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LONGTIME BEHAVIOR OF SOLUTIONS OF A NAVIER-STOKES/CAHN-HILLIARD SYSTEM

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Abstract. We study a diffuse interface model for the flow of two viscous incompressible Newtonian fluids of the same density in a bounded domain. The fluids are assumed to be macroscopically immiscible, but a partial mixing in a small interfacial region is assumed in the model. Moreover, diffusion of both components is taken into account. This leads to a coupled Navier-Stokes/Cahn-Hilliard system, which can describe the evolution of droplet formation and collision during the flow. We review some results on existence, uniqueness and regularity of weak and strong solutions in two and three space dimensions. Moreover, we prove stability of local minima of the energy and show existence of a weak global attractor, which is strong if $d = 2$.

1. Introduction. In the present contribution we study the Navier-Stokes/Cahn-Hilliard system:

$$\partial_t v + v \cdot \nabla v - \operatorname{div}(\nu(c) Dv) + \nabla p = -\varepsilon \operatorname{div}(\nabla c \otimes \nabla c) \quad \text{in } Q, \tag{1.1}$$

$$\operatorname{div} v = 0 \quad \text{in } Q, \tag{1.2}$$

$$\partial_t c + v \cdot \nabla c = m \Delta \mu \quad \text{in } Q, \tag{1.3}$$

$$\mu = \varepsilon^{-1} \phi(c) - \varepsilon \Delta c \quad \text{in } Q, \tag{1.4}$$

together with the boundary and initial conditions

$$v|_{\partial\Omega} = 0 \quad \text{on } S, \tag{1.5}$$

$$\partial_n c|_{\partial\Omega} = \partial_n \mu|_{\partial\Omega} = 0 \quad \text{on } S, \tag{1.6}$$

$$(v, c)|_{t=0} = (v_0, c_0) \quad \text{in } \Omega, \tag{1.7}$$

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where $Q = \Omega \times (0,\infty)$, $S = \partial\Omega \times (0,\infty)$ and $\Omega \subset \mathbb{R}^d$, $d = 2,3$, is a bounded domain with C^3-boundary. Moreover, $\nu \in C^2(\mathbb{R})$ with $\nu(s) \geq \nu_0 > 0$ for all $s \in \mathbb{R}$.

This system arises in a so-called diffuse interface model for the two-phase flow of two viscous, incompressible fluids, which are macroscopically immiscible. Such models take a partial mixing of the fluids on a small length scale proportional to $\varepsilon > 0$ into account. We refer to Gurtin et al. [10] for a derivation of this model and to Anderson and McFadden [5] for a review on diffuse interface models.

Here v is the mean velocity of the mixture, $Dv = \frac{1}{2}(\nabla v + \nabla v^T)$, p is the pressure, c is an order parameter related to the concentration of the fluids (e.g. the concentration difference or the concentration of one component), and Ω is a suitable bounded domain. Moreover, $\nu(c) > 0$ is the viscosity of the mixture, $\varepsilon > 0$ is a (small) parameter, which will be related to the "thickness" of the interfacial region, and $\phi = \Phi'$ for some suitable energy density Φ specified below. It is assumed that the densities of both components as well as the density of the mixture are constant and for simplicity equal to one. We note that capillary forces due to surface tension are modeled by an extra contribution $\varepsilon\nabla c \otimes \nabla c$ in the stress tensor leading to the term on the right-hand side of (1.1). Moreover, we note that in the modeling diffusion of the fluid components is taken into account. Therefore $m\Delta\mu$ is appearing in (1.3), where $m > 0$ is the mobility coefficient, which is assumed to be constant.

Here (1.5) is the usual no-slip boundary condition for viscous fluids, n is the exterior normal on $\partial\Omega$, $\partial_n\mu|_{\partial\Omega} = 0$ means that there is no flux of the components through the boundary, and $\partial_n c|_{\partial\Omega} = 0$ describes a "contact angle" of $\pi/2$ of the diffused interface and the boundary of the domain.

The total energy of the system above is given by $E(c,v) = E_{\text{free}}(c) + E_{\text{kin}}(v)$, where

$$E_{\text{free}}(c) = \frac{1}{2}\int_\Omega \varepsilon|\nabla c(x)|^2\,dx + \int_\Omega \varepsilon^{-1}\Phi(c(x))\,dx, \tag{1.8}$$

$$E_{\text{kin}}(v) = \frac{1}{2}\int_\Omega |v(x)|^2\,dx. \tag{1.9}$$

Here the Ginzburg-Landau energy $E_{\text{free}}(c)$ describes an interfacial energy associated with the region where c is not close to the minima of $\Phi(c)$ and $E_{\text{kin}}(v)$ is the kinetic energy of the fluid. The system is dissipative. More precisely, for sufficiently smooth solutions

$$\frac{d}{dt}E(c(t),v(t)) = -\int_\Omega \nu(c(t))|Dv(t)|^2\,dx - m\int_\Omega |\nabla\mu(t)|^2\,dx.$$

There are only a few results on the mathematical analysis of diffuse interface models in fluid mechanics and the system above. First results on existence of strong solutions, if $\Omega = \mathbb{R}^2$ and Φ is a suitably smooth double well potential, were obtained by Starovoĭtov [13]. More complete results were presented by Boyer [6] in the case that $\Omega \subset \mathbb{R}^d$ is a periodical channel and f is a suitably smooth double well potential. Moreover, (1.1)–(1.7) was also briefly discussed by Liu and Shen [11].

