitive tiling graphs (Theorem 20) and their duals (Corollary 21). Earlier in my master thesis I considered only graphs of two special regular tilings of  $\mathbb{H}^2$  (one of them is shown in Fig. 3). On the other hand, Lalley in [Lal] proves similar facts about Bernoulli percolation for Cayley graphs of cocompact Fuchsian groups of genus at least 2 and for some class of hyperbolic triangle groups (namely: for groups of presentation  $\langle c_1, c_2, c_3 \mid c_1^2 = c_2^{4m} = c_3^{4m} = c_1 c_2 c_3 = 1 \rangle$ , where  $m \geq 5$ ).

First of all I deduce from [BS01, Theorem 1.1], cited below, that, indeed, on the graphs I consider, there are three essential phases of Bernoulli bond percolation, mentioned in the introduction.

THEOREM 8 ([BS01]). *Let  $G$  be a transitive, nonamenable, planar graph with one end. Then*

$$0 < p_c(G) < p_u(G) < 1,$$

*for Bernoulli bond or site<sup>3</sup> percolation on  $G$ .*

Basing on Proposition 4 the following is obvious:

COROLLARY 9. *For any vertex-transitive tiling graph  $G$  we have*

$$0 < p_c(G) < p_u(G) < 1.$$

REMARK 10. In fact, for  $p \in [0; p_c]$  there are a.s. no infinite cluster in  $\omega^{(p)}$ , there are a.s.  $\infty$  of them for  $p \in (p_c; p_u)$  and exactly 1 for  $p \in [p_u; 1]$  (so we have three essential and pure phases, determined by the number of infinite clusters). The same is true about the dual  $G^\dagger$  (see Section 5 for notions of duality). These remarks can be easily deduced from Theorems 1.1, 3.7, 1.2 and 1.3 of [BS01] (see also proofs of Theorems 1.1 and 3.8 there; the fact that the event of existence of an infinite cluster is increasing should be used; for increasing and decreasing events, see [Grim], Chapter 2.1, especially Theorem 2.1).

REMARK 11. One can easily deduce from the proof of Proposition 2.1 from [BS01] that in fact any transitive, nonamenable, planar graph with one end can be realized as a vertex-transitive tiling graph in  $\mathbb{H}^2$ . Hence vertex-transitive tiling graphs are all the graphs known by [BS01, Thm. 1.1] to have three essential phases of Bernoulli bond percolation.

It turns out also that, in this setting, the property that all the infinite clusters of the random subgraph have one-point boundaries of ends does not depend on the embedding of the underlying whole graph in  $\mathbb{H}^2$ , but just on the abstract graph. This can be explained in terms of Gromov boundary<sup>4</sup>:  $\partial\mathbb{H}^2$  can be defined as the Gromov boundary of  $\mathbb{H}^2$ . On the other hand, when graph  $G$  is embedded by a quasi-isometry in  $\mathbb{H}^2$  (it is then closed in  $\mathbb{H}^2$ ), then by [BH], Chapter III.H, Theorem 1.9,  $G$  is hyperbolic (in the sense of Gromov) and from Theorem 3.9 from this chapter that quasi-isometry induces a homeomorphism of the Gromov boundaries of  $G$  and  $\mathbb{H}^2$ . Let  $\hat{G}$  be the compactification of an abstract graph  $G$  by its Gromov boundary  $\partial G$ . Then one can embed  $\hat{G}$  in  $\hat{\mathbb{H}}^2$  sending  $\partial G$  onto  $\partial\mathbb{H}^2$  by

<sup>3</sup>Bernoulli site percolation is performed by removing vertices of the graph (instead of edges in bond percolation).

<sup>4</sup>For basics on Gromov boundaries, see [BH], Chapter III.H.3.

the above homeomorphism. One can easily check that for a subset  $A$  of  $G \subseteq \mathbb{H}^2 \subseteq \widehat{\mathbb{H}^2}$  its ends and boundaries of ends are in principle the same as when  $A$  is considered a subset of  $\widehat{G}$ . It follows that phenomena of 1-dimensional clusters occurring on abstract  $\widehat{G}$  and on  $\widehat{\mathbb{H}^2}$  agree.

**4. Main theorem.** Before I prove the main theorem (Theorem 20), I need the following lemmas.

Let  $G$  be a vertex-transitive tiling graph and  $\omega^{(p)}$  be  $p$ -Bernoulli bond percolation process on  $G$  in the middle phase.

