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SYMMETRIZATION OF PROBABILITY MEASURES, PUSHFORWARD OF ORDER 2 AND THE BOOLEAN CONVOLUTION

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Abstract. We study relations between the Boolean convolution and the symmetrization and the pushforward of order 2. In particular we prove that if μ_1, μ_2 are probability measures on $[0,\infty)$ then $(\mu_1 \uplus \mu_2)^{\mathbf{s}} = \mu_1^{\mathbf{s}} \uplus \mu_2^{\mathbf{s}}$ and if ν_1, ν_2 are symmetric then $(\nu_1 \uplus \nu_2)^{(2)} = \nu_1^{(2)} \uplus \nu_2^{(2)}$. Finally we investigate necessary and sufficient conditions under which the latter equality holds.

1. Pushforward of order 2 versus symmetrization. Let $\mathcal{M}$ denote the class of probability measures on the real line $\mathbb{R}$. We will distinguish two subclasses of $\mathcal{M}$, namely $\mathcal{M}^{\mathbf{s}}$ consisting of symmetric measures (i.e. such that $\mu(-B) = \mu(B)$ for every Borel subset of $\mathbb{R}$) and $\mathcal{M}^{+}$ consisting of those measures which have support contained in the positive halfline $[0, +\infty)$.

For $\mu \in \mathcal{M}$ we define its two transforms:

$$G_\mu(z) := \int_{\mathbb{R}} \frac{d\mu(t)}{z-t}, \qquad M_\mu(z) := \frac{1}{z}\, G_\mu\Big(\frac{1}{z}\Big), \tag{1}$$

which are analytic functions on $\mathbb{C} \setminus \mathbb{R}$ (the former is called the *Cauchy transform* of μ). If μ has compact support then M_μ is well defined in a neighborhood of 0 and is the

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generating function for the moment sequence $\int_{\mathbb{R}} x^m\, d\mu(x)$, $m \geq 0$, of μ. Note that $\mu \in \mathcal{M}$ is symmetric if and only if its Cauchy transform is odd: $G_\mu(-z) = -G_\mu(z)$ and M_μ is even: $M_\mu(-z) = M_\mu(z)$.

For $\mu \in \mathcal{M}$ we define $\mu^{(2)}$ as the pushforward of μ by the function $x \mapsto x^2$, i.e.

$$\mu^{(2)}(B) := \mu\left(\{x : x^2 \in B\}\right). \tag{2}$$

For example, if μ is a convex combination $\sum_{i=1}^N p_i \delta_{x_i}$, then $\mu^{(2)} = \sum_{i=1}^N p_i \delta_{x_i^2}$.

For $\mu \in \mathcal{M}^+$ we define its *symmetrization* as the measure $\mu^{\mathbf{s}} \in \mathcal{M}^{\mathbf{s}}$ satisfying

$$\mu^{\mathbf{s}}(B) := \mu(\{x^2 : x \in B\}) \tag{3}$$

for every symmetric Borel set B. For example, if $\mu = \sum_{i=1}^N p_i \delta_{x_i}$, with $x_i \geq 0$, then $\mu^{\mathbf{s}} = \frac{1}{2}\sum_{i=1}^N p_i \left(\delta_{-\sqrt{x_i}} + \delta_{\sqrt{x_i}}\right)$. It was observed in [2] that

$$\mu^{\mathbf{s}} = \frac{1}{2}\left(\delta_{-1} + \delta_1\right) \boxtimes \mu^{\boxtimes \frac{1}{2}} \tag{4}$$

(where $\boxtimes$ denotes the multiplicative free convolution), whenever $\mu^{\boxtimes \frac{1}{2}}$ exists.

