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A LIMIT THEOREM FOR THE q-CONVOLUTION

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Abstract. The q-convolution is a measure-preserving transformation which originates from non-commutative probability, but can also be treated as a one-parameter deformation of the classical convolution. We show that its commutative aspect is further certified by the fact that the q-convolution satisfies all of the conditions of the generalized convolution (in the sense of Urbanik). The last condition of Urbanik's definition, the law of large numbers, is the crucial part to be proved and the non-commutative probability techniques are used.

1. Introduction. The classical convolution, expressed in terms of moment sequences, mimics the binomial formula for two commuting variables: if $ab = ba$, then

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} b^{n-k} a^k$$

and, analogously

$$m_n(\mu * \nu) = \sum_{k=0}^{n} \binom{n}{k} m_{n-k}(\nu) m_k(\mu)$$

where μ, ν denote two measures, $\mu * \nu$ is the classical convolution of μ and ν, and $m_n(\mu)$ denotes the n-th moment of μ.

If we consider two q-commuting variables, i.e. we assume that $ab = qba$ for some parameter $q > 0$, then the q-binomial formula appears:

$$(a+b)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q b^{n-k} a^k,$$

where $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ is the q-deformation of the usual binomial symbol (see Notation). So in parallel

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to the classical case, we can try to define the operation

$$m_n(\mu \star_q \nu) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q m_{n-k}(\nu) m_k(\mu) \tag{1}$$

and ask about its properties.

Such an operation, called the q-convolution, was introduced and studied by G. Carnovale and T. H. Koornwinder in 2001, [3]. It appeared as a (formal) one-parameter deformation of the classical convolution of functions on $\mathbb{R}$ which is associative and commutative (at least for some special class of measures) and which has a nice analogue of the Fourier transform. But from the very beginning the q-convolution had a 'flavour of non-commutativity', not only because it involved the q-calculus formulas (q-numbers, q-binomial formula, q-Jackson integral, etc.) which—as we saw in the example above—fit to the q-commuting variable setting. A more convincing argument is that the q-convolution was invented as a deformation of a convolution of functionals on a braided $*$-bialgebra, introduced by A. Kempf and S. Majid in [5].

In [8], we set (1) as the definition of an operation which transforms the sequences and we were wondering whether it really is a transformation that preserves moment sequences (measures), as it is in the classical case. This turned out to be the case if

- $q \in (0, 1)$,
- one takes the q-moment sequences $(\mu_n)_n$ instead of the usual moment sequences $(m_n(\mu))_n$ (the relation is $\mu_n = q^{n(n-1)/2} m_n(\mu)$, see Section 2.1),
- $(\mu_n)_n$ and $(\nu_n)_n$ correspond to measures supported on $[0, +\infty)$.

Note that the q-moment sequences are related to the notion of q-normality: $aa^* = qa^*a$ (studied by Ôta and Szafraniec in the papers [10, 11], for instance), so once again the non-commutative nature of the q-convolution is revealed.

However, recently, with B. Jasiulis-Gołdyn [4], we observed that the q-convolution resembles a lot the so-called *generalized convolution*, defined by K. Urbanik in [13]. This is a transformation of measures on $[0, +\infty)$, which satisfies the algebraic properties of the classical convolution: it is linear (with respect to convex combinations), dilation-invariant and weak-continuous and has a neutral element (Dirac delta at 0). This object has been studied quite intensively in the classical probability. When the definition of the generalized convolution is interpreted on the set of q-moment sequences, then it is easy to check that all but one of the conditions are satisfied. The aim of this note is to show that the q-convolution satisfies also the last condition of Urbanik's definition, called the *law of large numbers*.

The paper is organized as follows. In Section 2 we recall the definitions and some facts which we will need in the sequel (q-moment sequence, the q-convolution). Next section is devoted to the notion of a generalized convolution on the set of q-moment sequences. It also contains an easy proof of the fact that the q-convolution satisfies four (out of five) properties of the generalized convolution. The main result (Theorem 5) and its proof is the content of Section 4. More precisely, we show there that for each $n \in \mathbb{N}$ there exists the limit

$$\lim_{N\to+\infty}(T_{1/N}\delta_1^{\star_q N})_n,$$

where δ_1 denotes the Dirac delta at 1, $\star_q$ is the q-convolution and T_a is the dilation of measure (see Notation). The crucial step in the proof is the definition of an appropriate unital algebra $\mathcal{A}$ and a functional φ on it, which allows us to describe the q-moments of the measure $T_{1/N}\delta_1^{\star_q N}$, for any fixed N. The second step is the explicit calculation of the q-moments, where we use similar techniques as in the proof of the limit theorem for (p,q)-convolution (Theorem 6.1 from [8]), including standard quantitative arguments. In the last section (Section 5) we describe a measure which corresponds to the q-moment sequence $\left(\frac{[n]_q!}{n!}\right)_n$ from Theorem 5. It is related to a generalized Gamma convolution, studied by Berg in [1]. We shortly discuss the problem of whether the measure is unique.

