NONLOCAL AND ABSTRACT PARABOLIC EQUATIONS
AND THEIR APPLICATIONS
BANACH CENTER PUBLICATIONS, VOLUME 86
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2009

ANISOTROPIC GEOMETRIC FUNCTIONALS AND GRADIENT FLOWS

GIOVANNI BELLETTINI

Dipartimento di Matematica, Università di Roma "Tor Vergata"
via della Ricerca Scientifica, 00133 Roma, Italy
and
INFN Laboratori Nazionali di Frascati
via E. Fermi, 40, 00044 Frascati (Roma), Italy
E-mail: Giovanni.Bellettini@lnf.infn.it

LUCA MUGNAI

Max Planck Institute for Mathematics in the Sciences
Inselstr. 22, D-04103 Leipzig, Germany
E-mail: mugnai@mis.mpg.de

Abstract. We survey some recent results on the gradient flow of an anisotropic surface energy, the integrand of which is one-homogeneous in the normal vector. We discuss the reasons for assuming convexity of the anisotropy, and we review some known results in the smooth, mixed and crystalline case. In particular, we recall the notion of calibrability and the related facet-breaking phenomenon. Minimal barriers as weak solutions to the gradient flow in case of nonsmooth anisotropies are proposed. Furthermore, we discuss some relations between cylindrical anisotropies, the prescribed curvature problem and the capillarity problem. We conclude the paper by examining some higher order geometric functionals. In particular we discuss the anisotropic Willmore functional and compute its first variation in the smooth case.

1. Introduction. Mean curvature flow, namely the gradient flow of the area functional, received a lot of attention in the recent literature [Br], [Hu], [Gr], [EH], [ES], [CGG], [Il1]-[Il4], [Ec], mainly because of its relations with differential geometry, minimal surfaces and materials science. From the physical point of view its relevance becomes even more apparent once we extend the geometric evolution to the anisotropic setting [Sp], [BBR]. This extension has also a geometric meaning, which corresponds to looking at the

2000 *Mathematics Subject Classification*: Primary 53C44; Secondary 74E10.
Key words and phrases: gradient flow, anisotropy, minimal barrier.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc86-0-2

evolution from the point of view of Finsler geometry [BP1]. In this case the functional to flow is of the form

$$\int_{\partial E} \phi^o(\mathbf{n})\, d\mathcal{H}^{n-1}, \tag{1}$$

where the unit covector $\mathbf{n}(x)$ at $x \in \partial E$ identifies the tangent space to ∂E at x, and is in turn identified with the outer unit normal vector field to ∂E at x. Moreover, ϕ^o is a suitable function and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure in $\mathbf{R}^n$. We begin this paper by reviewing some recent results on the gradient flow of (1). The need of assuming ϕ^o well defined on the whole unit sphere is shown in Example 2.1, while the reason for assuming convexity is explained in Example 2.2. Neglecting these two assumptions may lead to vanishing denominators, or to forward-backward geometric evolution problems, that are out of the scope of the present paper. Even under these two assumptions, several nontrivial problems arise, for instance in the crystalline case, namely when the unit ball of ϕ^o_{ext} is a polyhedron, where ϕ^o_{ext} denotes the one-homogeneous extension of ϕ^o to all one-covectors. This case is of particular interest in materials science, as observed in [CHT]. We review some recent results on uniqueness of the crystalline flow (Section 4), on its existence under convexity assumptions, and on calibrability and its applications in the mixed case (Section 5). Again concerning crystalline mean curvature flow, a new observation is given in Section 7: once we have at our disposal a local in time existence result (for *convex* initial sets), we are able to define a weak evolution using the minimal barriers approach of De Giorgi [DeG], with no restrictions on the space dimension. We note that an approach based on the level set method seems not to be available, at the moment, in such generality (see however [Gi] for developments in this direction).

Nonsmooth anisotropic mean curvature flow has unexpected connections with the prescribed curvature problem, with the capillarity problem and with the total variation flow. Such connections are briefly described in Section 6.

