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TRIVIAL NONCOMMUTATIVE PRINCIPAL TORUS BUNDLES

STEFAN WAGNER

Department of Mathematical Sciences, University of Copenhagen
Universitetsparken 5, DK-2100 Copenhagen Ø, Denmark
E-mail: stefan.wagner@math.ku.dk

Abstract. A (smooth) dynamical system with transformation group $\mathbb{T}^n$ is a triple $(A, \mathbb{T}^n, \alpha)$, consisting of a unital locally convex algebra A, the n-torus $\mathbb{T}^n$ and a group homomorphism $\alpha : \mathbb{T}^n \to \operatorname{Aut}(A)$, which induces a (smooth) continuous action of $\mathbb{T}^n$ on A. In this paper we present a new, geometrically oriented approach to the noncommutative geometry of trivial principal $\mathbb{T}^n$-bundles based on such dynamical systems, i.e., we call a dynamical system $(A, \mathbb{T}^n, \alpha)$ a trivial noncommutative principal $\mathbb{T}^n$-bundle if each isotypic component contains an invertible element. Each trivial principal bundle $(P, M, \mathbb{T}^n, q, \sigma)$ gives rise to a smooth trivial noncommutative principal $\mathbb{T}^n$-bundle of the form $(C^\infty(P), \mathbb{T}^n, \alpha)$. Conversely, if P is a manifold and $(C^\infty(P), \mathbb{T}^n, \alpha)$ a smooth trivial noncommutative principal $\mathbb{T}^n$-bundle, then we recover a trivial principal $\mathbb{T}^n$-bundle. While in classical (commutative) differential geometry there exists up to isomorphy only one trivial principal $\mathbb{T}^n$-bundle over a given manifold M, we will see that the situation completely changes in the noncommutative world. Moreover, it turns out that each trivial noncommutative principal $\mathbb{T}^n$-bundle possesses an underlying algebraic structure of a $\mathbb{Z}^n$-graded unital associative algebra, which might be thought of an algebraic counterpart of a trivial principal $\mathbb{T}^n$-bundle. In the second part of this paper we provide a complete classification of this underlying algebraic structure, i.e., we classify all possible trivial noncommutative principal $\mathbb{T}^n$-bundles up to completion.

Introduction. The correspondence between geometric spaces and commutative algebras is a familiar and basic idea of algebraic geometry. Noncommutative Topology started with the famous Gelfand–Naimark Theorems: Every commutative C*-algebra is the algebra of continuous functions vanishing at infinity on a locally compact space and vice versa. In particular, a noncommutative C*-algebra may be viewed as "the algebra of continu-

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ous functions vanishing at infinity" on a "quantum space". The aim of Noncommutative Geometry is to develop the basic concepts of Topology, Measure Theory and Differential Geometry in algebraic terms and then to generalize the corresponding classical results to the setting of noncommutative algebras. The question whether there is a way to translate the geometric concept of a fibre bundle to Noncommutative Geometry is quite interesting in this context. In the case of vector bundles a refined version of the Theorem of Serre and Swan ([Swa62]) gives the essential clue: The category of vector bundles over a manifold M is equivalent to the category of finitely generated projective modules over $C^\infty(M)$. It is therefore reasonable to consider finitely generated projective modules over an arbitrary algebra A as "noncommutative vector bundles". The case of principal bundles is so far not treated in a satisfactory way. From a geometrical point of view it is not sufficiently well understood what should be a "noncommutative principal bundle". However, there is a well-developed abstract algebraic approach using the theory of Hopf algebras. An important handicap of this approach is the ignorance of any topological and geometrical aspects. This paper is part of my thesis concerned with a geometric approach to noncommutative principal bundles. A natural first step towards a theory of "noncommutative principal $\mathbb{T}^n$-bundles" is to determine the *trivial* objects, i.e., to determine the trivial noncommutative principal $\mathbb{T}^n$-bundles.

In Section 1 we introduce the concept of trivial noncommutative principal $\mathbb{T}^n$-bundles. A dynamical system $(A, \mathbb{T}^n, \alpha)$ is called a trivial noncommutative principal $\mathbb{T}^n$-bundle if each isotypic component contains an invertible element. This definition is inspired by the following observation: A principal bundle $(P, M, \mathbb{T}^n, q, \sigma)$ is trivial if and only if it admits a trivialization map. Such a trivialization map consists basically of n smooth functions $f_i : P \to \mathbb{T}$ satisfying $f_i(\sigma(p, z)) = f_i(p) \cdot z_i$ for all $p \in P$ and $z \in \mathbb{T}^n$. From an algebraical point of view this condition means that each isotypic component of the (naturally) induced dynamical system $(C^\infty(P), \mathbb{T}^n, \alpha)$ contains an invertible element. Conversely, we show that each trivial noncommutative principal $\mathbb{T}^n$-bundle of the form $(C^\infty(P), \mathbb{T}^n, \alpha)$ induces a trivial principal $\mathbb{T}^n$-bundle of the form $(P, P/\mathbb{T}^n, \mathbb{T}^n, \mathrm{pr}, \sigma)$. The crucial point here is to verify the freeness of the induced action of $\mathbb{T}^n$ on P.

In Section 2 we present various examples including noncommutative tori, topological dynamical systems and certain crossed product constructions.

Section 3 to Section 7 are devoted to a complete classification of trivial noncommutative principal $\mathbb{T}^n$-bundles up to completion. In fact, it turns out that each trivial noncommutative principal $\mathbb{T}^n$-bundle possesses an underlying algebraic structure of a $\mathbb{Z}^n$-graded unital associative algebra. This structure may be considered as an algebraic counterpart of a trivial noncommutative principal $\mathbb{T}^n$-bundle and can be classified with methods from the extension theory of groups. We further present some nice examples of these algebraically trivial principal $\mathbb{T}^n$-bundles.

Finally, in Section 8 we provide an outlook to non-trivial noncommutative principal $\mathbb{T}^n$-bundles.

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Preliminaries and notation. All manifolds appearing in this paper are assumed to be finite-dimensional, paracompact, second countable and smooth. For the necessary background on principal bundles we refer to [KoNo63]. All algebras are assumed to be complex. If A is an algebra, we write

$$\Gamma_A := \mathrm{Hom}_{\mathrm{alg}}(A, \mathbb{C})\backslash\{0\}$$

(with the topology of pointwise convergence on A) for the spectrum of A. For elements of $\mathbb{Z}^n$ we write $\mathbf{k} = (k_1, \ldots, k_n)$ and think of them as multi-indices. In particular, we write $e_i = (0, \ldots, 1, \ldots, 0)$ for the canonical basis of $\mathbb{Z}^n$ and $\mathbf{0} = (0, \ldots, 0)$ for its unit element. A (smooth) dynamical system with transformation group $\mathbb{T}^n$, or simply a (smooth) dynamical system is a triple $(A, \mathbb{T}^n, \alpha)$, consisting of a unital locally convex algebra A, the n-torus $\mathbb{T}^n$ and a group homomorphism $\alpha : \mathbb{T}^n \to \mathrm{Aut}(A)$, which induces a (smooth) continuous action of $\mathbb{T}^n$ on A. We write

$$A_{\mathbf{k}} := \{a \in A : (\forall z \in \mathbb{T}^n) \quad \alpha(z).a = z^{\mathbf{k}} \cdot a\}$$

for the isotypic component of $(A, \mathbb{T}^n, \alpha)$ corresponding to $\mathbf{k} \in \mathbb{Z}^n$. In order to simplify the notation we write "NCP" for "noncommutative principal". We also use concepts of classical group cohomology: If G is a group we say that a map $f : (\mathbb{Z}^n)^p \to G$ is normalized if

$$(\exists j) \quad \mathbf{k_j} = 0 \Rightarrow f(\mathbf{k_1}, \ldots, \mathbf{k_p}) = 1_G$$

and write $C^p(\mathbb{Z}^n, G)$ for the space of all normalized maps $(\mathbb{Z}^n)^p \to G$, the so called p-cochains. For a detailed background on group cohomology we refer to [Ma95], Chapter IV.

Contents

1. Trivial NCP torus bundles. In this section we present a geometrically oriented approach to the noncommutative geometry of trivial principal $\mathbb{T}^n$-bundles. In particular, we will see that this approach perfectly reproduces the classical geometry of trivial principal $\mathbb{T}^n$-bundles.

DEFINITION 1.1 (Trivial NCP $\mathbb{T}^n$-bundles). A (smooth) dynamical system $(A, \mathbb{T}^n, \alpha)$ is called a *(smooth) trivial NCP $\mathbb{T}^n$-bundle*, if each isotypic component $A_{\mathbf{k}}$ contains an invertible element.

REMARK 1.2. Note that if $(A, \mathbb{T}^n, \alpha)$ is a dynamical system and $\mathbf{g_1}, \ldots, \mathbf{g_l} \in \mathbb{Z}^n$ a finite set of generators of $\mathbb{Z}^n$ such that each isotypic component $A_{\mathbf{g_j}}$ contains an invertible element, then $(A, \mathbb{T}^n, \alpha)$ is already a trivial NCP $\mathbb{T}^n$-bundle. In particular, if $1 \leq i \leq n$ and $A_i := A_{e_i}$, then $(A, \mathbb{T}^n, \alpha)$ is a trivial NCP $\mathbb{T}^n$-bundle if and only if each A_i contains an invertible element.