In this article we review the results of [1] on existence, uniqueness, and regularity of weak and strong solutions of (1.1)–(1.7). Moreover, it was shown that any weak solution (v,c) of (1.1)–(1.7) converges to $(0,c_\infty)$, where c_∞ is a solution of the stationary Cahn-Hilliard equation. With similar techniques we will show in Section 2 stability of local minima of E. Finally, in Section 3, we show the existence of a weak global attractor, which is a strong global attractor if $d = 2$.

First of all, let us recall the class of free energy densities Φ used in [1].

ASSUMPTION 1.1. Let $\Phi \in C([a,b]) \cap C^2((a,b))$ such that $\phi = \Phi'$ satisfies

$$\lim_{s\to a} \phi(s) = -\infty, \qquad \lim_{s\to b} \phi(s) = \infty, \qquad \phi'(s) \geq -\alpha$$

for some $\alpha \in \mathbb{R}$. Furthermore, we assume that $\nu \in C^2([a,b])$ is a positive function.

We extend $\Phi(x)$ by $+\infty$ if $x \notin [a,b]$. Hence $E_{\text{free}}(c) < \infty$ implies $c(x) \in [a,b]$ for almost every $x \in \Omega$. We note that the previous assumption yields the decomposition

$$\Phi(s) = \Phi_0(s) - \frac{\alpha}{2}c^2, \qquad \phi(s) = \phi_0(s) - \alpha c. \tag{1.10}$$

Often c is a just the concentration difference of both components and $[a,b] = [-1,1]$. But it is mathematically useful to consider a general interval.

We note that (1.1) can be replaced by

$$\partial_t v + v \cdot \nabla v - \operatorname{div}(\nu(c)Dv) + \nabla g = \mu_0 \nabla c \tag{1.11}$$

with $g = p + \frac{1}{2}|\nabla c|^2 + \Phi(c) - \overline{\mu} c$ and $\mu = \mu_0 + \overline{\mu}$ such that $\int_\Omega \mu_0(x)\,dx = 0$ and $\overline{\mu} \in \mathbb{R}$ since $\mu \nabla c = \nabla(\frac{1}{2}|\nabla c|^2 + \Phi(c)) - \operatorname{div}(\nabla c \otimes \nabla c)$.

In the following we will for simplicity assume that $\varepsilon = 1$ and $m = 1$. But all results are valid for general $\varepsilon > 0, m > 0$.

Furthermore, let $Q_{(s,t)} = \Omega \times (s,t)$, $Q_t = Q_{(0,t)}$, and $Q = Q_{(0,\infty)}$. In the following $BC_w(0,T;X)$ denotes the set of all weakly continuous functions $f\colon I \to X$, where $I = [0,T]$ if $0 < T < \infty$ and $I = [0,\infty)$ if $T = \infty$. If $A \subset \mathbb{R}^N$, $N \in \mathbb{N}$, is a set, then

$$C^\infty_{(0)}(A) = \{f\colon A \to \mathbb{R} : f = F|_A, F \in C_0^\infty(\mathbb{R}^N), \operatorname{supp} F \cap A \text{ is compact}\}.$$

Here and in the following $C_0^\infty(\Omega)$ denotes the space of all smooth and compactly supported functions $f\colon \Omega \to \mathbb{R}$ for a domain $\Omega \subset \mathbb{R}^N$. Moreover,

$$H^1_{(0)}(\Omega) = \left\{ u \in H^1(\Omega) : \int_\Omega u\,dx = 0 \right\}$$

is the space equipped with the norm $\|u\|_{H^1_{(0)}(\Omega)} := \|\nabla u\|_{L^2(\Omega)}$ and $H^{-1}_{(0)}(\Omega) = H^1_{(0)}(\Omega)'$. Finally, $L^2_\sigma(\Omega) = \{f \in L^2(\Omega)^d : \operatorname{div} f = 0, n \cdot f|_{\partial\Omega} = 0\}$ and $W^1_{q,0}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\nabla\cdot\|_{L^q}}$. For complete definitions of the function spaces in the following we refer to [1].

DEFINITION 1.2. Let $0 < T \leq \infty$. A triple (v,c,μ) such that

$$\begin{aligned} &v \in BC_w(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;H^1_0(\Omega)^d),\\ &c \in BC_w(0,T;H^1(\Omega)),\ \phi(c) \in L^2_{\text{loc}}([0,T);L^2(\Omega)), \nabla\mu \in L^2(Q_T) \end{aligned}$$

is called a *weak solution* of (1.1)–(1.7) on $(0,T)$ if

$$-(v,\partial_t\psi)_{Q_T} - (v_0, \psi|_{t=0})_\Omega + (v\cdot\nabla v, \psi)_{Q_T} + (\nu(c)Dv, D\psi)_{Q_T} = (\mu\nabla c, \psi)_{Q_T}$$