LEMMA 12. *The limits in  $\partial\mathbb{H}^2$  of paths in  $\omega^{(p)}$  a.s. lie densely in  $\partial\mathbb{H}^2$ .*

*Proof.* This can be deduced from Theorem 4.1 and Lemma 4.3 of [BS01], which I quote here:

THEOREM 13 ([BS01]). *Let  $T$  be a vertex-transitive tiling of  $\mathbb{H}^2$  with finite-sided faces, let  $G$  be the graph of  $T$ , and let  $\omega$  be Bernoulli percolation on  $G$ . Almost surely, every infinite component of  $\omega$  contains a path that has a unique limit point in  $\partial\mathbb{H}^2$ .*

The following lemma is formulated for *invariant percolation process* on  $G$ , i.e. random subgraph process whose probability distribution is invariant on some vertex-transitive group action on  $G$ . Bernoulli bond percolation is an example of invariant percolation.

LEMMA 14 ([BS01]). *Let  $T$  be a vertex-transitive tiling of  $\mathbb{H}^2$  with finite-sided faces, let  $G$  be the graph of  $T$ , and let  $\omega$  be invariant percolation on  $G$ . Let  $Z$  be the set of points  $z \in \partial\mathbb{H}^2$  such that there is a path in  $\omega$  with limit  $z$ . Then a.s.  $Z = \emptyset$  or  $Z$  is dense in  $\partial\mathbb{H}^2$ .*

Basing on Theorem 13 and on Remark 10, in our situation there are a.s. some paths in  $\omega^{(p)}$  with limit points in  $\partial\mathbb{H}^2$  and hence set  $Z$  from Lemma 14 is a.s. dense in  $\partial\mathbb{H}^2$ . ■

REMARK 15. In special case of  $G$  being the graph of regular tiling of  $\mathbb{H}^2$  with right-angled pentagons and  $p > \frac{1}{2}$ , this lemma can also be proved in the following more elementary way, similar to the technique used in the proof of Theorem 1 in [Lal] (on p. 171):

I embed an infinite complete binary tree in the graph  $G$  (see Fig. 4). (It is done by using hyperbolic geometry.)

When I have such a tree  $T$  embedded in  $G$ , I can move it by an isometry  $\gamma$  preserving  $G$  so that  $\partial\gamma(T)$  will be included in arbitrary (small) arc  $\Phi$  of  $\partial\mathbb{H}^2$  (see Proposition 19). The random graph  $\omega^{(p)} \cap \gamma(T)$  is  $p$ -percolation process on the tree  $\gamma(T)$ , where the critical probability equals  $\frac{1}{2}$ . So for  $p > \frac{1}{2}$  we a.s. obtain an open infinite path in  $\omega^{(p)} \cap \gamma(T)$ ; such path in  $\gamma(T)$  has always a limit in  $\partial\gamma(T) \subseteq \Phi$ . Such limits lie a.s. densely in  $\partial\mathbb{H}^2$ .

LEMMA 16. *In the middle phase a.s. every halfplane meets infinitely many infinite clusters of  $\omega^{(p)}$ .*

REMARK 17. In this paper a *halfplane* is always closed.

Before proving the lemma let us consider a group  $\Gamma$  of isometries of  $\mathbb{H}^2$  which acts transitively on vertices of  $G$  (by the assumption on  $G$ ). One can easily see that  $\Gamma$  is a discrete subgroup of  $\text{Isom}(\mathbb{H}^2)$  (with the usual topology), because it preserves a tiling



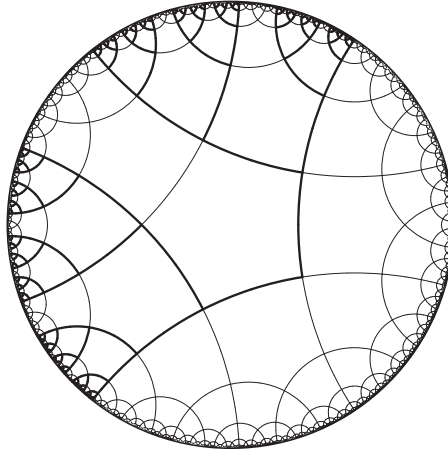


Fig. 4. Infinite complete binary tree embedded in  $G$  in Remark 15

of  $\mathbb{H}^2$ . Basing on that we are going to say something about the action of  $\Gamma$  on  $\mathbb{H}^2$  using basic theory of Fuchsian groups, which can be found in [K].