Proposition 1.3 (The underlying algebraic skeleton). *Let A be a complete unital locally convex algebra and $(A, \mathbb{T}^n, \alpha)$ a trivial NCP $\mathbb{T}^n$-bundle. If $a_{\mathbf{k}} \in A_{\mathbf{k}}$ is an invertible element for each $\mathbf{k} \in \mathbb{Z}^n$ and $B := A_{\mathbf{0}}$, then the following assertions hold:*

(a) *Each map*

$$\phi_{\mathbf{k}} : B \to A_{\mathbf{k}}, \quad b \mapsto a_{\mathbf{k}} b$$

is an isomorphism of locally convex B-modules. In particular, $A_{\mathbf{k}} = a_{\mathbf{k}} B$ for each $\mathbf{k} \in \mathbb{Z}^n$.

(b) *The space*

$$A_d = \bigoplus_{\mathbf{k} \in \mathbb{Z}^n} A_{\mathbf{k}} = \bigoplus_{\mathbf{k} \in \mathbb{Z}^n} a_{\mathbf{k}} B$$

is a dense $\mathbb{T}^n$-invariant subalgebra of A.

Proof. (a) An easy calculation shows that each map $\phi_{\mathbf{k}}$ is a morphism of locally convex B-modules, and therefore the statement follows from the fact that $a_{\mathbf{k}} \in A_{\mathbf{k}}$ is invertible.

(b) The space A_d is obviously $\mathbb{T}^n$-invariant by construction. To see that it is a subalgebra, take $\mathbf{k}, \mathbf{l} \in \mathbb{Z}^n$ and choose $a_{\mathbf{k}} \in A_{\mathbf{k}}$ and $a_{\mathbf{l}} \in A_{\mathbf{l}}$. Then

$$\alpha(z).(a_{\mathbf{k}} a_{\mathbf{l}}) = (\alpha(z).a_{\mathbf{k}})(\alpha(z).a_{\mathbf{l}}) = (z^{\mathbf{k}+\mathbf{l}}) \cdot a_{\mathbf{k}} a_{\mathbf{l}}$$

for all $z \in \mathbb{T}^n$ and we therefore conclude that $a_{\mathbf{k}} a_{\mathbf{l}} \in A_{\mathbf{k}+\mathbf{l}} \subseteq A_d$. The density statement is a consequence of the Big Peter and Weyl Theorem for Compact Abelian Groups (cf. [HoMo06], Theorem 3.51 and Theorem 4.22). ■

Proposition 1.4. *If A is a commutative unital locally convex algebra and $(A, \mathbb{T}^n, \alpha)$ a trivial NCP $\mathbb{T}^n$-bundle, then the map*

$$\sigma : \Gamma_A \times \mathbb{T}^n \to \Gamma_A, \quad \chi.z := \sigma(\chi, z) := \chi \circ \alpha(z)$$

defines a free action of $\mathbb{T}^n$ on the spectrum Γ_A of A.

Proof. An easy observation shows that σ defines an action of $\mathbb{T}^n$ on the spectrum Γ_A. The crucial part of the proof is to verify the freeness of the map σ, i.e., to show that the stabilizer of each element of Γ_A is trivial. For this, first choose in each isotypic component $A_{\mathbf{k}}$ an invertible element $a_{\mathbf{k}}$. Now, let $\chi \in \Gamma_A$ and $z \in \mathbb{T}^n$ such that $\chi.z = \chi \circ \alpha(z) = \chi$. Then

$$(\chi \circ \alpha(z))(a_{\mathbf{k}}) = \chi(\alpha(z).a_{\mathbf{k}}) = z^{\mathbf{k}} \cdot \chi(a_{\mathbf{k}}) = \chi(a_{\mathbf{k}})$$

implies that $z^{\mathbf{k}} = 1$ holds for all $\mathbf{k} \in \mathbb{Z}^n$ since each element $a_{\mathbf{k}}$ is invertible. We thus conclude that $z = (1, \ldots, 1)$, which proves the freeness of the map σ. ■

Remark 1.5. If P is a manifold, $p \in P$ and δ_p the corresponding point evaluation map on $C^\infty(P)$, then there is a unique smooth structure on the spectrum $\Gamma_{C^\infty(P)}$ of $C^\infty(P)$ for which the map

$$\Phi : P \to \Gamma_{C^\infty(P)}, \quad p \mapsto \delta_p,$$

becomes a diffeomorphism. A proof of this statement can be found in [Wa11], Lemma 6.1.5.

PROPOSITION 1.6. *If P is a manifold and $(C^\infty(P), \mathbb{T}^n, \alpha)$ a smooth trivial NCP $\mathbb{T}^n$-bundle, then the map*

$$\sigma : P \times \mathbb{T}^n \to P, \quad (\delta_p, z) \mapsto \delta_p \circ \alpha(z),$$

where we have identified P with the set of characters via the map Φ from Remark 1.5, *is smooth and defines a free and proper action of $\mathbb{T}^n$ on the manifold P. In particular, we obtain a principal bundle $(P, P/\mathbb{T}^n, \mathbb{T}^n, \mathrm{pr}, \sigma)$.*

Proof. The smoothness of the map σ is a consequence of [Wa11], Proposition 6.1.6 and its freeness follows directly from Proposition 1.4. The properness of σ is automatic, since $\mathbb{T}^n$ is compact. Finally, the Quotient Theorem implies that we obtain a principal bundle $(P, P/\mathbb{T}^n, \mathbb{T}^n, \mathrm{pr}, \sigma)$ (cf. [tD00], Kapitel VIII, Satz 21.6). ∎

LEMMA 1.7. *If P is a manifold and $\phi : C^\infty(P) \to C^\infty(P)$ an algebra automorphism, then there is a diffeomorphism $\tau : P \to P$ such that $\phi(f) := f \circ \tau^{-1}$ for all $f \in C^\infty(P)$.*

Proof. This statement follows from [Gue08], Lemma 2.95. ∎

REMARK 1.8 (*-automorphisms). In view of the previous lemma each algebra automorphism of $C^\infty(P)$ is automatically a *-automorphism.

PROPOSITION 1.9. *Let P be a manifold and $(C^\infty(P), \mathbb{T}^n, \alpha)$ a dynamical system. If $\mathbf{k} \in \mathbb{Z}^n$ and $f \in C^\infty(P)_{\mathbf{k}}$ is invertible, then $|f|$ is invariant under the action of $\mathbb{T}^n$, i.e.,*

$$\alpha(z).|f| = |f| \quad \textit{for all} \quad z \in \mathbb{T}^n.$$

Proof. If $z \in \mathbb{T}^n$, then $\alpha(z).f = z^{\mathbf{k}} \cdot f$. On the other hand, Lemma 1.7 implies that there exists a diffeomorphism $\tau_z : P \to P$ such that $\alpha(z).g = g \circ \tau_z^{-1}$ for all $g \in C^\infty(P)$. Hence, we get

$$\alpha(z).|f| = |\cdot| \circ f \circ \tau_z^{-1} = |z^{\mathbf{k}} \cdot f| = |f|. \ ∎$$

REMARK 1.10. Each principal bundle $(P, M, \mathbb{T}^n, q, \sigma)$ induces a smooth dynamical system $(C^\infty(P), \mathbb{T}^n, \alpha)$, where the smooth action of $\mathbb{T}^n$ on $C^\infty(P)$ is given by

$$\alpha : \mathbb{T}^n \times C^\infty(P) \to C^\infty(P), \quad \alpha(z, f)(p) := (z.f)(p) := f(\sigma(p, z)).$$

In fact, a proof for the smoothness of the map α can be found in [Wa11], Proposition 6.1.1.

We now come to the main theorem of this section.

THEOREM 1.11 (Trivial principal $\mathbb{T}^n$-bundles). *Let P be a manifold. Then the following assertions hold:*

(a) *If $(C^\infty(P), \mathbb{T}^n, \alpha)$ is a smooth trivial NCP $\mathbb{T}^n$-bundle, then the corresponding principal bundle $(P, P/\mathbb{T}^n, \mathbb{T}^n, \mathrm{pr}, \sigma)$ of Proposition* 1.6 *is trivial.*
(b) *Conversely, if $(P, M, \mathbb{T}^n, q, \sigma)$ is a trivial principal $\mathbb{T}^n$-bundle, then the corresponding smooth dynamical system of Remark* 1.10 *$(C^\infty(P), \mathbb{T}^n, \alpha)$ is a trivial NCP $\mathbb{T}^n$-bundle.*

Proof. (a) In view of Remark 1.2, we may choose for each $1 \leq i \leq n$ an invertible function $f_i \in C^\infty(P)_i$ with $\mathrm{im}(f_i) \subseteq \mathbb{T}$. Indeed, if $f_i \in C^\infty(P)_i$ is invertible, then the function

$$g_i : P \to \mathbb{C}, \quad p \mapsto \frac{f_i(p)}{|f_i(p)|},$$

is also invertible and satisfies $\mathrm{im}(g_i) \subseteq \mathbb{T}$. Moreover, Proposition 1.9 implies that $\alpha(z).g_i = z_i \cdot g_i$ holds for all $z \in \mathbb{T}^n$ and therefore that $g_i \in C^\infty(P)_i$. Next, we consider the map

$$\varphi : P \to P/\mathbb{T}^n \times \mathbb{T}^n, \quad p \mapsto (\mathrm{pr}(p), f_1(p), \ldots, f_n(p)).$$

Since $\varphi(p.z) = \varphi(p) \cdot z$ for all $z \in \mathbb{T}^n$, the map φ defines an equivalence of principal $\mathbb{T}^n$-bundles over $P/\mathbb{T}^n$. Thus, the principal bundle $(P, P/\mathbb{T}^n, \mathbb{T}^n, \mathrm{pr}, \sigma)$ of Proposition 1.6 is trivial.

