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REMARKS ON CONDITIONAL MOMENTS OF THE FREE DEFORMED POISSON RANDOM VARIABLES

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Abstract. We will show that the conditional first moment of the free deformed Poisson random variables ($q = 0$) corresponding to operators fulfilling the free relation is a linear function of the regression and the conditional variance also is a linear function of the regression. For this purpose we will first demonstrate some properties of the Wick product and then we will concentrate on the free deformed Poisson random variables.

1. Introduction. In this article we consider a linear mapping $\mathcal{H} \ni f \mapsto a_f \in \mathcal{A}$ from a real Hilbert space $\mathcal{H}$ into the algebra $\mathcal{A}$ of bounded operators acting on a real Hilbert space which satisfies the commutation relations

$$a_f a_g^* = \langle f, g\rangle I \qquad f, g \in \mathcal{H}. \tag{1}$$

For the real Hilbert space $\mathcal{H}$ with complexification $\mathcal{H}_c = \mathcal{H} \oplus i\mathcal{H}$ we define its full *Fock space* $\Gamma(\mathcal{H})$ as the closure with respect to the norm (2) of $\mathbb{C}\Omega \bigoplus_n \mathcal{H}_c^{\otimes n}$, where Ω is the vacuum vector. We introduce the scalar product

$$\langle f_1 \otimes \ldots \otimes f_n, g_1 \otimes \ldots \otimes g_m\rangle = \begin{cases} \prod_{j=1}^n \langle f_j, g_j\rangle & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \tag{2}$$

where $f_1 \otimes \ldots \otimes f_n \in \mathcal{H}_c^{\otimes n}$ and $g_1 \otimes \ldots \otimes g_m \in \mathcal{H}_c^{\otimes n}$. Given the full Fock space $\Gamma(\mathcal{H})$ and $f \in \mathcal{H}$ we define the annihilation operator $a_f : \Gamma(\mathcal{H}) \to \Gamma(\mathcal{H})$ and its adjoint with

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respect to scalar product, the creation operator, $a_f^*\colon \Gamma(\mathcal{H}) \to \Gamma(\mathcal{H})$, as follows:

$$a_f\Omega : = 0, \qquad a_f f_1 \otimes \ldots \otimes f_n := \langle f, f_1\rangle f_2 \otimes \ldots \otimes f_n, \tag{3}$$

$$a_f^*\Omega : = f, \qquad a_f^* f_1 \otimes \ldots \otimes f_n := f \otimes f_1 \otimes \ldots \otimes f_n. \tag{4}$$

These operators are bounded, satisfy the commutation relation (1), and $a_{f+g} = a_f + a_g$ (see [1]). It was also shown in [1] that the scalar product (2) is strictly positive. Analogously we define the right creation and annihilation operators, denoted by a^r and a^{r*}, respectively. Fix an orthonormal basis $\{e_i\}_{i\in I}$ of $\mathcal{H}_c$. We use the notation $\underline{i}$ to denote a multiple index, i.e., $\underline{i} = (i_1, \ldots, i_n) \in I^n$. The length of such a multiple index will be denoted by $|\underline{i}|$. The empty set $\emptyset$ is also regarded as a multiple index, of length zero. Let A denote the family of all multiple indices and A_n the subfamily of all multiple indices of length n ($n \geq 0$). We put

$$e_\emptyset := \Omega,\ e_{\underline{i}} := e_{i_1} \otimes \ldots \otimes e_{i_n}, \qquad \underline{i} = (i_1, \ldots, i_n) \in A_n\ (n \geq 1).$$

Then $\{e_{\underline{i}}\}$ is an orthonormal basis of the full Fock space $\Gamma(\mathcal{H})$. For a $\xi \in \Gamma(\mathcal{H})$ we have

$$\xi = \sum_{n\geq 0} \xi_n \quad \text{with} \quad \xi_n \in \mathcal{H}_c^{\otimes n}\ (n \geq 0).$$

Since $\{e_{\underline{i}}\}_{\underline{i}\in A_n}$ is an orthonormal basis of $\mathcal{H}_c^{\otimes n}$ with respect to the free scalar product, we have

$$\xi_n = \sum_{\underline{i}\in A_n} \langle e_{\underline{i}}, \xi_n\rangle e_{\underline{i}}$$

where the series converges in $\Gamma(\mathcal{H})$. Since

$$\langle e_{\underline{i}}, \xi_n\rangle = \langle e_{\underline{i}}, \xi\rangle, \qquad \underline{i} \in A,\ n \geq 0,$$

it follows that

$$\xi = \sum_{n\geq 0} \sum_{\underline{i}\in A_n} \langle e_{\underline{i}}, \xi_n\rangle e_{\underline{i}} = \sum_{\underline{i}\in A} \langle e_{\underline{i}}, \xi_n\rangle e_{\underline{i}}, \tag{5}$$

where the last series converges in $\Gamma(\mathcal{H})$. Analogously we can deduce (5) for any linearly independent vectors $\{e_i\}_{i\in I}$ (not necessarily an orthonormal basis).

2. Conditional expectation of Wick product. Let $\mathcal{A}$ be the algebra generated by the creation and the annihilation operators. Let W be a bounded linear operator $W : \Gamma(\mathcal{H}) \to \mathcal{A}$ such that for all $g, g_1, \ldots, g_n \in \mathcal{H}$

$$W_{g_1\otimes\ldots\otimes g_n}\Omega = g_1 \otimes \ldots \otimes g_n, \tag{6}$$

$$W_{g_1} \ldots W_{g_n}\Omega = g_1 \otimes \ldots \otimes g_n + \eta, \quad \text{where } \eta \in \bigoplus_{i=0}^{n-1} \mathcal{H}^{\otimes i}, \tag{7}$$

$$W_g = W_g^*, \tag{8}$$

$$W^*_{g_1\otimes\ldots\otimes g_n}\Omega = g_n \otimes \ldots \otimes g_1 + \eta, \quad \text{where } \eta \in \bigoplus_{i=0}^{n-1} \mathcal{H}^{\otimes i}. \tag{9}$$

Additionally, we assume that each $W_{g_1\otimes\ldots\otimes g_n}$ and $W^*_{g_1\otimes\ldots\otimes g_n}$ can be expressed as finite combinations of the some elements $a^*_{g_{\pi(1)}} \ldots a^*_{g_{\pi(m_1)}} a_{g_{\pi(1)}} \ldots a_{g_{\pi(m_2)}}$ where $m_1, m_2 \in \mathbb{N}$ and $\pi(k) \in \{1, \ldots, n\}$ ($1 \leq k \leq n$). Let $\mathcal{B} = \mathrm{alg}(\mathbf{1}, W_{f_1}, \ldots, W_{f_n})$ be the von Neumann algebra $\mathcal{B} \subset B(\Gamma(\mathcal{H}))$, generated by $\mathbf{1}, W_{f_1}, \ldots, W_{f_n}$, where $f_1, \ldots, f_n \in \mathcal{H}$ are fixed linearly independent vectors. We assume that the vacuum vector Ω is cyclic and separating for $\mathcal{B}$ and each element $x \in \mathcal{B}$ can be represented as

$$x = \sum_{c \in C} \alpha_c W_c \tag{10}$$

where $C = \bigcup_{s=0}^{\infty} C_s$, $C_0 = \{\Omega\}$, $C_n = \{\bigotimes_{k=1}^{n} f_{\pi(k)} : \pi(k) \in \{1, \ldots, n\} \text{ for } 1 \leq k \leq n\}$ and the series (10) converges in the weak-star operator topology.