for all $\psi \in C^\infty_{(0)}([0,T) \times \Omega)^d$ with $\operatorname{div}\psi = 0$,

$$\begin{aligned} -(c,\partial_t\varphi)_{Q_T} - (c_0,\varphi|_{t=0})_\Omega + (v\cdot\nabla c, \varphi)_{Q_T} &= -(\nabla\mu, \nabla\varphi)_{Q_T}\\ (\mu,\varphi)_{Q_T} &= (\phi(c),\varphi)_{Q_T} + (\nabla c, \nabla\varphi)_{Q_T} \end{aligned}$$

for all $\varphi \in C^\infty_{(0)}([0,T)\times\overline{\Omega})$, and if the (strong) energy inequality

$$E(v(t),c(t)) + \int_{Q_{(t_0,t)}} (\nu(c)|Dv|^2 + |\nabla\mu|^2)\, d(x,\tau) \leq E(v(t_0),c(t_0)) \tag{1.12}$$

holds for almost all $0 \leq t_0 < T$ including $t_0 = 0$ and all $t \in [t_0,T)$.

Since μ is uniquely determined by c via (1.4), we often simply call (v,c) a weak solution of (1.1)–(1.7). Finally, replacing $[0,\infty)$ by $[T,\infty)$, $T\in\mathbb{R}$, one defines weak solutions of (1.1)–(1.7) on $[T,\infty)$.

Theorem 1.3 (Global existence of weak solutions). *For every $v_0 \in L^2_\sigma(\Omega)$, $c_0 \in H^1(\Omega)$ with $c_0(x) \in [a,b]$ almost everywhere there is a weak solution (v,c,μ) of (1.1)–(1.7) on $(0,\infty)$. Moreover, if $d = 2$, then (1.12) holds with equality for all $0 \leq t_0 \leq t < \infty$. Finally, every weak solution on $(0,\infty)$ satisfies*

$$\nabla^2 c, \phi(c) \in L^2_{loc}([0,\infty); L^r(\Omega)), \qquad \frac{t^{\frac12}}{1+t^{\frac12}} c \in BUC(0,\infty; W^1_q(\Omega)) \tag{1.13}$$

where $r = 6$ if $d = 3$ and $1 < r < \infty$ is arbitrary if $d = 2$ and $q > 3$ is independent of the solution and initial data. If additionally $c_0 \in H^2_N(\Omega) := \{c \in H^2(\Omega) : \partial_n c|_{\partial\Omega} = 0\}$ and $-\Delta c_0 + \phi_0(c_0) \in H^1(\Omega)$, then $c \in BUC(0,\infty; W^1_q(\Omega))$.

We note that the regularity statement $t^{\frac12}/(1+t^{\frac12})c \in BUC(0,\infty; W^1_q(\Omega))$ with $q > d$ for *any* weak solution in the latter theorem is a crucial ingredient for obtaining higher regularity of weak solutions.

Proposition 1.4 (Uniqueness). *Let $0 < T \leq \infty$, $q = 3$ if $d = 3$ and let $q > 2$ if $d = 2$. Moreover, assume that $v_0 \in W^1_{q,0}(\Omega)^d \cap L^2_\sigma(\Omega)$ and let $c_0 \in H^1_{(0)}(\Omega) \cap C^{0,1}(\overline{\Omega})$ with $c_0(x) \in [a,b]$ for all $x \in \Omega$. If there is a weak solution (v,c,μ) of (1.1)–(1.7) on $(0,T)$ with $v \in L^\infty(0,T; W^1_q(\Omega)^d)$ and $\nabla c \in L^\infty(Q_T)$, then any weak solution (v',c',μ') of (1.1)–(1.7) on $(0,T)$ with the same initial values and $\nabla c' \in L^\infty(Q_T)$ coincides with (v,c,μ).*

For the following we denote $V^{1+j}_2(\Omega) = H^{1+j}(\Omega)^d \cap H^1_0(\Omega)^d \cap L^2_\sigma(\Omega)$, $j = 0,1$. Moreover, for $s \in (0,1)$ we define $V^{1+s}_2(\Omega) = (V^1_2(\Omega), V^2_2(\Omega))_{s,2}$, where $(.,.)_{s,q}$ denotes the real interpolation functor.

Theorem 1.5 (Regularity of weak solutions). *Let $c_0 \in H^2_N(\Omega)$ such that $E_{free}(c_0) < \infty$ and $-\Delta c_0 + \phi(c_0) \in H^1(\Omega)$.*

1. *Let $d = 2$ and let $v_0 \in V^{1+s}_2(\Omega)$ with $s \in (0,1]$. Then every weak solution (v,c) of (1.1)–(1.7) on $(0,\infty)$ satisfies*
$$v \in L^2(0,\infty; H^{2+s'}(\Omega)^d) \cap H^1(0,\infty; H^{s'}(\Omega)^d) \cap BUC([0,\infty); H^{1+s-\varepsilon}(\Omega)^d)$$
for all $s' \in [0,\frac12) \cap [0,s]$ and all $\varepsilon > 0$ as well as $\nabla^2 c, \phi(c) \in L^\infty(0,\infty; L^r(\Omega))$ for every $1 < r < \infty$. In particular, the weak solution is unique.