The map $\mu \mapsto \mu^{\mathbf{s}}$ is a bijection $\mathcal{M}^+ \to \mathcal{M}^{\mathbf{s}}$ and the map $\nu \mapsto \nu^{(2)}$ restricted to $\mathcal{M}^{\mathbf{s}}$ is its inverse. For $\mu \in \mathcal{M}^+$ we have

$$G_{\mu^{\mathbf{s}}}(z) = zG_\mu(z^2), \qquad M_{\mu^{\mathbf{s}}}(z) = M_\mu(z^2), \tag{5}$$

while for $\nu \in \mathcal{M}$

$$2zG_{\nu^{(2)}}(z^2) = G_\nu(z) - G_\nu(-z), \qquad 2M_{\nu^{(2)}}(z^2) = M_\nu(z) + M_\nu(-z). \tag{6}$$

The *Boolean convolution* is a binary operation on $\mathcal{M}$ which can be defined as: $\mu := \mu_1 \uplus \mu_2$ if and only if

$$\frac{1}{G_\mu(z)} = \frac{1}{G_{\mu_1}(z)} + \frac{1}{G_{\mu_2}(z)} - z, \tag{7}$$

or, equivalently,

$$\frac{1}{M_\mu(z)} = \frac{1}{M_{\mu_1}(z)} + \frac{1}{M_{\mu_2}(z)} - 1. \tag{8}$$

For $\mu \in \mathcal{M}$, $t > 0$ we define Boolean power $\mu^{\uplus t}$ by

$$\frac{1}{G_{\mu^{\uplus t}}(z)} := \frac{t}{G_\mu(z)} - (t-1)z \tag{9}$$

or

$$M_{\mu^{\uplus t}}(z) := \frac{M_\mu(z)}{(1-t)M_\mu(z) + t}. \tag{10}$$

It is clear that the class $\mathcal{M}^{\mathbf{s}}$ is closed under the Boolean convolutions and powers. The same is true for the class $\mathcal{M}^+$ (see Remark 2.7 and Theorem 6.2 in [1]).

THEOREM 1.1. *For $\mu_1, \mu_2, \mu \in \mathcal{M}^+$, $\nu_1, \nu_2, \nu \in \mathcal{M}^{\mathbf{s}}$ and $t > 0$ we have*

$$(\mu_1 \uplus \mu_2)^{\mathbf{s}} = \mu_1^{\mathbf{s}} \uplus \mu_2^{\mathbf{s}}, \qquad (\mu^{\uplus t})^{\mathbf{s}} = (\mu^{\mathbf{s}})^{\uplus t} \tag{11}$$

and

$$(\nu_1 \uplus \nu_2)^{(2)} = \nu_1^{(2)} \uplus \nu_2^{(2)}, \qquad \left(\nu^{\uplus t}\right)^{(2)} = \left(\nu^{(2)}\right)^{\uplus t}. \tag{12}$$

Proof. Putting $\mu := \mu_1 \uplus \mu_2$ we have

$$\frac{1}{G_{\mu^{\mathbf{s}}}(z)} = \frac{1}{z}\left(\frac{1}{G_{\mu_1}(z^2)} + \frac{1}{G_{\mu_2}(z^2)} - z^2\right) = \frac{1}{zG_{\mu_1}(z^2)} + \frac{1}{zG_{\mu_2}(z^2)} - z,$$

which is the reciprocal of the Cauchy transform of $\mu_1^{\mathbf{s}} \uplus \mu_2^{\mathbf{s}}$. Similarly, putting $\mu_t := \mu^{\uplus t}$ we have

$$\frac{1}{G_{\mu_t^{\mathbf{s}}}(z)} = \frac{1}{z}\left(\frac{t}{G_{\mu}(z^2)} - (t-1)z^2\right) = \frac{t}{zG_{\mu}(z^2)} - (t-1)z,$$

which is the reciprocal of the Cauchy transform of $(\mu^{\mathbf{s}})^{\uplus t}$.

