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A REMARK ON p-CONVOLUTION

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Abstract. We introduce a *p-product* of algebraic probability spaces, which is the definition of independence that is natural for the model of noncommutative Brownian motions, described in [10] (for $q = 1$). Using methods of the conditionally free probability (cf. [4, 5]), we define a related *p-convolution* of probability measures on $\mathbb{R}$ and study its relations with the notion of *subordination* (cf. [1, 8, 9, 13]).

1. Introduction. In the paper [10] a two-parameter family of noncommutative Gaussian operators, acting on the free Fock space was introduced. It was associated with Kesten laws and gave a continuous interpolation between free, monotone and Boolean Gaussian operators. In this paper we will describe a *p-product* of algebraic probability spaces, which can be treated as a definition of independence that is natural for the model described in [10] (for $q = 1$). To introduce the notion of *p-product* we will use well known framework of conditionally free probability [4, 5]. Using those methods we will also introduce a *p-convolution* of probability measures on $\mathbb{R}$ which will appear to be a natural convolution related to the model noncommutative Brownian motions described in [10]. Moreover we will study its relation with the notion of *subordination* described in [1, 8, 9, 13].

2. Conditionally free product. In this section we will recall from [4, 5] the well known conditionally free product of algebraic probability spaces. Let $\mathcal{A}_i$ be a unital *-algebra with the unit 1_i and two states φ_i, ψ_i on it, for $i = 1, 2$. The triple $(\mathcal{A}_i, \varphi_i, \psi_i)$ will be called the *algebraic probability space with two states*. The *conditionally free product* of the algebraic probability spaces $(\mathcal{A}_i, \varphi_i, \psi_i)$ is the pair $(\mathcal{A}, \Phi)$, where $\mathcal{A} = \mathcal{A}_1 * \mathcal{A}_2$ is the free

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product of algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ with identification of units and Φ is a state, the so called *conditionally free product* of pairs (φ_1, ψ_1) and (φ_2, ψ_2), given by

$$\Phi(a_1 a_2 \cdots a_n) = \varphi_{i_1}(a_1)\varphi_{i_2}(a_2)\cdots\varphi_{i_n}(a_n),$$

for any $a_k \in \mathcal{A}_{i_k}$, $i_k \in \{1,2\}$, $i_1 \neq i_2 \neq \ldots \neq i_n$, $n \in \mathbb{N}$, such that $\psi_{i_k}(a_k) = 0$ for $k = 1, \ldots, n$. The pair $(\mathcal{A}, \Phi)$ is an *algebraic probability space* (*with one state*) and we will write

$$(\mathcal{A}, \Phi) = (\mathcal{A}_1, \varphi_1, \psi_1) \boxed{\mathrm{c}} (\mathcal{A}_2, \varphi_2, \psi_2) \quad \text{and} \quad \Phi = (\varphi_1, \psi_1) \boxed{\mathrm{c}} (\varphi_2, \psi_2).$$

Now we can define the conditionally free convolution of probability measures with compact support. Let μ_i, ν_i be the distribution of $a_i \in \mathcal{A}_i$ with respect to φ_i, ψ_i, i.e. $\varphi_i(a_i^n) = \int_{\mathbb{R}} x^n \, d\mu_i(x)$ and $\psi_i(a_i^n) = \int_{\mathbb{R}} x^n \, d\nu_i(x)$ for $i = 1, 2$. Then the distribution μ of $a_1 + a_2 \in \mathcal{A}_1 * \mathcal{A}_2$ with respect to Φ is called *the conditionally free convolution* of pairs of probability measures (μ_i, ν_i) and we write $\mu = (\mu_1, \nu_1) \boxed{\mathrm{c}} (\mu_2, \nu_2)$.

3. *p*-product of *-algebras. Using the language of conditionally free probability we will present a construction of a *p-product* of *-algebras with a state. This product can be viewed as a definition of independence that is a natural independence related to the model of a two-parameter family of Gaussian operators given in [10]. A similar construction for the monotone product has been given by Franz in [6].

