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GROUP C*-ALGEBRAS SATISFYING KADISON'S CONJECTURE

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Abstract. We tackle R. V. Kadison's similarity problem (i.e. any bounded representation of any unital C*-algebra is similar to a *-representation), paying attention to the class of C*-unitarisable groups (those groups G for which the full group C*-algebra C$^*(G)$ satisfies Kadison's problem) and thereby we establish some stability results for Kadison's problem. Namely, we prove that $A \otimes_{\min} B$ inherits the similarity problem from those of the C*-algebras A and B, provided B is also nuclear. Then we prove that G/Γ is C*-unitarisable provided G is C*-unitarisable and Γ is a normal subgroup; and moreover, if G/Γ is amenable and Γ is C*-unitarisable, so is the whole group G (Γ a normal subgroup).

1. Introduction. This work pursues the work [HP, H] where we aim to further the study of Kadison's similarity problem (SP), see [Ka] or [P2], focusing on the class of group C*-algebras. Our study benefits somehow from Banach algebra techniques already considered in e.g. [H]. In particular, we would like to understand when the full group C*-algebra C$^*(G)$ satisfies (SP) and if (SP) of C$^*(G)$ passes to (or is inherited from) C$^*(\Gamma)$ and C$^*(G/\Gamma)$ with Γ a (normal) subgroup of a locally compact (or discrete) group G. Moreover, when C$^*(G_1 \times G_2)$ satisfies (SP) or, more generally, for which tensor products (SP) is inherited from the underlying C*-algebras.

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The origin of the similarity problems should perhaps be traced back to 1929 when John von Neumann introduced the notion of an amenable group and late 1940's with the work of Dixmier [Di] with the notion of unitarisable group. Remind that a locally compact group G is called unitarisable if any continuous and uniformly bounded (u.b.)

$$\sup_{g\in G} \|\pi(g)\| < \infty \tag{1}$$

representation π of G on a Hilbert space H is unitarisable (= similar to a unitary representation, i.e. there exists an invertible operator $S : H \to H$ such that $g \mapsto S^{-1}\pi(g)S$ is a unitary representation of G). Dixmier asked in 1950 whether

$$G \text{ unitarisable} \implies G \text{ amenable?} \tag{2}$$

This is still an open problem, but the converse was proved by Dixmier [Di] and H. Day independently, see also [P5]. Motivated by this, Kadison [Ka] conjectured in 1955 that C*-algebras satisfy the *similarity problem* (abbreviated as (SP)), where we say that a C*-algebra A satisfies (SP) if any bounded (not necessarily *-preserving) homomorphism $\pi : A \to B(H)$ from a C*-algebra A into the algebra of bounded operators in a Hilbert space H, is similar to a *-homomorphism, i.e. given such π there exists an invertible operator $S : H \to H$ such that $S\pi(\cdot)S^{-1}$ is a *-homomorphism, see e.g. [P5]. So Kadison's conjecture can be written as follows:

$$\pi : A \to B(H) \text{ bounded homomorphism} \implies \pi \text{ similar to a C}^*\text{-representation?} \tag{3}$$

We note that any *-homomorphism between C*-algebras is contractive and thus bounded. Also note that if $\pi : A \to B(H)$ is a *-representation of a C*-algebra A and $G \subseteq \mathcal{U}_A$ is a subgroup of the unitary group $\mathcal{U}_A$ of A, the restriction $\pi|_G : G \to B(H)$ is a unitary representation. More generally, if π is similar to a *-representation then its restriction $\pi|_G : G \to B(H)$ is similar to a unitary representation. Although there are examples of non-unitarisable groups, e.g. $\mathrm{SL}(2,\mathbb{R})$ [EM], free groups or some Burnside groups [OM], we still do not know if there exists a C*-algebra A (related or not with the above non-unitarisable groups) and a bounded homomorphism $\pi : A \to B(H)$ which is not similar to a *-representation!

C*-algebras without tracial states [H1] (e.g. $B(l^2(\mathbb{N}))$) and nuclear C*-algebras (e.g. commutative and finite-dimensional ones) do satisfy (SP). It is worth mentioning that deep results of Connes [C2] and Haagerup [H2] show that amenable C*-algebras are precisely the nuclear ones. Type II_1 factors with property Γ also satisfy (SP), as in [Ch]. In sharp contrast, Kadison's conjecture (3) is still open when A is, for example, the reduced (or full) group C*-algebra of the non-abelian free group $\mathbb{F}_n$ on n generators, although these discrete groups are well known to be non-unitarisable, see [P2].

