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A NOISE OF NEW TYPE AND ITS GENERALIZED FUNCTIONALS

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Abstract. The purpose of this paper is to introduce a new noise denoted by $P'(u)$. It has the space parameter u, being compared with the usual noise depending on the time t. We first explain why such a noise arises naturally. Then, we come to the analysis of functionals of this new noise. We shall emphasize the significance of generalized functionals of $P'(u)$, in particular, linear and quadratic.

1. Preliminaries. The main purpose of this report is to introduce a new noise depending on the space parameter. For this purpose we shall follow the following steps.

1. We first give a clear interpretation on the notion of *noise*.
2. Then, we remind the well-known noises. They depend on the time parameter t.
3. There arises a question if there exists any noise depending on other parameter, say the space parameter. We shall show that a new noise with space parameter does exist.
4. Then, we shall come to the analysis of functionals of new noise. There we can see the significant difference from the time dependent noises.

Before coming to these steps, we have to explain what a noise means.

Given a random complex phenomenon, we wish to find a system of *idealized elemental random variables* (abbr. i.e.r.v.) which are independent, atomic, infinitesimal random variables such that the system has the same information as the given random phenomenon. In addition, the members of the system are parametrized by a certain ordered set. We often call such a system a *noise*.

Once the phenomenon in question is expressed as a function of a noise (which is a variable system), then we understand that synthesis is done. Having done so, the function is ready to be analyzed. We thus follow the standard steps of stochastic analysis:

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Reduction *Synthesis* *Analysis*

The setup of the analysis involves the choice of the class of functionals. In fact, we propose to take a class of *generalized functionals* of a noise and come to the discussion of *operators* acting on them.

2. Noise

2.1. Two classes of noise. We classify the well-known noises according to the choice of parameter. A noise usually consists of continuously many *idealized* random variables, and they are parametrized by an ordered set.

REMARK. We say *idealized* random variables, because if they were ordinary random variables as many as continuum, then the probability distribution of the system in question could not be an abstract Lebesgue space; that is, it is impossible to carry on the calculus with the help of the Lebesgue type integral. It is therefore necessary and natural to have idealized (generalized) random variables.

a) *Time* dependent noise

(Gaussian) White Noise: $\dot{B}(t)$, Poisson Noise: $\dot{P}(t)$.

They are so popular, so that there is no need to explain.

b) *Space* dependent noise

$P'(u)$, $u > 0$.

This is the system exactly what we are going to discuss in this report.

2.2. The birth of noises. Consider a linear parameter. To fix the idea we take the unit interval $I = [0, 1]$. In case the *reduction* is successful, we are given a noise with parameter set I.

Case I. Let $\mathbf{\Delta}^n = \{\Delta_j^n : 1 \le j \le 2^n\}$ be the partition of I. To fix the idea, we assume that $|\Delta_j^n| = 2^{-n}$.

To each subinterval Δ_j^n we associate a random variable X_j^n. Assume that $\{X_j^n\}$ are i.i.d. (independent identically distributed) with mean 0 and finite variance v.

Let n be larger. Then, we can appeal to the central limit theorem to have a standard Gaussian distribution $N(0, 1)$ as the limit of the distribution of

$$S_n = \Big(\sum_1^{2^n} X_j^n\Big)/\sqrt{2^n v}.$$

If we take a subinterval $[a, b]$ of I, then the same trick gives us a Gaussian distribution $N(0, b - a)$.

We may therefore consider S_n as an approximation of a Brownian motion and each X_j^n approximates an elemental infinitesimal random variable. This means that we are given a (Gaussian) white noise.

Case II. The interval I and its partition Δ^n are the same as in the case I. The independent random variables X_j^n are i.i.d., but all are subject to a simple probability distribution such that

$$P(X_j^n = 1) = p_n, \quad P(X_j^n = 0) = 1 - p_n.$$

Let p_n be smaller as n is getting larger keeping the relation

$$2^n p_n = \lambda$$

for some positive constant $\lambda > 0$. Then, by the *law of small probabilities of Poisson* (this term comes from [7]) we are given the Poisson distribution $P(\lambda)$ with intensity λ.

Here are important notes.

1) We have a freedom to choose the constant λ whatever we wish, so far as it is positive.

2) The positive constant λ is the expectation of a random variable $P(\lambda)$ to which the sum converges in law. It can be viewed as *scale* or *space* variable.

We now understand that a noise, which is taken to be a realization of the randomness due to reduction, can eventually create a space variable.

