

QUANTUM DETAILED BALANCE CONDITIONS WITH TIME REVERSAL: THE FINITE-DIMENSIONAL CASE

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Abstract. We classify generators of quantum Markov semigroups \mathcal{T} on $\mathcal{B}(\mathfrak{h})$, with \mathfrak{h} finite-dimensional and with a faithful normal invariant state ρ satisfying the standard quantum detailed balance condition with an anti-unitary time reversal θ commuting with ρ , namely $\mathrm{tr}(\rho^{1/2}x\rho^{1/2}\mathcal{T}_t(y)) = \mathrm{tr}(\rho^{1/2}\theta y^*\theta\rho^{1/2}\mathcal{T}_t(\theta x^*\theta))$ for all $x, y \in \mathcal{B}(\mathfrak{h})$ and $t \geq 0$.

Our results also show that it is possible to find a standard form for the operators in the Lindblad representation of the generators extending the standard form of generators of quantum Markov semigroups satisfying the usual quantum detailed balance condition with non-symmetric multiplications $x \mapsto \rho^s x \rho^{1-s}$ ($s \in [0, 1]$, $s \neq 1/2$) whose generators must commute with the modular group associated with ρ . This supports our conclusion that the most appropriate non-commutative version of the classical detailed balance condition is the above standard quantum detailed balance condition with an anti-unitary time reversal.

1. Introduction. A Markov semigroup on a commutative algebra satisfies the classical detailed balance condition, also called reversibility, when it is symmetric in the L^2 space of an invariant measure.

Several non-commutative versions have been proposed. The best known one, for a norm continuous Quantum Markov Semigroup (QMS) \mathcal{T} on the algebra $\mathcal{B}(\mathfrak{h})$ of linear

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bounded operators on a complex separable Hilbert space \mathfrak{h} , with a faithful normal invariant state ρ , was introduced by Alicki [3] (see also Frigerio, Gorini, Kossakowski, and Verri [12]): a QMS \mathcal{T} generated by \mathcal{L} satisfies the quantum detailed balance condition (QDB) if

$$\mathrm{tr}(\rho x \mathcal{L}(y)) - \mathrm{tr}(\rho \mathcal{L}(x)y) = 2i \mathrm{tr}(\rho [K, x]y), \quad (1)$$

for some self-adjoint operator $K \in \mathcal{B}(\mathfrak{h})$ and all $x, y \in \mathcal{B}(\mathfrak{h})$.

This implies that the conditionally completely positive map $\tilde{\mathcal{L}}$ defined by $\tilde{\mathcal{L}}(x) = \mathcal{L}(x) - 2i[K, x]$ coincides with the adjoint of \mathcal{L} with respect to the pre-scalar product $\langle x, y \rangle_0 := \mathrm{tr}(\rho x^*y)$ on $\mathcal{B}(\mathfrak{h})$. As a consequence, both \mathcal{L} and $\tilde{\mathcal{L}}$ commute with the modular group $(\sigma_t)_{t \in \mathbb{R}}$ associated with ρ given by $\sigma_t(a) = \rho^{it} a \rho^{-it}$ ([12], Proposition 2.1), which is quite a restrictive condition (see e.g. [9], Theorems 3, 4 and 8, and also [14]). Moreover, the norm continuous semigroup $\tilde{\mathcal{T}} = (\tilde{\mathcal{T}}_t)_{t \geq 0}$ generated by $\tilde{\mathcal{L}}$, called the *dual semigroup* of \mathcal{T} , satisfies the corresponding equation $\mathrm{tr}(\rho x \tilde{\mathcal{T}}_t(y)) = \mathrm{tr}(\rho \tilde{\mathcal{T}}_t(x)y)$ for all $t \geq 0$. The picture does not change if we consider pre-scalar products $\langle x, y \rangle_s := \mathrm{tr}(\rho^{1-s} x^* \rho^s y)$ with $s \in [0, 1]$, $s \neq 1/2$, as shown in [7], Section 8.

Complete positivity of the maps $\tilde{\mathcal{T}}_t$ of the dual semigroup follows immediately without any additional condition if we consider the dual semigroup with respect to the above pre-scalar product on $\mathcal{B}(\mathfrak{h})$ with $s = 1/2$ as in [4, 11]. The QDB condition obtained with the dual quantum Markov semigroup $\tilde{\mathcal{T}}$ defined with respect to this scalar product is called *standard* detailed balance condition (SQDB) (see [5]).

Agarwal's original definition of quantum detailed balance [2], however, as well as its generalisations studied by Majewski [13], also keeps into account the parity of the observables x, y . This goal is achieved in physical applications by introducing an anti-unitary time reversal θ and considering time-reversed observables $\theta x^* \theta$. It is reasonable to assume that θ commutes with ρ because ρ is typically a function of the energy of the system and thus it is invariant under time reversal (see [16]), i.e. $\theta \rho^* \theta = \theta \rho \theta = \rho$. The quantum detailed balance condition with time reversal θ (QDB- θ) is then defined by

$$\mathrm{tr}(\rho x \mathcal{T}_t(y)) = \mathrm{tr}(\rho \theta y^* \theta \mathcal{T}_t(\theta x^* \theta)) \quad (2)$$

for all $x, y \in \mathcal{B}(\mathfrak{h})$ and $t \geq 0$. This, however, implies again commutation with the modular group that can be avoided by considering, as we proposed in [9, 10], the standard quantum detailed balance condition with time reversal θ (SQDB- θ) defined by

$$\mathrm{tr}(\rho^{1/2} x \rho^{1/2} \mathcal{T}_t(y)) = \mathrm{tr}(\rho^{1/2} \theta y^* \theta \rho^{1/2} \mathcal{T}_t(\theta x^* \theta)) \quad (3)$$

for all $x, y \in \mathcal{B}(\mathfrak{h})$ and $t \geq 0$.

There are then essentially four types of quantum detailed balance conditions depending on the choice of the pre-scalar product with $s \neq 1/2$ or $s = 1/2$ and on the inclusion or exclusion of an anti-unitary time reversal θ in the definition. In a series of papers (see e.g. [7, 9, 10]) we studied the structure of generators satisfying these four types of quantum detailed balance conditions. The results give several reasons to think that the best non-commutative version is obtained by considering the standard quantum detailed balance condition with an anti-unitary time reversal.

This paper brings another argument to support this conclusion. Indeed, we give a full classification and representation in a standard form of generators of quantum Markov

semigroups on $\mathcal{B}(\mathfrak{h})$, with \mathfrak{h} n -dimensional, satisfying the standard quantum detailed balance condition with an anti-unitary time reversal, which generalises similar results for generators, commuting with the modular group, of QMSs satisfying the QDB- θ condition (see e.g. [9], Theorem 8).

The paper is organised as follows. In Section 2 we review our results on the structure of generators of QMSs satisfying one of the above quantum versions of the detailed balance condition. Standard forms of their generators, when the Hilbert space is finite-dimensional and the invariant state ρ is generic (in a sense that will be clarified later), are given in Section 3, Theorem 10 and Section 4, Theorem 14.

As a consequence, we show that the convex cone of generators of QMSs satisfying the SQDB- θ (resp. QDB- θ) condition, when looked at as a real manifold, has dimension growing as $n^4/2$ (resp. $3n^2/2$).

We would like to stress here that understanding the quantum versions of the classical detailed balance conditions is the first step in the study of their generalisations for systems out of equilibrium like the Accardi-Imafuku dynamical detailed balance condition defined in [1].

2. Notation and preliminaries. Let \mathfrak{h} be a complex separable Hilbert space and consider a norm continuous quantum Markov semigroup (QMS) $\mathcal{T} = (\mathcal{T}_t)_{t \geq 0}$ on $\mathcal{B}(\mathfrak{h})$ with a faithful normal invariant state ρ with spectral representation $\sum_{j \geq 1} \rho_j |e_j\rangle\langle e_j|$ with respect to an orthonormal basis $(e_j)_{j \geq 1}$ of \mathfrak{h} .