We may extend (SP) to operator algebras (cf. (12)), not necessarily self-adjoint norm closed subalgebras of some $B(H)$, or Banach algebras, see (11). Pisier [P1] proved that the disc algebra $A(\mathbb{D})$ does not satisfy the (generalised) similarity problem.

The rest of the paper is organised as follows. In Section 2 we provide some background of the notions of amenable groups and group C*-algebras (reduced, full and 'big' group algebra) for locally compact groups, the multiplier algebra, nuclear C*-algebras and completely bounded maps. In Section 3, we introduce the notions of completely bounded map as in (9) and the total reduction property, as in (10), for operator algebras. Then for Banach algebras we put forward the notions of the similarity property in (11) and the generalised similarity problem in (12). We also conclude that for C*-algebras, the total reduction property, similarity property and generalised similarity property, the derivation problem and the completely bounded property are all equivalent.

Finally in Section 4 we present our main results. After reviewing the concepts of weak similarity problems (WSP) for von Neumann algebras and C*-unitarisable groups (see Definitions 4.1 and 4.6), we relate the (SP) of a C*-algebra A with the (WSP) of its bidual A^{**} (see Propositions 4.2 and 4.3) and draw some more or less standard conclusions regarding the unitarisable groups G and the (SP) of their cousin C*-algebras, namely of the reduced $\mathrm{C}^*_{\mathrm{red}}(G)$, full $\mathrm{C}^*(G)$, big $\mathcal{A}(G)$ and of the von Neumann $\mathrm{vN}(G)$ group algebras, in Propositions 4.7 and 4.8. Regarding the stability results of (SP): in Proposition 4.4 we show that $A \otimes_{\min} B$ inherits the (SP) from those of A and B, assuming B to be nuclear, in particular $G_1 \times G_2$ is C*-unitarisable provided G_1 is C*-unitarisable and G_2 is amenable. In Proposition 4.9 we relate the C*-unitarisability of G with that of G/Γ and of Γ, for Γ a (normal) subgroup of G.

2. Preliminaries

2.1. Group C*-algebras $\mathbf{C}^*_{\mathbf{red}}(G)$, $\mathbf{C}^*(G)$ and $\mathcal{A}(G)$. Let G be a locally compact group and μ the (left) Haar measure of G. Recall that the notion of amenable group was introduced by von Neumann in 1929, and says that G is amenable if there exists a left invariant mean on G, i.e. if there exists a positive linear functional $m : L^\infty(G) \to \mathbb{C}$ such that $m(1) = 1$ and $m(f) = m({}_g f)$ for any $g \in G$, where ${}_g f(t) = f(g^{-1}t)$, and $L^\infty(G)$ is the Banach space of all essentially bounded functions $G \to \mathbb{C}$ with respect to the Haar measure.

Next, we let $C_c(G)$ be the space of complex valued continuous functions on G with compact support. Consider $L^2(G)$ the Hilbert space of square integrable functions with respect to μ. We also recall the convolution product as follows:

$$(f * g)(t) = \int_G f(s)g(s^{-1}t)\, d\mu(s). \tag{4}$$

Let Δ be the modular function on G ($\Delta \equiv 1$ for discrete groups). Then $f^*(t) = \Delta(t^{-1})\overline{f(t^{-1})}$ equips $C_c(G)$ with an involution *. For an integrable function $f \in C_c(G)$, $\|f\|_1 := \int_G |f(t)|\, d\mu(t)$ equips $C_c(G)$ with a structure of a normed algebra. The convolution algebra $L^1(G)$ is the *-Banach algebra obtained by completion of $C_c(G)$ in that norm. Any unitary representation π of G can be lifted to a *-representation $\overline{\pi}$ of $L^1(G)$ on the same Hilbert space. The (full) group C*-algebra $\mathrm{C}^*(G)$ of G is the C*-enveloping algebra of $L^1(G)$, i.e. the completion of $L^1(G)$ with respect to the largest C*-norm:

$$\|f\|_{\mathrm{C}^*(G)} := \sup_{\pi} \|\overline{\pi}(f)\|, \tag{5}$$

where π ranges over all unitary representations of G on Hilbert spaces. The reduced C*-algebra $\mathrm{C}^*_{\mathrm{red}}(G)$ is the C*-algebra generated by the left regular representation $\lambda(G)$ in $B(L^2(G))$ and defined as follows:

$$\lambda_g(f)(t) = f(g^{-1}t), \quad g, t \in G,\ f \in L^2(G). \tag{6}$$

The left regular representation gives rise to a natural C*-morphism $\mathrm{C}^*(G) \to \mathrm{C}^*_{\mathrm{red}}(G)$ which is an isomorphism if and only if G is amenable. Also let $\mathrm{vN}(G)$ be the von Neumann algebra generated by $\lambda(G)$ in $B(L^2(G))$. In general we have:

$$\|f\|_{\mathrm{C}^*_{\mathrm{red}}(G)} \leq \|f\|_{\mathrm{C}^*(G)} \leq \|f\|_{L^1(G)}. \tag{7}$$