DEFINITION 18. There are three kinds of orientation preserving isometries of  $\mathbb{H}^2$  other than identity: so-called hyperbolic, parabolic and elliptic. This classification is based on how many fixed points in  $\partial\mathbb{H}^2$  such isometry has (it makes sense, since every isometry of  $\mathbb{H}^2$  extends continuously in a unique way to a homeomorphism of  $\widehat{\mathbb{H}^2}$ ). Such isometries with exactly two fixed points in  $\partial\mathbb{H}^2$  are *hyperbolic*, one fixed point—*parabolic* and no fixed points—*elliptic*. One may think of hyperbolic and elliptic isometries as analogues of translations and rotations in Euclidean plane, respectively. Some basics of these notions are present in [K].

A *Fuchsian group* is a discrete subgroup of  $\text{Isom}(\mathbb{H}^2)$  consisting only of orientation preserving isometries of  $\mathbb{H}^2$ . The *limit set* of a Fuchsian group  $\Gamma_F$  is the boundary in  $\partial\mathbb{H}^2$  of orbit  $\Gamma_F x_0$  of some point  $x_0 \in \mathbb{H}^2$  (one can observe that it does not depend on the choice of  $x_0$ ).

Let  $\Gamma_F$  be the Fuchsian group of all orientation preserving isometries in  $\Gamma$ . (The index of this subgroup in  $\Gamma$  is at most 2.) We claim that  $\Gamma_F$  acts cocompactly on  $\mathbb{H}^2$ , which means that there exists a compact set  $K \subseteq \mathbb{H}^2$  such that the family  $\Gamma_F K$  covers  $\mathbb{H}^2$ . Indeed, since  $\Gamma$  itself acts cocompactly on  $\mathbb{H}^2$ , then if we take  $K \cup \gamma K$  (for  $K$  the witness for  $\Gamma$ ), where  $\gamma$  is some orientation changing isometry  $\gamma \in \Gamma$ , we have covering of  $\mathbb{H}^2$  by  $\Gamma_F(K \cup \gamma K)$ .

Next we observe that the limit set of  $\Gamma_F$  is the whole  $\partial\mathbb{H}^2$ . If it were not, then the open set  $\widehat{\mathbb{H}^2} \setminus (\Gamma_F x_0 \cup \partial\Gamma_F x_0)$  for some orbit  $\Gamma_F x_0$  meets  $\partial\mathbb{H}^2$ , so some halfplane would be disjoint with this orbit, which is impossible because of cocompactness of  $\Gamma_F$ . So, by Theorem 3.4.4 from [K], the set of fixed points in  $\partial\mathbb{H}^2$  of hyperbolic translations in  $\Gamma_F$  is dense in  $\partial\mathbb{H}^2$ .

This gives us the following fact:

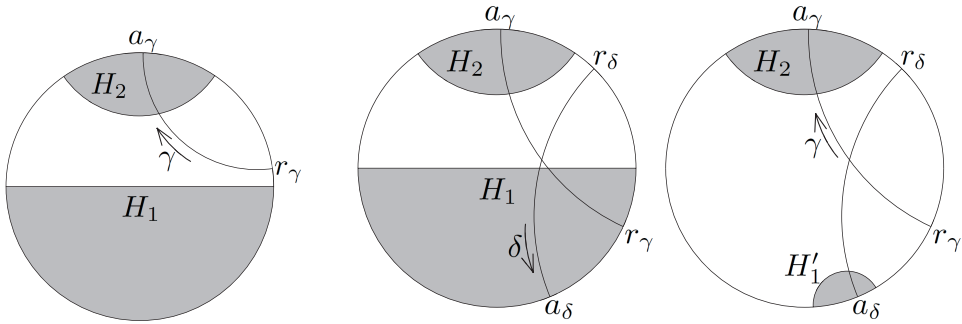


Fig. 5. Proof of Proposition 19

**PROPOSITION 19.** *Every halfplane  $H_1$  in  $\mathbb{H}^2$  can be mapped into any halfplane  $H_2$  by some isometry in  $\Gamma_F$  (and hence in  $\Gamma$ ).*