As already observed in [CHT], also evolutions of higher order functionals may be of some interest in materials science, see also [BGN]; moreover, such evolutions may appear in differential geometry [D], [KS]. These flows in general are more difficult to analyze with respect to anisotropic mean curvature flow, one reason being the lackness of the comparison principle in the standard form. Among higher order geometric evolutions, the surface diffusion flow is the one where the normal velocity of the hypersurface equals minus the laplacian of the mean curvature κ. Under this evolution, the surface area of the flowing manifold decreases, while the enclosed volume remains constant. The laplacian of the mean curvature arises, also as the leading term in the first variation of the Willmore functional

$$\int_{\partial E} \kappa^2\, d\mathcal{H}^{n-1}. \tag{2}$$

The functional (2) is a particular case of a second order functional of the form

$$\int_{\partial E} \Phi(\mathbf{n}, B)\, d\mathcal{H}^{n-1}, \tag{3}$$

where, if B denotes the second fundamental form of ∂E, $\Phi(\mathbf{n}, B) := (\text{tr}(B))^2$. The question then arises of what can be a reasonable generalization of the surface laplacian of the mean curvature in the anisotropic (also crystalline) setting [CRCT], [DG], [EG], [TC]. In the effort of answering this question it seems reasonable to consider, for a smooth uniformly convex function $\phi^o_{\rm ext}$, the first variation of the anisotropic Willmore functional, which is still a functional the form (3). This computation is performed in Section 8, and gives rise to a possible definition of the anisotropic surface diffusion flow. We conclude this introduction by mentioning that the crystalline case is not covered by this computation.

2. The map ϕ^o. We indicate by $(\cdot,\cdot)$ the standard scalar product between two vectors or between two covectors in $\mathbf{R}^n$, and we set $|\cdot|^2 = (\cdot,\cdot)$. Let Ω be an open subset of $\mathbf{R}^n$, $n \geq 1$. For $p \in \Omega$, the tangent space $\mathrm{T}_p(\Omega)$ to Ω at p, and its cotangent space $(\mathrm{T}_p(\Omega))^*$, are isomorphic to $\mathbf{R}^n$. Let

$$\phi^o : \mathrm{dom}(\phi^o) = \{\xi^* \in (\mathrm{T}_p(\Omega))^* : \phi^o(\xi^*) < +\infty\} \subseteq \{\xi^* \in (\mathrm{T}_p(\Omega))^* : |\xi^*| = 1\} \to [0, +\infty)$$

be a nonnegative continuous function with the following property: if $\phi^o_{\rm ext}$ stands for the one-homogeneous extension of ϕ^o to the cone $\{\lambda \mathrm{dom}(\phi^o) : \lambda \geq 0\}$, then the set $\{\phi^o_{\rm ext} \leq 1\} \subset (\mathrm{T}_p(\Omega))^*$ is *star-shaped* with respect to the origin.

If $(\phi^o_{\rm ext})^2 \in \mathcal{C}^1(\mathrm{dom}(\phi^o_{\rm ext}))$, we set, on $\mathrm{dom}(\phi^o_{\rm ext})$,

$$T_{\phi^o_{\rm ext}} := \frac{1}{2}\nabla\left((\phi^o_{\rm ext})^2\right), \tag{4}$$

where $\nabla\left((\phi^o_{\rm ext})^2\right)$ is identified with the gradient of the scalar field $(\phi^o_{\rm ext})^2$, hence a vector field. Therefore $T_{\phi^o_{\rm ext}}(\xi^*)$ is regarded as an element of $(\mathrm{T}_p(\Omega))^{**} \simeq \mathrm{T}_p(\Omega)$.

In what follows an important role will be played by the image of $\{\phi^o_{\rm ext} = 1\}$ through the gradient mapping (4), namely by the set $T_{\phi^o_{\rm ext}}(\{\phi^o_{\rm ext} = 1\})$.

Example 2.1 (Dirichlet integral). Let $n \geq 2$, $A \subseteq \mathbf{R}^{n-1}$ be an open set and $u \in \mathcal{C}^1_c(A)$. Then

$$\int_A |\nabla u|^2 \, dx = \int_{\mathrm{graph}(u)} \phi^o(\mathbf{n}) \, d\mathcal{H}^{n-1}, \tag{5}$$

where $\mathbf{n}$ is the unit normal to $\mathrm{graph}(u) \subset \Omega := A \times \mathbf{R}$ pointing for instance toward the epigraph of u and, setting $\xi^* = (\widehat{\xi^*}, \xi^*_n)$, $\widehat{\xi^*} = (\xi^*_1, \ldots, \xi^*_{n-1})$, we have

$$\mathrm{dom}(\phi^o_{\rm ext}) := \{\xi^* \in \mathbf{R}^n : \xi^*_n > 0\}, \qquad \phi^o_{\rm ext}(\xi^*) = |\widehat{\xi^*}|^2/\xi^*_n.$$

If $n = 2$, the set $\{\phi^o_{\rm ext} \leq 1\}$ is the (closed convex unbounded) epigraph of the function $\xi^*_2 = (\xi^*_1)^2$, which contains the origin in its boundary. Note that the natural regularization $|\widehat{\xi^*}|^2/\sqrt{(\xi^*_n)^2 + \epsilon^2}$ of $\phi^o_{\rm ext}(\xi^*)$ is not one-homogeneous (see [Ev] for related arguments on the evolution problem). Note also that if one writes ϕ^o as a supremum, then using (5) it may become possible to look for harmonic functions in A with the calibration method, see [ABD].