Any group morphism between two discrete groups $G_1 \to G_2$ can be lifted to a C*-algebra *-homomorphism $\mathrm{C}^*(G_1) \to \mathrm{C}^*(G_2)$ (see Rieffel's [R, Proposition 4.1]). In general this functoriality does not extend to locally compact groups.

The big group C*-algebra $A(G)$ associated to a locally compact group G is defined as follows. Let H be an infinite-dimensional separable Hilbert space and let G_H be the set of all unitary representations $\pi : G \to B(H)$ on that fixed Hilbert space H and $A(G) = \{J : G_H \to B(H)\}$ the set of maps from G_H to $B(H)$ satisfying some natural conditions as in [E, page 469], where we can define an involutive algebra structure. The weak topology on $A(G)$ is defined to be the smallest topology such that the functions $J \to \langle J(L)\xi, \psi\rangle$ are continuous, for all $J \in G_H$ and $\xi, \psi \in H$. As Banach algebras, $A(G)$ may be identified with $\mathrm{C}^*(G)^{**}$ the bidual of the full C*-algebra $\mathrm{C}^*(G)$. For every $g \in G$, let $\hat{g} : G_H \to B(H)$ be the map defined by $\hat{g}(\pi) = \pi(g)$. Then $g \to \hat{g}$ gives an imbedding of G into $A(G)$, with image $\hat{G}$ constituted by unitary operators, see [E, Theorem 2.3]. Moreover, the algebras $L^1(G)$ and $\mathrm{C}^*(G)$ are dense in $A(G)$, see [E, Corollary 3.2]. The *-subalgebra $\mathrm{Alg}(\hat{G})$ of all finite linear combinations of elements in $\hat{G}$ is dense in $A(G)$ relative to any of the topologies: weak and σ-weak, see [E, Theorem 7.2].

For locally compact groups, $\mathrm{C}^*(G)$ is non-unital, and so we will also be interested in the multiplier algebra $M(\mathrm{C}^*(G))$. Recall that for an algebra A, the double centraliser algebra $\Gamma(A)$ is a unital algebra defined as follows. A double centraliser of A is a pair (L, R) of maps from A to A satisfying $aL(b) = R(b)a$ for all $a, b \in A$. Then L and R are linear maps. We can embed A in $\Gamma(A)$ by using the map $a \mapsto (L_a, R_a)$ with $L_a(c) = ac$ and $R_a(c) = ca$. When A is a C*-algebra, then the multiplier algebra $M(A)$ of A is the set of bounded double centralisers (L, R), in which case $\|L\| = \|R\|$. In this way, $M(A)$ is a unital C*-algebra with the C*-norm $\|(L, R)\| = \|L\|$, and A is a strongly dense two sided ideal $M(A)$. We further remark that we have an embedding $j : G \to M(\mathrm{C}^*(G))$ of a locally compact group G into the multiplier algebra $M(\mathrm{C}^*(G))$, by mapping $g \mapsto (L_g, R_g)$ with $L_g(f) = \delta_g * f$ and $R_g(f) = f * \delta_g$.

2.2. Nuclear C*-algebras and completely bounded maps. For the minimal C*-norm, we first embed A and B in $B(H_1)$ and $B(H_2)$ as C*-algebras, respectively. Then for any $x = \sum a_i \otimes b_i$ in the algebraic tensor product $A \otimes B$, $\|x\|_{\min}$ is the norm on the space $B(H_1 \otimes H_2)$. The completion of this isometric *-homomorphism image of $A \otimes B$ into $B(H_1 \otimes H_2)$ is denoted by $A \otimes_{\min} B$. For the maximal tensor product we define

$$\|x\|_{\max} = \sup \|\pi(x)\|_{B(H)},$$

where π runs over all possible Hilbert spaces H and *-homomorphisms $\pi : A \otimes B \to B(H)$. The completion of $A \otimes B$ with the norm $\|\cdot\|_{\max}$ is denoted by $A \otimes_{\max} B$. For any C*-norm $\|\cdot\|$ on $A \otimes B$ we have

$$\|\cdot\|_{\min} \leq \|\cdot\| \leq \|\cdot\|_{\max}. \tag{8}$$

For two discrete groups G_1 and G_2, it is easily checked that (see [P4, page 149]): $C^*_{\rm red}(G_1) \otimes_{\min} C^*_{\rm red}(G_2) \simeq C^*_{\rm red}(G_1 \times G_2)$, and $C^*(G_1) \otimes_{\max} C^*(G_2) \simeq C^*(G_1 \times G_2)$.