Notation. All sequences appearing in the paper are indexed by $\mathbb{N}=\{0,1,2,\dots\}$, the set of non-negative integers. We denote by $\mathcal{P}^+$ the set of all probability measures supported on the non-negative half-line $[0,+\infty)$ and by $\mathcal{P}^+_{fm}$ the set of probability measures from $\mathcal{P}^+$, having all moments $m_n(\mu)=\int_0^{+\infty}t^n\mu(dt)$ finite. By δ_x we always denote the Dirac delta at point x.

For a probability measure $\mu\in\mathcal{P}^+$ and $a\geq 0$, we denote by $T_a\mu$ the *dilation* of μ, defined for a μ-measurable set $A\subset[0,+\infty)$ by the formula $(T_a\mu)(A)=\mu(A/a)$ when $a>0$ and $T_0\mu=\delta_0$. The dilation $T_a\mu$ can also be interpreted as the distribution of the random variable aX provided μ is the distribution of X.

For $\mu,\nu\in\mathcal{P}^+$, we denote by $\mu\circ\nu$ the *multiplicative convolution* of μ and ν (called also the *scale mixture* in classical probability), which is by definition

$$(\mu\circ\nu)(A)=\int_0^{+\infty}(T_s\mu)(A)\,\nu(ds).$$

An alternative definition is given in terms of probability distribution: if X,Y are two independent random variables with distributions μ_X and μ_Y respectively, then the *multiplicative convolution* $\mu_X\circ\mu_Y$ is the distribution of the random variable XY.

It follows directly from the definition that

$$\delta_a\circ\delta_b=\delta_{ab}\quad\text{and}\quad\delta_a\circ\mu=T_a\mu \tag{2}$$

for $a,b\geq 0$ and $\mu\in\mathcal{P}^+$. Moreover, the multiplicative convolution reflects the pointwise multiplication of the moment sequences: for measures μ and ν in $\mathcal{P}^+_{fm}$ we have

$$m_n(\mu\circ\nu)=m_n(\mu)\cdot m_n(\nu),\quad n\in\mathbb{N}. \tag{3}$$

Throughout the whole paper we assume that $0<q<1$. For such q we adopt the standard notation of the q-calculus (cf. [6]). For every $n\in\mathbb{N}$, we write

$$(a;q)_n=\prod_{k=0}^{n-1}(1-q^k a),\quad (a;q)_\infty=\lim_{n\to+\infty}(a;q)_n,$$

$$[n]_q=\frac{1-q^n}{1-q},\quad [n]_q!=\frac{(q;q)_n}{(1-q)^n},\quad \begin{bmatrix}n\\k\end{bmatrix}_q=\frac{[n]_q!}{[k]_q![n-k]_q!}.$$

For $q = 1$ these last three values are identified with the corresponding limits when $q \nearrow 1$, thus $[n]_1 = n$, $[n]_1! = n!$.

2. Preliminaries. In this section we collect basic information about the q-convolution and q-moment sequences.

2.1. *q*-moment sequences. The following generalization of the notion of moments will be useful in the sequel: a sequence $(\mu_n)_n$ is called a *q-moment sequence* if there exists a (positive Borel) measure μ on $\mathbb{R}$ such that

$$\mu_n = \mu_n^{(q)} = q^{n(n-1)/2} \int_{\mathbb{R}} t^n \, d\mu(t), \quad n \in \mathbb{N}. \tag{4}$$

Note that if $q = 1$, what we get is the standard notion of moment sequence.

As for the notation, since the parameter q is fixed, we waive the superscript (q) in $\mu_n^{(q)}$. Confusion could only appear when dealing simultaneously with moments and q-moments of a measure. To avoid it, we will always denote by μ_n the q-moments of a measure μ whereas for the standard moments of μ we reserve the notation $m_n(\mu)$. In particular, $\mu_n^{(1)} = m_n(\mu)$ for $n \in \mathbb{N}$.