We define the vacuum expectation state $\mathbb{E} : \mathcal{B} \to \mathbb{C}$ as $\mathbb{E}(X) := \langle X\Omega, \Omega \rangle$.

In particular, from (10) we get $W^*_{f_1 \otimes \ldots \otimes f_n} = \sum_{c \in C} \alpha_c W_c$. Using (9) and the fact that vectors $f_1, \ldots, f_n$ are linearly independent, we get $W^*_{f_1 \otimes \ldots \otimes f_n} = \sum_{c \in C, |c| \leq n} \alpha_c W_c$. It means that $W^*_{f_1 \otimes \ldots \otimes f_n}$ can be expressed as a finite linear combination of W_c, where $c \in C$.

Take any vector $g \in \mathcal{H}$, and denote the orthogonal projection of g onto the linear span of $f_1, \ldots, f_n$ by $P(g)$. Then we have

$$\langle g, f_i \rangle = \langle P(g), f_i \rangle \quad \forall i \in \{1, \ldots, n\}. \tag{11}$$

LEMMA 2.1. *If $m_1, m_2 \in \mathbb{N}$ and $f_i, g_j \in \mathcal{H}$ $(i, j \in \mathbb{N})$, then*

$$\begin{aligned} &\mathbb{E}\big(W_{f_{\pi(1)} \otimes \ldots \otimes f_{\pi(m_1)}} a_{g_1} \ldots a_{g_m} W_{f_{\pi'(1)} \otimes \ldots \otimes f_{\pi'(m_2)}}\big) \\ &= \mathbb{E}\big(W_{f_{\pi(1)} \otimes \ldots \otimes f_{\pi(m_1)}} a_{P(g_1)} \ldots a_{P(g_m)} W_{f_{\pi'(1)} \otimes \ldots \otimes f_{\pi'(m_2)}}\big) \end{aligned} \tag{12}$$

for any $\pi(i), \pi'(i) \in \{1, \ldots, n\}$, $i \in \mathbb{N}$.

Proof. Take any $W_{f_{\pi(1)} \otimes \ldots \otimes f_{\pi(m_1)}}$, $W_{f_{\pi'(1)} \otimes \ldots \otimes f_{\pi'(m_2)}}$. Then

$$\begin{aligned} E &:= \mathbb{E}\big(W_{f_{\pi(1)} \otimes \ldots \otimes f_{\pi(m_1)}} a^*_{g_1} \ldots a^*_{g_m} W_{f_{\pi'(1)} \otimes \ldots \otimes f_{\pi'(m_2)}}\big) \\ &= \big\langle W_{f_{\pi(1)} \otimes \ldots \otimes f_{\pi(m_1)}} a^*_{g_1} \ldots a^*_{g_m} W_{f_{\pi'(1)} \otimes \ldots \otimes f_{\pi'(m_2)}} \Omega, \Omega \big\rangle. \end{aligned}$$

If $m_1 = m_2 + m$, then from (9)

$$\begin{aligned} E &= \big\langle a^*_{g_1} \ldots a^*_{g_m} f_{\pi'(1)} \otimes \ldots \otimes f_{\pi'(m_2)}, f_{\pi(m_1)} \otimes \ldots \otimes f_{\pi(1)} \big\rangle \\ &= \big\langle g_1 \otimes \ldots \otimes g_m \otimes f_{\pi'(1)} \otimes \ldots \otimes f_{\pi'(m_2)}, f_{\pi(m_1)} \otimes \ldots \otimes f_{\pi(1)} \big\rangle. \end{aligned}$$

If we use (2) then

$$E = \prod_{j=1}^{m} \big\langle g_j, f_{\pi(m_1+1-j)} \big\rangle \prod_{j=1}^{m_2} \big\langle f_{\pi'(j)}, f_{\pi(m_2+1-j)} \big\rangle.$$

From (11) we get

$$\begin{aligned} E &= \prod_{j=1}^{m} \big\langle P(g_j), f_{\pi(m_1+1-j)} \big\rangle \prod_{j=1}^{m_2} \big\langle f_{\pi'(j)}, f_{\pi(m_2+1-j)} \big\rangle \\ &= \big\langle P(g_1) \otimes \ldots \otimes P(g_m) \otimes f_{\pi'(1)} \otimes \ldots \otimes f_{\pi'(m_2)}, f_{\pi(m_1)} \otimes \ldots \otimes f_{\pi(1)} \big\rangle \end{aligned}$$

Inverting the above steps we get

$$\begin{aligned} &\mathbb{E}\big(W_{f_{\pi(1)} \otimes \ldots \otimes f_{\pi(m_1)}} a^*_{g_1} \ldots a^*_{g_m} W_{f_{\pi'(1)} \otimes \ldots \otimes f_{\pi'(m_2)}}\big) \\ &\qquad = \mathbb{E}\big(W_{f_{\pi(1)} \otimes \ldots \otimes f_{\pi(m_1)}} a^*_{P(g_1)} \ldots a^*_{P(g_m)} W_{f_{\pi'(1)} \otimes \ldots \otimes f_{\pi'(m_2)}}\big). \end{aligned} \tag{13}$$

If $m_1 \neq m_2 + m$ then

$$\begin{aligned} E &= \mathbb{E}\big(W_{f_{\pi(1)}\otimes\ldots\otimes f_{\pi(m_1)}} a^*_{g_1} \ldots a^*_{g_m} W_{f_{\pi'(1)}\otimes\ldots\otimes f_{\pi'(m_2)}}\big) \\ &= \big\langle W_{f_{\pi(1)}\otimes\ldots\otimes f_{\pi(m_1)}} a^*_{g_1} \ldots a^*_{g_m} W_{f_{\pi'(1)}\otimes\ldots\otimes f_{\pi'(m_2)}} \Omega, \Omega\big\rangle \\ &= \big\langle a^*_{g_1} \ldots a^*_{g_m} W_{f_{\pi'(1)}\otimes\ldots\otimes f_{\pi'(m_2)}} \Omega, W^*_{f_{\pi(1)}\otimes\ldots\otimes f_{\pi(m_1)}} \Omega\big\rangle. \end{aligned} \tag{14}$$