To prove the second part one can put $\nu_1 := \mu_1^{\mathbf{s}}$, $\nu_2 := \mu_2^{\mathbf{s}}$. ∎

Example. Define

$$\begin{aligned}
\mathbf{m} &:= \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}\,dx && \text{on } [0,4] && \text{(the Marchenko-Pastur law)},\\
\mathbf{w} &:= \frac{1}{2\pi}\sqrt{4-x^2}\,dx && \text{on } [-2,2] && \text{(the Wigner law)},\\
\mathbf{a}_+ &:= \frac{1}{\pi\sqrt{x(4-x)}}\,dx && \text{on } [0,4] && \text{(the positive arcsine law)},\\
\mathbf{a} &:= \frac{1}{\pi\sqrt{4-x^2}}\,dx && \text{on } [-2,2] && \text{(the symmetric arcsine law)}.
\end{aligned}$$

Then

$$M_{\mathbf{m}}(z) = \frac{2}{1+\sqrt{1-4z}}\,, \qquad M_{\mathbf{w}}(z) = \frac{2}{1+\sqrt{1-4z^2}}\,,$$
$$M_{\mathbf{a}_+}(z) = \frac{1}{\sqrt{1-4z}}\,, \qquad M_{\mathbf{a}}(z) = \frac{1}{\sqrt{1-4z^2}}\,,$$

which leads to the relations: $\mathbf{m}^{\mathbf{s}} = \mathbf{w}$, $\mathbf{a}_+^{\mathbf{s}} = \mathbf{a}$, $\mathbf{m}^{\uplus 2} = \mathbf{a}_+$ and $\mathbf{w}^{\uplus 2} = \mathbf{a}$. Hence

$$(\mathbf{m}^{\mathbf{s}})^{\uplus 2} = (\mathbf{m}^{\uplus 2})^{\mathbf{s}} = \mathbf{a} \qquad \text{and} \qquad \left(\mathbf{w}^{\uplus 2}\right)^{(2)} = \left(\mathbf{w}^{(2)}\right)^{\uplus 2} = \mathbf{a}_+.$$

Remark. Note that in Theorem 1.1 we cannot replace $\uplus$ by the classical or free convolution. For example, if $\mu_1 := \frac{1}{2}(\delta_{-a} + \delta_a)$ and $\mu_2 := \frac{1}{2}(\delta_{-b} + \delta_b)$ then

$$\mu_1^{(2)} * \mu_2^{(2)} = \delta_{a^2+b^2} \quad \text{while} \quad (\mu_1 * \mu_2)^{(2)} = \frac{1}{2}\left(\delta_{(a+b)^2} + \delta_{(a-b)^2}\right).$$

For the free convolution let $\mathbf{m}_t := \mathbf{m}^{\boxplus t}$ and $\mathbf{w}_t := \mathbf{w}^{\boxplus t}$. These measures exist for all $t > 0$ (see [5, 3]) and

$$\mathbf{m}_t = \max\{1-t, 0\}\delta_0 + \frac{\sqrt{4t-(x-1-t)^2}}{2\pi x}\,dx,$$

with the absolutely continuous part supported on $\left[(1-\sqrt{t})^2, (1+\sqrt{t})^2\right]$,

$$\mathbf{w}_t = \frac{1}{2\pi t}\sqrt{4t-x^2}\,dx$$

on $[-2\sqrt{t}, 2\sqrt{t}]$. The moment generating functions are

$$M_{\mathbf{m}_t}(z) = \frac{2}{1+(1-t)z+\sqrt{(1-(1+t)z)^2-4tz^2}}$$

and

$$M_{\mathbf{w}_t}(z) = \frac{2}{1+\sqrt{1-4tz^2}}\,.$$

Therefore if $0 < t \neq 1$ then the measure $\left(\mathbf{m}^{\boxplus t}\right)^{\mathbf{s}} = \mathbf{m}_t^{\mathbf{s}}$ is different from $\left(\mathbf{m}^{\mathbf{s}}\right)^{\boxplus t} = \mathbf{w}_t$.