We recall (Parthasarathy [15], Theorem 30.10) that the generator \mathcal{L} of \mathcal{T} admits a Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) representation with respect to ρ of the form

$$\mathcal{L}(x) = i[H, x] - \frac{1}{2} \sum_k (L_k^* L_k x - 2L_k^* x L_k + x L_k^* L_k) \tag{4}$$

by means of operators $H = H^*$ and $(L_k)_{k \geq 1}$ in $\mathcal{B}(\mathfrak{h})$ such that

- (i) $\text{tr}(\rho L_k) = 0$ for each $k \geq 1$,
- (ii) $\sum_{k \geq 1} L_k^* L_k$ is strongly convergent,
- (iii) if $\sum_{k \geq 0} |c_k|^2 < \infty$ and $c_0 + \sum_{k \geq 1} c_k L_k = 0$ for complex scalars $(c_k)_{k \geq 0}$ then $c_k = 0$ for every $k \geq 0$.

A GKSL representation with the above properties will be called *special*.

Another family $\{H', L'_k : k \geq 1\}$ of bounded operators in $\mathcal{B}(\mathfrak{h})$ with H' self-adjoint satisfies equation (4) and conditions (i)–(iii) if and only if the lengths of the sequences $(L_k)_{k \geq 1}, (L'_k)_{k \geq 1}$ are equal and

$$H' = H + \alpha, \quad L'_k = \sum_j u_{kj} L_j$$

for some real scalar α and a unitary matrix $(u_{kj})_{k,j}$ on a Hilbert space \mathfrak{k} , called the multiplicity space of the completely positive part of \mathcal{L} .

Introducing the bounded operator

$$G = -iH - \frac{1}{2} \sum_k L_k^* L_k$$

on \mathfrak{h} , we can also write \mathcal{L} as

$$\mathcal{L}(a) = G^*a + \sum_k L_k^* a L_k + aG. \tag{5}$$

We call such a representation *special* if the operators $H = (G^* - G)/2i$ and L_k give a special GKSL representation of \mathcal{L} . In this case, G is unique up to a purely imaginary multiple of the identity operator. Indeed, if $G', \{L'_k\}$ give another special representation of \mathcal{L} , then $G' = -iH' - \frac{1}{2} \sum_k L_k'^* L'_k$ fulfils

$$G' = -iH - i\alpha - \frac{1}{2} \sum_{k,j,m} \bar{u}_{kj} u_{km} L_j^* L_m = G - i\alpha$$

for some $\alpha \in \mathbb{R}$, since the matrix $(u_{kj})_{kj}$ is unitary.

Following the terminology of [7], a special GKSL representation of \mathcal{L} is called *privileged* if H commutes with ρ and the operators L_ℓ satisfy $\rho L_\ell = \lambda_\ell L_\ell \rho$ for some $\lambda_\ell > 0$. In an equivalent way we can say that a special GKSL representation is privileged if its operator H commutes with ρ and the L_ℓ are eigenvectors of the modular group satisfying $\sigma_t(L_\ell) = \lambda_\ell^t L_\ell$.

A special GKSL representation of the generator \mathcal{L} of a norm-continuous QMS with respect to a given state ρ always exists. This is not the case for privileged representations. Indeed, we have the following (see [7], Theorem 4.3).

THEOREM 1. *The following conditions are equivalent:*

- (1) *the generator \mathcal{L} commutes with the modular group,*
- (2) *there exists a privileged GKSL representation of \mathcal{L} ,*
- (3) *the dual semigroup $\tilde{\mathcal{T}}$ with respect to pre-scalar products $\langle \cdot, \cdot \rangle_s$ with $s \neq 1/2$ is a QMS.*

We now list the characterisations of generators satisfying the QDB, SQDB, QDB- θ and SQDB- θ conditions referring to [7, 9, 10] for the proofs.

THEOREM 2. *The QMS \mathcal{T} satisfies the QDB condition (1) if and only if there exists a privileged GKSL representation of \mathcal{L} , by means of operators H, L_k , such that*

- (1) *$H = K + c$ for some $c \in \mathbb{R}$,*
- (2) *$\lambda_\ell^{-1/2} L_\ell^* = \sum_j u_{\ell j} L_j$ for some $\lambda_\ell > 0$ and some $U = (u_{\ell j})_{\ell j}$ unitary operator on \mathfrak{k} which is also symmetric (i.e. $u_{jk} = u_{kj}$ for all j, k).*

It can be shown that *all* privileged GKSL representations enjoy the above properties. The proof can be found in [7], Theorem 5.1 and also [9], Theorem 14. The unitary U is symmetric by the following algebraic argument. Noting that $\sigma_{i/2}(L_\ell) = \lambda_\ell^{-1/2} L_\ell$, we can write $\lambda_\ell^{-1/2} L_\ell^* = \rho^{1/2} L_\ell^* \rho^{-1/2}$ and the identity $\lambda_\ell^{-1/2} L_\ell^* = \sum_j u_{\ell j} L_j$ becomes

$$\rho^{1/2} L_\ell^* = \sum_j u_{\ell j} L_j \rho^{1/2}. \tag{6}$$

Taking the adjoint we find $L_\ell \rho^{1/2} = \sum_j \bar{u}_{\ell j} \rho^{1/2} L_j^*$, i.e., by a change of indices, $L_j \rho^{1/2} =$

$\sum_k \bar{u}_{jk} \rho^{1/2} L_k^*$. Replacing this in the right-hand side of (6) we have

$$\rho^{1/2} L_\ell^* = \sum_{j,k} u_{\ell j} \bar{u}_{jk} \rho^{1/2} L_k^*$$

and the linear independence of the operators $(\rho^{1/2} L_\ell^*)_{\ell \geq 1}$ implies that $U\bar{U}$ is the identity operator, i.e. $\bar{U} = U^*$ because U is unitary. This implies that U coincides with its transpose.

THEOREM 3. *The QMS \mathcal{T} satisfies the SQDB condition*

$$\text{tr}(\rho^{1/2} x \rho^{1/2} \mathcal{L}(y)) = \text{tr}(\rho^{1/2} \tilde{\mathcal{L}}(x) \rho^{1/2} y) \quad \text{and} \quad \mathcal{L}(x) - \tilde{\mathcal{L}}(x) = 2i[K, x] \quad (7)$$

for some self-adjoint operator K if and only if there exists a special GKSL representation of the generator \mathcal{L} by means of operators G , L_ℓ and there exists a unitary $(u_{jk})_{j,k}$ on \mathfrak{k} which is also symmetric (i.e. $u_{jk} = u_{kj}$ for all j, k) such that, for all $\ell \geq 1$, we have $\rho^{1/2} L_\ell^* = \sum_k u_{\ell k} L_k \rho^{1/2}$.

We refer to [10], Theorem 5 and Remark 4 for the proof (see also [9], Theorem 19).

We now consider quantum detailed balance conditions with an anti-unitary time reversal θ . Recall that it satisfies $\theta\theta^* = \theta^*\theta = \mathbf{1}$, $\theta^* = \theta^{-1}$ and $\langle v, \theta u \rangle = \langle u, \theta^* v \rangle$ for all $u, v \in \mathfrak{h}$.

THEOREM 4. *The QMS \mathcal{T} satisfies the QDB- θ -condition (2) if and only if there exists a privileged GKSL representation of the generator \mathcal{L} , by means of operators H and L_ℓ such that*

- (1) $H = \theta^{-1} H \theta + c$ for some $c \in \mathbb{R}$,
- (2) $\lambda_\ell^{-1/2} \theta^{-1} L_\ell^* \theta = \sum_j u_{\ell j} L_j$ for some $\lambda_\ell > 0$ and some $U = (u_{\ell j})_{\ell, j}$ unitary operator on \mathfrak{k} which is also self-adjoint.