*Proof.* Take arbitrary halfplanes  $H_1$  and  $H_2$ . Let  $\gamma \in \Gamma_F$  be a hyperbolic translation with attracting point  $a_\gamma \in \partial\mathbb{H}^2$  lying in the interior of the closed arc  $\partial H_2$ . If the repelling point  $r_\gamma \in \partial\mathbb{H}^2$  of  $\gamma$  is not in  $\partial H_1$ , then some multiply composition of  $\gamma$  moves  $H_1$  into  $H_2$  (see left picture in Fig. 5). Now if  $r_\gamma \in \partial H_1$ , then take any  $\delta \in \Gamma_F$  with repelling point  $r_\delta$  distinct from  $a_\gamma$  and  $r_\gamma$  and not lying in  $\partial H_1$ . It can be deduced from the proof of Theorem 2.4.3 (Case 1) of [K] that the attracting point  $a_\delta$  of  $\delta$  is as well different from  $r_\gamma$ . Hence again some multiply composition of  $\delta$  maps  $H_1$  to  $H'_1$  which is arbitrarily close to  $a_\delta$ , so that its boundary  $\partial H'_1$  does not include point  $r_\gamma$  (middle picture in Fig. 5). Then some multiply composition of  $\gamma$  pushes  $H'_1$  into  $H_2$  (see right picture in Fig. 5). Composition of these two compositions gives us desired isometry. ■

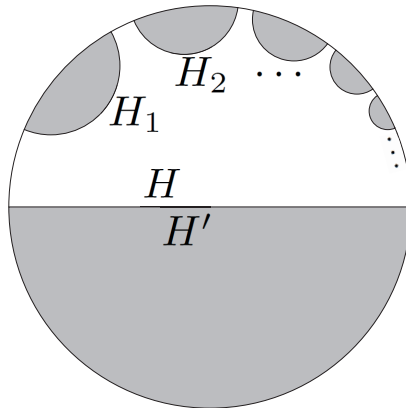


Fig. 6. Proof of Lemma 16

*Proof of Lemma 16.* Let us assume *a contrario* that there is a halfplane  $H$  which meets only finitely many infinite clusters of  $\omega^{(p)}$  with positive probability. In such situation the halfplane  $H' = \overline{H}^c$  includes entirely infinitely many infinite clusters (by Remark 10). Let

$H_1, H_2, \dots$  be a sequence of pairwise disjoint halfplanes all lying in  $H$ , and even more: such that distances between them are greater than twice the maximal hyperbolic length of an edge in  $G$ . By the above proposition we can move  $H'$  by some sequence of isometries  $\gamma_1, \gamma_2, \dots \in \Gamma$  into  $H_1, H_2, \dots$ , respectively (see Fig. 6).

Note that one can precisely say whether a halfplane contains infinitely many infinite clusters looking only at the behaviour of  $\omega^{(p)}$  on the edges intersecting with this halfplane. So the random event  $C(I)$  that in a halfplane  $I$  there are infinitely many infinite clusters depends only on these edges, for any halfplane  $I$ . There follows that events  $C(H_1), C(H_2), \dots$  are stochastically independent, because the underlying sets of edges are pairwise disjoint. Moreover, they have the same positive probability as  $C(H')$ , so the probability that none of them occurs is less or equal than  $(1 - P(C(H')))^n$  for any  $n \in \mathbb{N}$ , whence equal to 0. This gives us that a.s. some  $H_n$  contains infinitely many infinite clusters but so does  $H$ , because it includes  $H_n$ , a contradiction. This ends the proof of the lemma. ■

Now I state the main theorem.

**THEOREM 20.** *In the middle phase of Bernoulli bond percolation on any vertex-transitive tiling graph  $G$  a.s. all the ends of all the infinite clusters have one-point boundaries in  $\partial\mathbb{H}^2$ .*

*Proof.* The techniques used here are similar to those of Lalley used in [Lal]. Let  $\omega^{(p)}$  be  $p$ -Bernoulli bond percolation process on  $G$ , when  $p \in (p_c(G); p_u(G))$ . Let us assume *a contrario* that with positive probability there is an end  $e$  of an infinite cluster  $a$  of  $\omega^{(p)}$  with non-one-point boundary.