.2 *Let $d = 2,3$. Then for every weak solution (v,c,μ) of (1.1)–(1.7) on $(0,\infty)$ there is some $T > 0$ such that*
$$v \in L^2(T,\infty; H^{2+s}(\Omega)^d) \cap H^1(T,\infty; H^s(\Omega)^d) \cap BUC([T,\infty); H^{2-\varepsilon}(\Omega)^d)$$

for all $s \in [0, \frac{1}{2})$ *and all* $\varepsilon > 0$ *as well as* $\nabla^2 c, \phi(c) \in L^\infty(T, \infty; L^r(\Omega))$ *with* $r = 6$ *if* $d = 3$ *and* $1 < r < \infty$ *if* $d = 2$.

3. *If* $d = 3$ *and* $v_0 \in V_2^{s+1}(\Omega)$, $s \in (\frac{1}{2}, 1]$, *then there is some* $T_0 > 0$ *such that every weak solution* (v, c) *of (1.1)–(1.7) on* $(0, T_0)$ *satisfies*

$$v \in L^2(0, T_0; H^{2+s'}(\Omega)^d) \cap H^1(0, T_0; H^{s'}(\Omega)^d) \cap BUC([0, T_0]; H^{1+s-\varepsilon}(\Omega)^d)$$

for all $s' \in [0, \frac{1}{2})$ *and all* $\varepsilon > 0$ *as well as* $\nabla^2 c, \phi(c) \in L^\infty(0, T_0; L^6(\Omega))$. *In particular, the weak solution is unique on* $(0, T_0)$.

Finally, because of the regularity of any weak solution for large times, one is able to modify the proof in [4], based on the Łojasiewicz-Simon inequality, to show convergence to stationary solutions as $t \to \infty$.

THEOREM 1.6 (Convergence to stationary solution). *Assume that* $\Phi\colon (a, b) \to \mathbb{R}$ *is analytic and let* (v, c, μ) *be a weak solution of (1.1)–(1.7). Then* $(v(t), c(t)) \rightharpoonup_{t\to\infty} (0, c_\infty)$ *in* $H^{2-\varepsilon}(\Omega)^d \times H^2(\Omega)$ *for all* $\varepsilon > 0$ *and for some* $c_\infty \in H^2(\Omega)$ *with* $\phi(c_\infty) \in L^2(\Omega)$ *solving the stationary Cahn-Hilliard equation*

$$-\Delta c_\infty + \phi(c_\infty) = const. \qquad \text{in } \Omega, \tag{1.14}$$

$$\partial_n c_\infty|_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega, \tag{1.15}$$

$$\int_\Omega c_\infty(x)\, dx = \int_\Omega c_0(x)\, dx. \tag{1.16}$$

All the previous results are proved in [1].

2. Stability of local minima. We assume that $m_0 \in H^2_N(\Omega) = \{u \in H^2(\Omega) : \partial_n u|_{\partial\Omega} = 0\}$ with $m_0(x) \in (a, b)$ for all $x \in \Omega$ is a local minimum of $E_{\text{free}}(c)$ in the sense that there is some $\varepsilon_1 > 0$ such that

$$E_{\text{free}}(m_0) \le E_{\text{free}}(c) \quad \text{ if } \|c - m_0\|_{H^1(\Omega)\cap L^\infty(\Omega)} \le \varepsilon_0,\ \int_\Omega c\, dx = \int_\Omega m_0\, dx.$$

W.l.o.g. we can assume that $\int_\Omega m_0\, dx = 0$ since by a simple translation of c and Φ we can always reduce to this case. Furthermore, changing Φ by a constant, we can reduce to the case that $E_{\text{free}}(m_0) = 0$.

Moreover, we assume that $\varepsilon_1 > 0$ is chosen so small that

$$c(x) \in [a', b'] \quad \text{ for all } x \in \overline{\Omega}$$

for some $a < a' < b' < b$ and that the Łojasiewicz-Simon inequality

$$|E_{\text{free}}(c) - E_{\text{free}}(m_0)|^{1-\theta} \le C\|D\widetilde{E}_{\text{free}}(c)\|_{H^{-1}_{(0)}} \tag{2.1}$$

holds for all $\|c - m_0\|_{H^1_{(0)}} \le \varepsilon_1$, where $\theta \in (0, \frac{1}{2}]$, cf. [4, Proposition 6.3] and [1, Section 3.2]. Here $D\widetilde{E}_{\text{free}}$ denotes the Fréchet derivative of $\widetilde{E}_{\text{free}}\colon H^1_{(0)}(\Omega) \to \mathbb{R}$ and $\widetilde{E}_{\text{free}}$ denotes the functional obtained from E_{free} by replacing Φ by a suitable smooth $\widetilde{\Phi}$ with $\widetilde{\Phi}|_{[a',b']} = \Phi|_{[a',b']}$, cf. [4, Section 6] for details.

We want to show stability of this stationary solution $(0, m_0)$.