The definition of q-moment sequence is motivated by the following fact, which is a q-analogue of the classical Hamburger theorem (see [7] for details).

Proposition 1. *A sequence $(\mu_n)_n$ is a q-moment sequence for the measure μ on $\mathbb{R}$ if and only if for all $n \in \mathbb{N}$ and all scalars $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ the following inequality holds*

$$\sum_{i,j=0}^{n} q^{-ij} \alpha_i \bar{\alpha}_j \mu_{i+j} \geq 0. \tag{qPD}$$

We will denote by $\mathcal{M}_q^+$ the set of all q-moment sequences $(\mu_n)_n$ corresponding to measures from $\mathcal{P}_{fm}^+$. It can be described in the following way (cf. [7]).

Proposition 2. *For a sequence $(\mu_n)_n$ with $\mu_0 = 1$ the following conditions are equivalent*:

1. $(\mu_n)_n \in \mathcal{M}_q^+$, *i.e. $(\mu_n)_n$ is a q-moment sequence corresponding to a probability measure μ on $[0, +\infty)$,*
2. *both sequences $(\mu_n)_n$ and $(\mu_{n+1})_n$ satisfy the condition* (qPD),
3. $\left(q^{n(n-1)/2}\mu_n\right)_n$ *is a moment sequence corresponding to a probability measure μ on $[0, +\infty)$.*

2.2. *q*-convolution. The q-convolution was originally defined by Carnovale and Koornwinder in [3] in the following way:

Definition 1. Let f be a function on $\mathbb{R}$ such that all its *weighted moments*

$$\mu_n(f) := q^{n(n+1)/2} \int t^n f(t) \, d_q(t)$$

are finite and let g be a function on some subset of the complex plane $\mathbb{C}$. Then the *q-convolution* $f \star_q g$ is defined by

$$(f \star_q g)(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \mu_n(f)}{[n]_q!} (\partial_q^n g)(z)$$

(for $z \in \mathbb{C}$ such that the formula makes sense). Here $\partial_q^n g$ denotes the n-th q-derivative of a function g, where ∂_q is defined as $(\partial_q g)(x) = \frac{g(x)-g(qx)}{(1-q)x}$ and $\int f(t)\, d_q(t)$ denotes the Jackson integral (cf. [6]).

The weighted moments $\mu_n(f)$ should not be confused with q-moments. Indeed, if we denote by μ the measure $d\mu(t) = f(t)d_q(t)$, then

$$\mu_n(f) = q^{n(n+1)/2} \int t^n f(t)\, d_q(t) = q^n \cdot q^{n(n-1)/2} \int_{\mathbb{R}} t^n\, d\mu(t) = q^n \mu_n.$$

The q-convolution appeared as a deformation of a convolution in braided covector algebras, introduced by Kempf and Majid [5], adopted to the case of braided line $\mathbb{C}_q[x]$. Carnovale and Koornwinder showed in [3] that if f and g satisfy some analyticity conditions, then the convolution is well-defined, associative and commutative (see [3] for further details). For such functions we also have a nice formula

$$\mu_n(f \star_q g) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mu_k(f)\mu_{n-k}(g),$$

where $\mu_k(f)$ is the k-th weighted moment from Definition 1.

It is easy to check that the same formula holds if we replace the weighted moments by q-moments of the measure $d\mu(t) = f(t)\, d_q(t)$. This formula is now taken as a general definition of an operation on q-moment sequences.

DEFINITION 2. Let $0 < q < 1$ and let $(\mu_n)_n$, $(\nu_n)_n$ belong to $\mathcal{M}_q^+$. Then their *q-convolution* is the sequence

$$(\mu \star_q \nu)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mu_k \nu_{n-k}, \quad n \in \mathbb{N}.$$

What can we say about such an operation? The crucial fact concerning the behaviour of the q-convolution is Proposition 3.3 from [8].

PROPOSITION 3. *If $0 < q < 1$, then the q-convolution preserves the sequences from $\mathcal{M}_q^+$ in the sense that if $(\mu_n)_n$, $(\nu_n)_n$ belong to $\mathcal{M}_q^+$, then their q-convolution $((\mu \star_q \nu)_n)_n$ belongs to $\mathcal{M}_q^+$ too.*

It is worth noting that the result is no longer true if we change the range of the parameter q or if we enlarge the set of moment sequences. Indeed, it was shown in [8] that the q-convolution preserves neither the set $\mathcal{M}_q^+$ when $q > 1$ nor the set of the q-moment sequences corresponding to measures on the whole real line $\mathbb{R}$ when $q \in (0,1)$.