A C*-algebra A is called *nuclear* if for any C*-algebra B, there is a unique C*-norm on $A \otimes B$, i.e. $\| \cdot \|_{\max} = \| \cdot \|_{\min}$. Connes [C2] and Haagerup [H2] showed that nuclear C*-algebras are precisely the *amenable* ones. If A is finite-dimensional or commutative then A is nuclear. For discrete groups G, the reduced $C^*_{\rm red}(G)$ (or full $C^*(G)$) algebra is nuclear if and only if G is amenable [C1, C2].

3. Similarity problems for Banach algebras versus C*-algebras. Let A be a complex Banach algebra. A Banach ⟨left, bi⟩-module on A is a Banach space X which is an algebraic ⟨left, bi⟩-module on A such that the module actions are continuous. Note that X^* becomes a Banach ⟨left, bi⟩-module with respect to the dual actions $\langle a, f\rangle(b) = f(ba)$ and $\langle f, a\rangle(b) = f(ab)$ identically for every $a, b \in A$ and $f \in X^*$. A derivation from A into a Banach A-bimodule X is a bounded map $D : A \longrightarrow X$ such that $D(ab) = D(a)b + aD(b)$ for all $a, b \in A$. A derivation D is inner if there is $x \in X$ such that $D(a) = a.x - x.a$ for $a \in A$. A Banach algebra A is said to be amenable if for each Banach A-bimodule X, every derivation $D : A \longrightarrow X^*$ is inner. Amenable Banach algebras were introduced by B. E. Johnson in [J1] and they were well investigated in [Cur, GLW, J2]. It was established in [C2] by A. Connes that every amenable C*-algebra A (i.e., every closed self-adjoint of $B(H)$, the algebra of bounded operators on a given Hilbert space H) is nuclear; i.e., for all C*-algebra B, there exists a unique C*-norm for which the completion of the (algebraic) tensor product $A \otimes B$ is a C*-algebra. The converse has been shown by Haagerup in [H2].

Assume now that A is an operator algebra, that is, a Banach algebra which acts as an algebra of bounded operators on a Hilbert space K (in fact, A is a norm-closed subalgebra of the algebra $B(K)$ of bounded operators on K). Identifying the matrix space $\mathrm{M}_n(B(K))$ over $B(K)$ with $B(K^n)$, we let $\mathrm{M}_n(A)$ have the relative norm in $B(K^n)$. A bounded homomorphism $\phi : A \longrightarrow B$ is completely bounded (c.b. for short) if for every n, $\phi_n : \mathrm{M}_n(A) \longrightarrow \mathrm{M}_n(B)$ defined by $(a_{ij}) \mapsto (\phi(a_{ij}))$ is bounded, such that

$$\text{(c.b.)} \qquad \|\phi\|_{\rm cb} = \sup \|\phi_n\|_{\mathrm{M}_n(A) \to \mathrm{M}_n(B)} < \infty. \qquad (9)$$

The relevant point here for us is the result of Haagerup [H1] that says that a bounded homomorphism $\pi : A \to B(H)$ is similar to *-homomorphism if and only if π is completely bounded, see also [P2] and [H].

A representation π of A on a Hilbert space H, i.e., a bounded homomorphism $\pi : A \longrightarrow B(H)$, is said to be non-degenerated if $\overline{\pi(A)H} = H$. It is called irreducible when the only closed invariant subspaces of $\pi(A)$ are $\{0\}$ and H. A Hilbert A-module H is defined to be a left Banach module on A that is isomorphic to a Hilbert space. This is equivalent to the existence of a representation $\pi : A \longrightarrow B(H)$ of A on H. A Hilbert A-module is said to have the reduction property if for every closed submodule V of H, there is a closed submodule W such that $H = V \oplus W$. A Hilbert A-module H is said to have the complete reduction property if the amplified module H^∞ $\big(=H \otimes_2 l^2(\mathbf{N})\big)$ has the reduction property for $A^{(\infty)}$. An operator algebra A is said to have the total reduction property (TRP for short) if

$$\text{(TRP)} \qquad \text{every Hilbert } A\text{-module has the reduction property.} \tag{10}$$