THEOREM 2.1. *Let $R > 0$ be arbitrary and assume that $c_0 \in H^2_N(\Omega)$ with $-\Delta c_0 + \phi(c_0) \in H^1(\Omega)$ and $v_0 \in H^2(\Omega)^d \cap H^1_0(\Omega)^d \cap L^2_\sigma(\Omega)$ satisfy*

$$\|c_0\|_{H^2(\Omega)} + \| - \Delta c_0 + \phi(c_0)\|_{H^1(\Omega)} + \|v_0\|_{H^2(\Omega)} \leq R. \tag{2.2}$$

Then for every $0 < \varepsilon \leq \varepsilon_1$ there is some $\delta > 0$ such that, if

$$\|c_0 - m_0\|_{H^1} + E(c_0, v_0) \leq \delta \quad \text{and} \quad \int_\Omega c_0\, dx = \int_\Omega m_0\, dx,$$

then there is a unique weak solution (v, c) of (1.1)–(1.7) on $(0, \infty)$ such that

$$\|c - m_0\|_{L^\infty(0,\infty;H^1(\Omega)\cap L^\infty(\Omega))} + \|v\|_{BUC([0,\infty);H^1(\Omega))} \leq \varepsilon, \tag{2.3}$$

as well as $c \in L^\infty(0, \infty; W^2_r(\Omega))$, where $r = 6$ if $d = 3$ and $1 < r < \infty$ is arbitrary if $d = 2$, and $v \in BUC([0, \infty); H^s(\Omega)^d)$ for every $s < 2$.

In order to prove the latter theorem, we start with a refinement of the last statement of Theorem 1.3.

LEMMA 2.2. *Assume that either $\kappa(t) = t^{\frac{1}{2}}/(1 + t^{\frac{1}{2}})$ or $\kappa(t) \equiv 1$ and $c_0 \in H^2_N(\Omega)$ such that $-\Delta c_0 + \phi(c_0) \in H^1(\Omega)$. There are some $q > 3$ and $s > 1$ such that for any $R > 0$ there is some $C(R) > 0$ such that*

$$\|\kappa c\|_{BUC([0,\infty);W^s_q(\Omega))} \leq C(R)$$

for any weak solution (v, c) of (1.1)–(1.7) with

$$\begin{aligned} E(c_0, v_0) &\leq R \quad && \text{if } \kappa(t) = t^{\frac{1}{2}}/(1 + t^{\frac{1}{2}}), \\ E(c_0, v_0) + \|-\Delta c_0 + \phi(c_0)\|_{H^1(\Omega)} &\leq R && \text{if } \kappa(t) \equiv 1. \end{aligned}$$

Proof. If $s = 1$, then the statement is proved as part of the proof of [1, Theorem 1.4]. The proof relies on direct estimates using the bounds given by energy estimate (1.12), [1, Lemma 3.2], and interpolation. The assumption $c_0 \in H^2_N(\Omega)$ and $-\Delta c_0 + \phi(c_0) \in H^1(\Omega)$ is only needed for the regularity statements of [1, Lemma 3.2] if $\kappa(t) \equiv 1$. The uniform dependence of the constants as stated in the lemma can be easily checked going through the proofs. Moreover, the proof of [1, Theorem 1.4] also shows that there are some $s > 1$ and some (slightly smaller) $q > 3$ such that the statement of the lemma holds. ■

The key argument for the stability of the local minimum is contained in the proof of the following lemma and is based on the Łojasiewicz-Simon inequality (2.1).

LEMMA 2.3. *For every $0 < \varepsilon < \varepsilon_1$ there is some $\delta > 0$ such that, if (v, c) is a weak solution of (1.1)–(1.7) with*

$$\|c_0 - m_0\|_{H^1(\Omega)\cap L^\infty(\Omega)} + E(c_0, v_0) \leq \delta,$$

then

$$\sup_{0\leq t<\infty} \|c(t) - m_0\|_{H^1(\Omega)\cap L^\infty(\Omega)} \leq \varepsilon$$

Proof. First of all, we assume that $0 < \delta \leq \varepsilon_1$. Since $c \in BUC([0,\infty); W_q^{1+\varepsilon}(\Omega))$ and $W_q^{1+\varepsilon}(\Omega) \hookrightarrow H^1(\Omega) \cap L^\infty(\Omega)$ due to $q > 3$, we have

$$\sup_{0\leq t\leq T} \|c(t) - m_0\|_{H^1(\Omega)\cap L^\infty(\Omega)} < \varepsilon_1 \tag{2.4}$$

at least for some $T > 0$. Moreover, we have

$$\begin{aligned}\|c(t) - c_0\|_{H^1(\Omega)\cap L^\infty(\Omega)} &\leq \|c(t) - c_0\|^{1-\alpha}_{W_q^{1+\varepsilon}(\Omega)} \|c(t) - c_0\|^{\alpha}_{H^{-1}_{(0)}(\Omega)} \\ &\leq C(R)\|c(t) - c_0\|^{\alpha}_{H^{-1}_{(0)}(\Omega)}\end{aligned}$$

for some $0 < \alpha < 1$ by interpolation and since $\|c\|_{BUC([0,\infty);W_q^{1+\varepsilon}(\Omega))} \leq C(R)$. On the other hand, as long as (2.4) holds we have