One can verify by direct calculations that the operation $\star_q$ is associative and commutative, that is $(\mu \star_q \nu) \star_q \rho = \mu \star_q (\mu \star_q \rho)$ and $\mu \star_q \nu = \nu \star_q \mu$ for any $\mu, \nu, \rho \in \mathcal{M}_q^+$.

If $q \nearrow 1$, then q-moments become the (classical) moments of measures and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \longrightarrow \binom{n}{k},$$

so the 1-convolution coincides with the classical convolution of sequences. That is why the q-convolution can be called a q-deformation of the classical convolution.

3. Generalized convolution. The notion of the generalized convolution was introduced by K. Urbanik in [13] and studied in a series of papers published since 1964.

DEFINITION 3. An associative and commutative binary operation $\circledast$ on $\mathcal{P}^+$ is called a *generalized convolution on* $\mathcal{P}^+$ if it satisfies the following conditions:

(i) $\delta_0 \circledast \mu = \mu$ for all $\mu \in \mathcal{P}^+$ (δ_0 is the unit element),
(ii) $(c\mu_1 + (1-c)\mu_2) \circledast \nu = c(\mu_1 \circledast \nu) + (1-c)(\mu_2 \circledast \nu)$ whenever $\mu_1, \mu_2, \nu \in \mathcal{P}^+$ and $c \in (0,1)$,
(iii) $T_a(\mu \circledast \nu) = (T_a\mu) \circledast (T_a\nu)$ for any $\mu, \nu \in \mathcal{P}^+$ and $a \geq 0$,
(iv) if $\mu^{(n)} \xrightarrow{\mathtt{w}} \mu$ with $\mu^{(n)}, \mu \in \mathcal{P}^+$ ($n \in \mathbb{N}$), then $\mu^{(n)} \circledast \nu \xrightarrow{\mathtt{w}} \mu \circledast \nu$ for all $\nu \in \mathcal{P}^+$, ("$\xrightarrow{\mathtt{w}}$" denotes the weak convergence of probability measures),
(v) there exists a sequence $(c_n)_n$ of positive numbers such that the sequence $T_{c_n}\delta_1^{\circledast n}$ converges weakly to a measure different from δ_0 (*law of large numbers*).

Even though the q-convolution is an associative and commutative operation which "preserves measures" from $\mathcal{P}^+$ (in the sense of Proposition 3), there is little hope it could satisfy directly Urbanik's definition. The point is that it is well defined not on the whole $\mathcal{P}^+$, but only on $\mathcal{M}_q^+$. However, one can notice that $\mathcal{M}_q^+ \cong \mathcal{P}_{fm}^+/\sim$, where $\sim$ is the equivalence relation defined in the following way. Given two measures $\mu, \nu \in \mathcal{P}_{fm}^+$, μ is *equivalent* to ν ($\mu \sim \nu$) if they have all q-moments equal ($\mu_n = \nu_n$ for all $n \in \mathbb{N}$). This suggests the following interpretation of Urbanik's definition on $\mathcal{M}_q^+$:

1. all equalities for measures should be changed into equalities for respective cosets, that is, the equalities of all terms of q-moment sequences (or moment sequences, equivalently),
2. the weak convergence should be replaced by the convergence of all q-moments (or moments, equivalently); this will be denoted by "m-convergence",

Hence we arrive at the following definition.

DEFINITION 4. An associative and commutative binary operation $\circledast$ on $\mathcal{M}_q^+$ is called a *generalized convolution on* $\mathcal{M}_q^+$ if it satisfies the following conditions:

(i) $\delta_0 \circledast \mu = \mu$ for any $\mu \in \mathcal{M}_q^+$,
(ii) $(c\mu_1 + (1-c)\mu_2) \circledast \nu = c(\mu_1 \circledast \nu) + (1-c)(\mu_2 \circledast \nu)$ for any $\mu_1, \mu_2, \nu \in \mathcal{M}_q^+$, $c \in (0,1)$,
(iii) $T_a(\mu \circledast \nu) = (T_a\mu) \circledast (T_a\nu)$ for any $\mu, \nu \in \mathcal{M}_q^+$, $a \geq 0$,
(iv) $\forall \mu^{(n)}, \mu, \nu \in \mathcal{M}_q^+$: $\mu^{(n)} \xrightarrow{\mathtt{m}} \mu \implies \mu^{(n)} \circledast \nu \xrightarrow{\mathtt{m}} \mu \circledast \nu$,
(v) there exists a sequence $(c_n)_n$ of positive numbers such that the sequence $T_{c_n}\delta_1^{\circledast n}$ m-converges to a measure different from δ_0.