The proof is given in [8], Theorem 9. Self-adjointness of U follows from an argument similar to the above proof of symmetry (Theorem 2) recalling that θ , as an anti-unitary operator, is anti-linear.

THEOREM 5. *The QMS \mathcal{T} satisfies the SQDB- θ condition (3) if and only if there exists a special GKSL representation of \mathcal{L} , with operators G , L_ℓ , such that*

- (1) $\rho^{1/2} \theta^{-1} G^* \theta = G \rho^{1/2}$,
- (2) $\rho^{1/2} \theta^{-1} L_\ell^* \theta = \sum_j u_{\ell j} L_j \rho^{1/2}$ for a self-adjoint unitary $(u_{\ell j})_{\ell, j}$ on \mathfrak{k} .

We refer to [10], Theorem 8 for the proof.

Standard quantum detailed balance conditions do not imply commutation with the modular group, which is a purely algebraic constraint on the dynamics. However, when this algebraic condition holds, the SQDB and QDB, as well as the SQDB- θ and QDB- θ conditions, coincide (see e.g. [7], Theorem 7.1 and [4], Theorem 6.6).

There are two reasons convincing us that the anti-unitary time reversal θ must play some role. The first one is the physical motivation exposed in the introduction, namely keeping into account the parity of the observables ([2], [16]). The second one is the possibility to define, as discussed in [6], a natural notion of entropy production for QMS which is zero if and only if the SQDB- θ holds. Moreover, the following remark allows us to find a simple standard form for generators of QMSs satisfying the SQDB- θ condition.

REMARK 6. Let $V = (v_{jk})_{jk}$ be a unitary on the multiplicity space k of the completely positive part of \mathcal{L} and define $L_j := \sum_j v_{jk} M_k$. Then the identity $\rho^{1/2} \theta^{-1} L_\ell^* \theta = \sum_j u_{\ell j} L_j \rho^{1/2}$ becomes $\rho^{1/2} \theta^{-1} M_j^* \theta = \sum_k w_{jk} M_k \rho^{1/2}$ where $W = (w_{jk})_{jk}$ is the self-adjoint unitary $W = V^* U V$. In particular, since $U = (u_{\ell j})_{\ell j}$ is unitary and self-adjoint, for a suitable choice of V , W is a diagonal matrix with eigenvalues $+1$ and -1 and then either $\rho^{1/2} \theta^{-1} M_j^* \theta = M_j \rho^{1/2}$ or $\rho^{1/2} \theta^{-1} M_j^* \theta = -M_j \rho^{1/2}$ for all j .

Exploiting this simple remark, we shall describe explicitly the structure of generators \mathcal{L} of QMS satisfying the SQDB- θ condition, generalising the analogous result for QMS satisfying the QDB and QDB- θ condition (see [9], Theorem 8 and Section 4 here).

Throughout the paper, for the sake of simplicity and concreteness, we shall consider a finite-dimensional Hilbert space \mathbb{C}^n and QMS acting on the algebra $M_n(\mathbb{C})$ of $n \times n$ complex matrices. E_k^j denotes the rank one matrix $|e_k\rangle\langle e_j|$. We choose a canonical orthonormal basis $(e_k)_{k=1}^n$ where $\rho = \sum_{j=1}^n \rho_j E_j^j$ is diagonal. Clearly $0 < \rho_k < 1$ for all $k = 1, \dots, n$ since ρ is faithful.

The anti-unitary time reversal θ will be just conjugation with respect to this basis, i.e. $\theta \sum_j z_j e_j = \sum_j \bar{z}_j e_j$. As a consequence $\theta^{-1} X^* \theta$ is the transpose X^T and $\theta^{-1} X \theta = \bar{X}$, the matrix obtained by X complex conjugating its entries.

3. SQDB- θ condition. In this section, we find a standard form of a special GKSL representation of the generator \mathcal{L} of a QMS \mathcal{T} satisfying the SQDB- θ . Since θ and ρ commute, Theorem 5 says that this property holds if and only if the operators G, L_k satisfy

- (i) $G = \rho^{1/2} G^T \rho^{-1/2}$,
- (ii) $L_k = \sum_{j \in \mathcal{J}} u_{kj} \rho^{1/2} L_j^T \rho^{-1/2}$ for some unitary self-adjoint $U = (u_{kj})_{k,j \in \mathcal{J}}$.

In order to determine the structure of the operators G and L_k fulfilling (i) and (ii), we choose the basis $\{E_l^l, X_i^j, Y_i^j : l, i, j = 1, \dots, n, i < j\}$ of eigenvectors of the linear map on $M_n(\mathbb{C})$

$$\mathcal{R}(X) = \rho^{1/2} X^T \rho^{-1/2}, \tag{8}$$

given by

$$X_h^j := \rho^{1/2} (E_h^j + E_j^h), \quad Y_h^j = i \rho^{1/2} (E_j^h - E_h^j)$$

for $h < j$. In fact

$$\mathcal{R}(E_l^l) = E_l^l, \quad \mathcal{R}(X_h^j) = X_h^j, \quad \mathcal{R}(Y_h^j) = -Y_h^j \tag{9}$$

for all $l, h, j = 1, \dots, n, h < j$.

With respect to this basis we can write

$$H = \sum_{l=1}^n h_l E_l^l + \sum_{1 \leq k < j \leq n} (m_{kj} X_k^j + w_{kj} Y_k^j) \tag{10}$$

for some $h_l, m_{kj}, w_{kj} \in \mathbb{C}$.

LEMMA 7. *The operator H given by (10) is self-adjoint if and only if $h_l \in \mathbb{R}$ and*

$$\Im m_{kj} = \frac{\rho_k^{1/2} - \rho_j^{1/2}}{\rho_k^{1/2} + \rho_j^{1/2}} \Re w_{kj}, \quad \Im w_{kj} = \frac{\rho_j^{1/2} - \rho_k^{1/2}}{\rho_k^{1/2} + \rho_j^{1/2}} \Re m_{kj} \quad (11)$$

for all $l, j, k = 1, \dots, n$ with $k < j$.

Proof. Set $Z_k^j := m_{kj} X_k^j + w_{kj} Y_k^j = \rho_k^{1/2} (m_{kj} - iw_{kj}) E_k^j + \rho_j^{1/2} (m_{kj} + iw_{kj}) E_j^k$ for $k < j$. Since $H = \sum_{l=1}^n h_l E_l^l + \sum_{k < j} Z_k^j$ and

$$(Z_k^j)^* = \rho_k^{1/2} (\overline{m}_{kj} + i\overline{w}_{kj}) E_j^k + \rho_j^{1/2} (\overline{m}_{kj} - i\overline{w}_{kj}) E_k^j,$$

H is self-adjoint if and only if every h_l is real and

$$\begin{cases} \rho_k^{1/2} (m_{kj} - iw_{kj}) = \rho_j^{1/2} (\overline{m}_{kj} - i\overline{w}_{kj}) \\ \rho_j^{1/2} (m_{kj} + iw_{kj}) = \rho_k^{1/2} (\overline{m}_{kj} + i\overline{w}_{kj}) \end{cases}$$

for all $k < j$. This system is easily solvable giving the equations (11) as unique solution. ■

DEFINITION 8. Let $\{L_k\}_{k \in \mathcal{J}}$ be in a special representation of \mathcal{L} . We say that the operators L_k are expressed in a *standard form* with respect to ρ if $\mathcal{R}(L_k) = \epsilon_k L_k$ with $\epsilon_k \in \{-1, 1\}$ for all $k \in \mathcal{J}$. In this case, the special representation of \mathcal{L} is called *standard*.