If $A \subseteq B(K)$ is an operator algebra, then the total reduction property for A implies the complete reduction for A, which in turn implies the reduction property. If A is an operator algebra and $\pi : A \longrightarrow B(H)$ is a bounded representation, then H has the reduction property if and only if every submodule of H is the range of an idempotent operator in $\pi(A)'$, the commutant of $\pi(A)$. Such idempotents are called idempotent operator module projections. The total reduction property for operator algebras has been introduced in J. A. Gifford's thesis [G1]. This property is satisfied by amenable operator algebras. In fact, many problems raised in the amenability context, have natural extension to operator algebras with the total reduction property. In [P2, page 13], Pisier asks which unital Banach algebras A have the similarity property (SP), i.e., are such that for each bounded unital representation $\phi : A \to B(H)$,

$$\text{(SP)} \qquad \exists S \in B(H) \text{ invertible such that } a \mapsto S^{-1}\phi(a)S \text{ is a contraction.} \tag{11}$$

He gave several results answering partially this question. In his paper [P3], he also has raised the Generalised Similarity Problem (GSP): Which unital operator algebras have the following

$$\text{(GSP)} \quad \text{Any representation } \pi : A \longrightarrow B(H) \ (H \text{ an arbitrary Hilbert space) is c.b.} \tag{12}$$

In the C*-algebras case, (SP) is equivalent to (GSP). Gifford in [G2] has shown that a C*-algebra A has the total reduction property if and only if it satisfies (SP), i.e. each representation of A is similar to a *-homomorphism. Note that for C*-algebras a *-homomorphism is automatically a contraction, and Kadison conjectured in 1955 that all (unital) C*-algebras do satisfy (SP). It is shown in [H, Theorem 3.1] that an operator algebra A has (TRP) if both I and A/I have the (TRP), with I a closed two sided ideal of A. Since (TRP) is equivalent to (SP) for C*-algebras, we obtain the following

Proposition 3.1 ([H, Corollary 3.5]). *Let A be a* C*-*algebra and I a two sided ideal of A, such that both I and A/I satisfy* (SP). *Then A satisfies* (SP).

This result is still open for amenable groups. Kadison's similarity problem (i.e. (SP) for C*-algebra) is equivalent to another crucial problem (the *derivation problem*) in the cohomology theory of operator algebras, as follows. A celebrated theorem of Kirchberg [Ki] states that, for C*-algebras the following holds

$$A \text{ satisfies (SP)} \iff A \text{ satisfies (DP)}, \tag{13}$$

where we say that a C*-algebra A satisfies (DP) – the derivation problem – if any derivation

$$\text{(DP)} \qquad D : A \to B(H) \text{ is inner, i.e. } D(a) = ab - ba, \text{ for some } b \in A. \tag{14}$$

Note that we may check that a map $D : A \to B(H)$ is a derivation with respect to a representation π of a C*-algebra A on H if and only the map defined by

$$\pi_D(a) = \begin{bmatrix} \pi(a) & D(a) \\ 0 & \pi(a) \end{bmatrix} \in B(H \oplus H)$$

is a representation of A in $H \oplus H$. Moreover π_D is similar to a *-homomorphism if and only if D is inner. To see the 'if' part, we remark that if $D = \delta_T$, then

$$\pi_D(a) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \pi(a) & 0 \\ 0 & \pi(a) \end{bmatrix} \begin{bmatrix} 1 & -T \\ 0 & 1 \end{bmatrix}.$$

Therefore, π is similar to the *-homomorphism $\mathrm{id_A} \oplus \mathrm{id_A}$. In this way we have established the easier implication (SP) $\Longrightarrow$ (DP). For the proof of the 'only if' of the above statement and thus the implication (DP) $\Longrightarrow$ (SP), we advise the reader to consult Kirchberg's proof in [Ki].

We naturally also have a notion of derivation for a group representation and we say that a group G satisfies the (DP) if any derivation on G is inner, see [J1].

In the case of (non-self-adjoint) operator algebras, it is natural to look for similar results including possible connections of the total reduction property with (GSP) or (SP). Anyhow, it has been conjectured in [G2] that:

Conjecture 1. *Every non-self-adjoint operator algebra with total reduction property is isomorphic* (*as Banach algebra*) *to a* C**-algebra.*

Conjecture 2. *Every weakly closed complete reduction non-self-adjoint operator algebra* $A \subseteq B(K)$ *is similar to* C**-algebra* B*, i.e.* $\exists S \in B(K)$ *invertible such that* $A = S^{-1}BS$*.*