We will show (by direct calculations) that the q-convolution satisfies the first four conditions of the reformulated definition of the generalized convolution for moment sequences (see also [4]).

THEOREM 4. *The q-convolution satisfies the conditions* (i)–(iv) *of the generalized convolution on $\mathcal{M}_q^+$* (*Definition* 4).

Proof. As for (i), let us note that the q-moment sequence corresponding to δ_0 is $(\delta_{n,0})_n = (1, 0, 0, \dots)$ and thus obviously

$$(\mu \star_q \delta_0)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mu_k \delta_{n-k,0} = \mu_n.$$

To show (ii), we just need to observe that for q-moment sequences one has $(\alpha\mu + \beta\nu)_k = \alpha\mu_k + \beta\nu_k$ for any $k \in \mathbb{N}$. Using this we get

$$\begin{aligned} [(c\mu_1 + (1-c)\mu_2) \star_q \nu]_n &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (c\mu_1 + (1-c)\mu_2)_k \nu_{n-k} \\ &= c \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (\mu_1)_k \nu_{n-k} + (1-c) \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (\mu_2)_k \nu_{n-k} \\ &= c(\mu_1 \star_q \nu)_n + (1-c)(\mu_2 \star_q \nu)_n \end{aligned}$$

Condition (iii) follows from the fact that $(T_a\mu)_n = a^n \mu_n$. This implies that

$$[T_a(\mu \star_q \nu)]_n = a^n (\mu \star_q \nu)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (a^k \mu_k)(a^{n-k}\nu_{n-k}) = [(T_a\mu) \star_q (T_a\nu)]_n.$$

Finally, given a sequence $(\mu^{(n)})_n$ of elements from $\mathcal{M}_q^+$, converging in q-moments to $\mu \in \mathcal{M}_q^+$, we observe that the formula for $(\mu^{(n)} \circledast \nu)_k$ is a finite linear combination of q-moments $\mu_l^{(n)}$ and ν_l $(0 \le l \le k)$, so the convergence of q-moments is preserved for the convolution. This proves condition (iv). ∎

4. Law of large number for the q-convolution. Our aim in this section is to show that the q-convolution satisfies also the condition (v) of Definition 4 and thus that all the conditions of the generalized convolution on $\mathcal{M}_q^+$ hold. More precisely, we shall prove the following:

THEOREM 5 (law of large number for the q-convolution). *For every $n \in \mathbb{N}$ the following limit exists*

$$\lim_{N \to +\infty} (T_{1/N} \delta_1^{\star_q N})_n = \lim_{N \to +\infty} \big[T_{1/N} \underbrace{(\delta_1 \star_q \dots \star_q \delta_1)}_{N \ times}\big]_n \tag{5}$$

and equals $\frac{[n]_q!}{n!}$. Moreover, the sequence $\big(\frac{[n]_q!}{n!}\big)_n$ belongs to $\mathcal{M}_q^+$.

Before starting the proof we first define a unital algebra and $\mathcal{A}$ and a functional φ on $\mathcal{A}$, which allow us to describe the q-moments of the measure $T_{1/N}\delta_1^{\star_q N}$ for fixed N.

We would emphasize that the functional φ plays only the role of a tool for performing the combinatorial calculations and its positivity is not considered (neither the $*$-structure on $\mathcal{A}$, nor the positivity of the functional φ is necessary for that). The fact that the limit sequence corresponds to a positive measure on $[0, +\infty)$ follows (independently of the definition of φ) from the fact that the q-convolution preserves $\mathcal{M}_q^+$ (cf. Proposition 3).