Clearly every family of operators $\{L_k\}_{k \in \mathcal{J}}$ expressed in a standard form satisfies condition (ii) with $U = \text{diag}(\epsilon_1, \dots, \epsilon_{|\mathcal{J}|})$, where $|\mathcal{J}|$ denotes the cardinality of \mathcal{J} . Conversely, if L_k 's fulfil condition (ii) for some unitary self-adjoint matrix U , then we can suppose that they are expressed in a standard form. Indeed, by Remark 6, we have $U = V^* D V$ for some unitary matrix $V = (v_{ij})_{i,j \in \mathcal{J}}$ and some diagonal matrix D of the form

$$\text{diag}(\epsilon_1, \dots, \epsilon_{|\mathcal{J}|}), \quad \epsilon_i \in \{-1, 1\}. \quad (12)$$

Thus, replacing the L_k 's by operators $L'_k := \sum_{j \in \mathcal{J}} v_{kj} L_j$ if necessary, we can take U as in (12).

REMARK 9. Let $\{L_k\}_{k \in \mathcal{J}}$ and $\{L'_k\}_{k \in \mathcal{J}'}$ be operators giving the same special standard representation of \mathcal{L} such that

$$L'_k = \sum_{j \in \mathcal{J}} w_{kj} L_j \quad (13)$$

for some unitary matrix $W = (w_{kj})_{k \in \mathcal{J}', j \in \mathcal{J}}$. If $\mathcal{R}(L'_k) = \epsilon'_k L'_k$ and $\mathcal{R}(L_j) = \epsilon_j L_j$ with $\epsilon_j, \epsilon'_k \in \{-1, 1\}$ for all k, j , then

$$w_{kj} (\epsilon'_k - \epsilon_j) = 0 \quad \text{for all } j \in \mathcal{J}, k \in \mathcal{J}'. \quad (14)$$

Indeed, applying the operator \mathcal{R} to both sides of equation (13), we get

$$\epsilon'_k \sum_{j \in \mathcal{J}} w_{kj} L_j = \epsilon'_k L'_k = \sum_{j \in \mathcal{J}} w_{kj} \epsilon_j L_j,$$

which clearly gives condition (14) by the linear independence of L_j 's.

We can now characterise the Hamiltonian H and the standard form of operators L_k in a special GKSL representation of the generator under the SQDB- θ .

THEOREM 10. *The QMS \mathcal{T} satisfies the SQDB- θ with respect to ρ if and only if there exists a special standard GKSL representation of \mathcal{L} given by operators*

$$H = \sum_{l=1}^n h_l E_l^l + \sum_{h < j} (m_{hj} X_h^j + w_{hj} Y_h^j) \tag{15}$$

$$L_k = \sum_{l=1}^n v_l^{(k)} E_l^l + \sum_{h < j} \xi_{hj}^{(k)} X_h^j \quad \text{for } k \in \mathcal{J}^+ \subseteq \{1, \dots, \frac{n(n+1)}{2} - 1\} \tag{16}$$

$$L_k = \sum_{h < j} \zeta_{hj}^{(k)} Y_h^j \quad \text{for } k \in \mathcal{J}^- \subseteq \{\frac{n(n+1)}{2}, \dots, n^2 - 1\}, \tag{17}$$

with

$$\begin{aligned} & \{ (v_1^{(k)}, \dots, v_{n-1}^{(k)}, \xi_{12}^{(k)}, \dots, \xi_{1n}^{(k)}, \dots, \xi_{n-2, n-1}^{(k)}, \xi_{n-2, n}^{(k)}, \xi_{n-1, n}^{(k)}) : k \in \mathcal{J}^+ \} \\ & \{ (\zeta_{12}^{(k)}, \dots, \zeta_{1n}^{(k)}, \dots, \zeta_{n-2, n-1}^{(k)}, \zeta_{n-2, n}^{(k)}, \zeta_{n-1, n}^{(k)}) : k \in \mathcal{J}^- \} \end{aligned}$$

sets of linearly independent vectors in $\mathbb{C}^{n(n+1)/2-1}$ and $\mathbb{C}^{n(n-1)/2}$ respectively,

$$v_n^{(k)} = - \sum_{l=1}^{n-1} v_l^{(k)} \rho_l \rho_n^{-1} \quad \text{for } k \in \mathcal{J}^+, \tag{18}$$

$h_l \in \mathbb{R}$, m_{hj} given by (11) for all $h < j$, and

$$w_{hj} = \frac{1}{2} \operatorname{tr}(\rho_h^{-1/2} (G_0)_{hj} - \rho_j^{-1/2} (G_0)_{jh}) = \frac{i}{2} \operatorname{tr}(Y_h^j \rho^{-1/2} G_0 \rho^{-1/2}), \tag{19}$$

where $G_0 := -2^{-1} \sum_{k \in \mathcal{J}^+ \cup \mathcal{J}^-} L_k^* L_k$.

Proof. Assume that SQDB- θ holds and let $\{L_k, H : k \in \mathcal{J}\}$ be in a special GKSL representation of \mathcal{L} , with L_k 's expressed in a standard form. The Hamiltonian H is clearly given by (15) with $h_l \in \mathbb{R}$ and m_{hj}, w_{hj} satisfying equations (11) thanks to Lemma 7.

Since condition (ii) gives either $L_k = \rho^{1/2} L_k^T \rho^{-1/2} = \mathcal{R}(L_k)$ or $L_k = -\rho^{1/2} L_k^T \rho^{-1/2} = -\mathcal{R}(L_k)$, expanding L_k on the basis $\{E_l^l, X_i^j, Y_i^j : l, i, j = 1, \dots, n, i < j\}$ and recalling equation (9), we get

$$L_k = \rho^{1/2} L_k^T \rho^{-1/2} \Leftrightarrow L_k = \sum_{l=1}^n v_l^{(k)} E_l^l + \sum_{h < j} \xi_{hj}^{(k)} X_h^j, \tag{20}$$

$$L_k = -\rho^{1/2} L_k^T \rho^{-1/2} \Leftrightarrow L_k = \sum_{h < j} \zeta_{hj}^{(k)} Y_h^j \tag{21}$$

for some $v_l^{(k)}, \xi_{hj}^{(k)}, \zeta_{hj}^{(k)} \in \mathbb{C}$, with $l, h, j = 1, \dots, n$ and $h < j$. Assume that (20) and (21) hold for $k \in \mathcal{J}^+$ and $k \in \mathcal{J}^-$, respectively, with \mathcal{J}^+ and \mathcal{J}^- disjoint subsets of \mathcal{J} such that $\mathcal{J}^+ \cup \mathcal{J}^- = \mathcal{J}$. Therefore, condition $\operatorname{tr}(\rho L_k) = 0$ is clearly always satisfied if $k \in \mathcal{J}^-$, while it means

$$\sum_{l=1}^n \rho_l v_l^{(k)} = 0, \quad \text{i.e.} \quad v_n^{(k)} = - \sum_{l=1}^{n-1} v_l^{(k)} \rho_l \rho_n^{-1}$$

for each $k \in \mathcal{J}^+$. Finally, by the linear independence of $\{\mathbf{1}, L_k : k \in \mathcal{J}\}$, the inequalities

$$|\mathcal{J}| \leq n^2 - 1, \quad |\mathcal{J}^+| \leq n - 1 + n(n - 1)/2 = n(n + 1)/2 - 1, \quad |\mathcal{J}^-| \leq n(n - 1)/2,$$

hold and the vectors in $\mathbb{C}^{n(n+1)/2-1}$ and $\mathbb{C}^{n(n-1)/2}$ respectively

$$\begin{aligned} & \{(v_1^{(k)}, \dots, v_{n-1}^{(k)}, \xi_{12}^{(k)}, \dots, \xi_{1n}^{(k)}, \dots, \xi_{n-2n-1}^{(k)}, \xi_{n-2n}^{(k)}, \xi_{n-1n}^{(k)} : k \in \mathcal{J}^+\} \\ & \{(\zeta_{12}^{(k)}, \dots, \zeta_{1n}^{(k)}, \dots, \zeta_{n-2n-1}^{(k)}, \zeta_{n-2n}^{(k)}, \zeta_{n-1n}^{(k)} : k \in \mathcal{J}^-\} \end{aligned}$$

must be linearly independent.