Using assumptions (9) and (6) we can express $W^*_{f_{\pi(1)}\otimes\ldots\otimes f_{\pi(m_1)}}\Omega$ as a linear combination of $W_{f_{\pi(1)}\otimes\ldots\otimes f_{\pi(k)}}\Omega$ for some $k \leq m_1$. Let $W^*_{f_{\pi(1)}\otimes\ldots\otimes f_{\pi(m_1)}}\Omega = \sum_{b=0}^{m_1} W_{\sum_{\rho_b\in C_b} \alpha_{\rho_b}\rho_b}\Omega$, and $\alpha_{\rho_b} \in \mathbf{R}$. So, we get

$$\begin{aligned} E &= \big\langle a^*_{g_1} \ldots a^*_{g_m} W_{f_{\pi'(1)}\otimes\ldots\otimes f_{\pi'(m_2)}} \Omega, W^*_{f_{\pi(1)}\otimes\ldots\otimes f_{\pi(m_1)}} \Omega\big\rangle \\ &= \Big\langle a^*_{g_1} \ldots a^*_{g_m} W_{f_{\pi'(1)}\otimes\ldots\otimes f_{\pi'(m_2)}} \Omega, \sum_{b=0}^{m_1} W_{\sum_{\rho_b\in C_b} \alpha_{\rho_b}\rho_b}\Omega\Big\rangle \\ &= \Big\langle a^*_{g_1} \ldots a^*_{g_m} W_{f_{\pi'(1)}\otimes\ldots\otimes f_{\pi'(m_2)}} \Omega, \sum_{b=0}^{m_1} \sum_{\rho_b\in C_b} \alpha_{\rho_b}\rho_b\Big\rangle. \end{aligned} \tag{15}$$

If $b \neq m + m_2$ then it follows from (2) that

$$\Big\langle a^*_{g_1} \ldots a^*_{g_m} W_{f_{\pi'(1)}\otimes\ldots\otimes f_{\pi'(m_2)}} \Omega, \sum_{\rho_b\in C_b} \alpha_{\rho_b}\rho_b\Big\rangle = 0. \tag{16}$$

In particular, we can write

$$\begin{aligned} E' &:= \Big\langle a^*_{g_1} \ldots a^*_{g_m} W_{f_{\pi'(1)}\otimes\ldots\otimes f_{\pi'(m_2)}} \Omega, \sum_{\rho_b\in C_b} \alpha_{\rho_b}\rho_b\Big\rangle \\ &= \Big\langle a^*_{P(g_1)} \ldots a^*_{P(g_m)} W_{f_{\pi'(1)}\otimes\ldots\otimes f_{\pi'(m_2)}} \Omega, \sum_{\rho_b\in C_b} \alpha_{\rho_b}\rho_b\Big\rangle = 0. \end{aligned} \tag{17}$$

Denote by $J : \Gamma(\mathcal{H}) \to \Gamma(\mathcal{H})$ the reversing order operator, $J(g_1\otimes\ldots\otimes g_n) = g_n\otimes\ldots\otimes g_1$ and $J(\Omega) = \Omega$.

If $b = m + m_2$ then

$$\begin{aligned} E' &= \Big\langle a^*_{g_1} \ldots a^*_{g_m} W_{f_{\pi'(1)}\otimes\ldots\otimes f_{\pi'(m_2)}} \Omega, \sum_{\rho_b\in C_b} \alpha_{\rho_b}\rho_b\Big\rangle \\ &= \Big\langle a^*_{g_1} \ldots a^*_{g_m} W_{f_{\pi'(1)}\otimes\ldots\otimes f_{\pi'(m_2)}} \Omega, \sum_{\rho_b\in C_b} \alpha_{\rho_b} W_{\rho_b}\Omega\Big\rangle. \end{aligned}$$

From (9) and assumption $b = m + m_2$

$$\begin{aligned} E' &= \Big\langle a^*_{g_1} \ldots a^*_{g_m} W_{f_{\pi'(1)}\otimes\ldots\otimes f_{\pi'(m_2)}} \Omega, \sum_{\rho_b\in C_b} \alpha_{\rho_b} W^*_{J(\rho_b)}\Omega\Big\rangle \\ &= \sum_{\rho_b\in C_b} \alpha_{\rho_b} \big\langle a^*_{g_1} \ldots a^*_{g_m} W_{f_{\pi'(1)}\otimes\ldots\otimes f_{\pi'(m_2)}} \Omega, W^*_{J(\rho_b)}\Omega\big\rangle. \end{aligned}$$

From (13)

$$\begin{aligned}E' &= \sum_{\rho_b\in C_b}\alpha_{\rho_b}\big\langle a^*_{P(g_1)}\dots a^*_{P(g_m)}W_{f_{\pi'(1)}\otimes\ldots\otimes f_{\pi'(m_2)}}\Omega, W^*_{J(\rho_b)}\Omega\big\rangle\\ &= \Big\langle a^*_{P(g_1)}\dots a^*_{P(g_m)}W_{f_{\pi'(1)}\otimes\ldots\otimes f_{\pi'(m_2)}}\Omega, \sum_{\rho_b\in C_b}\alpha_{\rho_b}W^*_{J(\rho_b)}\Omega\Big\rangle\\ &= \Big\langle a^*_{P(g_1)}\dots a^*_{P(g_m)}W_{f_{\pi'(1)}\otimes\ldots\otimes f_{\pi'(m_2)}}\Omega, \sum_{\rho_b\in C_b}\alpha_{\rho_b}W_{\rho_b}\Omega\Big\rangle. \qquad (18)\end{aligned}$$

Then the right hand side of equation (15) can be expressed as

$$\begin{aligned}E &= \Big\langle a^*_{g_1}\dots a^*_{g_m}W_{f_{\pi'(1)}\otimes\ldots\otimes f_{\pi'(m_2)}}\Omega, \sum_{b=0}^{m_1}\sum_{\rho_b\in C_b}\alpha_{\rho_b}\rho_b\Big\rangle\\ &= \Big\langle a^*_{P(g_1)}\dots a^*_{P(g_m)}W_{f_{\pi'(1)}\otimes\ldots\otimes f_{\pi'(m_2)}}\Omega, \sum_{b=0}^{m_1}\sum_{\rho_b\in C_b}\alpha_{\rho_b}\rho_b\Big\rangle\\ &= \big\langle a^*_{P(g_1)}\dots a^*_{P(g_m)}W_{f_{\pi'(1)}\otimes\ldots\otimes f_{\pi'(m_2)}}\Omega, W^*_{f_{\pi(1)}\otimes\ldots\otimes f_{\pi(m_1)}}\Omega\big\rangle\\ &= \big\langle W_{f_{\pi(1)}\otimes\ldots\otimes f_{\pi(m_1)}}a^*_{P(g_1)}\dots a^*_{P(g_m)}W_{f_{\pi'(1)}\otimes\ldots\otimes f_{\pi'(m_2)}}\Omega, \Omega\big\rangle. \qquad (19)\end{aligned}$$

So, we have proved that for all $Y_1, Y_2 \in \{W_{\underline{e}}; \underline{e}\in C\}$

$$\mathbb{E}(Y_1 a^*_{g_1}\dots a^*_{g_m}Y_2) = \mathbb{E}(Y_1 a^*_{P(g_1)}\dots a^*_{P(g_m)}Y_2). \qquad (20)$$

In particular,

$$\begin{aligned}&\mathbb{E}\big(W^*_{f_{\pi'(1)}\otimes\ldots\otimes f_{\pi'(m_2)}}a^*_{g_1}\dots a^*_{g_m}W^*_{f_{\pi(1)}\otimes\ldots\otimes f_{\pi(m_1)}}\big)\\ &= \mathbb{E}\big(W^*_{f_{\pi'(1)}\otimes\ldots\otimes f_{\pi'(m_2)}}a^*_{P(g_1)}\dots a^*_{P(g_m)}W^*_{f_{\pi(1)}\otimes\ldots\otimes f_{\pi(m_1)}}\big) \qquad (21)\end{aligned}$$

because $W^*_{f_{\pi'(1)}\otimes\ldots\otimes f_{\pi'(m_2)}}, W^*_{f_{\pi(1)}\otimes\ldots\otimes f_{\pi(m_1)}}$ can be expressed as finite linear combinations of $W_{e_{\underline{i}}}$, $e_{\underline{i}}\in C$.