Example 2.2 (Perona-Malik functional). Let $n = 2$, $A \subseteq \mathbf{R}$ be an interval and $u \in \mathrm{Lip}(A)$. Then

$$\int_A \log(1 + (u_x)^2) \, dx = \int_{\mathrm{graph}(u)} \phi^o(\mathbf{n}) \, d\mathcal{H}^1,$$

where $\phi^o_{\rm ext}(\xi^*) = \log\big(1 + (\xi_1^*)^2/(\xi_2^*)^2\big)\,|\xi_2^*|$, and $\{\phi^o_{\rm ext} \le 1\}$ is a nonconvex (and nonconcave) unbounded set containing the origin in its interior. We refer to [FGP] (and references therein) for a discussion on the interesting phenomena related to the Perona-Malik equation, which is obtained as the gradient flow of the Perona-Malik functional.

2.1. *Anisotropies.* The degeneracy of ϕ^o at $\xi_n^* = 0$ (Example 2.1) leads to difficulties in the related evolution problems, see for instance [Ev], [BCN1]. For this reason from now on we assume that

$$\mathrm{dom}(\phi^o) = \{\xi^* \in (\mathrm{T}_p(\Omega))^* : |\xi^*| = 1\},$$

(in particular $\mathrm{dom}(\phi^o_{\rm ext}) = (\mathrm{T}_p(\Omega))^*$) and that the origin is in the interior of $\{\phi^o_{\rm ext} \le 1\}$.

Remark 2.1. Even if $n = 2$ and $(\phi^o_{\rm ext})^2 \in \mathcal{C}^1((\mathrm{T}_p(\Omega))^*)$, the map $\phi^o_{\rm ext}$ can be nonconvex, and therefore the set

$$T_{\phi^o}(\{\phi^o_{\rm ext} = 1\})$$

can be a curve with transverse self-intersections and cusps (Example 2.2). In this case, the related evolution problem is forward-backward, and suffers of instabilities and formation of microstructures. To avoid such a kind of phenomena we assume from now on that $\phi^o_{\rm ext}$ is *convex.* Convexity allows to define the gradient mapping $T_{\phi^o_{\rm ext}}$ in (4) without smoothness assumptions, provided we interpret $\nabla((\phi^o_{\rm ext})^2)$ as the subdifferential of $(\phi^o_{\rm ext})^2$: note that $T_{\phi^o_{\rm ext}}$ is in general multivalued, and is a maximal monotone graph. Finally, we will also assume for simplicity that $\phi^o_{\rm ext}$ is even (i.e., $\phi^o_{\rm ext}(\xi^*) = \phi^o_{\rm ext}(-\xi^*)$), even if this requirement could be often dropped.

Definition 2.1. A function ϕ^o satisfying all the above assumptions will be called an anisotropy.

From now on, given an anisotropy ϕ^o, if no confusion is possible and with an abuse of notation, we write ϕ^o in place of $\phi^o_{\rm ext}$. We set $B_{\phi^o} := \{\phi^o \le 1\}$. Sometimes, ϕ^o is referred to as the surface tension and assigns to each flat interface a positive weight. The interface is the tangent space to ∂E at $x \in \partial E$ (identified with $\mathbf{n}(x)^\perp$), and is the kernel of the linear map associated with the covector $\mathbf{n}(x)$. B_{ϕ^o} is sometimes called the Frank diagram.

Definition 2.2 (Wulff shape). We define the map $\phi : (\mathrm{T}_p(\Omega))^{**} \simeq \mathrm{T}_p(\Omega) \to [0, +\infty)$ as the anisotropy such that $\{\phi \le 1\} = T_{\phi^o}(B_{\phi^o})$.

Under our convexity assumption, ϕ^o and ϕ are dual to each other. We set $B_\phi := \{\phi \le 1\}$; sometimes B_ϕ is called the Wulff shape. We let $T_\phi := \frac{1}{2}\nabla(\phi^2)$. Under our assumptions one can show that $T_\phi T_{\phi^o} = T_{\phi^o} T_\phi = \mathrm{Id}$, and that $\phi(\xi) = \sup\{\langle \xi^*, \xi\rangle : \xi^* \in B_{\phi^o}\}$ for any vector $\xi \in \mathbf{R}^n$, where $\langle \cdot, \cdot \rangle$ is the duality between covectors and vectors.