Note however that formulae (11) would be true if we replaced the Boolean convolution $\uplus$ by $*$ (resp. by $\boxplus$) and the map $\mu \mapsto \mu^{\mathbf{s}}$ by the map $\mu \mapsto \mu*\widehat{\mu}$, with $t \in \mathbb{N}$ (resp. $\mu \mapsto \mu\boxplus\widehat{\mu}$, with $t \geq 1$), where $\widehat{\mu}$ denotes the *reflection* of μ, i.e. $\widehat{\mu}(B) := \mu(-B)$.

THEOREM 1.2. *Let $\nu \in \mathcal{M}$ and $0 < t \neq 1$. Then the equality*

$$\left(\nu^{\uplus t}\right)^{(2)} = \left(\nu^{(2)}\right)^{\uplus t}$$

holds if and only if ν is symmetric.

Proof. Put $\nu_1 := (\nu^{\uplus t})^{(2)}$, $\nu_2 := \left(\nu^{(2)}\right)^{\uplus t}$, $M := M_\nu(z)$ and $N := M_\nu(-z)$. Then

$$\begin{aligned} M_{\nu_1}(z^2) &= \frac{M}{2[(1-t)M+t]} + \frac{N}{2[(1-t)N+t]} \\ &= \frac{2(1-t)MN + t(M+N)}{2[(1-t)M+t][(1-t)N+t]} \\ &= \frac{2(1-t)^2MN(M+N) + 4t(1-t)MN + t(1-t)(M+N)^2 + 2t^2(M+N)}{2[(1-t)M+t][(1-t)N+t][(1-t)(M+N)+2t]}\,, \end{aligned}$$

while

$$\begin{aligned} M_{\nu_2}(z^2) &= \frac{M+N}{(1-t)(M+N)+2t} \\ &= \frac{2(1-t)^2MN(M+N) + 2t(1-t)(M+N)^2 + 2t^2(M+N)}{2[(1-t)M+t][(1-t)N+t][(1-t)(M+N)+2t]}\,. \end{aligned}$$

Therefore

$$M_{\nu_2}(z^2) - M_{\nu_1}(z^2) = \frac{t(1-t)(M-N)^2}{2[(1-t)M+t][(1-t)N+t][(1-t)(M+N)+2t]}\,,$$

which proves our statement. ∎

2. The case of nonsymmetric measures. In this part we are going to study circumstances in which the equality

$$\mu_1^{(2)} \uplus \mu_2^{(2)} = (\mu_1 \uplus \mu_2)^{(2)} \tag{13}$$

holds. Putting $\eta_1 := (\mu_1 \uplus \mu_2)^{(2)}$, $\eta_2 := \mu_1^{(2)} \uplus \mu_2^{(2)}$, $M_1 := M_{\mu_1}(z)$, $N_1 := M_{\mu_1}(-z)$, $M_2 := M_{\mu_2}(z)$ and $N_2 := M_{\mu_2}(-z)$ we have

$$M_{\eta_1}(z^2) = \frac{M_1M_2}{2\,(M_1+M_2-M_1M_2)} + \frac{N_1N_2}{2\,(N_1+N_2-N_1N_2)}\,, \tag{14}$$

$$M_{\eta_2}(z^2) = \frac{(M_1+N_1)(M_2+N_2)}{2(M_1+N_1+M_2+N_2)-(M_1+N_1)(M_2+N_2)}\,. \tag{15}$$

THEOREM 2.1. *Assume that $\mu_1 \in \mathcal{M}^{\mathrm{s}}$, $\mu_2 \in \mathcal{M}$ and that* (13) *holds. Then either $\mu_2 \in \mathcal{M}^{\mathrm{s}}$ or $\mu_1 = \delta_0$.*