One can prove a topological fact saying that always the boundary of an end is connected and compact (the proof is given in Appendix). So in our situation  $\partial e$  is a non-degenerate closed arc in  $\partial\mathbb{H}^2$ , or the whole  $\partial\mathbb{H}^2$ . Let  $\Phi$  be an open non-empty arc in  $\partial\mathbb{H}^2$ , included in  $\partial e$ . By Lemma 12 the limits of paths in  $\omega^{(p)}$  lie densely in  $\Phi$ . I consider two cases:

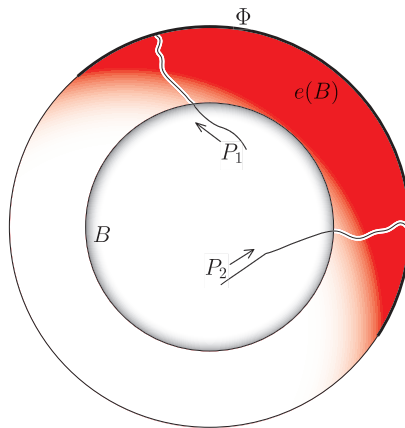


Fig. 7. Proof of Theorem 20, the first case

1) There are two paths  $P_1, P_2 \subseteq \omega^{(p)}$  not lying in  $a$  with distinct limits in  $\Phi$ .

Let us take a closed ball  $B$  in  $\mathbb{H}^2$ , meeting  $P_1$  and  $P_2$ . Then  $\partial e(B)$  (and also  $\overline{e(B)}^{\mathbb{H}^2}$ ) contains  $\Phi$ , and  $e(B)$  is connected, but  $P_1$  and  $P_2$  have limits in  $\Phi$  so they should cut  $e(B) \subseteq a$ , which is a contradiction (see Fig. 7).

2) In  $\Phi$  there are infinitely many limits of open paths lying in  $a$ .

Then, let us take two such paths  $P_1, P_2$  with distinct limits in  $\Phi$  and two others  $P'_1, P'_2$  with still other limits in  $\Phi$  such as in Fig. 8.

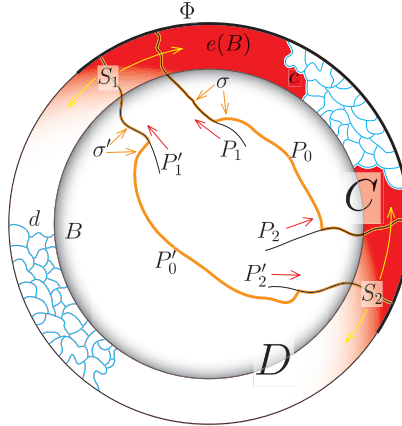


Fig. 8. Proof of Theorem 20, the second case

We can join  $P_1$  and  $P_2$  by an open path  $P_0$  in  $a$  and so  $P'_1$  and  $P'_2$  by  $P'_0$  in  $a$ . It provides paths  $\sigma, \sigma' \subseteq a$  shown in Fig. 8, which disconnects  $\mathbb{H}^2$  into components, two of which— $C$  and  $D$ —are shown in the figure. We can take two halfplanes lying in  $C$  and  $D$ , respectively. From Lemma 16 we know that each of them a.s. meets some infinite cluster other than  $a$ . So one of these clusters lies in  $C$  and the other in  $D$ —denote them by  $c$  and  $d$ , respectively. So  $\partial c \subseteq \partial C$  and  $\partial d \subseteq \partial D$ , which means that for a sufficiently large ball  $B$  the union of  $c$  and  $d$  disconnects  $\mathbb{H}^2 \setminus B$  into components, two of which are  $S_1$  and  $S_2$  containing respectively the tails of  $P_1, P'_1$  and  $P_2, P'_2$ . But the areas of  $S_i$  between  $P_i$  and  $P'_i$  for  $i = 1, 2$  meet  $e(B)$  (because their boundaries lie in  $\Phi$ ) so  $e(B)$  meet both  $S_1$  and  $S_2$ , which means that  $e(B)$  is not connected (because it is disjoint with  $c$  and  $d$ ), a contradiction.