Next step of our study is concerned with a general new noise depending on a space parameter.

Keep random variables X_j^n as above and divide the sum S_n into partial sums:

$$S_n = \sum_p S_n^p, \quad \text{where } S_n^p = \sum_{k(p)+1}^{k(p+1)} X_j^n,$$

with

$$1 = k(0) < k(1) < k(2) < \ldots < k(m) = 2^n.$$

We assume that $k(p+1) - k(p) \to \infty$ as $n \to \infty$ and that each ratio $\frac{k(p+1)-k(p)}{2^n}$ converges to $\frac{\lambda_j}{\lambda}$, respectively.

THEOREM 2.1. *Let S_n, S_n^k and λ be as above. Let $P(\lambda_k)$, $1 \le k \le m$, be mutually independent Poisson random variables with intensity λ_k, respectively. Then,*

i) *S_n and S_n^k converge to $P(\lambda)$ and $P(\lambda_k)$ in law, respectively.*
ii) *$P(\lambda_k)$ is a realization, in terms of distribution, of the limit (in law) of the S_n^k. Let the u_k's be linearly independent real numbers over Z, and let $P = \sum_k u_k P(\lambda_k)$. Then, if we know the values of P, we can determine the values of each $P(\lambda_k)$ which can be approximated in law by S_n^k.*

Proof. i) is easy to observe the characteristic function $\varphi_n^k(z)$ of S_n^k:

$$\varphi_n^k(z) = \Big(1 + \frac{\lambda}{2^n}(e^{iz} - 1)\Big)^{n_k}$$

which tends to

$$\varphi_k(z) = e^{\lambda_k(e^{iz}-1)}.$$

This proves the assertion.

ii) comes from the fact that the value $\sum u_k x_k$ determines all the values x_k. ∎

COROLLARY 2.1. *The characteristic function $\varphi(z)$ of P given by the above theorem is expressed in the form*

$$\varphi(z) = \exp\Big(\sum_k \lambda_k(e^{iu_k z} - 1)\Big).$$

Before we come to the next topic, we pause to see the following facts.

Remark. The type of probability distributions.

All the Gaussian distributions are the same type. (A constant, the exceptional Gaussian variable is excluded. It is just excluded in Case I.)

On the other hand, Poisson type distributions with different intensities are not the same type. This can be proved by the formula of characteristic function. Poisson type distribution means a distribution of $uP(\lambda)+c$, c may be ignored. Namely, we can compare two characteristic functions:

$$\varphi_1(z) = e^{\lambda_1(e^{iz}-1)},$$
$$\varphi_2(z) = e^{\lambda_2(e^{iz}-1)}.$$

These two functions of z cannot be exchanged by any affine transformation of z if $\lambda_1 \neq \lambda_2$.

We can say that

1) We have a freedom to choose intensity arbitrarily. Hence we can form, by the sum of i.i.d. random variables, continuously many Poisson type random variables with different type.

2) The intensity is a parameter, different from the time, which is viewed as a space parameter, The above construction shows it is additive in λ.

3) Multiplication by a constant to Poisson type variable, where the constant can be a label. So, take a constant $u = u(\lambda)$ as a label of the intensity. The function $u(\lambda)$ is therefore univalent. In view of this fact, we can form an inverse function $\lambda = \lambda(u)$ which is to be monotone.

With the remark made above, we change our eyes towards multi-dimensional view. We consider

$$P(\lambda) = (P(\lambda_k)),$$

and its characteristic function

$$\varphi(\mathbf{z}) = \prod_k e^{\lambda_k(e^{iz_k}-1)},$$

where $\mathbf{z} = (z_k)$.

We now wish to identify every component $P(\lambda_k)$, so that we give a label to each $P(\lambda_k)$, say different real number u_k. Now let us have passage from digital $\{k\}$ to real $u > 0$. The characteristic function $\varphi(\mathbf{z})$ turns into a functional of ξ in some function space E, expressed in the form

$$C(\xi) = \exp\Big[\int \lambda(u)(e^{iu\xi(u)} - 1)\, du\Big].$$

What we have done is that starting from a higher dimensional characteristic function of Poisson type distribution, we have its limit $C(\xi)$.

We now claim

Theorem 2.2. *Take a nuclear space E which is a dense subspace of $L^2([0,\infty))$. Then, $C(\xi)$ obtained above is a characteristic functional of a generalized stochastic process with parameter set $[0,\infty)$.*

The proof comes from [1], Chapter III, §4.