Now, since

$$G = -iH - 2^{-1} \sum_k L_k^* L_k = -i \left[\sum_{l=1}^n h_l E_l^l + \sum_{h < j} (m_{hj} X_h^j + w_{hj} Y_h^j) \right] + G_0,$$

in view of (9), condition (i), i.e. $G = \mathcal{R}(G)$, reads $-2i(\sum_{h < j} w_{hj} Y_h^j) + G_0 = \mathcal{R}(G_0)$, that is

$$2i \sum_{h < j} w_{hj} Y_h^j = G_0 - \rho^{1/2} G_0^T \rho^{-1/2}. \tag{22}$$

Note that

$$\text{tr}((E_l^m - E_m^l)(E_j^h - E_h^j)) = 2\delta_{mj}\delta_{lh} - 2\delta_{mh}\delta_{lj}$$

and so, left multiplying both sides of equation (22) by $(E_l^m - E_m^l)\rho^{-1/2}$ with $m < l$ and tracing, we get

$$-4i w_{ml} = \text{tr}((E_l^m - E_m^l)(\rho^{-1/2} G_0 - G_0^T \rho^{-1/2})).$$

Now, since $\text{tr}(E_l^m \rho^{-1/2} G_0) = \text{tr}(G_0^T \rho^{-1/2} E_m^l) = \text{tr}(E_m^l G_0^T \rho^{-1/2})$, we can write the previous relation as

$$\begin{aligned} w_{ml} &= \frac{i}{2} \text{tr}((E_l^m - E_m^l)\rho^{-1/2} G_0) = \frac{1}{2} \text{tr}(Y_m^l \rho^{-1/2} G_0 \rho^{-1/2}) \\ &= \frac{i}{2} (\rho_m^{-1/2} (G_0)_{ml} - \rho_l^{-1/2} (G_0)_{lm}). \blacksquare \end{aligned}$$

COROLLARY 11. *If \mathcal{T} satisfies the SQDB- θ , then every special standard representation of \mathcal{L} is of the form given in Theorem 10.*

Proof. Let $\{H, L_k, L_l : k \in \mathcal{J}^+, l \in \mathcal{J}^-\}$ be a special standard representation of \mathcal{L} defined as in (15)–(17) with $h_l \in \mathbb{R}$, m_{hj}, w_{hj} given for all $h < j$ by (11) and (19) respectively,

$$\begin{aligned} & \{(v_1^{(k)}, \dots, v_{n-1}^{(k)}, \xi_{12}^{(k)}, \dots, \xi_{1n}^{(k)}, \dots, \xi_{n-2n-1}^{(k)}, \xi_{n-2n}^{(k)}, \xi_{n-1n}^{(k)} : k \in \mathcal{J}^+\} \\ & \{(\zeta_{12}^{(k)}, \dots, \zeta_{1n}^{(k)}, \dots, \zeta_{n-2n-1}^{(k)}, \zeta_{n-2n}^{(k)}, \zeta_{n-1n}^{(k)} : k \in \mathcal{J}^-\} \end{aligned}$$

sets of linearly independent vectors in $\mathbb{C}^{n(n+1)/2-1}$ and $\mathbb{C}^{n(n-1)/2}$ respectively and let $v_n^{(k)}$ for $k \in \mathcal{J}^+$ be as in (18).

If we consider $\{\tilde{H}, \tilde{L}_k\}_{k \in \tilde{\mathcal{J}}}$ another special standard representation of \mathcal{L} , then we have $|\mathcal{J}^+| + |\mathcal{J}^-| = |\tilde{\mathcal{J}}|$ and

$$\tilde{L}_k = \sum_{j \in \mathcal{J}^+} u_{kj} L_j + \sum_{j \in \mathcal{J}^-} u_{kj} L_j, \quad \tilde{H} = H + \alpha$$

for some unitary matrix $U = (u_{kj})_{k \in \tilde{\mathcal{J}}, j \in \mathcal{J}}$ and real constant α . Since $\mathcal{R}(\tilde{L}_k) = \tilde{\epsilon}_k \tilde{L}_k$ with $\tilde{\epsilon}_k \in \{-1, 1\}$, and

$$\mathcal{R}(L_j) = \begin{cases} L_j & \text{if } j \in \mathcal{J}^+ \\ -L_j & \text{if } j \in \mathcal{J}^-, \end{cases}$$

defining $\tilde{\mathcal{J}}^+ := \{k \in \tilde{\mathcal{J}} : \tilde{\epsilon}_k = 1\}$ and $\tilde{\mathcal{J}}^- := \{k \in \tilde{\mathcal{J}} : \tilde{\epsilon}_k = -1\}$ we get $u_{kj} = 0$ for all $(k, j) \in (\tilde{\mathcal{J}}^+ \times \tilde{\mathcal{J}}^-) \cup (\tilde{\mathcal{J}}^- \times \tilde{\mathcal{J}}^+)$ thanks to Remark 9. This means that

$$\tilde{L}_k = \begin{cases} \sum_{j \in \mathcal{J}^+} u_{kj} L_j & \text{if } k \in \tilde{\mathcal{J}}^+ \\ \sum_{j \in \mathcal{J}^-} u_{kj} L_j & \text{if } k \in \tilde{\mathcal{J}}^-, \end{cases}$$

i.e.

$$\begin{aligned} \tilde{L}_k &= \sum_{l=1}^n \left(\sum_{j \in \mathcal{J}^+} u_{kj} v_l^{(j)} \right) E_l^l + \sum_{h < l} \left(\sum_{j \in \mathcal{J}^+} u_{kj} \xi_{hl}^{(j)} \right) X_h^j & \text{if } k \in \tilde{\mathcal{J}}^+, \\ \tilde{L}_k &= \sum_{h < l} \left(\sum_{j \in \mathcal{J}^+} u_{kj} \zeta_{hl}^{(j)} \right) Y_h^l & \text{if } k \in \tilde{\mathcal{J}}^-, \end{aligned}$$

with $\tilde{\mathcal{J}}^+ \subseteq \{1, \dots, \frac{n(n+1)}{2} - 1\}$ and $\tilde{\mathcal{J}}^- \subseteq \{\frac{n(n+1)}{2}, \dots, n^2 - 1\}$ by linear independence of \tilde{L}_k . Set now

$$\begin{aligned} \tilde{v}_l^{(k)} &:= \sum_{j \in \mathcal{J}^+} u_{kj} v_l^{(j)}, & \tilde{\xi}_{hl}^{(k)} &:= \sum_{j \in \mathcal{J}^+} u_{kj} \xi_{hl}^{(j)} & \text{for } k \in \tilde{\mathcal{J}}^+, \\ \tilde{\zeta}_{hl}^{(k)} &:= \sum_{j \in \mathcal{J}^+} u_{kj} \zeta_{hl}^{(j)} & & & \text{for } k \in \tilde{\mathcal{J}}^-. \end{aligned}$$

Therefore, by equation (18) we have

$$\tilde{v}_l^{(k)} = - \sum_{j \in \mathcal{J}^+} u_{kj} \sum_{l=1}^{n-1} v_l^{(j)} \rho_l \rho_n^{-1} = - \sum_{l=1}^{n-1} \tilde{v}_l^{(k)} \rho_l \rho_n^{-1}.$$

Moreover, by the invertibility of U ,

$$\begin{aligned} &\{(\tilde{v}_1^{(k)}, \dots, \tilde{v}_{n-1}^{(k)}, \tilde{\xi}_{12}^{(k)}, \dots, \tilde{\xi}_{1n}^{(k)}, \dots, \tilde{\xi}_{n-2n-1}^{(k)}, \tilde{\xi}_{n-2n}^{(k)}, \tilde{\xi}_{n-1n}^{(k)}) : k \in \tilde{\mathcal{J}}^+\} \\ &\{(\tilde{\zeta}_{12}^{(k)}, \dots, \tilde{\zeta}_{1n}^{(k)}, \dots, \tilde{\zeta}_{n-2n-1}^{(k)}, \tilde{\zeta}_{n-2n}^{(k)}, \tilde{\zeta}_{n-1n}^{(k)}) : k \in \tilde{\mathcal{J}}^-\} \end{aligned}$$

are sets of linearly independent vectors in $\mathbb{C}^{n(n+1)/2-1}$ and $\mathbb{C}^{n(n-1)/2}$, respectively.