This ends the proof. ■

## 5. Dual graphs

**COROLLARY 21.** *Theorem 20 also applies to the dual graph of any vertex-transitive tiling graph  $G$ .*

Let us introduce notions of duality:

**DEFINITION 22.** For any plane graph  $G$  one defines its *dual graph*  $G^\dagger$ : the set of vertices is the set of faces of  $G$  and two such vertices are joined by an edge, iff the corresponding faces are neighbours by an edge in  $G$ . (Note that to define the dual graph the plane

realization is needed, not only the abstract graph.) Such a dual graph is also a planar graph, because one can realize it in the plane placing its vertices inside the faces of the original graph  $G$  (called also the *primal graph*), and constructing the edges as some plane paths leading from any vertex of the dual graph to an interior point of an edge of the face including it, then to the vertex inside the second face touching this edge. We call the constructed edge the *dual edge* to the original edge, which is cut by it in exactly one point.

REMARK 23. The dual graph of a plane graph  $G$  does not need to be a simple graph, i.e. it may have multiple edges or loops.

I will consider the dual percolation process for  $\omega^{(p)}$ , which I define below:

DEFINITION 24. For a plane graph  $G$  of a polygonal tiling and for any edge  $e$  of  $G$  let  $e^\dagger \in G^\dagger$  denote the dual edge of  $e$  in the dual graph  $G^\dagger$ . (The operation  $e \mapsto e^\dagger$  is a bijection between  $E(G)$  and  $E(G^\dagger)$ .) Now for any subgraph  $H$  of  $G$  let the “dual subgraph”  $H^\dagger$  be the subgraph of  $G^\dagger$  such that  $V(H^\dagger) = V(G^\dagger)$  and  $E(H^\dagger) = \{e^\dagger : e \in E(G) \setminus E(H)\}$ .

Then the random subgraph  $\omega^{(p)\dagger}$  (“dual” to  $\omega^{(p)}$ ) is called the *dual percolation process* (dual to  $\omega^{(p)}$ ).

REMARK 25. Notice that  $\omega^{(p)\dagger}$  is in fact a  $(1 - p)$ -Bernoulli bond percolation on  $G^\dagger$ .

*Proof of Corollary 21.* For given vertex-transitive tiling graph  $G$  and its dual  $G^\dagger$ , use the fact that in the middle phase percolation on both the graphs we have infinitely many infinite clusters (see Remark 10). Then in setting of proof of Theorem 20 (with assumption *a contrario*,  $\Phi$ ,  $a$  and  $e$ ), but with  $a$ —component of  $\omega^{(p)\dagger}$  instead of  $\omega^{(p)}$ , we know by Lemma 12 that the limits of paths in  $\omega^{(p)}$  lie densely in  $\Phi$ . So there are two paths  $P_1, P_2 \subseteq \omega^{(p)}$  with distinct limits in  $\Phi$ . Then, similarly as in the first case in proof of the theorem, we have contradiction, because  $e(B) \subseteq a$  is connected subset of  $\omega^{(p)\dagger}$  and  $\overline{e(B)}^{\hat{\Phi}^2}$  contains  $\Phi$  (with limits of  $P_1, P_2$ , so these paths need to cut  $e(B)$ ). See Fig. 7. ■

**Appendix.** The aim of this appendix is proving the following lemma, used in the proof of Theorem 20.

LEMMA 26. For any topological space  $X$ , which is locally compact and  $T_{3\frac{1}{2}}$  (as in Definition 1) and for any compactification  $\hat{X}$  of it and for any  $a \subseteq X$  every end  $e$  of  $a$  has non-empty connected boundary.

REMARK 27. Recall that  $\partial X = \hat{X} \setminus X$  is always closed (and  $X$  is open) in  $\hat{X}$ .

There is classical topological notion of boundary (with other meaning than  $\partial$  in Definition 1). Due to it I will call this notion *topological boundary*.

Now let us consider a set  $A \subseteq X$  and its arbitrary end  $e$ . Notice that then for any compact  $K \subseteq X$  the set  $\partial e(K)$  is compact (as a subspace of  $\hat{X}$ ).

It is worth noting that in the above setting  $\partial e(K) \neq \emptyset$ . It is so because  $e(K)$  is not conditionally compact in  $X$ ; if it were,  $\overline{e(K)}^X$  would be compact and  $e(K \cup \overline{e(K)}^X) \subseteq e(K)$ , but  $e(K \cup \overline{e(K)}^X)$  and  $\overline{e(K)}^X$  are disjoint (by the definition of end), so  $e(K \cup \overline{e(K)}^X) = \emptyset$ , which contradicts the definition of component.