Let us consider the simplest case $N = 2$. Let $\mathcal{A}$ be the unital $*$-algebra generated by two elements a and b which q-commute (that is, $ab = qba$). Given two sequences

$(\mu_n)_n, (\nu_n)_n \in \mathcal{M}_q^+$, we define the functional $\varphi : \mathcal{A} \to \mathbb{C}$ by the formula

$$\varphi(a^m b^n) = q^{mn} \mu_m \nu_n, \quad m, n \in \mathbb{N}.$$

Note that this is the same as to say that $\varphi(b^n a^m) = \nu_n \mu_m$ for $m, n \in \mathbb{N}$ and, in particular, $\varphi(a^n) = \mu_n$ and $\varphi(b^n) = \nu_n$. Observe also that then

$$\varphi((a+b)^n) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \varphi(b^k a^{n-k}) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mu_k \nu_{n-k} = (\mu \star_q \nu)_n.$$

Similarly, we can take an N-tuple of q-moment sequences $(\mu^{(n)})_{n=1}^N$, corresponding to measures supported in $[0, +\infty)$ and the N-tuple of elements $a_1, \dots, a_N$, which q-commute in monotonical order, that is

$$a_i a_j = q a_j a_i \quad \text{whenever} \quad 1 \le i < j \le N, \tag{6}$$

and which generate a unital algebra $\mathcal{A}$. Then, if we put

$$\varphi(a_N^{m_N} \cdots a_1^{m_1}) = \mu_{m_1}^{(1)} \cdot \ldots \cdot \mu_{m_N}^{(N)} \tag{7}$$

and extend it to a linear function on $\mathcal{A}$, we can show that

$$\varphi((a_1 + \ldots + a_N)^n) = \big(\mu^{(1)} \star_q \mu^{(2)} \star_q \ldots \star_q \mu^{(N)}\big)_n.$$

(Note that the elements a_i in the definition of φ (Eq. 7) are in decreasing order of indices.) Indeed, for $1 \le j \le N-1$, we have $(a_1 + \ldots + a_j) a_{j+1} = q a_{j+1} (a_1 + \ldots + a_j)$ and we can proceed by induction. The j-th step is the following

$$\begin{aligned}
\varphi((a_1 + \ldots + a_j)^n) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \varphi\big(a_j^k (a_1 + \ldots + a_{j-1})^{n-k}\big) \\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mu_k^{(j)} \big(\mu^{(1)} \star_q \ldots \star_q \mu^{(j-1)}\big)_{n-k} \\
&= \big(\mu^{(j)} \star_q \mu^{(1)} \star_q \ldots \star_q \mu^{(j-1)}\big)_n = \big(\mu^{(1)} \star_q \ldots \star_q \mu^{(j-1)} \star_q \mu^{(j)}\big)_n.
\end{aligned}$$

The last equality follows from the commutativity of the q-convolution.

Now we are ready to prove the main theorem.

Proof of Theorem 5. We consider the functional φ defined by formula (7) with all sequences $\mu^{(1)}, \dots, \mu^{(N)}$ being equal to the sequence $(q^{\binom{n}{2}})_n$. The latter is the q-moment sequence corresponding to the Dirac measure at 1, and by a little abuse of notation we shall denote this sequence as δ_1. Since the dilation transforms the q-moments like moments, that is $(T_a \mu)_n = a^n \mu_n$, hence for a fixed $n \in \mathbb{N}$ we have

$$\big(T_{1/N}(\delta_1 \star_q \ldots \star_q \delta_1)\big)_n = \frac{1}{N^n} \varphi\big((a_1 + \ldots + a_N)^n\big) = \frac{1}{N^n} \sum_{1 \le j_i \le N} \varphi(a_{j_1} \cdots a_{j_n}).$$

To compute the value of the functional φ on the element $a_{j_1} \cdots a_{j_n}$ we need to have the a_i's in decreasing order. For this purpose, we use the q-commutation (6): $a_i a_j = q a_j a_i$ whenever $1 \le i < j \le N$, and we observe that each time we take an element a_{j_k} to the left of an element which has a lower index, we get a factor q. Of course, the number of different elements in $a_{j_1} \cdots a_{j_n}$ can be smaller than n.