Of course, this is equivalent to

$$\begin{aligned}&\mathbb{E}\big(W_{f_{\pi(1)}\otimes\ldots\otimes f_{\pi(m_1)}}a_{g_1}\dots a_{g_m}W_{f_{\pi'(1)}\otimes\ldots\otimes f_{\pi'(m_2)}}\big)\\ &\qquad = \mathbb{E}\big(W_{f_{\pi(1)}\otimes\ldots\otimes f_{\pi(m_1)}}a_{P(g_1)}\dots a_{P(g_m)}W_{f_{\pi'(1)}\otimes\ldots\otimes f_{\pi'(m_2)}}\big). \ \blacksquare\end{aligned}$$

Take any $Y_1, Y_2\in\mathcal{B}$, and by (10), write $Y_1 = \sum_{p\in C}\alpha_p^{(1)}W_p$ and $Y_2 = \sum_{i\in C}\alpha_i^{(2)}W_i$. Then we have

$$\begin{aligned}&\mathbb{E}\Big(\sum_{p\in C}\alpha_p^{(1)}W_p a_{g_1}\dots a_{g_m}\sum_{i\in C}\alpha_i^{(2)}W_i\Big)\\ &= \sum_{p\in C}\alpha_p^{(1)}\sum_{i\in C}\alpha_i^{(2)}\mathbb{E}(W_p a_{g_1}\dots a_{g_m}W_i)\\ &= \sum_{p\in C}\alpha_p^{(1)}\sum_{i\in C}\alpha_i^{(2)}\mathbb{E}(W_p a_{P(g_1)}\dots a_{P(g_m)}W_i)\\ &= \mathbb{E}\Big(\sum_{p\in C}\alpha_p^{(1)}W_p a_{P(g_1)}\dots a_{P(g_m)}\sum_{i\in C}\alpha_i^{(2)}W_i\Big). \qquad (22)\end{aligned}$$

COROLLARY 2.2. $\mathbb{E}(Y_1 a_{g_1}\dots a_{g_m}Y_2) = \mathbb{E}(Y_1 a_{P(g_1)}\dots a_{P(g_m)}Y_2)$ *for all* $Y_1, Y_2\in\mathcal{B}$.

Lemma 2.3. *If $m_1, m_2, m_3, m_4 \in \mathbb{N}$, $f_i, g_j \in \mathcal{H}$ $(i, j \in \mathbb{N})$ then*

$$\begin{aligned}&\mathbb{E}\big(W_{f_{\pi(1)}\otimes\ldots\otimes f_{\pi(m_3)}} a^*_{g_1}\ldots a^*_{g_{m_1}} a_{g_{m_1+1}}\ldots a_{g_{m_1+m_2}} W_{f_{\pi'(1)}\otimes\ldots\otimes f_{\pi'(m_4)}}\big)\\ &= \mathbb{E}\big(W_{f_{\pi(1)}\otimes\ldots\otimes f_{\pi(m_3)}} a^*_{P(g_1)}\ldots a^*_{P(g_{m_1})} a_{P(g_{m_1+1})}\ldots a_{P(g_{m_1+m_2})} W_{f_{\pi'(1)}\otimes\ldots\otimes f_{\pi'(m_4)}}\big) \quad (23)\end{aligned}$$

for any $\pi(i), \pi'(i) \in \{1,\ldots,n\}$, $i \in \mathbb{N}$.

Proof. The proof is by induction on m_1. If $m_1 = 0$ then we apply Lemma 2.1 to get the result. Suppose that (23) is true for some m_1. Take any $W_{f_{\pi(1)}\otimes\ldots\otimes f_{\pi(m_3)}}$, $W_{f_{\pi(1)}\otimes\ldots\otimes f_{\pi(m_4)}}$ with $m_3 = m_4 - m_2 + m_1 + 1$, and denote $f_{\pi'(1)}\otimes\ldots\otimes f_{\pi'(m_4)}$ by Φ. Then we have

$$\begin{aligned}E'' &:= \mathbb{E}\big(W_{f_{\pi(1)}\otimes\ldots\otimes f_{\pi(m_3)}} a^*_g a^*_{g_1}\ldots a^*_{g_{m_1}} a_{g_{m_1+1}}\ldots a_{g_{m_1+m_2}} W_\Phi\big)\\ &= \big\langle W_{f_{\pi(1)}\otimes\ldots\otimes f_{\pi(m_3)}} a^*_g a^*_{g_1}\ldots a^*_{g_{m_1}} a_{g_{m_1+1}}\ldots a_{g_{m_1+m_2}} W_\Phi\Omega, \Omega\big\rangle.\end{aligned}$$

By (6) and (9)

$$\begin{aligned}E'' &= \big\langle a^*_{g_1}\ldots a^*_{g_{m_1}} a_{g_{m_1+1}}\ldots a_{g_{m_1+m_2}}\Phi, a_g f_{\pi(m_3)}\otimes\ldots\otimes f_{\pi(1)}\big\rangle\\ &= \langle g, f_{\pi(m_3)}\rangle\big\langle a^*_{g_1}\ldots a^*_{g_{m_1}} a_{g_{m_1+1}}\ldots a_{g_{m_1+m_2}}\Phi, f_{\pi(m_3-1)}\otimes\ldots\otimes f_{\pi(1)}\big\rangle\\ &= \langle g, f_{\pi(m_3)}\rangle\big\langle a^*_{g_1}\ldots a^*_{g_{m_1}} a_{g_{m_1+1}}\ldots a_{g_{m_1+m_2}} W_\Phi\Omega, W^*_{f_{\pi(1)}\otimes\ldots\otimes f_{\pi(m_3-1)}}\Omega\big\rangle\\ &= \langle g, f_{\pi(m_3)}\rangle\mathbb{E}\big(W_{f_{\pi(1)}\otimes\ldots\otimes f_{\pi(m_3-1)}} a^*_{g_1}\ldots a^*_{g_{m_1}} a_{g_{m_1+1}}\ldots a_{g_{m_1+m_2}} W_\Phi\big). \quad (24)\end{aligned}$$