From now on, we will assume that the *-algebra $\mathcal{A}_i$ has a decomposition $\mathcal{A}_i = \mathbb{C}1_i \oplus \mathcal{A}_i^\circ$, in the sense of the direct sum of vector spaces, such that $\mathcal{A}_i^\circ$ is a *-subalgebra of $\mathcal{A}_i$, for $i = 1, 2$. This assumption is quite restrictive but for our purpose, which is study of a p-convolution of the probability measures, is suitable. One of the simplest example of that kind of algebra is the algebra $\mathbb{C}[X]$ of all polynomials of a variable X. Moreover, note that if $\mathcal{A}_1$ and $\mathcal{A}_2$ have a decomposition as above, then also $\mathcal{A}_1 * \mathcal{A}_2$ does, i.e.

$$\mathcal{A}_1 * \mathcal{A}_2 = \mathbb{C}1 \oplus (\mathcal{A}_1 * \mathcal{A}_2)^\circ,$$

where $(\mathcal{A}_1 * \mathcal{A}_2)^\circ = \mathcal{A}_1^\circ * \mathcal{A}_2^\circ = \{a_1 a_2 \ldots a_n : a_k \in \mathcal{A}_{i_k}^\circ,\ i_1 \neq i_2 \neq \ldots \neq i_n,\ n \geqslant 1\}$.

On the algebra $\mathcal{A}_i$ we can put a functional δ_i given by $\delta_i(a + \lambda 1_i) = \lambda$, for $a \in \mathcal{A}_i^\circ$, $\lambda \in \mathbb{C}$. Obviously, δ_i is well defined and is a state on $\mathcal{A}_i$ since we assume that $\mathcal{A}_i^\circ$ is a *-subalgebra. Using this state we define

$$\widetilde{\varphi}_i = p\varphi_i + (1-p)\delta_i,$$

where $p \in [0,1]$. As a convex combination of states $\widetilde{\varphi}_i$ is also a state that we will call the *p-deformation of* φ_i.

Definition 3.1. The pair $(\mathcal{A}, \Phi)$ given by

$$(\mathcal{A}, \Phi) = (\mathcal{A}_1, \varphi_1, \widetilde{\varphi}_1) \boxed{\mathrm{c}} (\mathcal{A}_2, \varphi_2, \varphi_2)$$

will be called the *p-product* of the algebraic probability spaces $(\mathcal{A}_1, \varphi_1)$ and $(\mathcal{A}_2, \varphi_2)$ and denoted by $(\mathcal{A}, \Phi) = (\mathcal{A}_1, \varphi_1) \triangleright_p (\mathcal{A}_2, \varphi_2)$. The copies of $\mathcal{A}_1, \mathcal{A}_2$ contained in $\mathcal{A}$ will be called *p-independent*.

Example 3.2. Let $a \in \mathcal{A}_1^\circ$ and $b \in \mathcal{A}_2^\circ$. We will use the above definition to calculate some mixed moments of variables a and b in state Φ. For this matter let $a^\circ = a - \widetilde{\varphi}_1(a)1$

and $b^\circ = b - \varphi_2(b)1$. Obviously, $\widetilde{\varphi}_1(a^\circ) = \varphi_2(b^\circ) = 0$, from which and from Definition 3.1 it follows that $\Phi(a) = \varphi_1(a)$ and $\Phi(b) = \varphi_2(b)$. Moreover

$$\begin{aligned}\Phi(ab) &= \Phi(a^\circ b^\circ) + \Phi(a^\circ)\varphi_2(b) + \widetilde{\varphi}_1(a)\Phi(b^\circ) + \widetilde{\varphi}_1(a)\varphi_2(b)\\ &= \varphi_1(a^\circ)\varphi_2(b^\circ) + \varphi_1(a^\circ)\varphi_2(b) + \widetilde{\varphi}_1(a)\varphi_2(b^\circ) + \widetilde{\varphi}_1(a)\varphi_2(b) = \varphi_1(a)\varphi_2(b).\end{aligned}$$

Similarly $\Phi(ba) = \varphi_1(a)\varphi_2(b)$. In the same way we compute the moments of third order:

$$\begin{aligned}\Phi(bab) &= \Phi(b^\circ ab) + \varphi_2(b)\Phi(ab)\\ &= \Phi(b^\circ a^\circ b) + \widetilde{\varphi}_1(a)\Phi(b^\circ b) + \varphi_2(b)\Phi(ab)\\ &= \Phi(b^\circ a^\circ b^\circ) + \varphi_2(b)\Phi(b^\circ a^\circ) + \widetilde{\varphi}_1(a)\Phi(b^\circ b) + \varphi_2(b)\Phi(ab)\\ &= \varphi_1(a^\circ)\varphi_2^2(b^\circ) + \varphi_2(b)\varphi_2(b^\circ)\varphi_1(a^\circ) + \widetilde{\varphi}_1(a)\varphi_2(b^\circ b) + \varphi_2^2(b)\varphi_1(a)\\ &= p\varphi_1(a)\varphi_2(b^2) + (1-p)\varphi_1(a)\varphi_2^2(b),\end{aligned}$$

but $\Phi(aba) = \varphi_1(a^2)\varphi_2(b)$ (we omit similar calculations).