For the element $a_{j_1}\cdots a_{j_n}$ there exist a non-negative integer l and sequences $\underline{\xi} = (\xi_1,\ldots,\xi_l)$ and $\underline{k} = (k_1,\ldots,k_l)$ such that $1 \le l \le n$, $1 \le \xi_l < \ldots < \xi_1 \le N$, $k_i \ge 1$, $k_1 + \ldots + k_l = n$, and

$$a_{j_1}\cdots a_{j_n} = q^{\alpha(j_1,\ldots,j_n)} a_{\xi_1}^{k_1}\cdots a_{\xi_l}^{k_l}.$$

Note that the exponent $\alpha(j_1,\ldots,j_n)$ is a non-negative integer and $q^{\alpha(j_1,\ldots,j_n)} \in (0,1)$. Of course, there are several sequences $a_{j_1}\cdots a_{j_n}$ which, after reordering, give the same sequence $a_{\xi_1}^{k_1}\cdots a_{\xi_l}^{k_l}$, but with different exponents α. That is why, for l, $\underline{k} = (k_1,\ldots,k_l)$ and $\underline{\xi} = (\xi_1,\ldots,\xi_l)$ as above, we define the set $J(l;\underline{k},\underline{\xi})$ as the set of all sequences $(j_1,\ldots,j_n)$ such that $\{j_1,\ldots,j_n\} = \{\xi_1,\ldots,\xi_l\}$ and $k_i = |\{s : j_s = \xi_i\}|$ for $i = 1,\ldots,l$.

This allows us to change the summation in the following way

$$\frac{1}{N^n}\sum_{1\le j_i\le N}\varphi(a_{j_1}\cdots a_{j_n}) = \frac{1}{N^n}\sum_{l=1}^{n}\sum_{\underline{k}}\sum_{\underline{\xi}} c_{l;\underline{k},\underline{\xi}}\varphi(a_{\xi_1}^{k_1}\cdots a_{\xi_l}^{k_l}),$$

where the constant $c_{l;\underline{k},\underline{\xi}}$ is to be computed as the sum of all $q^{\alpha(j_1,\ldots,j_n)}$ over $J(l;\underline{k},\underline{\xi})$. This means that it can be estimated as follows

$$c_{l;\underline{k},\underline{\xi}} = \sum_{J(l;\underline{k},\underline{\xi})} q^{\alpha(j_1,\ldots,j_n)} \le \sum_{J(l;\underline{k},\underline{\xi})} 1 = \frac{n!}{k_1!\cdots k_l!} =: C_{l,k}$$

and the estimation no longer depends on ξ.

Next, we show that the part of the sum with $l < n$ can be neglected in the limit (as $N \to +\infty$). Indeed, take $l < n$ and fix some $\underline{k} = (k_1,\ldots,k_l)$ such that $k_i \ge 1$ and $k_1 + \ldots + k_l = n$. In this case there exists some i_0 such that $k_{i_0} \ge 2$ and there are at most $n-1$ different elements ξ_k in $\underline{\xi}$. So we have the following estimate:

$$\begin{aligned}\Big|\frac{1}{N^n}\sum_{\underline{\xi}} c_{l;\underline{k},\underline{\xi}}\varphi(a_{\xi_1}^{k_1}\cdots a_{\xi_l}^{k_l})\Big| &\le \frac{C_{l,k}}{N^n}\sum_{\underline{\xi}}|\varphi(a_{\xi_1}^{k_1}\cdots a_{\xi_l}^{k_l})| = \frac{C_{l,k}}{N^n}\sum_{\underline{\xi}} q^{\binom{k_1}{2}}\ldots q^{\binom{k_l}{2}}\\ &\le \frac{C_{l,k}}{N^n}\cdot\big|\{\underline{\xi} = (\xi_1,\ldots,\xi_l) : 1 \le \xi_l < \ldots < \xi_1 \le N\}\big| \le \frac{C_{l,k}}{N^n}\cdot\binom{N}{n-1} \le \frac{C_{l,k,n}}{N},\end{aligned}$$

with a constant $C_{l,k,n}$ independent of N.

We showed that, indeed, each term under the sum over l, such that $l < n$, is of order at most $\frac{1}{N}$ and thus vanishes in the limit. Therefore, only this part of the sum where $j_1,\ldots,j_n$ are all different, and where $\underline{\xi} = (\xi_1 > \ldots > \xi_n)$, remains in the limit:

$$\frac{1}{N^n}\sum_{1\le j_i\le N}\varphi(a_{j_1}\cdots a_{j_n}) \approx \frac{1}{N^n}\sum_{\underline{\xi}=(\xi_1>\ldots>\xi_n)} c_{n;\underline{k},\underline{\xi}}\varphi(a_{\xi_1}\cdots a_{\xi_n}).$$

Here, the sign $\approx$ means that both sides have the same limit as $N \to +\infty$.