(b) Conversely, let $(P, M, \mathbb{T}^n, q, \sigma)$ be a trivial principal $\mathbb{T}^n$-bundle and

$$\varphi : P \to M \times \mathbb{T}^n, \quad p \mapsto (q(p), f_1(p), \ldots, f_n(p)),$$

be an equivalence of principal $\mathbb{T}^n$-bundles over M. We first note that each function $f_i \in C^\infty(P)$ is invertible. Furthermore, the $\mathbb{T}^n$-equivariance of φ implies that $f_i \in C^\infty(P)_i$ for all $i \in I_n$. Therefore each isotypic component $C^\infty(P)_i$ contains invertible elements, and we conclude from Remark 1.2 that $(C^\infty(P), \mathbb{T}^n, \alpha)$ is a trivial NCP $\mathbb{T}^n$-bundle. ∎

REMARK 1.12. Note that Theorem 1.11 remains valid in the topological category.

2. Examples of trivial NCP torus bundles. In this part of the paper we present a bunch of examples of (smooth) trivial NCP $\mathbb{T}^n$-bundles.

EXAMPLE 2.1 (Noncommutative n-tori). Let θ be a real skew-symmetric $n \times n$ matrix. The *noncommutative n-torus* A^n_θ is the universal unital C*-algebra generated by unitaries $U_1, \ldots, U_n$ with

$$U_r U_s = \exp(2\pi i \theta_{rs}) U_s U_r \quad \text{for all} \quad 1 \leq r, s \leq n.$$

Moreover, there is a continuous action α of $\mathbb{T}^n$ on A^n_θ by algebra automorphisms, which is on generators given by

$$\alpha(t).U^{\mathbf{k}} := t.U^{\mathbf{k}} := t^{\mathbf{k}} \cdot U^{\mathbf{k}} \quad \text{for} \quad \mathbf{k} \in \mathbb{Z}^n,$$

where

$$U^{\mathbf{k}} := U_1^{k_1} \cdots U_n^{k_n}.$$

In particular, $(A^n_\theta)_{\mathbf{k}} = \mathbb{C} \cdot U_{\mathbf{k}}$ shows that the triple $(A^n_\theta, \mathbb{T}^n, \alpha)$ is a trivial NCP $\mathbb{T}^n$-bundle.

EXAMPLE 2.2 (Smooth noncommutative n-tori). The *smooth noncommutative n-torus* $\mathbb{T}^n_\theta$ is the unital subalgebra of smooth vectors for the action α of the previous example. Its elements are given by (norm-convergent) sums

$$a = \sum_{\mathbf{k} \in \mathbb{Z}^n} a_{\mathbf{k}} U^{\mathbf{k}}, \quad \text{with} \quad (a_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^n} \in S(\mathbb{Z}^n).$$

Further, a deeper analysis shows that the induced action of $\mathbb{T}^n$ on $\mathbb{T}^n_\theta$ is smooth. Thus, the triple $(\mathbb{T}^n_\theta, \mathbb{T}^n, \alpha)$ is a smooth trivial NCP $\mathbb{T}^n$-bundle. For details we refer to [Wa11], Appendix E.

REMARK 2.3. Suppose we are in the situation of Example 2.2. If $\theta = 0$, then we get the trivial NCP $\mathbb{T}^n$-bundle $(C^\infty(\mathbb{T}^n), \mathbb{T}^n, \alpha)$. The corresponding trivial principal bundle of Theorem 1.11 (a) is the (trivial) principal $\mathbb{T}^n$-bundle over a single point $\{*\}$, i.e.,

$$(\mathbb{T}^n, \{*\}, \mathbb{T}^n, q, \sigma_{\mathbb{T}^n})$$

for

$$q : \mathbb{T}^n \to \{*\}, \quad z \mapsto *.$$

Therefore, one should think of noncommutative n-tori as deformations of the trivial principal $\mathbb{T}^n$-bundle over a single point.

CONSTRUCTION 2.4 (ℓ^1-crossed products). Let $(A, \|\cdot\|,^*)$ be an involutive Banach algebra and $(A, \mathbb{Z}^n, \alpha)$ a dynamical system. Note that this means that $\mathbb{Z}^n$ acts by isometries of A. We write $F(\mathbb{Z}^n, A)$ for the vector space of functions $f : \mathbb{Z}^n \to A$ with finite support and define a multiplication on this space by

$$(f \star g)(\mathbf{k}) := \sum_{\mathbf{l} \in \mathbb{Z}^n} f(\mathbf{l})\alpha(\mathbf{l}, g(\mathbf{k} - \mathbf{l})).$$

Moreover, an involution is given by

$$f^*(\mathbf{k}) := \alpha(\mathbf{k}, (f(-\mathbf{k}))^*).$$

These two operations are continuous for the L^1-norm

$$\|f\|_1 := \sum_{\mathbf{k} \in \mathbb{Z}^n} \|f(\mathbf{k})\|,$$

and the completion of $F(\mathbb{Z}^n, A)$ in this norm is again an involutive Banach algebra denoted by $\ell^1(A \rtimes_\alpha \mathbb{Z}^n)$.

REMARK 2.5. If $A = \mathbb{C}$, then $\ell^1(A \rtimes_\alpha \mathbb{Z}^n)$ is just the algebra $\ell^1(\mathbb{Z}^n)$.

LEMMA 2.6. *If $(A, \|\cdot\|,^*)$ is an involutive Banach algebra and $(A, \mathbb{Z}^n, \alpha)$ a dynamical system, then the map*

$$\widehat{\alpha} : \mathbb{T}^n \times \ell^1(A \rtimes_\alpha \mathbb{Z}^n) \to \ell^1(A \rtimes_\alpha \mathbb{Z}^n), \quad (\widehat{\alpha}(z, f))(\mathbf{k}) := (z.f)(\mathbf{k}) := z^{\mathbf{k}} \cdot f(\mathbf{k})$$

defines a continuous action of $\mathbb{T}^n$ on $\ell^1(A \rtimes_\alpha \mathbb{Z}^n)$ by algebra automorphisms.

Proof. Obviously $\widehat{\alpha}$ defines an action. Moreover,

$$\begin{aligned}((z.f) \star (z.g))(\mathbf{k}) &= \sum_{\mathbf{l} \in \mathbb{Z}^n} ((z.f)(\mathbf{l}))(\alpha(\mathbf{l}, (z.g)(\mathbf{k} - \mathbf{l}))) \\ &= \sum_{\mathbf{l} \in \mathbb{Z}^n} (z^{\mathbf{l}} \cdot f(\mathbf{l}))(z^{\mathbf{k}-\mathbf{l}} \cdot \alpha(\mathbf{l}, g(\mathbf{k} - \mathbf{l}))) \\ &= z^{\mathbf{k}} \cdot \sum_{\mathbf{l} \in \mathbb{Z}^n} f(\mathbf{l})\alpha(\mathbf{l}, g(\mathbf{k} - \mathbf{l})) = (z.(f * g))(\mathbf{k}),\end{aligned}$$

$$\|z.f\|_1 = \sum_{\mathbf{k} \in \mathbb{Z}^n} \|(z.f)(\mathbf{k})\| = \sum_{\mathbf{k} \in \mathbb{Z}^n} \|z^{\mathbf{k}} \cdot f(\mathbf{k})\| = \sum_{\mathbf{k} \in \mathbb{Z}^n} \|f(\mathbf{k})\| = \|f\|_1$$

and

$$((z.f)^*)(\mathbf{k}) = \alpha(\mathbf{k}, ((z.f)(-\mathbf{k}))^*) = z^{\mathbf{k}} \cdot \alpha(\mathbf{k}, (f(-\mathbf{k}))^*) = (z.f^*)(\mathbf{k})$$

show that each element $z \in \mathbb{T}^n$ acts as an automorphism of $(A, \|\cdot\|,^*)$.

To see the continuity of $\widehat{\alpha}$, we choose $f \in F(\mathbb{Z}^n, A)$, $\epsilon > 0$ and a neighbourhood U of the unity of $\mathbb{T}^n$ such that $|z^{\mathbf{k}} - 1| \leq \epsilon$ for all $\mathbf{k} \in \operatorname{supp}(f)$ and $z \in U$. We therefore obtain

$$\|z.f - f\|_1 = \sum_{\mathbf{k} \in \mathbb{Z}^n} |z^{\mathbf{k}} - 1| \, \|f(\mathbf{k})\| = \sum_{\mathbf{k} \in \operatorname{supp}(f)} |z^{\mathbf{k}} - 1| \, \|f(\mathbf{k})\| \leq \epsilon \|f\|_1$$

for all $z \in U$. Since $F(\mathbb{Z}^n, A)$ is dense in $\ell^1(A \rtimes_\alpha \mathbb{Z}^n)$, $\widehat{\alpha}$ defines a continuous action of $\mathbb{T}^n$ on $\ell^1(A \rtimes_\alpha \mathbb{Z}^n)$ by algebra automorphisms. ■

PROPOSITION 2.7. *If* $(A, \|\cdot\|, ^*)$ *is an involutive Banach algebra and* $(A, \mathbb{Z}^n, \alpha)$ *a dynamical system, then the triple*

$$(\ell^1(A \rtimes_\alpha \mathbb{Z}^n), \mathbb{T}^n, \widehat{\alpha})$$

defines a dynamical system.