A first example of anisotropy is the Riemannian one, where B_{ϕ^o} (and hence B_ϕ) is an ellipsoid. A wide generalization of this is the *regular* case (smooth spatially homogeneous even Finsler case), namely when B_{ϕ^o} is a smooth uniformly convex body (and then the same holds for B_ϕ [Sc]).

Example 2.3. The *crystalline* case refers to the situation where B_{ϕ^o} (and hence B_ϕ) is a polyhedron. It will be analyzed in Sections 4 and 5.

EXAMPLE 2.4. The *mixed* case, where for instance B_{ϕ^o} can be strictly convex but not smooth, or smooth but not strictly convex. Also the degree of smoothness plays a role, as can be seen by the unit balls of the l^p-norms, for $p \in (1, +\infty)$. The mixed case is of interest, for instance, when B_ϕ is the cartesian product of a convex body with a segment: a useful case is when B_ϕ is the portion of a circular cylinder (hence B_{ϕ^o} is a "double" cone), which will be discussed in Section 6.

3. Some geometric functionals. Associated with an anisotropy ϕ^o, there are several geometric functionals that could be studied, together with their gradient flow.

EXAMPLE 3.1 (Anisotropic mean curvature flow, regular case). The first functional is the anisotropic perimeter of a set of locally finite perimeter in $\mathbf{R}^n$ (see Section 1), namely

$$P_\phi(E) := \int_{\partial E} \phi^o(\mathbf{n})\ d\mathcal{H}^{n-1}, \tag{6}$$

where $\mathbf{n}$ is identified with the (set-theoretic) outward unit normal to the (set-theoretic) boundary ∂E of E. When ϕ^o is the euclidean norm $P_\phi(E)$ reduces to the usual perimeter $P(E)$ of E. When ϕ^o is regular, the gradient flow of P_ϕ has been studied by several authors [GG], [BP1], [GGIS], and gives rise to the anisotropic mean curvature flow, where, roughly speaking, the velocity field $\mathbf{V}$ equals the anisotropic mean curvature $\kappa_\phi \mathbf{n}_\phi =: \mathbf{H}_\phi$, and hence the normal velocity $\mathbf{V}\cdot\mathbf{n}$ equals $\phi^o(\mathbf{n})\kappa_\phi$. We recall for convenience of the reader the definitions of κ_ϕ and of the vector field $\mathbf{n}_\phi$ (sometimes called the Cahn-Hoffman vector field), when ∂E is smooth enough. We let

$$\mathbf{n}^*_\phi(x) := \frac{\mathbf{n}(x)}{\phi^o(\mathbf{n}(x))}, \qquad \mathbf{n}_\phi(x) := T_{\phi^o}(\mathbf{n}^*_\phi(x)), \qquad x \in \partial E.$$

Then, for any $C \subseteq \mathbf{R}^n$ and $x \in \mathbf{R}^n$, we set $d_\phi(x, C) := \inf_{y\in C} \phi(y - x)$, and

$$d_\phi(x) := d_\phi(x, E) - d_\phi(x, \mathbf{R}^n \setminus E), \qquad x \in \mathbf{R}^n.$$

Since it turns out that $\nabla d_\phi = \mathbf{n}^*_\phi$ on ∂E, we have that ∇d_ϕ (identified if necessary with the differential of d_ϕ) is a covector field extending $\mathbf{n}^*_\phi$ also outside ∂E, in a suitable neighbourhood U of ∂E. We then define

$$\mathbf{N}_\phi := T_{\phi^o}(\nabla d_\phi) \qquad \text{in } U, \tag{7}$$

which is an extension in U of the vector field $\mathbf{n}_\phi$. Finally, we set

$$\kappa_\phi := \mathrm{div}\mathbf{N}_\phi \qquad \text{on } \partial E,$$

where the divergence is taken in the ambient space $\mathbf{R}^n$. We recall [BP1] that the evolution law $\mathbf{V} = \mathbf{H}_\phi$ is a consequence of the expression of the first variation of P_ϕ, which takes the following form. If $\alpha_\lambda(x) := x + \lambda X(x)$ is a smooth diffeomorphism of $\mathbf{R}^n$, with X the vector field (with compact support) describing the initial deformation, and $E_\lambda := \alpha_\lambda(E)$, then

$$\frac{d}{d\lambda} P_\phi(E_\lambda)_{|\lambda=0} = \int_{\partial E} \langle \mathbf{H}^*_\phi, X\rangle \phi^o(\mathbf{n})\ d\mathcal{H}^{n-1}, \tag{8}$$

where $\mathbf{H}^*_\phi := \kappa_\phi \mathbf{n}^*_\phi$.