As partial results, Gifford has proven that every operator algebra $A \subseteq B(K)$ with total reduction property which is a closed subalgebra of an abelian C*-algebra is a C*-algebra and every operator algebra $A \subseteq B(K)$ with total reduction property which is a closed subalgebra of the algebra of compact operators on K is similar to a C*-algebra. It is clear that if Conjecture 2 is true, then Conjecture 1 holds for weakly closed operator algebras. We finish this section by giving recent partial answers to the above conjectures:

Theorem 3.2 ([H]). *Let* $A \subseteq B(K)$ *be an operator algebra with* (TRP)*.*

1) *If* A *is isomorphic to a* C**-algebra, then* A *is similar to a* C**-algebra.*
2) *Let* B *be the weak-closure of the algebra generated by* A' *and* A''*. Then* B *is similar to a* C**-algebra.*

4. Main results

4.1. Similarity properties for C*-algebras vs their duals. Similarity properties for C*-algebras vs their duals

A von Neumann algebra is of course a C*-algebra, but besides bounded representations of N as a C*-algebra we may also consider the ones that are bounded and weakly-continuous, giving rise to the following definition.

Definition 4.1 ([HP]). We say that a von Neumann algebra N satisfies the *weak similarity problem*, for short (WSP), if for any bounded and weak*-continuous representation $\pi : N \to B(H)$, the operator $S^{-1}\pi(\cdot)S$ is a *-homomorphism for some invertible operator S.

It is clear by definition that N satisfies (WSP) if N satisfies (SP).

Proposition 4.2. *A* C^**-algebra* A *satisfies* (SP) *if and only if* A^{**} *has* (WSP).

Proof. Let $\pi : A^{**} \to B(H)$ be a w*-continuous representation on A^{**}. There is an invertible operator $S \in B(H)$ such that $S^{-1}\pi S$ is a *-homomorphism on A. Notice that the algebra isomorphism of $B(H)$ defined by $T \mapsto S^{-1}TS$ is w*-continuous. It follows that $S^{-1}\pi S$ is a *-homomorphism on A^{**}. For the converse, let $\pi : A \to B(H)$ be a bounded representation. Its extension to A^{**} is w^*-continuous and bounded on A^{**}. Then there is an invertible operator $S \in B(H)$ such that $S^{-1}\pi S$ is a *-homomorphism on A^{**}. The restriction of this *-homomorphism is also a *-homomorphism on A. This completes the proof. ∎

Proposition 4.3. *A* C^**-algebra* A *satisfies* (SP) *if and only if the bicommutant satisfies* (WSP).

Proof. Let $\pi : A'' \to B(H)$ be a w*-continuous bounded homomorphism. Its restriction on A is similar to a *-homomorphism; this means that there exists an invertible operator S in $B(H)$ such that $S^{-1}\pi S$ is a *-homomorphism. The homomorphism $T \mapsto S^{-1}TS$ is w*-continuous on $B(H)$, hence $S^{-1}\pi S$ is a *-homomorphism on A''. This completes the proof. ∎

In particular, if $C^*_{\text{red}}(G)$ has (SP), then $\text{vN}(G)$ has (WSP).

Proposition 4.4. *Let* A *and* B *be unital* C^**-algebras such that* A *has* (SP) *and* B *is nuclear. Then* $A \otimes_{\min} B$ *has* (SP).

Proof. It suffices to show that $A \otimes_{\min} B$ has (GSP), i.e. every bounded representation of $A \otimes_{\min} B$ is completely bounded. Consider a bounded representation $\pi : A \otimes B \to B(H)$. Let π_A and π_B be the representations, respectively, on A and B, defined by $\pi_A(a) = a \otimes 1$ and $\pi_B(b) = 1 \otimes b$. Since π_B is completely bounded, it is an invertible operator $S \in B(H)$ such that $\rho_B = S^{-1}\pi_B S$ is a *-homomorphism. Put $\rho_A = S^{-1}\pi_A S$. It is completely bounded. Then, first, $\rho_A \otimes \rho_B$ is completely bounded on $A \otimes_{\min} B$ and since $\rho_A(A)$ is in the commutant of $\rho_B(B)$, $\rho_A \otimes \rho_B$ has the image in $\rho_B(B)' \otimes \rho_B(B)$.

Second, we can easily check that the restriction of the linear map $p : B(H) \otimes B(H) \to B(H)$ defined by $a \otimes b \mapsto ab$ is a *-homomorphism from $\rho_B(B)' \otimes_{\max} \rho_B(B)$. Since $\rho_B(B)' \otimes_{\max} \rho_B(B) = \pi_B(B)' \otimes_{\min} \rho_B(B)$ and $\rho = p \circ (\rho_A \otimes \rho_B)$, ρ is completely bounded. We deduce that $\pi = S^{-1}\rho S$ which is completely bounded. ∎

4.2. Stability results for group C*-algebras. Let A be a C^*-algebra and $U(A)$ its unitary group. When equipped with the discrete topology, we denote this (locally compact) group by $U_d(A)$.