Proof. The claim is a direct consequence of Lemma 2.6. ■

EXAMPLE 2.8. If $(A, \|\cdot\|, ^*)$ is an involutive Banach algebra and $(A, \mathbb{Z}^n, \alpha)$ a dynamical system, then the dynamical system $(\ell^1(A \rtimes_\alpha \mathbb{Z}^n), \mathbb{T}^n, \widehat{\alpha})$ is a trivial NCP $\mathbb{T}^n$-bundle. Indeed, for $\mathbf{k} \in \mathbb{Z}^n$ define

$$\delta_{\mathbf{k}}(\mathbf{l}) := \begin{cases} 1_A & \text{for} \quad \mathbf{l} = \mathbf{k} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$z.\delta_{\mathbf{k}} = z^{\mathbf{k}} \cdot \delta_{\mathbf{k}} \quad \text{and} \quad \delta_{\mathbf{k}} \star \delta_{\mathbf{k}}^* = \delta_{\mathbf{k}}^* \star \delta_{\mathbf{k}} = \mathbf{1}$$

show that $\delta_{\mathbf{k}}$ is an invertible element of $\ell^1(A \rtimes_\alpha \mathbb{Z}^n)$ lying in the isotypic component $\ell^1(A \rtimes_\alpha \mathbb{Z}^n)_{\mathbf{k}}$.

CONSTRUCTION 2.9 (The enveloping C^*-algebra). If $(A, \|\cdot\|, ^*)$ is an involutive Banach algebra, then any involutive representation $(\pi, \mathcal{H})$ of A, for some Hilbert space $\mathcal{H}$, satisfies

$$\|\pi(a)\|_{\text{op}} \leq \|a\|.$$

This follows from the fact π is norm-decreasing since it shrinks spectra and $B(\mathcal{H})$ is a C^*-algebra. The supremum over all such involutive representations $(\pi, \mathcal{H})$ is bounded, i.e.,

$$\|a\|_{\sup} = \sup_{(\pi, \mathcal{H})} \|\pi(a)\|_{\text{op}} \leq \|a\|,$$

and thus defines a seminorm on A. If this is not already a norm, we divide A by its kernel to get a normed algebra. Since $\|\pi(a^*a)\|_{\text{op}} = \|\pi(a)\|_{\text{op}}^2$ for each $(\pi, \mathcal{H})$, this is a C^*-norm. The completion of A in this norm is a C^*-algebra and called the *enveloping* C^**-algebra.*

REMARK 2.10 (C^*-crossed products). If $(A, \|\cdot\|, ^*)$ is an involutive Banach algebra and $(A, \mathbb{Z}^n, \alpha)$ a dynamical system, then the enveloping C^*-algebra of $\ell^1(A \rtimes_\alpha \mathbb{Z}^n)$ is denoted by $C^*(A \rtimes_\alpha \mathbb{Z}^n)$ and is called the C^**-crossed product* associated to $(A, \mathbb{Z}^n, \alpha)$.

EXAMPLE 2.11. If $(A, \|\cdot\|, ^*)$ is an involutive Banach algebra and $(A, \mathbb{Z}^n, \alpha)$ a dynamical system, then the action $\widehat{\alpha}$ of Lemma 2.6 extends to a continuous action of $\mathbb{T}^n$ on the C^*-crossed product $C^*(A \rtimes_\alpha \mathbb{Z}^n)$ by algebra automorphisms. For details we refer to the paper [Ta74]. In particular, the corresponding dynamical system

$$(C^*(A \rtimes_\alpha \mathbb{Z}^n), \mathbb{T}^n, \widehat{\alpha})$$

is a trivial NCP $\mathbb{T}^n$-bundle. This follows exactly as in Example 2.8.

EXAMPLE 2.12 (Topological dynamical systems). To each topological dynamical system (X, φ), i.e., to each pair (X, φ), consisting of a compact Hausdorff space X and a homeomorphism $\varphi : X \to X$, one can associate a C^*-dynamical system. Indeed, choose $A = C(X)$ and define an action α of $\mathbb{Z}$ on $C(X)$ by

$$(\alpha(k).f)(x) := f(\varphi^{-k}(x)).$$

By Example 2.11, the associated C^*-crossed product $C^*(C(X) \rtimes_\alpha \mathbb{Z})$ is a trivial NCP $\mathbb{T}$-bundle.

REMARK 2.13 (2-cocycles). A 2-cocycle on $\mathbb{Z}^n$ with values in $\mathbb{T}$ is a map $\omega : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{T}$ satisfying $\omega(\mathbf{0}, \mathbf{0}) = 1$ and

$$\omega(\mathbf{k}, \mathbf{l})\omega(\mathbf{k} + \mathbf{l}, \mathbf{m}) = \omega(\mathbf{k}, \mathbf{l} + \mathbf{m})\omega(\mathbf{l}, \mathbf{m})$$

for all $\mathbf{k}, \mathbf{l}, \mathbf{m} \in \mathbb{Z}^n$. We write $Z^2(\mathbb{Z}^n, \mathbb{T})$ for the space of all 2-cocycle on $\mathbb{Z}^n$ with values in $\mathbb{T}$.

CONSTRUCTION 2.14 (ℓ^1-spaces associated to 2-cocycles). Let $(A, \| \cdot \|, ^*)$ be an involutive Banach algebra and ω a 2-cocycle in $Z^2(\mathbb{Z}^n, \mathbb{T})$. The involutive Banach algebra $\ell^1(A \times_\omega \mathbb{Z}^n)$ is defined by introducing a twisted multiplication

$$(f \star g)(\mathbf{k}) := \sum_{\mathbf{l} \in \mathbb{Z}^n} f(\mathbf{l})g(\mathbf{k} - \mathbf{l})\omega(\mathbf{l}, \mathbf{k} - \mathbf{l})$$

and an involution

$$(f^*)(\mathbf{k}) := \overline{\omega(\mathbf{k}, -\mathbf{k})} \cdot f(-\mathbf{k}).$$

The cocycle property ensures that the multiplication is associative.

EXAMPLE 2.15. Let $(A, \| \cdot \|, ^*)$ be an involutive Banach algebra and ω a 2-cocycle in $Z^2(\mathbb{Z}^n, \mathbb{T})$. Similarly to Lemma 2.6, we see that the map

$$\widehat{\alpha} : \mathbb{T}^n \times \ell^1(A \times_\omega \mathbb{Z}^n) \to \ell^1(A \times_\omega \mathbb{Z}^n), \quad \widehat{\alpha}(z, f))(\mathbf{k}) := (z.f)(\mathbf{k}) := z^{\mathbf{k}} \cdot f(\mathbf{k}),$$

defines a continuous action of $\mathbb{T}^n$ on $\ell^1(A \times_\omega \mathbb{Z}^n)$ by algebra automorphisms. Moreover, the corresponding dynamical system

$$(\ell^1(A \times_\omega \mathbb{Z}^n), \mathbb{T}^n, \widehat{\alpha})$$

turns out to be a trivial NCP $\mathbb{T}^n$-bundle (cf. Example 2.8).

REMARK 2.16. Let $\theta \in \mathrm{Alt}^2(\mathbb{Z}^n, \mathbb{R})$ be a skew-symmetric real matrix and consider the 2-cocycle

$$\omega : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{T}, \quad \omega(\mathbf{k}, \mathbf{l}) := \exp(i\theta(\mathbf{k}, \mathbf{l})).$$

Then we obtain a (2-step nilpotent) Lie group $H := \mathbb{T} \times_\omega \mathbb{Z}^n$ which is a central extension of $\mathbb{Z}^n$ by the circle group $\mathbb{T}$. The circle $\mathbb{T}$ acts continuously on the group algebra $L^1(H)$ by translations in the first argument. The corresponding Fourier decomposition leads to

$$L^1(H)_{\mathrm{d}} = \bigoplus_{\mathbf{k} \in \mathbb{Z}} L^1(H)_{\mathbf{k}} \cong \bigoplus_{\mathbf{k} \in \mathbb{Z}} \chi_{\mathbf{k}} \cdot \ell^1(\mathbb{C} \times_\omega \mathbb{Z}^n),$$

where $\ell^1(\mathbb{C} \times_\omega \mathbb{Z}^n)$ denotes a ℓ^1-version of Example 2.1.

Example 2.17. The enveloping C^*-algebra of $\ell^1(A \times_\omega \mathbb{Z}^n)$ is denoted by $C^*(A \times_\omega \mathbb{Z}^n)$ and is called the *twisted group C^*-algebra of G by ω*. The action $\widehat{\alpha}$ of Example 2.15 extends to a continuous action of $\mathbb{T}^n$ on $C^*(A \times_\omega \mathbb{Z}^n)$ by algebra automorphisms (cf. Example 2.11). The corresponding dynamical system

$$(C^*(A \times_\omega \mathbb{Z}^n), \mathbb{T}^n, \widehat{\alpha})$$

is a trivial NCP $\mathbb{T}^n$-bundle as well.

3. Factor systems for trivial NCP torus bundles. In this short section we introduce a "cohomology theory" for trivial NCP $\mathbb{T}^n$-bundles, which is inspired by the classical cohomology theory of groups. The corresponding cohomology spaces will be crucial for the classification part of this paper.

Definition 3.1. Let $n \in \mathbb{N}$ and B be a unital algebra.

(a) We write $C_B : B^\times \to \mathrm{Aut}(B)$ for the *conjugation action* of $B^\times$ on B.