By induction and (11)

$$\begin{aligned}E'' &= \langle P(g), f_{\pi(m_3)}\rangle\mathbb{E}\big(W_{f_{\pi(1)}\otimes\ldots\otimes f_{\pi(m_3-1)}} a^*_{P(g_1)}\ldots a^*_{P(g_{m_1})} a_{P(g_{m_1+1})}\ldots a_{P(g_{m_1+m_2})} W_\Phi\big)\\ &= \langle P(g), f_{\pi(m_3)}\rangle\big\langle a^*_{P(g_1)}\ldots a^*_{P(g_{m_1})} a_{P(g_{m_1+1})}\ldots a_{P(g_{m_1+m_2})}\Phi, f_{\pi(m_3-1)}\otimes\ldots\otimes f_{\pi(1)}\big\rangle\\ &= \big\langle a^*_{P(g_1)}\ldots a^*_{P(g_{m_1})} a_{P(g_{m_1+1})}\ldots a_{P(g_{m_1+m_2})}\Phi, a_{P(g)} f_{\pi(m_3)}\otimes\ldots\otimes f_{\pi(1)}\big\rangle\\ &= \langle a^*_{P(g)} a^*_{P(g_1)}\ldots a^*_{P(g_{m_1})} a_{P(g_{m_1+1})}\ldots a_{P(g_{m_1+m_2})} W_\Phi\Omega, W^*_{f_{\pi(m_3)}\otimes\ldots\otimes f_{\pi(1)}}\Omega\rangle\\ &= \mathbb{E}\big(W_{f_{\pi(1)}\otimes\ldots\otimes f_{\pi(m_3)}} a^*_{P(g)} a^*_{P(g_1)} a^*_{P(g_{m_1})} a_{P(g_{m_1+1})}\ldots a_{P(g_{m_1+m_2})} W_\Phi\big). \quad (25)\end{aligned}$$

If $m_3 \neq m_4 - m_2 + m_1 + 1$ the proof is analogous to the proof of Lemma 2.1 for the case $m_1 \neq m + m_2$. ∎

Take any $Y_1, Y_2 \in \mathcal{B}$, $Y_1 = \sum_{p\in C}\alpha^{(1)}_p W_{e_{\underline{p}}}$ and $Y_2 = \sum_{i\in C}\alpha^{(2)}_i W_i$. We have

$$\begin{aligned}&\mathbb{E}\big(Y_1 a^*_{g_1}\ldots a^*_{g_{m_1}} a_{g_{m_1+1}}\ldots a_{g_{m_1+m_2}} Y_2\big)\\ &= \mathbb{E}\Big(\sum_{p\in C}\alpha^{(1)}_p W_p a^*_{g_1}\ldots a^*_{g_{m_1}} a_{g_{m_1+1}}\ldots a_{g_{m_1+m_2}}\sum_{i\in C}\alpha^{(2)}_i W_i\Big)\\ &= \sum_{p\in C}\alpha^{(1)}_p\sum_{i\in C}\alpha^{(2)}_i\mathbb{E}\big(W_p a^*_{g_1}\ldots a^*_{g_{m_1}} a_{g_{m_1+1}}\ldots a_{g_{m_1+m_2}} W_i\big)\\ &= \mathbb{E}\big(Y_1 a^*_{P(g_1)}\ldots a^*_{P(g_{m_1})} a_{P(g_{m_1+1})}\ldots a_{P(g_{m_1+m_2})} Y_2\big). \quad (26)\end{aligned}$$

Corollary 2.4.

$$\begin{aligned}&\mathbb{E}\big(Y_1 a^*_{g_1}\ldots a^*_{g_{m_1}} a_{g_{m_1+1}}\ldots a_{g_{m_1+m_2}} Y_2\big)\\ &\qquad= \mathbb{E}\big(Y_1 a^*_{P(g_1)}\ldots a^*_{P(g_{m_1})} a_{P(g_{m_1+1})}\ldots a_{P(g_{m_1+m_2})} Y_2\big) \quad (27)\end{aligned}$$

for all $Y_1, Y_2 \in \mathcal{B}$.

Fix an orthonormal basis $\{e'_{\underline{i}}\}$ of $\mathcal{H}^{\otimes n}$. Let $\mathcal{G}$ be the von Neumann algebra generated by $\{W_{e'_{\underline{i}}}; \underline{i} \in A \text{ and } \alpha_{e'_{\underline{i}}} \in \mathbf{R}\}$ (where A denotes the family of all multiple indices defined in the first part). Additionally we assume that each element of $\mathcal{G}$ can be expressed in the form $\sum_{\underline{i}\in A} \alpha_{e'_{\underline{i}}} W_{e'_{\underline{i}}}$.

THEOREM 2.5. *Let $P(e'_{\underline{k}}) = P(e'_1 \otimes \ldots \otimes e'_k) = P(e'_1) \otimes \ldots \otimes P(e'_k)$, then*

$$\mathbb{E}\Big(\sum_{\underline{i}\in A} \alpha_{e'_{\underline{i}}} W_{e'_{\underline{i}}} | \mathcal{B}\Big) = \sum_{e'_{\underline{i}}\in A} \alpha_{e'_{\underline{i}}} W_{P(e'_{\underline{i}})} \tag{28}$$

where the above series converges in the weak-star operator topology. Here $\mathbb{E}(X|\mathcal{B})$ is the conditional expectation of the element $X \in \mathcal{G}$ onto $\mathcal{B}$. Recall that a (non-commutative) conditional expectation on the probability space $L^2(\mathcal{G}, \mathbb{E})$ with respect to the subalgebra $\mathcal{B} \subset \mathcal{G}$ is a projection onto the corresponding $L^2(\mathcal{B}, \mathbb{E})$.

Proof. The assumptions ensure that each element $W_{g_1\otimes\ldots\otimes g_n}$ can be expressed as a linear combination of $a^*_{g_{\pi(1)}} \ldots a^*_{g_{\pi(m_1)}} a_{g_{\pi(1)}} \ldots a_{g_{\pi(m_2)}}$ where $m_1, m_2 \in \mathbb{N}$ and $\pi(k) \in \{1, \ldots, n\}$ $(1 \le k \le n)$. Corollaries 2.2 and 2.4 give

$$\mathbb{E}(Y_1 W_{g_1\otimes\ldots\otimes g_m} Y_2) = \mathbb{E}(Y_1 W_{P(g_1)\otimes\ldots\otimes P(g_m)} Y_2) \tag{29}$$

for all $Y_1, Y_2 \in \mathcal{B}$, and of course $W_{P(g_1)\otimes\ldots\otimes P(g_m)} \in \mathcal{B}$. Analogously we deduce that

$$\begin{aligned}\mathbb{E}\big(&Y_1 W^*_{g_{t_1}\otimes\ldots\otimes g_{t_{m'}}} W_{g_{t'_1}\otimes\ldots\otimes g_{t'_{m''}}} Y_2\big) \\ &= \mathbb{E}\big(Y_1 W^*_{P(g_{t_1})\otimes\ldots\otimes P(g_{t_{m'}})} W_{P(g_{t'_1})\otimes\ldots\otimes P(g_{t'_{m''}})} Y_2\big)\end{aligned} \tag{30}$$

where $t_{m'}, t'_{m''} \in \mathbb{N}$ and $m', m'' \in \mathbb{N}$. Now, for any element $\zeta = \sum_{\underline{i}\in A} \alpha_{e'_{\underline{i}}} W_{e'_{\underline{i}}} \in \mathcal{G}$ we have

$$\mathbb{E}(Y_1 \zeta Y_2) = \sum_{\underline{i}\in A} \alpha_{e'_{\underline{i}}} \mathbb{E}(Y_1 W_{e'_{\underline{i}}} Y_2) = \sum_{\underline{i}\in A} \alpha_{e'_{\underline{i}}} \mathbb{E}(Y_1 W_{P(e'_{\underline{i}})} Y_2) = \mathbb{E}\Big(Y_1 \sum_{\underline{i}\in A} \alpha_{e'_{\underline{i}}} W_{P(e'_{\underline{i}})} Y_2\Big) \tag{31}$$

for all $Y_1, Y_2 \in \mathcal{B}$.