Proposition 4.5. *If the group* $U_d(A)$ *is unitarisable then* A *satisfies* (SP).

Proof. Let $\tilde{\pi} : A \to B(H)$ be a bounded representation of A and $\pi := \tilde{\pi}|_{U_d(A)}$. Then π is a uniformly bounded representation of $U_d(A)$ because $\|\pi(g)\| \leq \|\tilde{\pi}\|$ for any $g \in U_d(A)$. Thus there exists an invertible operator $S : H \to H$ such that $S^{-1}\pi S$ is a unitary representation of $U_d(A)$, as $U_d(A)$ is unitarisable. Since every $a \in A$ is a linear combination of four unitaries in $U_d(A)$, we easily conclude that $S^{-1}\tilde{\pi} S$ is a *-homomorphism of A. ∎

Note that the finite-dimensional (full matrix) algebra $M_{2\times 2}(\mathbb{C})$ shows that the converse of Proposition 4.5 is false.

We may easily see that if we have a surjective homomorphism $\varphi : A \to B$ between C*-algebras, then if A satisfies Kadison's (SP), so does B. In particular we apply this for the natural surjective homomorphism $\lambda : \mathrm{C}^*(G) \to \mathrm{C}^*_{\mathrm{red}}(G)$ whenever $\mathrm{C}^*(G)$ has the (SP). We now propose the following definition.

Definition 4.6 ([HP]). A (locally compact) group G is C*-*unitarisable* if the C*-algebra $\mathrm{C}^*(G)$ satisfies (SP).

If G is amenable, then $\mathrm{C}^*(G)$ is a nuclear C*-algebra [H2] and thus $\mathrm{C}^*(G)$ satisfies (SP), and therefore G is a C*-unitarisable group. Also if we use Pisier [P5, Theorem 0.9], then it is clear that for a given discrete group G, a bounded representation $\pi : \mathrm{C}^*(G) \to B(H)$ of G on a Hilbert space H is similar to a *-homomorphism if and only if the restriction $\pi|_G$ is a unitarisable representation of the group G on H. We thus conclude that every unitarisable discrete group G is C*-unitarisable.

We summarise further easy consequences of the definitions in the following result.

Proposition 4.7. *Let G be a discrete group. Let us consider the following statements*:

1) *G is unitarisable.*
2) *G is* C*-*unitarisable.*
3) $\mathrm{C}^*_{\mathrm{red}}(G)$ *satisfies* (SP).

Then 1) $\Longrightarrow$ 2) $\Longrightarrow$ 3).

Proof. 1) $\Rightarrow$ 2). Let $\varphi : \mathrm{C}^*(G) \to B(H)$ be a continuous representation of the group C*-algebra $\mathrm{C}^*(G)$. Then we define a representation $\pi : G \to B(H)$ of the group G by setting $\pi(g) = \varphi(\delta_g)$.

As G is unitarisable, there exists an invertible operator $S : H \to H$ such that $S^{-1}\pi S$ is a unitary representation of G. Notice that since G is discrete, π can be extended to a representation $\tilde{\pi} : L^1(G) \to B(H)$, hence $S^{-1}\tilde{\pi}S \equiv S^{-1}\varphi S$ on $L^1(G)$ and in particular $S^{-1}\varphi S$ is a *-homomorphism on $\mathrm{C}^*(G)$.

2) $\Rightarrow$ 3). Let $\pi : \mathrm{C}^*_{\mathrm{red}}(G) \to B(H)$ be a representation. The left regular representation λ of G provides a natural *-homomorphism $\lambda : \mathrm{C}^*(G) \to \mathrm{C}^*_{\mathrm{red}}(G)$. Since $\mathrm{C}^*(G)$ satisfies (SP), there is an invertible $S \in B(H)$ such that $S^{-1}\lambda \circ \pi(.)S$ is a *-homomorphism. As λ is surjective, $S^{-1}\pi(.)S$ is a *-homomorphism. This show that $\mathrm{C}^*_{\mathrm{red}}(G)$ satisfies (SP). Of course we can use the argument before Definition 4.6 to prove this implication. ■

As an immediate consequence of Propositions 4.2 and 4.7 we have the following

Corollary 4.8. *Let G be a discrete and unitarisable group. Then the big group algebra $\mathcal{A}(G)$ satisfies* (SP).

We have the following stability result.