Let us observe that for $p = 1$ the above moments agree with the mixed moments of free random variables, and for $p = 0$ with those of monotone random variables.

4. p-convolution and transforms. In this section we will introduce and study a *p-convolution* of probability measures with compact support, i.e. we will try to find a measure, which would be the distribution of the sum of two random variables which come from p-independent *-subalgebras.

Let μ and ν be probability measures on $\mathbb{R}$ with compact support. Moreover, let $\mathcal{A}_i = \mathbb{C}[X_i]$ be the *-algebra of polynomials of the variable X_i and φ_i be a state on $\mathcal{A}_i$ such that

$$\varphi_1(X_1^n) = \int_{\mathbb{R}} x^n\, d\mu(x), \qquad \varphi_2(X_2^n) = \int_{\mathbb{R}} x^n\, d\nu(x),$$

for $n \geqslant 0$ and $i = 1, 2$. Obviously, $\widetilde{\varphi}_1(1) = 1$ and $\widetilde{\varphi}_1(X_1^n) = p\varphi_1(X_1^n)$ for $n \geqslant 1$. If we denote the distribution of X_1 with respect to $\widetilde{\varphi}_1$ by $\widetilde{\mu}$, then

$$\widetilde{\mu} = p\mu + (1-p)\delta_0,$$

i.e. $\widetilde{\mu}$ is a convex combination of the measures μ and δ_0.

DEFINITION 4.1. Let μ and ν be probability measures on $\mathbb{R}$ with compact support. Using the above notation, we define the *p-convolution* of μ and ν as follows

$$\mu \rhd_p \nu = (\mu, \widetilde{\mu}) \boxed{\mathrm{c}} (\nu, \nu),$$

which is the distribution of the sum $X_1 + X_2 \in \mathbb{C}[X_1] * \mathbb{C}[X_2]$ with respect to the state $\Phi = (\varphi_1, \widetilde{\varphi}_1) \boxed{\mathrm{c}} (\varphi_2, \varphi_2)$.

A symmetric version of the above convolution (i.e. $\mu \boxed{\mathrm{r}} \nu = (\mu, \widetilde{\mu}) \boxed{\mathrm{c}} (\nu, \widetilde{\nu})$) was introduced by Bożejko in [2] and also studied in [3].

Definition 4.1 allows us to use the conditionally free techniques and related transforms to study the p-convolution. Let us recall the well known formula for the R-transform of a pair of measures (μ, ν) (cf. [4, 7]) given by

$$R_{(\mu,\nu)}(z) = G_\nu^{-1}(z) - F_\mu(G_\nu^{-1}(z)), \tag{1}$$

where $F_\mu = 1/G_\mu$ is the reciprocal (inverse in the sense of multiplication of functions) Cauchy transform of μ and G_ν^{-1} is the inverse (in the sense of function composition)

Cauchy transform of ν. Moreover, we know that the R-transform is additive with respect to the conditionally free convolution. Using that fact and Definition 4.1 we get

$$R_{(\mu\triangleright_p\nu,\widetilde{\mu}\boxplus\nu)}(z) = R_{(\mu,\widetilde{\mu})}(z) + R_{(\nu,\nu)}(z). \tag{2}$$

From the identities (1) and (2) and the fact that the R-transform of a single measure ($R_\mu(z) = R_{\mu,\mu}(z)$) is additive with respect to free convolution (cf. [14]) we get the formula

$$F_{\mu\triangleright_p\nu}(z) = F_\mu(G_{\widetilde{\mu}}^{-1}(G_{\widetilde{\mu}\boxplus\nu}(z))). \tag{3}$$

This formula will be useful in the proofs of the next two theorems that describe the properties of p-convolution in terms of transforms. The first of these theorems is a rather simple connection of p-convolution with the notion of *subordination* described in [1, 8, 9, 13].