(b) We call a map $S \in C^1(\mathbb{Z}^n, \mathrm{Aut}(B))$ an *outer action* of $\mathbb{Z}^n$ on B if there exists

$$\omega \in C^2(\mathbb{Z}^n, B^\times) \quad \text{with} \quad \delta_S = C_B \circ \omega,$$

where

$$\delta_S(\mathbf{k}, \mathbf{l}) := S(\mathbf{k})S(\mathbf{l})S(\mathbf{k}+\mathbf{l})^{-1}.$$

(c) On the set of outer actions we define an equivalence relation by

$$S \sim S' \quad \Leftrightarrow \quad (\exists h \in C^1(\mathbb{Z}^n, B^\times))\ S' = (C_B \circ h) \cdot S$$

and call the equivalence class $[S]$ of an outer action S a *$\mathbb{Z}^n$-kernel*.

(d) For $S \in C^1(\mathbb{Z}^n, \mathrm{Aut}(B))$ and $\omega \in C^2(\mathbb{Z}^n, B^\times)$ let

$$(d_S\omega)(\mathbf{k}, \mathbf{l}, \mathbf{m}) := S(\mathbf{k})(\omega(\mathbf{l}, \mathbf{m}))\omega(\mathbf{k}, \mathbf{l}+\mathbf{m})\omega(\mathbf{k}+\mathbf{l}, \mathbf{m})^{-1}\omega(\mathbf{k}, \mathbf{l})^{-1}.$$

Lemma 3.2. *Let $n \in \mathbb{N}$ and B be a unital algebra and consider the group $C^1(\mathbb{Z}^n, B^\times)$ with respect to pointwise multiplication. This group acts on the set*

$$C^1(\mathbb{Z}^n, \mathrm{Aut}(B)) \quad by \quad h.S := (C_B \circ h) \cdot S$$

and on the product set

$$C^1(\mathbb{Z}^n, \mathrm{Aut}(B)) \times C^2(\mathbb{Z}^n, B^\times) \quad by \quad h.(S, \omega) := (h.S, h *_S \omega)$$

for

$$(h *_S \omega)(\mathbf{k}, \mathbf{l}) := h(\mathbf{k})S(\mathbf{k})(h(\mathbf{l}))\omega(\mathbf{k}, \mathbf{l})h(\mathbf{k}+\mathbf{l})^{-1}.$$

The stabilizer of (S, ω) is given by

$$C^1(\mathbb{Z}^n, B^\times)_{(S,\omega)} = Z^1(\mathbb{Z}^n, Z(B)^\times)_S$$

which depends only on $[S]$, but not on ω, and the following assertions hold:

(a) *The subset*

$$\{(S, \omega) \in C^1(\mathbb{Z}^n, \mathrm{Aut}(B)) \times C^2(\mathbb{Z}^n, B^\times) : \delta_S = C_B \circ \omega\}$$

is invariant.

(b) *If* $\delta_S = C_B \circ \omega$*, then* $\operatorname{im}(d_S\omega) \subseteq Z(B)^\times$.

(c) *If* $\delta_S = C_B \circ \omega$ *and* $h.(S.\omega) = (S', \omega')$*, then* $d_{S'}\omega' = d_S\omega$.

Proof. A proof of this Lemma can be found in [Wa11], Lemma 7.3.2. ■

DEFINITION 3.3. Let $n \in \mathbb{N}$ and B be a unital algebra. The elements of the set

$$Z^2(\mathbb{Z}^n, B) := \{(S, \omega) \in C^1(\mathbb{Z}^n, \operatorname{Aut}(B)) \times C^2(\mathbb{Z}^n, B^\times) : \delta_S = C_B \circ \omega, d_S\omega = 1\}$$

are called *factor systems* for the pair $(\mathbb{Z}^n, B)$ (or simply (n, B)). By Lemma 3.2, the set $Z^2(\mathbb{Z}^n, B)$ is invariant under the action of $C^1(\mathbb{Z}^n, B^\times)$ and we write

$$H^2(\mathbb{Z}^n, B) := Z^2(\mathbb{Z}^n, B)/C^1(\mathbb{Z}^n, B^\times)$$

for the corresponding cohomology space.

4. Classification of algebraically trivial NCP torus bundles. The main goal of this section is to present a complete classification of the underlying algebraic structure of a trivial NCP $\mathbb{T}^n$-bundle, i.e., to classify all possible trivial NCP $\mathbb{T}^n$-bundles up to completion. First, Proposition 1.3 leads to the following definition.

DEFINITION 4.1 (Algebraically trivial NCP $\mathbb{T}^n$-bundles). A $\mathbb{Z}^n$-graded unital associative algebra

$$A = \bigoplus_{\mathbf{k} \in \mathbb{Z}^n} A_{\mathbf{k}}$$

with $B := A_{\mathbf{0}}$ is called an *algebraically trivial NCP* $\mathbb{T}^n$*-bundle with base* B, if each grading space $A_{\mathbf{k}}$ contains an invertible element.

We now provide a construction that associates to each algebraically trivial NCP $\mathbb{T}^n$-bundle A a class in $H^2(\mathbb{Z}^n, B)$ for $B = A_{\mathbf{0}}$.

CONSTRUCTION 4.2 (Characteristic classes). Let $(A, \mathbb{T}^n, \alpha)$ be a trivial NCP $\mathbb{T}^n$-bundle. The set

$$A_h^\times := \bigcup_{\mathbf{k} \in \mathbb{Z}^n} A_{\mathbf{k}}^\times$$

of homogeneous units is a subgroup of $A^\times$ containing $B^\times \cdot 1_A \cong B^\times$. We thus obtain an extension

$$1 \longrightarrow B^\times \longrightarrow A_h^\times \longrightarrow \mathbb{Z}^n \longrightarrow 1$$

of groups. In particular, $A_h^\times$ is equivalent to a crossed product of the form $B^\times \times_{(S,\omega)} \mathbb{Z}^n$ for a factor system $(S, \omega) \in Z^2(\mathbb{Z}^n, B)$. In this way each algebraically trivial NCP $\mathbb{T}^n$-bundle A induces a *characteristic class*

$$\chi(A) := [(S, \omega)] \in H^2(\mathbb{Z}^n, B).$$

LEMMA 4.3. *Each algebraically trivial NCP* $\mathbb{T}^n$*-bundle* A *possesses a characteristic class* $\chi(A) \in H^2(\mathbb{Z}^n, B)$.

Proof. Indeed, this statement immediately follows from Construction 4.2. ■

Definition 4.4. Two algebraically trivial NCP $\mathbb{T}^n$-bundles A and A' with base B are called *equivalent* if there is an algebra isomorphism $\varphi : A \to A'$ satisfying $\varphi(A_{\mathbf{k}}) = A'_{\mathbf{k}}$ for all $\mathbf{k} \in \mathbb{Z}^n$. If A and A' are equivalent algebraically trivial NCP $\mathbb{T}^n$-bundles, then we write $[A]$ for the corresponding equivalence class.

Proposition 4.5. *Let A and A' be two equivalent algebraically trivial NCP $\mathbb{T}^n$-bundles. Then their corresponding characteristic classes coincide, i.e.,*

$$\chi(A) = \chi(A') \in H^2(\mathbb{Z}^n, B).$$

Proof. If A and A' are equivalent, then the same holds for their corresponding extensions of Construction 4.2. Thus, the claim follows from [Ma95], Chapter IV, Section 4. ■

Definition 4.6 (Set of equivalence classes). Let $n \in \mathbb{N}$ and B be a unital algebra. We write $\mathrm{Ext}(\mathbb{Z}^n, B)$ for the set of all equivalence classes of algebraically trivial NCP $\mathbb{T}^n$-bundles with base B.

Lemma 4.7. *Let B be a unital algebra. Then the map*

$$\chi : \mathrm{Ext}(\mathbb{Z}^n, B) \to H^2(\mathbb{Z}^n, B), \quad [A] \mapsto \chi(A)$$

is well-defined.

Proof. The statement immediately follows from Proposition 4.5. ■

In the remaining part of this section we show that the map χ is a bijection.

Construction 4.8. Let $n \in \mathbb{N}$ and B be a unital algebra. Further, let

$$A := \bigoplus_{\mathbf{k} \in \mathbb{Z}^n} B v_{\mathbf{k}}$$

be a vector space with basis $(v_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^n}$. For a factor system $(S, \omega) \in Z^2(\mathbb{Z}^n, B)$, we define a multiplication map

$$m_{(S,\omega)} : A \times A \to A$$

given on homogeneous elements by

$$m_{(S,\omega)}(b v_{\mathbf{k}}, b' v_{\mathbf{l}}) := b(S(\mathbf{k})(b'))\omega(\mathbf{k}, \mathbf{l}) v_{\mathbf{k}+\mathbf{l}}, \tag{1}$$

and write $A_{(S,\omega)}$ for the vector space A endowed with the multiplication (1). A short calculation shows that $A_{(S,\omega)}$ is a $\mathbb{Z}^n$-graded unital associative algebra with $A_{\mathbf{0}} = B$ and unit $v_{\mathbf{0}}$. Moreover, each grading space A_i, $i \in I_n$, contains invertible elements with respect to to this multiplication. Indeed, if $i \in I_n$ and $b \in B^\times$, then the inverse of bv_i is given by

$$S(i)^{-1}(b^{-1}\omega(i, -i)^{-1}) v_{-i}.$$

Thus, $A_{(S,\omega)}$ is an algebraically trivial NCP $\mathbb{T}^n$-bundle with base B and characteristic class $\chi(A_{(S,\omega)}) = [(S, \omega)]$ (cf. Remark 1.2).