$$-\frac{d}{dt}E(c(t), v(t))^\theta \geq C\|\nabla\mu(t)\|_{L^2(\Omega)} = C\|\partial_t c\|_{H^{-1}_{(0)}(\Omega)}$$

for all $t \in (0,T)$ by the same calculation as in [1, Section 7, Proof of Theorem 1.7]. This implies

$$\sup_{0\leq t\leq T} \|c(t) - c_0\|_{H^{-1}_{(0)}(\Omega)} \leq CE(c_0, v_0)^\theta,$$

where C is independent of T. Hence

$$\|c(t) - c_0\|_{H^1(\Omega)\cap L^\infty(\Omega)} \leq \frac{\varepsilon}{2}$$

for all $0 \leq t \leq T$ provided that $E(c_0, v_0) \leq \delta$ for sufficiently small δ independent of $T > 0$. Altogether, we have

$$\sup_{0\leq t\leq T} \|c(t) - m_0\|_{H^1(\Omega)\cap L^\infty(\Omega)} \leq \varepsilon < \varepsilon_1$$

for all $T > 0$ such that (2.4) holds. But this implies that we can choose $T = \infty$ in the latter estimate. Otherwise there would be some $0 < T < \infty$ such that

$$\varepsilon < \sup_{0\leq t\leq T} \|c(t) - m_0\|_{H^1(\Omega)\cap L^\infty(\Omega)} < \varepsilon_1.$$

But for this $T > 0$ the previous estimates show that

$$\sup_{0\leq t\leq T} \|c(t) - m_0\|_{H^1(\Omega)\cap L^\infty(\Omega)} \leq \varepsilon,$$

which would be a contradiction. ∎

Next we show smallness and regularity of v if $\delta > 0$ is sufficiently small.

LEMMA 2.4. *Let $R > 0$ and let (v, c) be a weak solution of (1.1)–(1.7) on $(0,\infty)$ with initial values $(v_0, c_0) \in (H_0^1(\Omega)^d \cap L^2_\sigma(\Omega)) \times H^2_N(\Omega)$ such that (2.2) and*

$$\sup_{0\leq t<\infty} \|c(t) - m_0\|_{H^1} \leq \varepsilon_1$$

hold. Then for every $\varepsilon > 0$ there is some $\delta > 0$ such that $v \in BUC([0,\infty); H^1(\Omega)^d)$ and

$$\|v\|_{BUC([0,\infty);H^1(\Omega))} \leq \varepsilon$$

provided that $E(c_0, v_0) \leq \delta$.

Proof. Let $f = \mu_0 \nabla c$. Then $v \in L^\infty(0,\infty; L^2_\sigma(\Omega)) \cap L^2(0,\infty; H^1(\Omega)^d)$ is a weak solution of

$$\partial_t v + v \cdot \nabla v - \operatorname{div}(\nu(c) Dv) + \nabla p = f \quad \text{in } Q, \tag{2.5}$$
$$\operatorname{div} v = 0 \quad \text{in } Q, \tag{2.6}$$
$$v|_{\partial\Omega} = 0 \quad \text{on } S, \tag{2.7}$$
$$v|_{t=0} = v_0 \quad \text{in } \Omega, \tag{2.8}$$

Because of [1, Theorem 4.6], there is some ε_2 such that, if

$$\|v_0\|_{H^1} + \|f\|_{L^2(Q)} \leq \varepsilon_2,$$

then (2.5)–(2.8) has a unique solution $v' \in L^2(0,\infty; H^2(\Omega)^d) \cap H^1(0,\infty; L^2_\sigma(\Omega))$. Here

$$\|f\|_{L^2(Q)} \leq \|\mu_0\|_{L^2(0,\infty;L^6(\Omega))} \|\nabla c\|_{L^\infty(0,\infty;L^3(\Omega))} \leq C(R) E(c_0, v_0)^{\frac{1}{2}}$$

because of Lemma 2.2. Therefore $\|v_0\|_{H^1} + \|f\|_{L^2(Q)} \leq \varepsilon_2$ provided that $\|v_0\|_{H^1} + E(c_0, v_0) \leq \delta$ for some $\delta > 0$ sufficiently small.

Moreover, because of [1, Proposition 4.8], the weak solution v of (2.5)–(2.8) coincides with the (strong) solution v'. Finally, from the contraction mapping argument in the proof of [1, Theorem 4.6] it can be easily seen that $\|v\|_{BUC([0,\infty);H^1(\Omega))} \leq \varepsilon$ provided that $\|v_0\|_{H^1} + E(c_0, v_0)$ is sufficiently small. Since

$$\|v_0\|_{H^1} \leq \|v_0\|_{H^2}^{\frac{1}{2}} \|v_0\|_{L^2}^{\frac{1}{2}} \leq C R^{\frac{1}{2}} E(c_0, v_0)^{\frac{1}{4}},$$

the same is true if $E(c_0, v_0) \leq \delta$ for some sufficiently small $\delta > 0$. ∎

Proof of Theorem 2.1. The estimate (2.3) follows from Lemma 2.2 and Lemma 2.3 for sufficiently small δ. That $c \in L^\infty(0,\infty; W^2_r(\Omega))$ and $v \in BUC([0,\infty); H^s(\Omega))$ for every $s < 2$ follows from the regularity results for the system (2.5)–(2.8) in the same way as in the proof of [1, Lemma 6.2]. Finally, uniqueness follows from Proposition 1.4. ∎

3. Existence of a global attractor. In this section we show existence of a weak global attractor using the concepts and results of Cheskidov and Foias [8]. We will show existence of a weak global attractor in the space

$$X_0 = \{(v, c) \in L^2_\sigma(\Omega) \times H^1_0(\Omega) : c(x) \in [a, b] \text{ a.e.}\}.$$

First of all, we show existence of a bounded absorbing set in X_0.