3. A new noise. We have found a new noise (in [6]). Here we shall not mention the results. But, we have explained why and how we are led to the new noise depending on the space variable u.

Having been motivated by the result in the literature [4] we start out with a functional $C^P(\xi)$, where the variable ξ runs through a certain nuclear space E. More precisely, E is a subspace of the Schwartz space $\mathcal{S}$ consisting of ξ's such that $\xi(u) = 0$ for $u \le 0$. In fact such an E is defined as a factor space of $\mathcal{S}$, so that the topology is naturally introduced.

$$C^P(\xi) = \exp\Big[\int (e^{iu\xi(u)} - 1)\, dn(u)\Big], \tag{3.1}$$

where $dn(u)$ is a measure on $(0, \infty)$, which is specified later.

To fix the idea, we assume that the measure $dn(u)$ is equivalent to the Lebesgue measure, i.e. it is of the form $dn(u) = \lambda(u)\, du$ with $\lambda(u)$ positive a.e. and the integral defining C^P is integrable.

Theorem 3.1. *Under these assumptions, the functional $C^P(\xi)$ is a characteristic functional.*

Proof. i) $C^P(\xi)$ is continuous in ξ. Details of the proof has been given in [6]. The topology introduced to E is slightly stronger than that in the case of white noise.

ii) $C^P(0) = 1$.

iii) Positive definiteness is shown by noting the fact that

$$\exp[(e^{izu} - 1)\lambda]$$

is a characteristic function. ∎

Hence, by the Bochner–Minlos theorem, there exists a probability measure ν^P on E^* such that

$$C^P(\xi) = \int_{E^*} e^{i\langle x, \xi\rangle}\, d\nu^P(x). \tag{3.2}$$

We introduce a notation $P'(u, \lambda(u))$ or write it simply $P'(u)$. With this notation we understand that ν^P-almost all $x \in E^*$ is a sample function of $P'(u)$.

Theorem 3.2. *$P'(u)$ has independent value at every point u.*

Proof. The proof is easy if the integral in (3.1) is understood to be extended over the support of ξ. Hence, if the supports of ξ_1 and ξ_2 are disjoint, then we have

$$C^P(\xi_1 + \xi_2) = C^P(\xi_1) C^P(\xi_2).$$

Hence the assertion follows. ∎

The bilinear form $\langle P', \xi\rangle$ is a random variable with mean

$$\int u\xi(u)\lambda(u)\, du$$

and variance

$$\int u^2 \xi(u)^2 \lambda(u)\, du.$$

Hence $\langle P', \xi\rangle$ extends to $\langle P', f\rangle$ with $f \in L^2((0, \infty), \lambda\, du)$. If uf and ug are orthogonal in $L^2(\lambda\, du)$, then $\langle P', f\rangle$ and $\langle P', g\rangle$ are uncorrelated. Thus, we can form a random measure and hence, we can define the space $\mathcal{H}_1(P)$ like $\mathcal{H}_1$ in the case of Gaussian white noise. The space $\mathcal{H}_1(P)$ can also be extended to a space $\mathcal{H}_1\varrho(P)$ of generalized linear functionals of $P'(u)$'s. Note that there we can give an *identity* to $P'(u)$ for any u.

Our conclusion is that the single noise $P'(u)$ of Poisson type with the parameter u can be found, cf. [6].

REMARK. The case where the parameter u runs through the negative interval $(-\infty, 0)$ can be discussed in a similar manner. It is, however, noted that the single point mass at $u = 0$ is omitted.

4. Representation of generalized functionals of $P'(u)$'s. We now come back to the discussion in §3, where we introduced the notation $P'(u)$ and the space $\mathcal{H}_1^{(-1)}(P)$ of the generalized linear functionals. (For $P'(u)$ the symbol P_{du} was used in [2]).

We note that the generalized stochastic process $P'(u)$ has independent values *at every point* u.

With this note we come to a representation of linear functionals.

Compute

$$\frac{1}{i}\frac{d}{dt}C^P(\xi + t\eta)|_{t=0} = \int e^{iu\xi(u)}\eta(u)\lambda(u)\, du \cdot C^P(\xi).$$

Up to the common factor $C^P(\xi)$, we are given a linear function of Poisson noise expressed in the form

$$\int e^{iu\xi(u)}u\eta(u)\lambda(u)\, du.$$

By subtracting off the constant (expectation) we have

$$\int (e^{iu\xi(u)} - 1)u\eta(u)\lambda(u)\, du.$$

This is linear in η. We are, therefore, given a linear space

$$\mathbf{F}_1 = \mathrm{span}\Big\{\int (e^{iu\xi(u)} - 1)u\eta(u)\lambda(u)\, du : \eta \in E\Big\},$$

which is isomorphic to $\mathcal{H}_1(P)$.