Uniqueness. Let us suppose that there exists some variable $Y \in \mathcal{B}$ (other than $W_{P(g_1)\otimes\ldots\otimes P(g_m)}$) such that

$$\mathbb{E}(Y_1 W_{g_1\otimes\ldots\otimes g_m} Y_2) = \mathbb{E}(Y_1 Y Y_2) \tag{32}$$

for all $Y_1, Y_2 \in \mathcal{B}$. From assumption (10) we get $Y = \sum_{\underline{p}\in C} \alpha_{\underline{p}} W_{\underline{p}} = \sum_{i=0} \sum_{\underline{i}\in C_i} \alpha_{\underline{i}} W_{\underline{i}}$. By taking any $\sum_{\underline{b}\in C_b} \alpha_{\underline{b}} W_{\underline{b}}$ and $b \ne m$, and putting $Y_2 = \mathbf{I}$ in equation (32), for $Y_1 = \big(\sum_{\underline{b}\in C_b} \alpha_{\underline{b}} W_{\underline{b}}\big)^*$ the left hand side of equation (32) becomes

$$\mathbb{E}(Y_1 W_{g_1\otimes\ldots\otimes g_m} Y_2) = \Big\langle W_{g_1\otimes\ldots\otimes g_m}\Omega, \sum_{\underline{b}\in C_b} \alpha_{\underline{b}} W_{\underline{b}}\Omega\Big\rangle = 0. \tag{33}$$

On the other hand,

$$\mathbb{E}(Y_1 Y) = \Big\langle \sum_{i=0} \sum_{\underline{i}\in C_i} \alpha_{\underline{i}} W_{\underline{i}}\Omega, \sum_{\underline{b}\in C_b} \alpha_{\underline{b}} W_{\underline{b}}\Omega\Big\rangle = \Big\langle \sum_{\underline{b}\in C_b} \alpha_{\underline{b}} \underline{b}, \sum_{\underline{b}\in C_b} \alpha_{\underline{b}} \underline{b}\Big\rangle = 0. \tag{34}$$

The last equality follows from (33).

Since the vectors $\underline{b} \in C_b$ are linearly independent, we have $\alpha_{\underline{b}} = 0$ for all $\underline{b} \in C_b$. This gives $Y = \sum_{\underline{i}\in C_m} \alpha_{\underline{i}} W_{\underline{i}}$. Rewriting all $\alpha_{\underline{i}}$ and $W_{\underline{m}}$ (where $\underline{i} \in C_m$) as α_s and $W_{\underline{m}^s}$ where

$s \in \{1, \dots, n\}^m$ and putting $Y_1 = W^*_{\underline{m}^{s'}}$, $Y_2 = \mathbf{I}$ in equation (32), yields

$$\mathbb{E}(Y_1 W_{g_1 \otimes \dots \otimes g_m}) = \mathbb{E}(Y_1 Y)\langle W_{g_1 \otimes \dots \otimes g_m}, W_{\underline{m}^{s'}} \rangle = \sum_{s \in \{1,\dots,n\}^m} \alpha^s_{\underline{m}} \langle W_{\underline{m}^s}, W_{\underline{m}^{s'}} \rangle \tag{35}$$

for all $s' \in \{1, \dots, n\}^m$, which is equivalent to the equation

$$\begin{bmatrix} \langle W_{g_1 \otimes \dots \otimes g_m}, W_{\underline{m}^{(1,\dots,1)}} \rangle \\ \vdots \\ \langle W_{g_1 \otimes \dots \otimes g_m}, W_{\underline{m}^{(n,\dots,n)}} \rangle \end{bmatrix} = \underbrace{\begin{bmatrix} \langle W_{\underline{m}^{(1,\dots,1)}}, W_{\underline{m}^{(1,\dots,1)}} \rangle & \dots & \langle W_{\underline{m}^{(n,\dots,n)}}, W_{\underline{m}^{(1,\dots,1)}} \rangle \\ \vdots & \ddots & \vdots \\ \langle W_{\underline{m}^{(1,\dots,1)}}, W_{\underline{m}^{(n,\dots,n)}} \rangle & \dots & \langle W_{\underline{m}^{(n,\dots,n)}}, W_{\underline{m}^{(n,\dots,n)}} \rangle \end{bmatrix}}_{D} \begin{bmatrix} \alpha^{(1,\dots,1)}_{\underline{m}} \\ \vdots \\ \alpha^{(n,\dots,n)}_{\underline{m}} \end{bmatrix}. \tag{36}$$

Since the vectors $W_{\underline{m}^s}$ ($s \in \{1, \dots, n\}^m$) are linearly independent, we obtained that $\det D \neq 0$. This gives us that $\alpha^s_{\underline{m}}$ is an explicit set ($s \in \{1, \dots, n\}^m$). Using this result and (31) we deduce that $\mathbb{E}(Y_1 \zeta Y_2) = \mathbb{E}(Y_1 \zeta_{\mathcal{B}} Y_2)$ can be true only for one element from $\zeta_{\mathcal{B}} \in \mathcal{B}$.

So, we can define an operator $\mathcal{E} : \mathcal{G} \to \mathcal{B}$ ($\zeta \to \mathcal{E}(\zeta)$) by the relation

$$\mathbb{E}(Y_1 \zeta Y_2) = \mathbb{E}(Y_1 \mathcal{E}(\zeta) Y_2) \tag{37}$$

for all $Y_1, Y_2 \in \mathcal{B}$ and $\zeta \in \mathcal{G}$. It is easy to see that the operator $\mathcal{E}$ is linear. Now, we shall prove that the operator $\mathcal{E}(X)$ is a positive contraction.

Positiveness. Take any $\zeta = \sum_{\underline{i} \in A} \alpha_{e'_{\underline{i}}} W_{e'_{\underline{i}}} \in \mathcal{G}$, then

$$\begin{aligned} \mathcal{E}(\zeta^* \zeta) &= \mathcal{E}\Big(\Big(\sum_{\underline{i} \in A} \alpha_{e'_{\underline{i}}} W^*_{e'_{\underline{i}}}\Big)\Big(\sum_{\underline{i} \in A} \alpha_{e'_{\underline{i}}} W_{e'_{\underline{i}}}\Big)\Big) \\ &= \mathcal{E}\Big(\sum_{\underline{j} \in A} \sum_{\underline{i} \in A} \alpha_{e'_{\underline{j}}} \alpha_{e'_{\underline{i}}} W^*_{e'_{\underline{j}}} W_{e'_{\underline{i}}}\Big) = \sum_{\underline{j} \in A} \sum_{\underline{i} \in A} \alpha_{e'_{\underline{j}}} \alpha_{e'_{\underline{i}}} \mathcal{E}(W^*_{e'_{\underline{j}}} W_{e'_{\underline{i}}}) \\ &= \sum_{\underline{j} \in A} \sum_{\underline{i} \in A} \alpha_{e'_{\underline{j}}} \alpha_{e'_{\underline{i}}} W^*_{P(e'_{\underline{j}})} W_{P(e'_{\underline{i}})} = \Big(\sum_{\underline{i} \in A} \alpha_{e'_{\underline{i}}} W^*_{P(e'_{\underline{i}})}\Big)\Big(\sum_{\underline{i} \in A} \alpha_{e'_{\underline{i}}} W_{P(e'_{\underline{i}})}\Big) \end{aligned} \tag{38}$$

which follows from (30).