Proposition 4.9. *If $n \in \mathbb{N}$ and B is a unital algebra, then each element $[(S, \omega)] \in H^2(\mathbb{Z}^n, B)$ can be realized by an algebraically trivial NCP $\mathbb{T}^n$-bundles A with*

$$A_{\mathbf{0}} = B \quad \textit{and} \quad \chi(A) = [(S, \omega)].$$

Proof. This statement is a consequence of Construction 4.8. In fact, if $[(S,\omega)]$ represents a class in $H^2(\mathbb{Z}^n, B)$, then $A_{(S,\omega)}$ satisfies the requirements of the proposition. ■

PROPOSITION 4.10. *Let $n \in \mathbb{N}$ and B be a unital algebra. Further, let A be an algebraically trivial NCP $\mathbb{T}^n$-bundle with $A_0 = B$. Then A is equivalent to an algebraically trivial NCP $\mathbb{T}^n$-bundle of the form $A_{(S,\omega)}$ for some factor system $(S,\omega) \in Z^2(\mathbb{Z}^n, B)$.*

Proof. Let A be an algebraically trivial NCP $\mathbb{T}^n$-bundle with $A_{\mathbf{0}} = B$. We consider the corresponding short exact sequence of groups

$$1 \longrightarrow A_{\mathbf{0}}^\times \longrightarrow A_h^\times \longrightarrow \mathbb{Z}^n \longrightarrow 1$$

and choose a section $\sigma : \mathbb{Z}^n \to A_h^\times$. Now, a short calculation shows that the map

$$\varphi : \Big(A = \bigoplus_{\mathbf{k}\in\mathbb{Z}^n} A_{\mathbf{k}}, m_A\Big) \to \Big(\bigoplus_{\mathbf{k}\in\mathbb{Z}^n} Bv_{\mathbf{k}}, m_{(C_B\circ\sigma,\delta_\sigma)}\Big),$$

given on homogeneous elements by

$$\varphi(a_{\mathbf{k}}) := a_{\mathbf{k}}\sigma(\mathbf{k})^{-1}v_{\mathbf{k}},$$

defines an equivalence of algebraically trivial NCP $\mathbb{T}^n$-bundles. ■

PROPOSITION 4.11. *Let $n \in \mathbb{N}$ and B be a unital algebra. Further, let (S,ω) and (S',ω') be two factor systems in $Z^2(\mathbb{Z}^n, B)$ with $[(S,\omega)] = [(S',\omega')]$. Then the corresponding algebraically trivial NCP $\mathbb{T}^n$-bundles $A_{(S,\omega)}$ and $A_{(S',\omega')}$ are equivalent.*

Proof. First recall that the condition $[(S',\omega')] = [(S,\omega)]$ is equivalent to the existence of an element $h \in C^1(\mathbb{Z}^n, B^\times)$ with

$$h.(S,\omega) = (S',\omega'),$$

Now, a short observation shows that the map

$$\varphi : \Big(\bigoplus_{\mathbf{k}\in\mathbb{Z}^n} Bv_{\mathbf{k}}, m_{(S',\omega')}\Big) \to \Big(\bigoplus_{\mathbf{k}\in\mathbb{Z}^n} Bv_{\mathbf{k}}, m_{(S,\omega)}\Big),$$

given on homogeneous elements by

$$\varphi(bv_{\mathbf{k}}) = bh(k)v_{\mathbf{k}},$$

is an automorphism of vector spaces leaving the grading spaces invariant. We further have

$$\begin{aligned} m_{(S,\omega)}(\varphi(bv_{\mathbf{k}}), \varphi(b'v_{\mathbf{l}})) &= m_{(S,\omega)}(bh(k)v_{\mathbf{k}}, b'h(l)v_{\mathbf{l}}) \\ &= bh(\mathbf{k})S(\mathbf{k})(b'h(\mathbf{l}))\omega(\mathbf{k},\mathbf{l})v_{\mathbf{k}+\mathbf{l}} \\ &= b[C_B(h(\mathbf{k}))(S(\mathbf{k})(b'))]h(\mathbf{k})S(\mathbf{k})(h(\mathbf{l}))\omega(\mathbf{k},\mathbf{l})v_{\mathbf{k}+\mathbf{l}} \\ &= b(h.S)(\mathbf{k})(b')(h *_S \omega)(\mathbf{k},\mathbf{l})h(\mathbf{k}+\mathbf{l})v_{\mathbf{k}+\mathbf{l}} \\ &= \varphi\big(b(h.S)(\mathbf{k})(b')(h *_S \omega)(\mathbf{k},\mathbf{l})v_{\mathbf{k}+\mathbf{l}}\big) \\ &= \varphi\big(m_{(S',\omega')}(bv_{\mathbf{k}}, b'v_{\mathbf{l}})\big). \end{aligned}$$

Thus, the map φ actually defines an equivalence of algebraically trivial NCP $\mathbb{T}^n$-bundles. ■

We are now ready to state and prove the main theorem of this section.

Theorem 4.12. *Let $n \in \mathbb{N}$ and B be a unital algebra. Then the map*

$$\chi : \mathrm{Ext}(\mathbb{Z}^n, B) \to H^2(\mathbb{Z}^n, B), \quad [A] \mapsto \chi(A)$$

is a well-defined bijection.

Proof. Lemma 4.7 implies that the map χ is well-defined. The surjectivity of χ follows from Proposition 4.9. Hence, it remains to show that the map χ is injective. For this let A and A' be two algebraically trivial NCP $\mathbb{T}^n$-bundles satisfying $A_{\mathbf{0}} = A'_{\mathbf{0}} = B$ and $\chi(A) = \chi(A')$. By Proposition 4.10 we may assume that $A = A_{(S,\omega)}$ and $A' = A_{(S',\omega')}$ for two factor systems (S, ω) and (S', ω') in $Z^2(\mathbb{Z}^n, B)$ with $[(S', \omega')] = [(S, \omega)]$. Thus, the claim follows from Proposition 4.11. ■

5. $\mathbb{Z}^n$-kernels. In the previous section we saw that the set of all equivalence classes of algebraically trivial NCP $\mathbb{T}^n$-bundles with a prescribed fixalgebra B is classified by the cohomology space $H^2(\mathbb{Z}^n, B)$. Moreover, Proposition 4.5 in particular implies that equivalent algebraically trivial NCP $\mathbb{T}^n$-bundles correspond to the same $\mathbb{Z}^n$-kernel. This leads to the following definition.

Definition 5.1 (Equivalence classes of $\mathbb{Z}^n$-kernels). Let $n \in \mathbb{N}$ and B be a unital algebra. We write $\mathrm{Ext}(\mathbb{Z}^n, B)_{[S]}$ for the set of equivalence classes of algebraically trivial NCP $\mathbb{T}^n$-bundles with base B corresponding to the $\mathbb{Z}^n$-kernel $[S]$.

Note that the set $\mathrm{Ext}(\mathbb{Z}^n, B)_{[S]}$ may be empty. The aim of this section is to show how to classify this set and give conditions for its non-emptiness. The following proposition basically states that if $\mathrm{Ext}(\mathbb{Z}^n, B)_{[S]}$ is non-empty, then it is classified by the second group cohomology space $H^2(\mathbb{Z}^n, Z(B)^\times)_{[S]}$ (cf. [Ma95], Chapter IV, Section 4).

Proposition 5.2. *Let $n \in \mathbb{N}$ and B be a unital algebra. Further, let $[S]$ be a $\mathbb{Z}^n$-kernel with* $\mathrm{Ext}(\mathbb{Z}^n, B)_{[S]} \neq \emptyset$. *Then the map*

$$H^2(\mathbb{Z}^n, Z(B)^\times)_{[S]} \times \mathrm{Ext}(\mathbb{Z}^n, B)_{[S]} \to \mathrm{Ext}(\mathbb{Z}^n, B)_{[S]}$$

given by

$$([\beta], [A_{(S,\omega)}]) \mapsto [A_{(S,\omega\cdot\beta)}]$$

is a well-defined simply transitive action.

Proof. A proof of this statement can be found in [Wa11], Corollary 7.3.23. ■

Remark 5.3 (Commutative fixed point algebras). Let $n \in \mathbb{N}$ and suppose that B is a commutative unital algebra. Then the adjoint representation of B is trivial and a factor system (S, ω) for $(\mathbb{Z}^n, B)$ consists of a module structure given by a homomorphism $S : \mathbb{Z}^n \to \mathrm{Aut}(B)$ and an element $\omega \in C^2(\mathbb{Z}^n, B^\times)$. Thus, (S, ω) defines an algebraically trivial NCP $\mathbb{T}^n$-bundle $A_{(S,\omega)}$ if and only if $d_S\omega = 1_B$, i.e., $\omega \in Z^2(\mathbb{Z}^n, B^\times)$. In this case we write A_ω for this algebraically trivial NCP $\mathbb{T}^n$-bundle.

Further $S \sim S'$ if and only if $S = S'$. Hence a $\mathbb{Z}^n$-kernel $[S]$ is the same as a $\mathbb{Z}^n$-module structure S on B and $\mathrm{Ext}(\mathbb{Z}^n, B)_S := \mathrm{Ext}(\mathbb{Z}^n, B)_{[S]}$ is the class of all algebraically trivial NCP $\mathbb{T}^n$-bundles with fixalgebra B corresponding to the $\mathbb{Z}^n$-module structure on B given by S.