LEMMA 3.1. *There is some $R > 0$ such that*

$$B_R(0) = \{(v, c) \in X_0 : \|v\|_{L^2} + \|c\|_{H^1} \leq R\}$$

is an absorbing set in the sense that for any $R' > 0$ and any weak solution (v, c) of (1.1)–(1.7) on $(0,\infty)$ with $\|v|_{t=0}\|_{L^2} + \|c|_{t=0}\|_{H^1} \leq R'$ there is some $t_0 > 0$ depending only on R' such that

$$\|v(t)\|_{L^2} + \|c(t)\|_{H^1} \leq R \quad \textit{for all } t \geq t_0.$$

Proof. W.l.o.g. let $\int_\Omega c_0 \, dx = 0 \in (a, b)$. First of all, since $\phi \in C((a, b))$, $\lim_{s\to b} \phi(s) = \infty$, and $\lim_{s\to a} \phi(s) = -\infty$, there is some $m_0 > 0$ such that

$$\phi(s)s \geq -m_0 \quad \text{for all } s \in (a, b).$$

Therefore (1.3) (with $\varepsilon = 1$) and (1.6) imply

$$(\mu_0, c)_\Omega = \|\nabla c\|^2_{L^2(\Omega)} + \int_\Omega \phi(c)c\,dx \geq \|\nabla c\|^2_{L^2(\Omega)} - |\Omega| m_0.$$

Hence

$$\|\nabla c\|^2_{L^2(\Omega)} \leq C\|\nabla \mu\|_{L^2(\Omega)}\|\nabla c\|_{L^2(\Omega)} + C',$$

where C, C' are independent of (v, c). Thus

$$E(c(t), v(t)) \leq C(\|\nabla \mu(t)\|^2_{L^2(\Omega)} + \|v(t)\|^2_{L^2(\Omega)}) + C'$$

for some constants $C, C' > 0$ independent of c, which implies that

$$M(t) := \max\{E(c(t), v(t) - C', 0\} \leq C(\|\nabla \mu(t)\|^2_{L^2(\Omega)} + \|v(t)\|^2_{L^2(\Omega)}) \in L^1(0, \infty).$$

Moreover, $M(t) \leq M(s)$ for almost every $0 \leq s < \infty$ and all $t \in [s, \infty)$ since the same is true for $E(c(t), v(t))$. Therefore

$$tM(t) \leq \int_0^t M(s)\,ds \leq CE(c_0, v_0) \leq C''(R')$$

holds for all $t > 0$, which yields

$$\|\nabla c(t)\|^2_{L^2(\Omega)} + \|v(t)\|^2_{L^2(\Omega)} \leq CE(c_0, v_0)t^{-1} + C'$$

uniformly in $t > 0$.

Now let $R = 2C'$. Then there is some t_0 depending only on $E(c_0, v_0)$ such that

$$\|\nabla c(t)\|^2_{L^2(\Omega)} + \|v(t)\|^2_{L^2(\Omega)} \leq R \quad \text{for all } t \geq t_0.$$

This finishes the proof. ∎

Because of the latter lemma, we can restrict ourselves to $X = \{(v, c) \in X_0 : \|v\|_{L^2} + \|c\|_{H^1} \leq R\}$ for the study of the asymptotic behavior of the system. (Here $R > 0$ is the same as in the previous lemma.) We equip X_0 with either the strong or the weak topology metrized by d_s, d_w, respectively. (Note that the weak topology on X is metrizable since X_0' is separable.) We define an evolutionary system $\mathcal{E}$ as in [8]. To this end let

$$\mathcal{I} = \{I : I = [T, \infty) \text{ for some } T \in \mathbb{R} \text{ or } I = (-\infty, \infty)\}.$$

Moreover, if $I \in \mathcal{I}$, then $\mathcal{F}(I)$ denotes the set of all $f \colon I \to X$. Then an evolutionary system $\mathcal{E}$ as defined in [8] is a mapping such that $\mathcal{E}(I) \subseteq \mathcal{F}(I)$ satisfying the following conditions:

1. $\mathcal{E}([0, \infty)) \neq \emptyset$.
2. $\mathcal{E}(s + I) = \{u : u(\cdot - s) \in \mathcal{E}(I)\}$ for all $s \in \mathbb{R}$.
3. $\{u|_{I_2} : u \in \mathcal{E}(I_1)\} \subseteq \mathcal{E}(I_2)$ for all $I_1, I_2 \in \mathcal{I}$ such that $I_2 \subseteq I_1$.
4. $\mathcal{E}((-\infty, \infty)) = \{u : u|_{[T,\infty)} \in \mathcal{E}([T, \infty))$ for all $T \in \mathbb{R}\}$.