Finally, since $\tilde{H} = H + \alpha$, it clearly has the form of equation (15) with real coefficients $\tilde{h}_l = h_l + \alpha$ for all $l = 1, \dots, n$ and $\tilde{m}_{hj} = m_{hj}$, $\tilde{w}_{hj} = w_{hj}$ for all $h < j$. ■

REMARK 12. Theorem 10 shows that, in the generic case when the ρ_i 's are all different, generators of QMSs with a faithful invariant state ρ satisfying the SQDB- θ condition are parametrised by

- (a) a set of $\frac{n(n+1)}{2} - 1 = \frac{(n-1)(n+2)}{2}$ linearly independent vectors in $\mathbb{C}^{(n-1)(n+1)/2}$,
- (b) a set of $n(n-1)/2$ linearly independent vectors in $\mathbb{C}^{n(n-1)/2}$,
- (c) $n-1$ real parameters h_l for the Hamiltonian H , since one of the h_l can be eliminated by adding a constant multiple of the identity operator without changing \mathcal{L} , constants m_{hj} are completely determined by w_{hj} through (11) (for the eigenvalues of ρ are all different) and any w_{hj} is given by G_0 through (19).

Two sets of linearly independent vectors $u = (u_1, \dots, u_d)$, $u' = (u'_1, \dots, u'_d)$ in \mathbb{C}^d (with $d = (n-1)(n+2)/2$ or $d = n(n-1)/2$) lead to the same generator if there exists a unitary operator U on \mathbb{C}^d such that $Uu = u'$. Therefore the L_k 's with $k \in \mathcal{J}^+$ (resp.

$k \in \mathcal{J}^-$) are parametrized by the linear manifold $GL(d, \mathbb{C})/U(d)$ of invertible matrices modulo unitary matrices with $d = (n - 1)(n + 2)/2$ (resp. $d = n(n - 1)/2$).

The real dimension $GL(d, \mathbb{C})$ is clearly $2d^2$. Moreover, norm preservation of d vectors and scalar products preservation of pairs of $d(d - 1)$ pairs of vectors give d and $d(d - 1)$ respectively identities involving the real and imaginary parts of these vectors. Thus we remain with $2d^2 - d(d - 1) - d = d^2$ free real parameters.

It follows then that the convex cone of generators of QMSs satisfying the SQDB- θ condition, when the invariant state ρ is non-degenerate, is a manifold with real dimension

$$\left(\frac{(n + 2)(n - 1)}{2}\right)^2 + \left(\frac{n(n - 1)}{2}\right)^2 + (n - 1) = \frac{n^2(n^2 - 1)}{2}.$$

4. QDB- θ condition in the generic case. In this section we analyse the QDB- θ condition for \mathcal{T} under the hypothesis that the faithful invariant density ρ is *generic*, i.e. its eigenvalues are all different and so are their ratios ρ_j/ρ_k for $j \neq k$. The time reversal θ is still the usual conjugation on \mathbb{C}^n .

We recall that, by Theorem 4, the QDB- θ condition holds if and only if there exists a special GKSL representation of \mathcal{L} , by operators H, L_k such that

- (i) $[H, \rho] = 0$,
- (ii) $\rho L_k = \lambda_k L_k \rho$ for some $\lambda_k \in \mathbb{R}$,
- (iii) $H = \overline{H}$,
- (iv) $\lambda_k^{-1/2} L_k = \sum_j u_{kj} L_j^T$ for some $\lambda_k > 0$ and some unitary self-adjoint matrix $U = (u_{kj})_{k,j \in \mathcal{J}}$.

A special representation satisfying (i) and (ii) is called *privileged*; moreover, see e.g. [7], the existence of a privileged representation of \mathcal{L} is equivalent to the commutation of every \mathcal{T}_t with the modular automorphism σ_{-i} defined by $\rho, \sigma_{-i}(x) = \rho x \rho^{-1}$ for $x \in M_n(\mathbb{C})$. In particular, condition (ii) means that each L_k is an eigenvector of the modular automorphism with eigenvalue $\lambda_k = \rho_j \rho_l^{-1}$ for each $j, l \in \{1, \dots, n\}$ for which $\langle e_j, L_k e_l \rangle \neq 0$; it follows that, in the generic case, there can exist a unique pair (j, l) with $j \neq l$ such that $\langle e_j, L_k e_l \rangle \neq 0$.

This remark allows us to characterise privileged GKSL representations of \mathcal{L} .

PROPOSITION 13. *Every privileged GKSL representation of \mathcal{L} with respect to the generic and faithful state ρ is of the form*

$$\begin{aligned} \mathcal{L}(a) = & i[H, a] - \frac{1}{2} \sum_{k \in \mathcal{J}^+} (L_k^* L_k a - 2L_k^* a L_k + a L_k^* L_k) \\ & - \frac{1}{2} \sum_{j < l} \gamma_{jl}^+ (E_j^j a - 2E_j^l a E_l^j + a E_j^j) - \frac{1}{2} \sum_{j < l} \gamma_{lj}^- (E_l^l a - 2E_l^j a E_j^l + a E_l^l), \end{aligned}$$

with $\gamma_{jl}^+, \gamma_{lj}^- \geq 0$ for all $j < l$,

$$\begin{aligned} H = \sum_{i=1}^n h_i E_i^i, & \quad h_i \in \mathbb{R} \quad \text{for all } i = 1, \dots, n, \\ L_k = \sum_{j=1}^n z_{kj} E_j^j & \quad \text{for } k \in \mathcal{J}_1 \subseteq \{1, \dots, n - 1\}, \end{aligned}$$

where

$$z_{kn} = - \sum_{i=1}^{n-1} z_{ki} \rho_i \rho_n^{-1}$$

and $\{z_k : k \in \mathcal{J}_1\}$ ($z_k := (z_{k1}, \dots, z_{kn-1})$) is a set of linearly independent vectors in \mathbb{C}^{n-1} .

Moreover, the state ρ is invariant if and only if the constants γ_{jl}^\pm satisfy

$$\begin{cases} \sum_{l=2}^n (\rho_l \gamma_{l1}^- - \rho_1 \gamma_{l1}^+) = 0 \\ \sum_{j=1}^{i-1} (\rho_i \gamma_{ij}^- - \rho_j \gamma_{ji}^+) = \sum_{l=i+1}^n (\rho_l \gamma_{li}^- - \rho_i \gamma_{il}^+) \end{cases} \quad (23)$$

for $i = 2, \dots, n-1$.

Proof. Let $\{H, L_k : k \in \mathcal{J}\}$ be operators of a special GKSL representation of \mathcal{L} ; the linear independence of $\{\mathbf{1}, L_k : k \in \mathcal{J}\}$ immediately gives $\text{card}(\mathcal{J}) \leq n^2 - 1$. Moreover, if we write H with respect to the basis $\{E_l^j\}_{l,j=1}^n$, its commutation with ρ is clearly equivalent to having $H = \sum_{j=1}^n h_j E_j^j$, with $h_j \in \mathbb{R}$ for $H = H^*$.