Proposition 4.2. *Let $F_\mu(z) = 1/G_\mu(z)$ and $K_\mu(z) = z - F_\mu(z)$ be the reciprocal Cauchy transform and the K-transform of a measure μ, respectively. Then for compactly supported measures μ and ν we have the following formula*

$$F_{\mu\triangleright_p\nu}(z) = F_\mu(z - K_\nu(z - K_{\widetilde{\mu}}(z - K_\nu(z - K_{\widetilde{\mu}}(z - \ldots))))), \tag{4}$$

where $K_{\widetilde{\mu}}(z) = (pzK_\mu(z))/(z-(1-p)K_\mu(z))$ is the K-transform of the measure $\widetilde{\mu} = p\mu+(1-p)\delta_0$. The right hand side of the above formula should be understood as a uniform limit on compact subsets of the upper complex half-plane $\mathbb{C}^+$.

Proof. First we will compute the Cauchy transform of $\widetilde{\mu}$, the support of which obviously is also compact. We have

$$G_{\widetilde{\mu}}(z) = \sum_{n=0}^{\infty} \frac{\widetilde{\mu}(n)}{z^{n+1}} = p\sum_{n=0}^{\infty} \frac{\mu(n)}{z^{n+1}} + \frac{1-p}{z} = pG_\mu(z) + (1-p)G_{\delta_0}(z),$$

where $\widetilde{\mu}(n)$ and $\mu(n)$ are the n-th moments of $\widetilde{\mu}$ and μ, respectively. So, $G_{\widetilde{\mu}}$ is a convex combination of G_μ and G_{δ_0} from which we can calculate a formula for $K_{\widetilde{\mu}}$, i.e.

$$\begin{aligned} K_{\widetilde{\mu}}(z) &= z - \frac{1}{pG_\mu(z) + (1-p)G_{\delta_0}(z)} = z - \frac{1}{\frac{p}{z-K_\mu(z)} + \frac{1-p}{z}} \\ &= \frac{pzK_\mu(z)}{z-(1-p)K_\mu(z)}. \end{aligned}$$

Now we can prove formula (4). From [8] we know that $G_{\widetilde{\mu}\boxplus\nu}(z) = G_{\widetilde{\mu}}(F_1(z))$, where F_1 is the so called *subordination function* with respect to $G_{\widetilde{\mu}}$, which has the form

$$F_1(z) = z - K_\nu(z - K_{\widetilde{\mu}}(z - K_\nu(z - K_{\widetilde{\mu}}(z - \ldots)))),$$

which, together with (3), ends the proof. ■

Let us observe that for $p = 0$ the transform $K_{\widetilde{\mu}}$ is equal to zero on $\mathbb{C}^+$. In that case, identity (4) takes the form

$$F_{\mu\triangleright_0\nu}(z) = F_\mu(z - K_\nu(z)) = F_\mu(F_\nu(z)).$$

We can see that the convolution $\triangleright_0$ agrees with the monotone convolution $\triangleright$ introduced in [11, 12]. On the other hand, for $p = 1$ the equivalence of the convolutions $\triangleright_1$ and $\boxplus$, not so obvious, is also true and follows from the identity

$$F_{\mu\boxplus\nu}(z) = F_\mu(z - K_\nu(z - K_\mu(z - K_\nu(z - K_\mu(z - \ldots))))),$$

proven in [8]. A more intuitive relation between free and p-convolution for $p = 1$ is provided by the following theorem.

THEOREM 4.3. *Let $R_\mu(z) = G_\mu^{-1}(z) - 1/z$ be the R-transform of a the measure μ. Then for compactly supported measures μ and ν the identity*

$$R_{\mu\triangleright_p\nu}(z) = R_\mu(z) + R_\nu\big(pH_{\delta_0}(z) + (1-p)H_\mu(z)\big) \tag{5}$$

holds, where $H_\mu(z) = 1/G_\mu^{-1}(z)$.