We know that $\langle x, \xi\rangle$, $\xi \in E$, is viewed as a sample of a random variable $\langle P', \xi\rangle$, the characteristic function of which is given by $\varphi^{P'}(z) = C^P(z\xi)$. Hence its mean value is $\int u\xi(u)\lambda(u)\, du$ and its variance is $\int u^2\xi(u)^2\lambda(u)\, du$.

There is a bijection:

$$\xi \Longleftrightarrow \langle P', \xi\rangle,\ \xi \in E.$$

This can be extended to

$$f \Longleftrightarrow \langle P', f\rangle,\ uf \in L^2((0,\infty), \lambda)\, du.$$

To establish a general theory of representation of random functions, we can appeal to the theory of Reproducing Kernel Hilbert Space with kernel $C^P(\xi - \eta)$, or the T-transform, which is an analogue of the Fourier transform. See [4].

By using the representation, we can now rigorously define the random measure $p(du)$ (which was briefly mentioned before) such that for any Borel subset $B \subset (0, \infty)$ with finite $d\lambda$ measure

$$p(B) = \langle P', I_B \rangle,$$

is defined, where I_B is the indicator function of a Borel set B.

Stochastic integrals based on $p(du)$ are defined in the usual manner, and the collections of the stochastic integrals form a Hilbert space which is in agreement with $\mathcal{H}_1(P)$.

Also, as in the Gaussian case, we can define the extended space $\mathcal{H}_1^{(-1)}(P)$, where each $P'(u)$ is a member of the *total* system of $\mathcal{H}_1^{(-1)}(P)$. Again, we note that $P'(u)$ corresponds to the kernel which is the δ-function, except $u = 0$. There the $P'(u)$ has rigorous, definite identity.

We can also consider multi-linear case in a similar manner, and the spaces $\mathbf{F}_n$ with $n \geq 2$ can be defined similarly, and of course, there is no need to have functionals renormalized.

5. Generalized functionals. Like Gaussian case, we can define spaces of generalized functionals of degree $n(\geq 2)$: $\mathbf{F}_n^{(-n)}$ and discuss their representations.

As soon as we come to nonlinear functionals of the $P'(u)$, we generally need renormalizations as in the case of $\dot{B}(t)$. There is one aspect to understand the necessity of renormalization as follows.

If we consider the second variation of general functionals, we are given similar expressions of *normal functionals* in the Lévy's sense [9]. Again, we use the expression in terms of Reproducing Kernel Hilbert Space defined by $C^P(\xi)$,

$$\int f(u)(e^{i\xi(u)u} - 1)^2 u^2 \eta(u)^2 \lambda(u)^2 \, du \\ + \int\int F(u,v)(e^{i\eta(u)u} - 1)(e^{i\eta(v)v} - 1) uv\eta(u)\eta(v)\lambda(u)\lambda(v) \, du \, dv,$$

where f and F satisfy integrability conditions, and where $F(u,v)$ is symmetric. Such an expression comes from "passage from digital to an analogue". The idea is the same as in the Gaussian case, so the details are omitted. See [4]. Usually (i.e. we often meet in the applications) quadratic functionals of Poisson noise are normal.

The most important fact is that the first term of the above equation is obtained after *renormalization* is applied.

There are normal functionals of higher degree that are significant and can be defined similarly but in somewhat complicated manner.

6. Applications

(1) We can now speak of the decomposition of a compound Poisson process, that is the jump finding problem. We have discussed it in [6], so we just note the problem to be applied and do not repeat the details here. See [6].

(2) Applications of our theory to statistics. In particular, we are interested in the study of statistical data that are subject to a stable distribution.

We can estimate the exponent α of the stable distribution. It is recommended to use the method explained in [11].

An important assumption is that this stable distribution can be embedded in a stable stochastic process with the same exponent α. We should carefully examine the environment and/or history so that this assumption is acceptable.

The Lévy measure for a stable process is of the form

$$dn(u) = c|u|^{-\alpha-1}\,du,$$

so that the theory stated in (1) above is acceptable.

Then, statistical theory follows.

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