We fix now $k \in \mathcal{J}$ and suppose $\rho L_k = \lambda_k L_k \rho$ for some $\lambda_k \in \mathbb{R}$; this means

$$\rho_j \langle e_j, L_k e_l \rangle = \lambda_k \rho_l \langle e_j, L_k e_l \rangle \quad \text{for all } j, l = 1, \dots, n.$$

Since $L_k \neq 0$, there exists at least a pair (j, l) such that $\langle e_j, L_k e_l \rangle \neq 0$, and so $\lambda_k = \rho_j \rho_l^{-1}$. If $j = l$ (i.e. $\lambda_k = 1$), L_k must be diagonal with respect to the basis $\{E_i^j\}_{i,j}$, since the eigenvalues of ρ are all distinct; therefore, we get

$$L_k = \sum_{j=1}^n z_{kj} E_j^j, \quad z_{kj} \in \mathbb{C}, \quad (24)$$

with $0 = \text{tr}(\rho L_k) = \sum_{j=1}^n \rho_j z_{kj}$, namely

$$z_{kn} = - \sum_{j=1}^{n-1} \frac{\rho_j}{\rho_n} z_{kj}.$$

We denote by \mathcal{J}_1 the set of indexes $k \in \mathcal{J}$ such that the corresponding L_k has the form (24); the linear independence condition gives then $\text{card}(\mathcal{J}_1) \leq n-1$ with $\{z_k : k \in \mathcal{J}_1\}$ linearly independent vectors in \mathbb{C}^{n-1} , where we defined $z_k := (z_{k1}, \dots, z_{kn-1})$.

If, otherwise, $j \neq l$, as already mentioned before, the assumption on ρ_i 's forces to have a unique pair (j, l) satisfying $\lambda_k = \rho_j \rho_l^{-1}$, so that L_k is a multiple of E_{jl}^l . We set

$$L_k = \begin{cases} \alpha_{lj}^+ E_j^l =: L_{lj}^+ & \text{if } j > l \\ \alpha_{jl}^- E_j^l =: L_{jl}^- & \text{if } j < l, \end{cases} \quad (25)$$

with $\alpha_{jl}^\pm \in \mathbb{C}$. Note that the operators L_{jl}^+, L_{jl}^- are linearly independent for all $j < l$, and they satisfy $\text{tr}(\rho L_{jl}^+) = 0 = \text{tr}(\rho L_{jl}^-)$. Moreover, up to a suitable choice of the phase factor for each L_{jl}^\pm , we can suppose $\alpha_{jl}^\pm \in \mathbb{R}$. We define

$$\gamma_{jl}^+ := (\alpha_{jl}^+)^2, \quad \gamma_{jl}^- := (\alpha_{jl}^-)^2 \quad \text{for all } j < l. \quad (26)$$

Conversely, it can easily be shown that the representation given by the operators $H = \sum_{j=1}^n h_j E_j^j$ with $h_j \in \mathbb{R}$, L_k and L_{jl}^\pm defined as in (24), (25) respectively, with $z_{kn} = -\sum_{j=1}^{n-1} \rho_j \rho_n^{-1} z_{kj}$, $\alpha_{jl}^\pm \in \mathbb{R}$ and γ_{jl}^\pm satisfying (26), is a privileged representation of \mathcal{L} with respect to ρ .

Finally, we analyse the condition of the invariance of ρ . Since ρ is faithful and the equalities $[H, \rho] = 0$, $\rho L_k^* L_k = L_k^* L_k \rho$, $L_k \rho L_k^* = \lambda_k^{-1} L_k L_k^* \rho$ hold for every privileged representation, it follows that ρ is an invariant state if and only if

$$\sum_{k \in \mathcal{J}} (L_k^* L_k - \lambda_k^{-1} L_k L_k^*) = 0.$$

Since $\lambda_k = 1$ for all $k \in \mathcal{J}_1$, this gives

$$\begin{aligned} 0 &= \sum_{j < l} \left((L_{jl}^+)^* L_{jl}^+ - \frac{\rho_j}{\rho_l} L_{jl}^+ (L_{jl}^+)^* + (L_{jl}^-)^* L_{jl}^- - \frac{\rho_l}{\rho_j} L_{jl}^- (L_{jl}^-)^* \right) \\ &= \sum_{j < l} \left\{ \gamma_{jl}^+ \left(E_j^j - \frac{\rho_j}{\rho_l} E_l^l \right) + \gamma_{lj}^- \left(E_l^l - \frac{\rho_l}{\rho_j} E_j^j \right) \right\} \\ &= \sum_{j < l} \left\{ \left(\gamma_{jl}^+ - \frac{\rho_l}{\rho_j} \gamma_{lj}^- \right) E_j^j + \left(\gamma_{lj}^- - \frac{\rho_j}{\rho_l} \gamma_{jl}^+ \right) E_l^l \right\}, \end{aligned}$$

i.e.

$$\begin{cases} \sum_{l=2}^n \left(\gamma_{1l}^+ - \frac{\rho_l}{\rho_1} \gamma_{l1}^- \right) = 0 \\ \sum_{l=i+1}^n \left(\gamma_{il}^+ - \frac{\rho_l}{\rho_i} \gamma_{li}^- \right) + \sum_{j=1}^{i-1} \left(\gamma_{ij}^- - \frac{\rho_j}{\rho_i} \gamma_{ji}^+ \right) = 0 \quad \text{for } i = 2, \dots, n-1 \\ \sum_{j=1}^{n-1} \left(\gamma_{nj}^- - \frac{\rho_j}{\rho_n} \gamma_{jn}^+ \right) = 0. \end{cases} \quad (27)$$

Denoting by R_i for $i = 1, \dots, n$ the left-hand side of the i -th equation in the above system, we can easily see that $R_n = -\rho_1 \rho_n^{-1} (R_1 + \sum_{i=2}^{n-1} R_i)$, and so the last equation in (27) is superfluous. Therefore, the condition of invariance can be read as

$$\begin{cases} \sum_{l=2}^n \left(\gamma_{1l}^+ - \frac{\rho_l}{\rho_1} \gamma_{l1}^- \right) = 0 \\ \sum_{j=1}^{i-1} \left(\gamma_{ij}^- - \frac{\rho_j}{\rho_i} \gamma_{ji}^+ \right) = \sum_{l=i+1}^n \left(\frac{\rho_l}{\rho_i} \gamma_{li}^- - \gamma_{il}^+ \right) \end{cases}$$

for $i = 2, \dots, n-1$. ■

We can now characterise generators of QMSs satisfying the QDB- θ . Recall that, by Theorem 4, this condition is fulfilled if and only if there exists a privileged GKSL representation of \mathcal{L} , by means of operators H and L_k , such that: (1) $H = \bar{H}$, (2) $\lambda_k^{-1/2} L_k = \sum_j u_{kj} L_j^T$ for some $\lambda_k > 0$ and some unitary self-adjoint matrix $U = (u_{kj})_{k,j \in \mathcal{J}}$.