According to Proposition 5.2, these equivalence classes correspond to cohomology classes of cocycles, so that the map

$$H^2(\mathbb{Z}^n, B^\times)_S \to \mathrm{Ext}(\mathbb{Z}^n, B)_S, \quad [\omega] \mapsto [A_\omega]$$

is a well-defined bijection.

We now give a condition that ensures the non-emptiness of the set $\mathrm{Ext}(\mathbb{Z}^n, B)_{[S]}$ for a given $\mathbb{Z}^n$-kernel S. We first need the following definition:

DEFINITION 5.4. Let $n \in \mathbb{N}$ and B be a unital algebra. Further, let S be an outer action of $\mathbb{Z}^n$ on B and choose $\omega \in C^2(\mathbb{Z}^n, B^\times)$ with $\delta_S = C_B \circ \omega$. The cohomology class

$$\nu(S) := [d_S\omega] \in H^3(\mathbb{Z}^n, Z(B)^\times)_S$$

does not depend on the choice of ω and is constant on the equivalence class of S, so that we may also write $\nu([S]) := \nu(S)$ (cf. [Wa11], Proposition 7.3.25). We call $\nu(S)$ the *characteristic class* of S.

The next theorem provides a group theoretic criterion for the non-emptiness of the set $\mathrm{Ext}(\mathbb{Z}^n, B)_{[S]}$.

THEOREM 5.5. *Let $n \in \mathbb{N}$ and B be a unital algebra. If $[S]$ is a $\mathbb{Z}^n$-kernel, then*

$$\nu([S]) = \mathbf{1} \quad \Leftrightarrow \quad \mathrm{Ext}(\mathbb{Z}^n, B)_{[S]} \neq \emptyset.$$

Proof. Again, a proof of this statement can be found in [Wa11], Theorem 7.3.27. ■

6. Some useful results on the second group cohomology of $\mathbb{Z}^n$. Let $n \in \mathbb{N}$ and B be a commutative unital algebra. In view of the previous two subsections we now present some useful results on the second cohomology groups $H^2(\mathbb{Z}^n, B^\times)$. We start with some general facts on the set of equivalence classes $\mathrm{Ext}(\mathbb{Z}^n, B^\times) \cong H^2(\mathbb{Z}^n, B^\times)$ of central extensions of $\mathbb{Z}^n$ by the abelian group $B^\times$. Note that since $\mathbb{Z}^n$ is free abelian, the set $\mathrm{Ext}_{\mathrm{ab}}(\mathbb{Z}^n, B^\times)$ of equivalence classes of central extensions of $\mathbb{Z}^n$ by the abelian group $B^\times$ which are abelian consists of a single element. Further, let $\mathrm{Alt}^2(\mathbb{Z}^n, B^\times)$ denote the set of biadditive alternating maps from $\mathbb{Z}^n$ to $B^\times$ and define for each $[\omega] \in H^2(\mathbb{Z}^n, B^\times)$ a map $f_\omega \in \mathrm{Alt}^2(\mathbb{Z}^n, B^\times)$ by

$$f_\omega(\mathbf{k}, \mathbf{l}) := \omega(\mathbf{k}, \mathbf{l}) - \omega(\mathbf{l}, \mathbf{k}).$$

PROPOSITION 6.1. *Let $n \in \mathbb{N}$ and B be a commutative unital algebra. Moreover, let $B^\times$ be a trivial $\mathbb{Z}^n$-module. Then the following assertions hold:*

(a) *The map*

$$\Phi : H^2(\mathbb{Z}^n, B^\times) \to \mathrm{Alt}^2(\mathbb{Z}^n, B^\times), \quad [\omega] \mapsto f_\omega$$

is an isomorphism of abelian groups.

(b) *Moreover, the map*

$$\mathrm{Alt}^2(\mathbb{Z}^n, B^\times) \to (B^\times)^{n(n-1)/2}, \quad f \mapsto (f(e_i, e_j))_{1 \le i < j \le n},$$

is an isomorphism of abelian groups.

Proof. This statement is a special case of [Ne07], Proposition II.4. ■

Next, we consider the general case of abelian extensions of $\mathbb{Z}^2$ by the unit group of a commutative algebra B. Recall that if $(B^\times, S)$ is a $\mathbb{Z}^2$-module, then $\mathrm{Ext}(\mathbb{Z}^2, B^\times)_S \cong H^2(\mathbb{Z}^2, B^\times)_S$ denotes the set of equivalence classes of all $B^\times$-extensions of $\mathbb{Z}^2$ for which the associated $\mathbb{Z}^2$-module structure on $B^\times$ is S. Moreover, we write

$$\sigma_\omega : \mathbb{Z}^2 \to B^\times \times_\omega \mathbb{Z}^2, \quad (\mathbf{k}, \mathbf{l}) \mapsto (1_B, (\mathbf{k}, \mathbf{l}))$$

for the section of the extension of $\mathbb{Z}^2$ by $B^\times$ corresponding to the cocycle $\omega \in Z^2(\mathbb{Z}^2, B^\times)_S$.

Proposition 6.2. *Let B be a commutative unital algebra and $(B^\times, S)$ a $\mathbb{Z}^2$-module. Further, let $\mathbf{e}$ and $\mathbf{e}'$ be generators of $\mathbb{Z}^2$. Then the map*

$$\beta : H^2(\mathbb{Z}^2, B^\times)_S \to B^\times / C, \quad [\omega] \mapsto b_\omega(\mathbf{e}, \mathbf{e}') + C,$$

is a well-defined isomorphism of abelian groups, where

$$b_\omega(\mathbf{e}, \mathbf{e}') := \sigma_\omega(\mathbf{e})\sigma_\omega(\mathbf{e}')\sigma_\omega(\mathbf{e})^{-1}\sigma_\omega(\mathbf{e}')^{-1}$$

and

$$C := \left\langle b' - S(\mathbf{e})b' - b + S(\mathbf{e}')b : b, b' \in B^\times \right\rangle.$$

Proof. The proof will be divided into four steps.

(i) We first show that the map β is well defined. Therefore let ω and ω' be two cocycles with $[\omega] = [\omega']$ and note that according to [Ma95], Chapter IV, Section 4 there exists an element $h \in C^1(\mathbb{Z}^2, B^\times)$ such that the map

$$\varphi : B^\times \times_{\omega'} \mathbb{Z}^2 \to B^\times \times_\omega \mathbb{Z}^2, \quad (b, \mathbf{k}) \mapsto (bh(\mathbf{k}), \mathbf{k}),$$

is an equivalence of extensions. In particular we get $\varphi \circ \sigma_{\omega'} = h \cdot \sigma_\omega$, which leads to

$$b_{\omega'}(\mathbf{e}, \mathbf{e}') = b_\omega(\mathbf{e}, \mathbf{e}') + h(\mathbf{e}') - S(\mathbf{e})h(\mathbf{e}') - h(\mathbf{e}) + S(\mathbf{e}')h(\mathbf{e}).$$

(ii) Next, let $[\omega], [\omega'] \in H^2(\mathbb{Z}^2, B^\times)_S$. Then a short calculation leads to

$$\beta([\omega + \omega']) = \beta([\omega]) + \beta([\omega']).$$

Hence, β is a homomorphism of abelian groups.

(iii) Now, let $[\omega] \in H^2(\mathbb{Z}^2, B^\times)_S$ with $\beta([\omega]) = 0$, i.e.,

$$b_\omega(\mathbf{e}, \mathbf{e}') = b' - S(\mathbf{e})b' - b + S(\mathbf{e}')b$$

for some $b, b' \in B^\times$, and consider the section $\sigma : \mathbb{Z}^2 \to B^\times \times_\omega \mathbb{Z}^2$ given for $k, l \in \mathbb{Z}$ by

$$\sigma(\mathbf{e}^k + (\mathbf{e}')^l) := (b^{-1}\sigma_\omega(\mathbf{e}))^k((b')^{-1}\sigma_\omega(\mathbf{e}'))^l.$$

Then

$$\sigma(\mathbf{e})\sigma(\mathbf{e}') = \sigma(\mathbf{e}')\sigma(\mathbf{e})$$

implies that σ is a homomorphism of groups. In particular the extension

$$q : B^\times \times_\omega \mathbb{Z}^2 \to \mathbb{Z}^2, \quad (b, (k, l)) \mapsto (k, l),$$

splits. Thus, $[\omega] = 0 \in H^2(\mathbb{Z}^2, B^\times)_S$, which means that the map β is injective.

(iv) To show that β is surjective, we associate to each $b_0 \in B^\times$ a group extension G_{b_0} of $\mathbb{Z}^2$ by $B^\times$ in the following way: Take G_{b_0} to be the set of all symbols

$$b \diamond mu \diamond nv, \quad b \in B^\times, \quad m, n \in \mathbb{Z},$$

with multiplication given by

$$u \diamond b = \mathbf{e}.b \diamond u, \quad v \diamond b = \mathbf{e}'.b \diamond v, \quad v \diamond u = b_0 \diamond u \diamond v.$$

It is an easy exercise to show that this multiplication is always associative and makes the set of symbols a group. This completes the proof. ∎

7. Examples of algebraically trivial NCP torus bundles. We finally give a bunch of examples of algebraically trivial NCP $\mathbb{T}^n$-bundles.