Proposition 4.9 ([HP]). *Let G be a discrete group.*

1) *If G is* C*-*unitarisable, then for every normal subgroup Γ, we have that G/Γ is* C*-*unitarisable.*

2) *Let* Γ *be a normal subgroup of* G*. Assume that* Γ *is* C^**-unitarisable and* G/Γ *is amenable. Then* G *is* C^**-unitarisable.*

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References

[Ch] E. Christensen, *Similarities of* II_1 *factors with property* Γ, J. Operator Theory 15 (1986), 281–288.

[C1] A. Connes, *Classification of injective factors. Cases* II_1, II_∞, III_λ, $\lambda \neq 1$, Ann. of Math. (2) 104 (1976), 73–115.

[C2] A. Connes, *On the cohomology of operator algebras*, J. Functional Analysis 28 (1978), 248–253.

[Cur] P. C. Curtis, Jr, R. J. Loy, *The structure of amenable Banach algebras*, J. London Math. Soc. (2) 40 (1989), 89–104.

[Di] J. Dixmier, *Les moyennes invariantes dans les semi-groups et leurs applications*, Acta Sci. Math. Szeged 12 (1950), 213–227.

[EM] E. Ehrenpreis, F. I. Mautner, *Uniformly bounded representations of groups*, Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 231–233.

[H] R. El Harti, *Reduction operator algebra and similarity problem*, J. Oper. Matrices 4 (2010), 559–572.

[HP] R. El Harti, P. R. Pinto, *Stability results for* C^**-unitarisable groups*, Ann. Funct. Anal. 2 (2011), no. 2, 1–9.

[E] J. A. Ernest, *A new group algebra for locally compact groups*, Amer. J. Math. 86 (1964), 467–492.

[GLW] F. Ghahramani, R. J. Loy, G. A. Willis, *Amenability and weak amenability of second conjugate Banach algebras*, Proc. Amer. Math. Soc. 124 (1996), 1489–1497.

[G1] J. A. Gifford, *Operator algebras with a reduction property*, Ph.D. Thesis, Australian National University, 1997.

[G2] J. A. Gifford, *Operator algebras with a reduction property*, J. Aust. Math. Soc. 80 (2006), 297–315.

[H1] U. Haagerup, *Solution of the similarity problem for cyclic representations of* C^**-algebras*, Ann. of Math. (2) 118 (1983), 215–240.

[H2] U. Haagerup, *All nuclear* C^**-algebras are amenable*, Invent. Math. 74 (1983), 305–319.

[J1] B. E. Johnson, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc. no. 127 (1972).

[J2] B. E. Johnson, *Approximate diagonals and cohomology of certain annihilator Banach algebras*, Amer. J. Math. 94 (1972), 685–698.

[Jol] P. Jolissaint, *Central sequences in the factor associated with Thompson's group* F, Ann. Inst. Fourier (Grenoble) 48 (1998), 1093–1106.

[Ka] R. V. Kadison, *On the orthogonalization of operator representations*, Amer. J. Math. 77 (1955), 600–620.

[Ki] E. Kirchberg, *The derivation problem and the similarity problem are equivalent*, J. Operator Theory 36 (1996), 59–62.

[L] E. C. Lance, *On nuclear C^*-algebras*, J. Functional Analysis 12 (1972), 157–176.

[OM] N. Monod, N. Ozawa, *The Dixmier problem, lamplighters and Burnside groups*, J. Funct. Anal. 258 (2010), 255–259.

[P1] G. Pisier, *A polynomially bounded operator on Hilbert space which is not similar to a contraction*, J. Amer. Math. Soc. 10 (1997), 351–369.

[P2] G. Pisier, *Similarity Problems and Completely Bounded Maps*, Lecture Notes in Math. 1618, Springer, Berlin, 2001.

[P3] G. Pisier, *Similarity problems and length*, International Conference on Mathematical Analysis and its Applications (Kaohsiung, 2000), Taiwanese J. Math. 5 (2001), 1–17.

[P4] G. Pisier, *Introduction to Operator Space Theory*, London Math. Soc. Lecture Note Ser. 294, Cambridge Univ. Press, Cambridge, 2003.

[P5] G. Pisier, *Are unitarizable groups amenable?*, in: Infinite Groups: Geometric, Combinatorial and Dynamical Aspects, Progr. Math. 248, Birkhäuser, Basel, 2005, 323–362.

[Pop] F. Pop, *The similarity problem for tensor products of certain C*-algebras*, Bull. Austral. Math. Soc. 70 (2004), 385–389.

[R] M. A. Rieffel, *Induced representations of C^*-algebras*, Advances in Math. 13 (1974), 176–257.