Contractiveness. Take any $\zeta = \sum_{\underline{i} \in A} \alpha_{e'_{\underline{i}}} W_{e'_{\underline{i}}} \in \mathcal{G}$, then

$$\begin{aligned} \|\mathcal{E}(\zeta)\|^2 &= \langle \mathcal{E}(\zeta)\Omega, \mathcal{E}(\zeta)\Omega \rangle = \Big\langle \sum_{\underline{i} \in A} \alpha_{e'_{\underline{i}}} W_{P(e'_{\underline{i}})}\Omega, \sum_{\underline{i} \in A} \alpha_{e'_{\underline{i}}} W_{P(e'_{\underline{i}})}\Omega \Big\rangle \\ &= \Big\langle \sum_{\underline{i} \in A} \alpha_{e'_{\underline{i}}} P(e'_{\underline{i}}), \sum_{\underline{i} \in A} \alpha_{e'_{\underline{i}}} P(e'_{\underline{i}}) \Big\rangle \overset{\text{(I)}}{=} \Big\langle \sum_{\underline{i} \in A} \alpha_{e'_{\underline{i}}} e'_{\underline{i}}, \sum_{\underline{i} \in A} \alpha_{e'_{\underline{i}}} P(e'_{\underline{i}}) \Big\rangle \\ &= \langle \zeta, \mathcal{E}(\zeta) \rangle \overset{\text{(II)}}{\leq} \|\zeta\| \, \|\mathcal{E}(\zeta)\|. \end{aligned} \tag{39}$$

Inequality (II) is obtained from the Schwarz inequality. Equality (I) follows from

$$\big\langle P(l_1)\otimes\ldots\otimes P(l_n), P(l'_1)\otimes\ldots\otimes P(l'_n)\big\rangle$$
$$=\prod_{j=1}^{n}\langle P(l_j),P(l'_j)\rangle=\prod_{j=1}^{n}\langle l_j,P(l'_j)\rangle=\big\langle l_1\otimes\ldots\otimes l_n, P(l'_1)\otimes\ldots\otimes P(l'_n)\big\rangle \quad (40)$$

where $l_1,\ldots,l_n,l'_1,\ldots,l'_n\in\mathcal{H}$.

It only remains to prove that

$$\mathcal{E}(b'\zeta b'')=b'\mathcal{E}(\zeta)b'' \quad (41)$$

for all $b',b''\in\mathcal{B}$ and $\zeta\in\mathcal{G}$. Take any $Y_1,Y_2\in\mathcal{B}$, then

$$\mathbb{E}(\underbrace{Y_1b'}_{\in\mathcal{B}}\mathcal{E}(\zeta)\underbrace{b''Y_2}_{\in\mathcal{B}})=\mathbb{E}(Y_1\underbrace{b'\zeta b''}_{\in\mathcal{G}}Y_2)=\mathbb{E}(Y_1\mathcal{E}(b'\zeta b'')Y_2). \quad (42)$$

So, we have proved that the operation $\mathcal{E}$ defined by equation (37) is a conditional expectation. Using (31) one concludes the theorem. ∎

3. Conditional moments. The following definition has been introduced in [3]:

Definition 3.1. *Free deformed Poisson random variables.* Consider (noncommutative) random variables as elements of the algebra generated by the self-adjoint operators $X_f=a_f^*a_f+\sqrt{\lambda}(a_f^*+a_f)+\lambda\mathbf{I}$, with $\lambda>0$. Define the Wick product

$$\Psi_{f\otimes f_1\otimes\ldots\otimes f_n}=X_f\Psi_{f_1\otimes\ldots\otimes f_n}/\sqrt{\lambda}-\langle f,f_1\rangle\Psi_{f\otimes f_2\otimes\ldots\otimes f_n}/\sqrt{\lambda}$$
$$-\langle f,f_1\rangle\Psi_{f_2\otimes\ldots\otimes f_n}-\sqrt{\lambda}\,\Psi_{f_1\otimes\ldots\otimes f_n} \quad (43)$$

where $\Psi_\Omega=\mathbf{I}$ and $\Psi_\emptyset=0$.

From (43) we have $\Psi_f=(X_f-\lambda\mathbf{I})/\sqrt{\lambda}$ (so, condition (8) is satisfied for Ψ instead of W). Condition (6) follows from the definition of Ψ. Condition (7) follows from the equation $\Psi_f=(X_f-\lambda\mathbf{I})/\sqrt{\lambda}$, because $\Psi_{f_1}\ldots\Psi_{f_n}=a_{f_1}^*\ldots a_{f_n}^*+\omega$ where ω is some polynomial (composition) of $a_{f_1}^*\ldots a_{f_k}^*a_{f_1}\ldots a_{f_l}$, where the number of creators is not greater than $n-1$. From (43) by induction we deduce that $\Psi_{f_1\otimes\ldots\otimes f_n}$ is a polynomial in $\Psi_{f_1},\ldots,\Psi_{f_n}$, so if we use (1) we obtain that $\Psi_{f_1\otimes\ldots\otimes f_n}$ can be expressed as a finite combination of some elements $a_{f_{\pi(1)}}^*\ldots a_{f_{\pi(m_1)}}^*a_{f_{\pi(1)}}\ldots a_{f_{\pi(m_2)}}$, where $m_1,m_2\in\mathbb{N}$ and $\pi(k)\in\{1,\ldots,n\}$ $(1\le k\le n)$. Analogously we deduce that (9) holds.

We define the right deformed Poisson random variables $X_f^r=a_f^{r*}a_f^r+\sqrt{\lambda}(a_f^{r*}+a_f^r)+\lambda\mathbf{I}$ and their Wick product

$$\Psi^r_{f_1\otimes\ldots\otimes f_n\otimes f}=X_f^r\Psi^r_{f_1\otimes\ldots\otimes f_n}/\sqrt{\lambda}-\langle f,f_n\rangle\Psi^r_{f_1\otimes\ldots\otimes f_{n-1}\otimes f}/\sqrt{\lambda}$$
$$-\langle f,f_n\rangle\Psi^r_{f_1\otimes\ldots\otimes f_{n-1}}-\sqrt{\lambda}\Psi^r_{f_1\otimes\ldots\otimes f_n} \quad (44)$$

where $\Psi^r_\Omega=\mathbf{I}$ and $\Psi^r_\emptyset=0$.