Let us now focus on the constants (for $l = n$)

$$c_{n;\underline{k},\underline{\xi}} = \sum_{J(n;\underline{k},\underline{\xi})} q^{\alpha(j_1,\ldots,j_n)}.$$

In this case $J(n;\underline{k},\underline{\xi})$ is just the set of all permutations of the set $\{\xi_1,\ldots,\xi_n\}$, which can be parametrized by all permutations of n-elements: σ is such that $\sigma(\xi_i) = j_i$. Moreover, $\alpha(j_1,\ldots,j_n) = |\sigma|$ is the number of inversions in the permutation. So finally we have (cf. [8])

$$c_{n;\underline{k},\underline{\xi}} = \sum_{\sigma\in S_n} q^{|\sigma|} = [n]_q!$$

and

$$\begin{aligned}\left(T_{1/N}\delta_1^{\star_q N}\right)_n &\approx \frac{1}{N^n} \sum_{\xi=(\xi_1>\ldots>\xi_n)} c_{n;\underline{k},\underline{\xi}}\varphi(a_{\xi_1}\cdots a_{\xi_n}) \\ &= \frac{1}{N^n}\sum_{\xi}\sum_{\sigma\in S_n} q^{|\sigma|}\varphi(a_{\xi_1}\cdots a_{\xi_l}) = \frac{1}{N^n}\sum_{\sigma\in S_n} q^{|\sigma|}\sum_{\xi}\mu_1^n \\ &= \frac{1}{N^n}[n]_q!\binom{N}{n} \longrightarrow \frac{[n]_q!}{n!} \qquad \text{as} \quad N\to+\infty.\end{aligned}$$

This way we showed that the limit (5) exists. Since, according to Proposition 3, the q-convolution preserves $\mathcal{M}_q^+$, the limit sequence also belongs to $\mathcal{M}_q^+$. ■

5. Limit measure. We showed that the limit sequence described in Theorem 5 belongs to $\mathcal{M}_q^+$, so it is a q-moment sequence corresponding to some (possibly not unique) measure λ on $[0,+\infty)$. By Proposition 2, the moments of λ are

$$m_n(\lambda) = q^{-\binom{n}{2}}\frac{[n]_q!}{n!} = \frac{[n]_{1/q}!}{n!}\,.$$

The next theorem presents a measure λ corresponding to this sequence. It is based on the results due to Berg ([1]).

THEOREM 6 (a limit measure for q-convolution). *The measure $T_{q/(1-q)}\check{J}_q$, with $\check{J}_q$ described below, corresponds to the moment sequence*

$$m_n(\lambda) = q^{-\binom{n}{2}}\frac{[n]_q!}{n!}, \quad n\in\mathbb{N}.$$

Proof. If we denote by I_q the generalized Gamma convolution, which is the measure on $(0,+\infty)$ with the density function

$$i_q(x) = \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)/2}}{(q;q)_n}\exp(-xq^{-n})\chi_{(0,+\infty)}(x),$$

then the moments of I_q are

$$m_n(I_q) = \frac{n!}{(q,q)_n}\,.$$

Defining

$$\check{J}_q = \frac{i_q(1/y)dy}{y\log(1/q)}\,,$$

Berg [1] showed that this is a probability measure and its moments are

$$m_n(\check{J}_q) = \frac{(q,q)_n}{n!}\,q^{-(n+1)n/2}.$$

In our case, using formulae (2) and (3), we have

$$\begin{aligned}m_n(\lambda) &= q^{-\binom{n}{2}}\frac{[n]_q!}{n!} = \left(\frac{q}{1-q}\right)^n\cdot q^{-\binom{n+1}{2}}\frac{(q;q)_n}{n!} \\ &= m_n(\delta_{q/(1-q)})\cdot m_n(\check{J}_q) = m_n(\delta_{q/(1-q)}\circ\check{J}_q) = m_n(T_{q/(1-q)}\check{J}_q).\ \blacksquare\end{aligned}$$

To the best of our knowledge, it is still an open question whether the moment sequence $\{m_n(\lambda)\}_n$ determines the measure in a unique way. Since the limit measure described in the previous theorem has unbounded support, this need to be the case for all measures (if more than one) related to the limit sequence in question. Using the Stirling formula, one easily checks that

$$\sum_n \frac{1}{\sqrt[2n]{m_n(\lambda)}} < \infty,$$

so the Carleman criterion does not give a decisive answer in this case. A discussion in [1] on the application of the Krein criterion to the measure $\check{J}_q$ reveals that none of the two possibilities (determined or indeterminate) is excluded.

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