For every $t \geq 0$ we define the mapping $R(t) \colon \mathcal{P}(X) \to \mathcal{P}(X)$ by

$$R(t)A := \{u(t) : u(0) \in A, u \in \mathcal{E}([0, \infty))\}.$$

We recall that a set $A \subset X$ is called a $d_\bullet$-attracting set ($\bullet = s, w$) if it attracts X uniformly in the $d_\bullet$-metric, i.e., for any $\varepsilon > 0$ there is some $t_0 > 0$ such that

$$R(t)X \subseteq B_\bullet(A, \varepsilon) \quad \text{for all } t \geq t_0,$$

where $B_\bullet(A,\varepsilon) = \{x \in X : d_\bullet(x,A) < \varepsilon\}$. $A_\bullet \subseteq X$ is a $d_\bullet$-global attractor ($\bullet = s, w$) if $A_\bullet$ is a minimal $d_\bullet$-closed $d_\bullet$-attracting set, cf. [8, Definition 2.3].

Now we define $\mathcal{E}$ by

$$\begin{aligned}&\mathcal{E}([T,\infty)) = \\ &\{(v,c) \in \mathcal{F}([T,\infty)) : (v,c) \text{ is a weak solution of (1.1)–(1.7) on } [T,\infty)\}.\end{aligned}$$

for $T \in \mathbb{R}$ and

$$\mathcal{E}((-\infty,\infty)) = \{u \in \mathcal{F}((-\infty,\infty)) : u|_{[T,\infty)} \in \mathcal{E}([T,\infty)) \text{ for all } T \in \mathbb{R}\}.$$

It is easy to check that $\mathcal{E}$ is an evolutionary system in the sense above.

THEOREM 3.2. *Let $d = 2, 3$. Then there exists a weak global attractor $\mathcal{A}_w$ that is a maximal invariant set and satisfies*

$$\mathcal{A}_w = \{u_0 \in X_w : u_0 = u(0) \text{ for some } u \in \mathcal{E}((-\infty,\infty))\}.$$

Moreover, if $d = 2$, then $\mathcal{A}_w$ is a d_s-compact and strong global attractor.

Proof. Since $\mathcal{E}([0,\infty)) \subseteq C([0,\infty); H_w)$, the first part follows from Theorem 2.11, Corollary 2.12, and Theorem 2.14 in [8].

The second part follows from [8, Theorem 2.16] if we show that $R(t)$ is asymptotically d_s-compact (if $d = 2$) in the sense that for any sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that $t_n \to_{n\to\infty} \infty$ and any $x_n \in R(t_n)X$ the sequence $(x_n)_{n\in\mathbb{N}}$ is relatively compact with respect to d_s. To this end, we show that there is some $M > 0$ such that

$$\|v\|_{BUC([1,\infty);H^1(\Omega))} \leq M \tag{3.1}$$

for any $(v,c) \in \mathcal{E}([0,\infty))$. Combining this with Lemma 2.2, it easily follows that $R(t)$ is asymptotically d_s-compact.

In order to show (3.1), let $(v,c) \in \mathcal{E}([0,\infty))$ be arbitrary. Since

$$\begin{aligned}|\{t \in [0,1] : \|\nabla v(t)\|^2_{L^2(\Omega)} \geq \lambda\}| &\leq \frac{1}{\lambda}\int_0^1 \|\nabla v(t)\|^2_{L^2(\Omega)}\,dt \\ &\leq \frac{CE(c(0),v(0))}{\lambda} \leq \frac{C(R)}{\lambda}\end{aligned}$$

for all $\lambda > 0$ because of (1.12), there is some $t_0 \in [\frac{1}{2}, 1]$ such that

$$\|v(t_0)\|_{H^1} \leq C(R)$$

for some $C(R)$ independent of $(v,c) \in \mathcal{E}([0,\infty))$. Moreover, because of Lemma 2.2, $c \in BUC([t_0,\infty); W^1_q(\Omega))$ and

$$\|c\|_{BUC([t_0,\infty);W^1_q(\Omega))} \leq C(R)$$

where $q > 3$. Therefore

$$\|\mu_0\nabla c\|_{L^2(Q_{t_0})} \leq \|\mu_0\|_{L^2(t_0,\infty;L^6(\Omega))}\|c\|_{BUC([t_0,\infty);W^1_3(\Omega))} \leq C(R).$$

Hence, using the regularity results for the linear Stokes system with viscosity $\nu(c) \in BUC([t_0,\infty); W^1_q(\Omega))$, cf. [1, Proposition 4.5], one can show in the same way as in the proof of [1, Lemma 6.2] that $v \in BUC([t_0,\infty);H^1(\Omega)^d)$ and

$$\|v\|_{BUC([1,\infty);H^1(\Omega))} \leq \|v\|_{BUC([t_0,\infty);H^1(\Omega))} \leq C(R)$$

for some constant $C(R)$ independent of $(v,c) \in \mathcal{E}([0,\infty))$, which implies (3.1). ■

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