Example 7.1 (The classical picture). If $B = C^\infty(M)$ and $S = \mathbf{1}$, then there exists up to isomorphy only one algebraically trivial NCP $\mathbb{T}^n$-bundle for which A is commutative. A possible realization is given by

$$A := \bigoplus_{\mathbf{k} \in \mathbb{Z}^n} C^\infty(M)\chi_{\mathbf{k}},$$

where $\chi_{\mathbf{k}}$ denotes the character of $\mathbb{T}^n$ corresponding to $\mathbf{k} \in \mathbb{Z}^n$. It is the algebraic skeleton of the smooth trivial NCP $\mathbb{T}^n$-bundle $(C^\infty(M \times \mathbb{T}^n), \mathbb{T}^n, \alpha)$, which is induced from the trivial principal $\mathbb{T}^n$-bundle

$$(M \times \mathbb{T}^n, M, \mathbb{T}^n, \mathrm{pr}_M, \sigma_{\mathbb{T}^n}).$$

This example perfectly reflects the commutative world, in which there exists up to isomorphy only one trivial principal $\mathbb{T}^n$-bundle over a given manifold M. On the other hand, it shows that the situation completely changes in the noncommutative world, where there exist plenty of algebraically trivial NCP $\mathbb{T}^n$-bundles with base $C^\infty(M)$ (cf. Proposition 6.1).

Definition 7.2 (The algebraic noncommutative n-torus). A $\mathbb{Z}^n$-graded unital associative algebra

$$A = \bigoplus_{\mathbf{k} \in \mathbb{Z}^n} A_{\mathbf{k}}$$

is called an *algebraic noncommutative n-torus*, if each grading space $A_{\mathbf{k}}$ is one-dimensional and each nonzero element of $A_{\mathbf{k}}$ is invertible. In [OP95], these algebras are called *twisted group algebras.*

Remark 7.3. Note that algebraic noncommutative n-tori are the algebraic counterpart of Example 2.1.

Example 7.4. Let $n \in \mathbb{N}$ and $B = \mathbb{C}$. The algebraic noncommutative n-tori are exactly the algebraically trivial NCP $\mathbb{T}^n$-bundles with base $\mathbb{C}$ corresponding to the trivial $\mathbb{Z}^n$-module structure on $\mathbb{C}$. Indeed, if A is an algebraic noncommutative n-torus, then a short observation shows that the short exact sequence

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow A_h^\times \longrightarrow \mathbb{Z}^n \longrightarrow 1$$

of groups is central. By Remark 5.3 and Proposition 6.1, these algebras are classified by

$$H^2(\mathbb{Z}^n, \mathbb{C}^\times) \cong (\mathbb{C}^\times)^{n(n-1)/2}.$$

Lemma 7.5. *Let A be an algebraic noncommutative n-torus. Then*

(a) *Each unit of A is graded, i.e., $A^\times = A_h^\times$.*
(b) *Each automorphism of A is graded, i.e., $\operatorname{Aut}(A) = \operatorname{Aut}_{\rm gr}(A)$.*

Proof. The proofs of these statements can be found in [Ne07], Appendix A. ■

Proposition 7.6. *Let $n, m \in \mathbb{N}$ and $B := B_f$ be the algebraic noncommutative n-torus corresponding to the cocycle $f \in Z^2(\mathbb{Z}^n, \mathbb{C}^\times)$. Further, let $(S, \omega) \in Z^2(\mathbb{Z}^m, B)$ be a factor system. Then $A_{(S,\omega)}$ is an algebraic noncommutative $(m+n)$-torus if and only if S leaves the grading spaces of B invariant and the cocycle ω takes values in $B_{\mathbf{0}}^\times \cong \mathbb{C}^\times$.*

Proof. Let

$$\bigoplus_{\mathbf{k}\in\mathbb{Z}^n} \mathbb{C}v_{\mathbf{k}}, \quad \text{resp.,} \quad \bigoplus_{\mathbf{l}\in\mathbb{Z}^m} Bw_{\mathbf{l}}$$

be the underlying vector space of B, resp., of A.

("$\Leftarrow$") We first write $c_{\mathbf{l},\mathbf{k}} \in \mathbb{C}^\times$ for the constant satisfying $S(\mathbf{l})v_{\mathbf{k}} = c_{\mathbf{l},\mathbf{k}}v_{\mathbf{k}}$. Since the cocycle ω takes values in $\mathbb{C}^\times$, the map S is a group homomorphism and thus the calculation

$$\begin{aligned}(v_{\mathbf{k}}w_{\mathbf{l}})(v_{\mathbf{k}'}w_{\mathbf{l}'}) &= v_{\mathbf{k}}S(\mathbf{l})v_{\mathbf{k}'}\omega(\mathbf{l},\mathbf{l}')w_{\mathbf{l}+\mathbf{l}'} = c_{\mathbf{l},\mathbf{k}'}\omega(\mathbf{l},\mathbf{l}')v_{\mathbf{k}}v_{\mathbf{k}'}w_{\mathbf{l}+\mathbf{l}'}\\ &= c_{\mathbf{l},\mathbf{k}'}\omega(\mathbf{l},\mathbf{l}')f(\mathbf{k},\mathbf{k}')v_{\mathbf{k}+\mathbf{k}'}w_{\mathbf{l}+\mathbf{l}'}\end{aligned}$$

shows that $A_{(S,\omega)}$ is an algebraic noncommutative $(m+n)$-torus corresponding to the cocycle

$$f' : \mathbb{Z}^{m+n} \times \mathbb{Z}^{m+n} \to \mathbb{C}^\times, \quad ((\mathbf{k},\mathbf{l}),(\mathbf{k}',\mathbf{l}')) \mapsto c_{\mathbf{l},\mathbf{k}'}\omega(\mathbf{l},\mathbf{l}')f(\mathbf{k},\mathbf{k}').$$

("$\Rightarrow$") Conversely, if $A_{(S,\omega)}$ is an algebraic noncommutative $(m+n)$-torus, then

$$(v_{\mathbf{k}}w_{\mathbf{l}})w_{\mathbf{l}'} = v_{\mathbf{k}}\omega(\mathbf{l},\mathbf{l}')w_{\mathbf{l}+\mathbf{l}'} \in A_{\mathbf{k},\mathbf{l}+\mathbf{l}'}$$

and Lemma 7.5 (a) imply that $\omega(\mathbf{l},\mathbf{l}') \in B_{\mathbf{0}}^\times \cong \mathbb{C}^\times$ for all $\mathbf{l},\mathbf{l}' \in \mathbb{Z}^m$. Moreover,

$$w_{\mathbf{l}'}(v_{\mathbf{k}}w_{\mathbf{l}}) = S(\mathbf{l}')v_{\mathbf{k}}\omega(\mathbf{l}',\mathbf{l})w_{\mathbf{l}+\mathbf{l}'} = \omega(\mathbf{l}',\mathbf{l})S(\mathbf{l}')v_{\mathbf{k}}w_{\mathbf{l}+\mathbf{l}'} \in A_{\mathbf{k},\mathbf{l}+\mathbf{l}'}$$

and Lemma 7.5 (b) imply that the map S must leave each grading spaces of B invariant. ■

Lemma 7.7. *Each automorphism of the matrix algebra $\mathrm{M}_n(\mathbb{C})$ is inner.*

Proof. This is a corollary of the well-known Skolem–Noether Theorem. ■

Example 7.8. Let $B = \mathrm{M}_m(\mathbb{C})$. In view of Lemma 7.7, each outer action of $\mathbb{Z}^n$ on $\mathrm{M}_m(\mathbb{C})$ is equivalent to $S = \mathbf{1}$. In particular, $\operatorname{Ext}(\mathbb{Z}^n, B)_{[S]} \neq \emptyset$ and Proposition 5.2 implies that the equivalence classes of algebraically trivial NCP $\mathbb{T}^n$-bundles with base $\mathrm{M}_m(\mathbb{C})$ are classified by

$$H^2(\mathbb{Z}^n, \mathbb{C}^\times) \cong (\mathbb{C}^\times)^{n(n-1)/2}.$$

EXAMPLE 7.9 (Direct sums). Let A and A' be two algebraically trivial NCP $\mathbb{T}^n$-bundles with base B and B', respectively. Then the direct sum $A \oplus A'$ is an algebraically trivial NCP $\mathbb{T}^n$ bundle with base $B \oplus B'$.

EXAMPLE 7.10 (Tensor products). Let A be an algebraically trivial NCP $\mathbb{T}^n$-bundle with base B and A' an algebraically trivial NCP $\mathbb{T}^m$-bundle with base B' Then their tensor product $A \otimes A'$ is an algebraically trivial NCP $\mathbb{T}^{n+m}$-bundle with base $B \otimes B'$.

8. Perspectives. In this paper we have presented a geometrically oriented approach to the noncommutative geometry of trivial principal $\mathbb{T}^n$-bundles. Therefore it is just natural to ask if there is also a reasonable, geometrically oriented approach to the noncommutative geometry of non-trivial principal $\mathbb{T}^n$-bundles. In classical differential geometry each principal bundle (P, M, G, q, σ) is locally trivial. Inspired by this fact, we have introduced an appropriate method of localizing algebras in a "smooth" way. The step from the trivial to the non-trivial case is then carried out by saying that a (smooth) dynamical system $(A, \mathbb{T}^n, \alpha)$ is a *NCP* $\mathbb{T}^n$*-bundle* if "localization" around characters of the fixalgebra Z of the induced action of $\mathbb{T}^n$ on the center C_A of A leads to trivial NCP $\mathbb{T}^n$-bundles. In particular, this approach covers the classical theory of principal $\mathbb{T}^n$-bundles and further examples are given by sections of algebra bundles endowed with certain actions of $\mathbb{T}^n$ by algebra automorphisms. For details we refer to [Wa11].

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