Proof. Putting $M := M_1 = N_1$ we have

$$M_{\eta_1}(z^2) - M_{\eta_2}(z^2) = \frac{M^2(M-1)(M_2-N_2)^2}{2(M+M_2-MM_2)(M+N_2-MN_2)\,(2M+M_2+N_2-MM_2-MN_2)},$$

which yields our statement. ∎

From now on we assume that $\mu_1 = \delta_{x_0}$, with $x_0 \neq 0$. Then $M_1(z) = 1/(1-x_0z)$, $N_1(z) = M_1(-z) = 1/(1+x_0z)$ and

$$M_{\eta_1}(z^2) - M_{\eta_2}(z^2) = \frac{x_0z(M_2+N_2)[2M_2-2N_2-x_0z(M_2+N_2-2M_2N_2)]}{2(1-x_0zM_2)(1+x_0zN_2)[2-x_0^2z^2(M_2+N_2)]}.$$

Therefore we have

THEOREM 2.2. *Assume that $\mu_1 = \delta_{x_0}$ with $x_0 \neq 0$. Then* (13) *holds if and only if*

$$2M - 2N - x_0z(M+N-2MN) = 0, \tag{16}$$

where $M := M_{\mu_2}(z)$, $N := M_{\mu_2}(-z)$.

COROLLARY 2.3. *If $\mu_1 = \delta_{x_0}$, $x_0 \neq 0$, μ_2 has compact support and if* (13) *holds then the mean of μ_2 is* 0.

Proof. Since μ_2 has compact support, M_{μ_2} is well defined as an analytic function in a neighborhood of 0, with $M_{\mu_2}(0) = 1$. It is sufficient to differentiate both sides of (16) at $z = 0$ to see that $M'_{\mu_2}(0) = 0$. ∎

Finally, we confine ourselves to a very particular case.

THEOREM 2.4. *Assume that $\mu_1 = \delta_{x_0}$, $\mu_2 = p\delta_{x_1} + (1-p)\delta_{x_2}$, with $x_0 \neq 0$, $x_1 \neq x_2$, $0 < p < 1$. Then* (13) *holds if and only if*

$$px_1 + (1-p)x_2 = 0 \tag{17}$$

and

$$x_0 + 2x_1 + 2x_2 = 0. \tag{18}$$

Note that (17) is a consequence of Corollary 2.3.

Proof. Since

$$M_2 := M_{\mu_2}(z) = \frac{p}{1-x_1z} + \frac{1-p}{1-x_2z}, \qquad N_2 := M_{\mu_2}(-z) = \frac{p}{1+x_1z} + \frac{1-p}{1+x_2z},$$

we have

$$\begin{aligned} &2M_2 - 2N_2 - x_0z(M_2+N_2-2M_2N_2) \\ &= \frac{2z\big[2px_1 + 2(1-p)x_2 + x_0z^2p(1-p)(x_1-x_2)^2 - 2z^2x_1x_2\big((1-p)x_1+px_2\big)\big]}{(1-x_1z)(1+x_1z)(1-x_2z)(1+x_2z)} \\ &= \frac{4z\big[px_1+(1-p)x_2\big] + 2z^3\big[x_0p(1-p)(x_1-x_2)^2 - 2x_1x_2\big((1-p)x_1+px_2\big)\big]}{(1-x_1z)(1+x_1z)(1-x_2z)(1+x_2z)} \end{aligned}$$

This rational function is equal to 0 if and only if $px_1 + (1-p)x_2 = 0$ (which implies, in particular, that $x_1 \cdot x_2 < 0$) and

$$x_0p(1-p)(x_1-x_2)^2 - 2x_1x_2\big((1-p)x_1+px_2\big) = 0. \tag{19}$$

By (17) we have $p(x_1 - x_2) = -x_2$ and $(1-p)(x_1 - x_2) = x_1$ so the left hand side of (19) can be written as

$$-x_1x_2\big(x_0 + 2(1-p)x_1 + 2px_2\big) = -x_1x_2(x_0 + 2x_1 + 2x_2),$$

which concludes the proof. ∎

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