Proof. Observe that formula (3) can be viewed as follows

$$G^{-1}_{\mu\triangleright_p\nu}(z) = G^{-1}_{\widetilde{\mu}\boxplus\nu}(G_{\widetilde{\mu}}(G_\mu^{-1}(z))).$$

Using the above identity and the additivity of the R-transform with respect to the free convolution, we get

$$\begin{aligned} R_{\mu\triangleright_p\nu}(z) &= G^{-1}_{\mu\triangleright_p\nu}(z) - \frac{1}{z} = G^{-1}_{\widetilde{\mu}\boxplus\nu}(G_{\widetilde{\mu}}(G_\mu^{-1}(z))) - \frac{1}{z} \\ &= G^{-1}_{\widetilde{\mu}}(G_{\widetilde{\mu}}(G_\mu^{-1}(z))) + G^{-1}_{\nu}(G_{\widetilde{\mu}}(G_\mu^{-1}(z))) - \frac{1}{G_{\widetilde{\mu}}(G_\mu^{-1}(z))} - \frac{1}{z} \\ &= R_\mu(z) + R_\nu(G_{\widetilde{\mu}}(G_\mu^{-1}(z))). \end{aligned}$$

Now we will use the fact that $G_{\widetilde{\mu}}$ is a convex combination of G_μ and G_{δ_0}, so

$$\begin{aligned} R_{\mu\triangleright_p\nu}(z) &= R_\mu(z) + R_\nu\big(pG_\mu(G_\mu^{-1}(z)) + (1-p)G_{\delta_0}(G_\mu^{-1}(z))\big) \\ &= R_\mu(z) + R_\nu\Big(pz + \frac{1-p}{G_\mu^{-1}(z)}\Big) \\ &= R_\mu(z) + R_\nu(pH_{\delta_0}(z) + (1-p)H_\mu(z)), \end{aligned}$$

since $H_{\delta_0}(z) = z$. ■

Substituting $p = 1$ to formula (5), and using the fact that $H_{\delta_0}(z) = z$, we get $R_{\mu\triangleright_1\nu}(z) = R_\mu(z) + R_\nu(z)$. So, the convolution $\triangleright_1$ agrees with the free convolution $\boxplus$. On the other hand, for $p = 0$ formula (5) takes the form

$$H_{\mu\triangleright_0\nu}(z) = H_\nu(H_\mu(z)), \tag{6}$$

which is equivalent to $F_{\mu\triangleright_0\nu}(z) = F_\mu(F_\nu(z))$ and again confirm the equivalence of convolutions $\triangleright_0$ and $\triangleright$.

EXAMPLE 4.4. We will use Theorem 4.3 to calculate the p-convolution of the Wigner law $d\mu = \frac{1}{2\pi}\sqrt{4-x^2}\,dx$ with a single point measure $\nu = \delta_a$. The R-transforms of those measures are $R_\mu(z) = z$ and $R_\nu(z) = a$. From Theorem 4.3 we have

$$R_{\mu\triangleright_p\nu}(z) = z + a,$$

which means that the p-convolution $\mu\triangleright_p\nu$ is also the Wigner law shifted by a, i.e. with the density $d(\mu\triangleright_p\nu) = \frac{1}{2\pi}\sqrt{4-(x+a)^2}\,dx$.

EXAMPLE 4.5. Now we compute the p-convolution of two two-point measures. To simplify the calculations we assume that

$$\mu = \frac{1}{2}(\delta_0 + \delta_1), \qquad \nu = \frac{1}{2}(\delta_{-1} + \delta_1).$$

Nevertheless, this approach could be applied to compute the p-convolution of any pair of two-point measures, where the first one has an atom at zero.

First we can easily calculate that

$$K_{\widetilde{\mu}}(z) = \frac{pz}{2z+p-2}\,, \qquad K_\nu(z) = \frac{1}{z}\,.$$

From Proposition 4.2 we know that $G_{\mu\triangleright_p\nu}(z) = G_\mu(F_1(z))$, where $F_1(z)$ is a function, that satisfies the equation $F_1(z) = z - K_\nu(z - K_{\widetilde{\mu}}(F_1(z)))$. After solving it we can get the explicit formula of $F_1(z)$, from where we deduce that

$$G_{\mu\triangleright_p\nu}(z) = \frac{N(z) + \big(2 - p - 2pz - 2(1-p)z^2\big)\sqrt{-2p + p^2 + (1 + z - z^2)^2}}{2p(p-2)(2z - z^2 - 2z^3 + z^4)}\,,$$

with $N(z) = (1-p)\big(2 - p + (2+p)z + (3p-4)z^2 - 2(1+p)z^3 + z^4\big)$. Applying the Stieltjes inversion formula to $G_{\mu\triangleright_p\nu}$ we can obtain an explicit formula for the density of the measure $\mu \triangleright_p \nu$.

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