Similarly,  $\partial e$  itself is non-empty as an intersection of family of compact sets from the definition, whose each finite subfamily has, by an easy exercise, non-empty intersection.

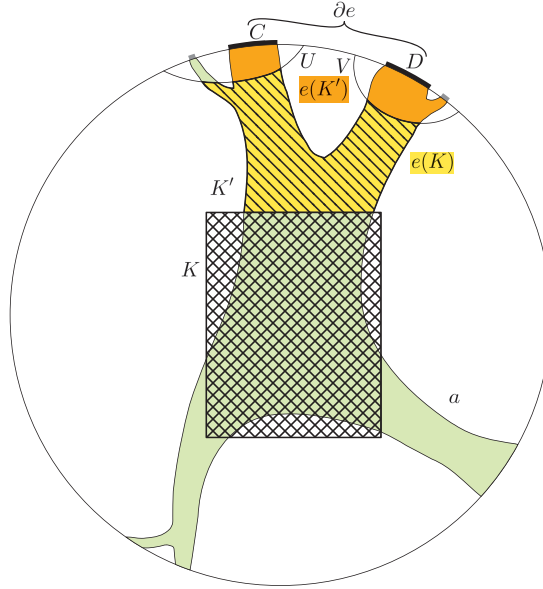


Fig. 9. Proof of Lemma 26

*Proof of Lemma 26.* The set  $\partial e$  is non-empty by the above remark, so it remains to show the connectivity.

Let us assume *a contrario* that  $\partial e$  is not connected. Then it is a sum of two closed disjoint non-empty sets  $C, D \in \hat{X}$ :

$$\partial e = C \cup D.$$

Because  $\hat{X}$  is normal, there exist disjoint open neighbourhoods  $U$  and  $V$  in  $\hat{X}$  of the sets respectively  $C$  and  $D$ .

CLAIM 28. *There is a compact set  $K \subseteq X$  such that  $\partial e(K) \subseteq U \cup V$ .*

*Proof.* Let us consider the family  $\{U \cup V, (\partial e(K))^c : K \subseteq X, K \text{—compact}\}$ . It is an open cover of  $\hat{X}$ , because

$$\hat{X} = (U \cup V) \cup (\partial e)^c = (U \cup V) \cup \bigcup_{\substack{K \subseteq X \\ K \text{—compact}}} (\partial e(K))^c.$$

Hence there is a finite subcover  $\{U \cup V, (\partial e(K_1))^c, \dots, (\partial e(K_n))^c\}$  for some compact  $K_1, \dots, K_n \subseteq X$ . Let us take  $K = \bigcup_{i=1}^n K_i$ . Then

$$\partial e(K) \subseteq \bigcap_{i=1}^n \partial e(K_i), \quad \text{so} \quad \bigcup_{i=1}^n (\partial e(K_i))^c \subseteq (\partial e(K))^c$$

and  $\{U \cup V, (\partial e(K))^c\}$  is also a cover of  $\hat{X}$ . Hence  $\partial e(K) \subseteq U \cup V$  as we desired. ■

CLAIM 29. *There exists  $K'$ —a superset of  $K$  such that even  $e(K') \subseteq U \cup V$ .*

*Proof.* The set  $\overline{e(K)}^{\hat{X}} \setminus (U \cup V)$  is a compact subset of  $X$ , because it is closed in  $\hat{X}$  and disjoint with  $\partial \hat{X}$ .

So let  $K' = K \cup (\overline{e(K)}^{\hat{X}} \setminus (U \cup V))$  be a compact subset of  $X$ . Then

$$e(K') \subseteq e(K) \setminus K' \subseteq e(K) \setminus (\overline{e(K)}^{\hat{X}} \setminus (U \cup V)) \subseteq U \cup V,$$

and

$$\partial e(K') \subseteq \partial e(K) \subseteq U \cup V,$$

but on the other hand

$$C \cup D = \partial e \subseteq \partial e(K').$$

It follows that  $\partial e(K')$  intersects both  $U$  and  $V$ . Hence  $e(K') \subseteq U \cup V$  is not connected, which contradicts the definition of end. ■

This finishes the proof of the lemma. ■

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