THEOREM 14. *The QMS \mathcal{T} satisfies the QDB- θ with respect to the generic, faithful and invariant state ρ if and only if there exists a privileged GKSL representation of \mathcal{L} given as in Proposition 13 with positive constants γ_{jl}^\pm fulfilling*

$$\rho_j \gamma_{jl}^+ = \rho_l \gamma_{lj}^- \quad \text{for all } 1 \leq j < l \leq n-1. \quad (28)$$

Proof. Consider a privileged representation

$$\begin{aligned} \mathcal{L}(a) &= i[H, a] - \frac{1}{2} \sum_{k \in \mathcal{J}_1} (L_k^* L_k a - 2L_k^* a L_k + a L_k^* L_k) \\ &\quad - \frac{1}{2} \sum_{j < l} \gamma_{jl}^+ (E_j^j a - 2E_j^l a E_l^j + a E_j^j) - \frac{1}{2} \sum_{j < l} \gamma_{lj}^- (E_l^l a - 2E_l^j a E_j^l + a E_l^l), \end{aligned}$$

given as in Proposition 13, with γ_{jl}^\pm satisfying (23). Clearly we have $H = \bar{H}$. We recall that, by setting

$$L_{jl}^+ = \sqrt{\gamma_{jl}^+} E_l^j, \quad L_{jl}^- = \sqrt{\gamma_{lj}^-} E_j^l, \quad \lambda_{jl} = \rho_l \rho_j^{-1} \quad \text{for all } j < l,$$

there holds

$$\begin{aligned} \rho L_k &= L_k \rho && \text{for all } k \in \mathcal{J}_1 \\ \rho L_{jl}^+ &= \lambda_{jl} L_{jl}^+ \rho, \quad \rho L_{jl}^- = \lambda_{jl}^{-1} L_{jl}^- \rho && \text{for all } j < l. \end{aligned}$$

Therefore, condition (2) of Theorem 4 can be rewritten as

$$\left(\begin{array}{c|c|c} \mathbf{1}_{|\mathcal{J}_1|} & 0 & 0 \\ \hline 0 & \Lambda & 0 \\ \hline 0 & 0 & \Lambda^{-1} \end{array} \right) \begin{pmatrix} L. \\ L.^+ \\ L.^- \end{pmatrix} = U \begin{pmatrix} L.^T \\ (L.^+)^T \\ (L.^-)^T \end{pmatrix} \quad (29)$$

for some unitary self-adjoint matrix $U = (u_{kj})_{k,j \in \mathcal{J}}$, where $\mathbf{1}_{|\mathcal{J}_1|}$ denotes the $|\mathcal{J}_1| \times |\mathcal{J}_1|$ -identity matrix and we define

$$\begin{aligned} \Lambda &= \text{diag}(\lambda_{12}^{-1/2}, \dots, \lambda_{1n}^{-1/2}, \lambda_{23}^{-1/2}, \dots, \lambda_{2n}^{-1/2}, \dots, \lambda_{n-1n}^{-1/2}) \\ L. &= (L_1, \dots, L_{|\mathcal{J}_1|})^T, \\ L.^{\pm} &= (L_{12}^{\pm}, \dots, L_{1n}^{\pm}, L_{23}^{\pm}, \dots, L_{2n}^{\pm}, \dots, L_{n-1n}^{\pm})^T. \end{aligned}$$

Since $(L_{jl}^+)^T = \sqrt{\gamma_{jl}^+} E_j^l$ and $(L_{jl}^-)^T = \sqrt{\gamma_{lj}^-} E_l^j$ for every $j < l$, condition (29) implies $\gamma_{jl}^+ = 0$ if and only if $\gamma_{lj}^- = 0$; otherwise we have

$$\lambda_{jl}^{-1/2} L_{jl}^+ = \sqrt{\frac{\rho_j \gamma_{jl}^+}{\rho_l \gamma_{lj}^-}} (L_{jl}^-)^T, \quad \lambda_{jl}^{1/2} L_{jl}^- = \sqrt{\frac{\rho_l \gamma_{lj}^-}{\rho_j \gamma_{jl}^+}} (L_{jl}^+)^T$$

for all $(j, l) \in N := \{(h, k) : 1 \leq h < k \leq n \text{ such that } \gamma_{hk}^+ \neq 0\}$.

As a consequence, since $L_k^T = L_k$ for all $k \in \mathcal{J}_1$, equation (29) holds if and only if

$$U = \left(\begin{array}{c|c|c} \mathbf{1}_{|\mathcal{J}_1|} & 0 & 0 \\ \hline 0 & 0 & U_0 \\ \hline 0 & U_0^{-1} & 0 \end{array} \right)$$

where

$$U_0 := \text{diag} \left(\sqrt{\frac{\rho_1 \gamma_{12}^+}{\rho_2 \gamma_{21}^-}}, \dots, \sqrt{\frac{\rho_1 \gamma_{1n}^+}{\rho_n \gamma_{n1}^-}}, \dots, \sqrt{\frac{\rho_{n-1} \gamma_{n-1n}^+}{\rho_n \gamma_{nn-1}^-}} \right), \quad (30)$$

with the convention that terms corresponding to $(j, l) \notin N$ are set equal to 1. Hence, U is unitary and self-adjoint if and only if U_0 is unitary, i.e.

$$\frac{\rho_j}{\rho_l} \frac{\gamma_{jl}^+}{\gamma_{lj}^-} = 1 \quad \text{for all } (j, l) \in N.$$

Finally, by condition (23) it is enough to have the previous equality only for $(j, l) \in N$ with $1 \leq j < l \leq n - 1$. This is clearly equivalent to (28). ■

Analysing the proof of the previous theorem, we can immediately see that under the QDB- θ condition the matrix U_0 given by (30) is the identity matrix, since the constants γ_{jl}^+ and γ_{lj}^- satisfy $\rho_j \gamma_{jl}^+ = \rho_l \gamma_{lj}^-$ for all $j < l$ (by (28) and (23)). Therefore, we can rewrite Theorem 14 in the following way.

COROLLARY 15. *The QMS \mathcal{T} satisfies the QDB- θ if and only if there exists a privileged representation of \mathcal{L} given by operators*

$$\begin{aligned} H &= \sum_{j=1}^n h_j E_j^j, & h_j &\in \mathbb{R} \quad \text{for all } j = 1, \dots, n, \\ L_k &= \sum_{j=1}^n z_{kj} E_j^j & \text{for } k &\in \mathcal{J}_1 \subseteq \{1, \dots, n - 1\}, \\ L_{jl}^+ &= \sqrt{\gamma_{jl}^+} E_l^j, & L_{jl}^- &= \sqrt{\rho_j \rho_l^{-1}} (L_{jl}^+)^T \quad \text{for all } (j, l) \in N, \end{aligned}$$

where γ_{jl}^+ are positive constants for all $j < l$, $N = \{(j, l) : j < l, \gamma_{jl}^+ \neq 0\}$,

$$z_{kn} = - \sum_{i=1}^{n-1} z_{ki} \rho_i \rho_n^{-1}$$

and $\{(z_{k1}, \dots, z_{kn-1}) : k \in \mathcal{J}_1\}$ is a set of linearly independent vectors in \mathbb{C}^{n-1} .

REMARK 16. Corollary 15 shows that the generator of a QMS satisfying the QDB- θ condition admits a privileged GKSL representation determined by

- (a) $n - 1$ real parameters h_j for the Hamiltonian H , since H is uniquely determined up to a real multiple of the identity operator;
- (b) $n - 1$ complex parameters z_{k1}, \dots, z_{kn-1} for each L_k with $k \in \mathcal{J}_1$, and $|\mathcal{J}_1| \leq n - 1$;
- (c) $n(n - 1)/2$ positive parameters $\{\gamma_{jl}^+ : j < l\}$.

Moreover, two sets $\{z_k = (z_{k1}, \dots, z_{kn-1}) : k \in \mathcal{J}_1\}$ and $\{z'_k = (z'_{k1}, \dots, z'_{kn-1}) : k \in \mathcal{J}_1\}$ of vectors in \mathbb{C}^{n-1} lead to a privileged GKSL representation of the same generator if and only if there exists a unitary U on \mathbb{C}^{n-1} such that $U z_k = z'_k$ for all $k = 1, \dots, n - 1$. By the arguments of Remark 12 we remain then with $(n - 1)^2$ free real parameters.

Thus the convex cone of generators of QMS satisfying the QDB- θ condition is a manifold with real dimension

$$(n - 1) + (n - 1)^2 + \frac{n(n - 1)}{2} = \frac{3n(n - 1)}{2}.$$

This, compared with the computation of Remark 12, shows that the QMS satisfying the SQDB- θ condition are much more numerous.

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