Simple calculations show that

$$\Psi_\xi\Psi^r_\eta=\Psi^r_\eta\Psi_\xi\quad\forall\xi,\eta\in C. \quad (45)$$

In particular (45) holds for all finite linear combinations of C. If we apply both sides of (45) to Ω, then we get

$$\Psi_\xi \eta = \Psi^r_\eta \xi \quad \forall \xi, \eta \in C. \tag{46}$$

Let $\mathcal{B} = \mathrm{alg}(\mathbf{1}, \Psi_{f_1}, \ldots, \Psi_{f_n})$ be the algebra generated by $\mathbf{1}, \Psi_{f_1}, \ldots, \Psi_{f_n}$, and respectively $\mathcal{B}^r$ be the algebra generated by $\mathbf{1}, \Psi^r_{f_1}, \ldots, \Psi^r_{f_n}$. From (45) it follows that $\mathcal{B}^r \subset \mathcal{B}'$ (commutant of $\mathcal{B}$). Now let $T \in \mathcal{B}'$. Then, in particular, for any $\eta \in C$ we have $T\Psi_\eta = \Psi_\eta T$. This applied to Ω gives $T\Psi_\eta\Omega = T\eta = \Psi_\eta T\Omega = \Psi^r_{T\Omega}\eta$, which follows from (46). T can be linearly extended to the dense subspace of C, so $T \in \mathcal{B}^r$. Similarly $(\mathcal{B}^r)' = \mathcal{B}$ and $\mathcal{B} = (\mathcal{B})''$. Hence $\mathcal{B}$ is a von Neumann algebra. Using this we can easily conclude that the vacuum vector Ω is cyclic and separating for the algebra $\mathcal{B}$.

Given linearly independent vectors $f_1, \ldots, f_n \in \mathcal{H}$, we denote by $\mathcal{H}_{(f_1,\ldots,f_n)}$ the linear span of $f_1, \ldots, f_n$. Then, as a consequence of (5), each element $\xi \in \Gamma(\mathcal{H}_{(f_1,\ldots,f_n)})$ can be written as

$$\xi = \sum_{n\geq 0} \sum_{\underline{f} \in C_n} \alpha_{\underline{f}} \underline{f}. \tag{47}$$

So, for this ξ and any $\underline{f'} \in C$ we have

$$\Psi^r_{\underline{f'}} \xi = \sum_{n\geq 0} \sum_{\underline{f} \in C_n} \Psi^r_{\underline{f'}} \alpha_{\underline{f}} \underline{f} \overset{(46)}{=} \sum_{n\geq 0} \sum_{\underline{f} \in C_n} \alpha_{\underline{f}} \Psi_{\underline{f}} \underline{f'} = \Psi_\xi \underline{f'} \tag{48}$$

where the last series converges in $\Gamma(\mathcal{H}_{(f_1,\ldots,f_n)})$. By linearity of Ψ the above equation holds on the dense subspace C.

If $x \in \mathcal{B}$ then

$$x\underline{f'} = x\Psi^r_{\underline{f'}}\Omega = \Psi^r_{\underline{f'}} x\Omega = \Psi_{x\Omega}\underline{f'}. \tag{49}$$

So, this yields that each element of $\mathcal{B}$ can be introduced in the form of equation (10). Analogously we prove the above property for $\mathcal{G}$.

Proposition 3.2.

$$\mathbb{E}(X_{f_0} | X_{f_1}, \ldots, X_{f_n}) = \sum_{i=1}^n \alpha_i X_{f_i} \tag{50}$$

$$\mathbb{E}\big(X^2_{f_0} | X_{f_1}, \ldots, X_{f_n}\big) = \Big(\sum_{i=1}^n \alpha_i X_{f_i}\Big)^2 + \lambda c \Big(\sum_{i=1}^n \alpha_i X_{f_i}\Big) \tag{51}$$

where $c = (\|f_0\|^2 - \|P(f_0)\|^2)$.

Proof. This follows from Theorem 2.5 and (43). Write the orthogonal projection of f_0 onto the span of $f_1, \ldots, f_k$ as the linear combination $P(f_0) = \sum_{i=1}^n \alpha_i f_i$. Then (since $(X_{f_0} - \lambda\mathbf{I})\sqrt{\lambda} = \Psi_{f_0}$)

$$\begin{aligned}\mathbb{E}\big((X_{f_0} - \lambda\mathbf{I})\sqrt{\lambda} | X_{f_1}, \ldots, X_{f_n}\big) &= \mathbb{E}(\Psi_{f_0} | X_{f_1}, \ldots, X_{f_n}) \\ &= \Psi_{P(f_0)} = \Big(\sum_{i=1}^n \alpha_i X_{f_i} - \lambda\mathbf{I}\Big)/\sqrt{\lambda}.\end{aligned}$$

Now we compute (equation (43))

$$\Psi_{f_0\otimes f_0} = X_{f_0}\Psi_{f_0}/\sqrt{\lambda} - \|f_0\|^2\Psi_{f_0}/\sqrt{\lambda} - \|f_0\|^2\mathbf{I} - \sqrt{\lambda}\Psi_{f_0}$$
$$= X_{f_0}(X_{f_0} - \lambda\mathbf{I})/\lambda - \|f_0\|^2(X_{f_0} - \lambda\mathbf{I})/\lambda - \|f_0\|^2\mathbf{I} - (X_{f_0} - \lambda\mathbf{I}).$$

So we have

$$\mathbb{E}\big(X_{f_0}(X_{f_0} - \lambda\mathbf{I})/\lambda - \|f_0\|^2(X_{f_0} - \lambda\mathbf{I})/\lambda - \|f_0\|^2\mathbf{I} - (X_{f_0} - \lambda\mathbf{I})|X_{f_1},\dots,X_{f_n}\big)$$
$$= \mathbb{E}\big(\Psi_{f_0\otimes f_0}|X_{f_1},\dots,X_{f_n}\big) = \Psi_{P(f_0)\otimes P(f_0)} = \frac{1}{\lambda}\Big(\sum_{i=1}^n \alpha_i X_{f_i}\Big)\Big(\sum_{i=1}^n \alpha_i X_{f_i} - \lambda\mathbf{I}\Big)$$
$$- \frac{\|P(f_0)\|^2}{\lambda}\Big(\sum_{i=1}^n \alpha_i X_{f_i} - \lambda\mathbf{I}\Big) - \|P(f_0)\|^2\mathbf{I} - \Big(\sum_{i=1}^n \alpha_i X_{f_i} - \lambda\mathbf{I}\Big)$$
$$= \frac{1}{\lambda}\Big(\sum_{i=1}^n \alpha_i X_{f_i}\Big)^2 - (2 + \|P(f_0)\|^2)\Big(\sum_{i=1}^n \alpha_i X_{f_i}\Big) + \lambda\mathbf{I}.$$

This proves that

$$\mathbb{E}(X_{f_0}^2|X_{f_1},\dots,X_{f_n}) = \Big(\sum_{i=1}^n \alpha_i X_{f_i}\Big)^2 + \lambda(\|f_0\|^2 - \|P(f_0)\|^2)\Big(\sum_{i=1}^n \alpha_i X_{f_i}\Big). \ \blacksquare$$

COROLLARY 3.3. *The conditional variance is a linear function of* $X_{f_1},\dots,X_{f_n}$

$$\mathrm{Var}(X_{f_0}|X_{f_1},\dots,X_{f_n}) = c\lambda\Big(\sum_{i=1}^n \alpha_i X_{f_i}\Big) \tag{52}$$

where $c = (\|f_0\|^2 - \|P(f_0)\|^2)$.

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