REMARK 3.1. It is interesting to note that, if we adopt definition (4), the vector field $T_{\phi^o}(\mathbf{n}^*_\phi)$ is well defined also in the nonconvex (smooth) case, for instance in Example 2.2. without any convexification. In this case, definition (4) could be preferable to the one given by $\mathrm{argmax}\{\langle \mathbf{n}^*_\phi(x), \xi\rangle : \xi \in B_\phi\}$.

EXAMPLE 3.2 (Prescribed curvature functionals). An interesting generalization of (6) is the anisotropic prescribed curvature functional with given contact angle at $\partial\Omega$, namely, for a regular ϕ^o,

$$\int_{\Omega\cap\partial E} \phi^o(\mathbf{n})\, d\mathcal{H}^{n-1} - \int_{\Omega\cap E} g\, dx + \int_{\partial\Omega\cap\partial E} \mu\, d\mathcal{H}^{n-1}, \tag{9}$$

where $g : \mathbf{R}^n \to \mathbf{R}$ is a given bounded sufficiently regular function, and $\mu : \partial\Omega \to [-1, 1]$ is given. Sufficiently smooth minimizers of (9) have anisotropic mean curvature κ_ϕ equal to g inside Ω, and contact angle suitably related to μ and ϕ^o on $\partial\Omega$. When $\Omega = \mathbf{R}^n$ (and $\mu \equiv 0$) the corresponding gradient flow is the forced anisotropic mean curvature flow, $\mathbf{V} = (\kappa_\phi + g)\mathbf{n}_\phi$. Variational properties of (9) via a convexification argument based on the coarea formula were considered in [BPV1], [BPV2], [BP] as well as a numerical study of the corresponding convex algorithms. See also Section 6. Mean curvature flow with some kind of boundary conditions on $\partial\Omega$ was considered in [Hu1], [MT]. As shown in [ATW], the variational analysis of (8) is at the basis of the *weak* definition of anisotropic mean curvature flow via the minimizing movements method.

The next example generalizes the geometric evolutions obtained from the functional (6). For $m \geq 1$, denote by $SBV(\Omega; \mathbf{R}^m)$ the space of $\mathbf{R}^m$-valued special functions of bounded variation in Ω [AFP].

EXAMPLE 3.3 (Free boundaries). Let X be a subset of $SBV(\Omega; \mathbf{R}^m)$, and $\psi : \mathbf{R}^m \times \mathbf{R}^m \times \mathbf{R}^n \to [0, +\infty]$ be a suitable function (see [AFP]). Consider the functional

$$\int_{J_u} \psi(u^-, u^+, \mathbf{n}_u)\, d\mathcal{H}^{n-1}, \tag{10}$$

where $u \in X$, $u^\pm$ are the two traces of u on its jump set J_u, oriented by a unit normal covector field $\mathbf{n}_u$, and $\psi = +\infty$ if $u \notin X$. Suitable choices of X and ψ lead to interesting generalizations of P_ϕ. For instance, when $m = 1$, and X consists of locally constant functions u, i.e., $\nabla^a u = 0$ almost everywhere in Ω, where $\nabla^a u$ denotes the absolutely continuous part of the measure derivative of u, (10) gives the functional of anisotropic partitions of Ω [AFP], [AT]. If $n = 2$ therefore, triple junctions are allowed, when more than two phases intersect. A study on the flow of crystalline partitions in the plane was considered in [BCN]. Another choice was considered in [Ch], in the effort of modelling a multhiphase planar problem associated with polycrystalline materials with a Wulff shape that, in each phase, is the rotation of a fixed Wulff shape [GN]. Let $\phi^o : \mathbf{R}^2 \to [0, +\infty)$ be an anisotropy. Take $X := \{u \in \mathrm{SBV}(\mathbf{R}^2; \mathbf{S}^1) \;:\; \nabla^a u = 0, \mathcal{H}^1(J_u) < +\infty\}$, where $\mathbf{S}^1 := \{y \in \mathbf{R}^2 : |y| = 1\}$. Then, roughly speaking, the integrand function in (10), written in complex notation, takes the form $\phi^o(e^{i[\frac{u^+ + u^-}{2} + \arg(\mathbf{n}_u)]})$. That is, we evaluate ϕ^o on the rotation of $\mathbf{n}_u$ of $(u^+ + u^-)/2$, which is an angle depending on the values of u off the interface.

4. Anisotropic mean curvature flow for crystalline or mixed ansotropies. Assume that the anisotropy is crystalline or mixed, so that the maps T_{ϕ^o} and T_ϕ may be multivalued (necessarily in the crystalline case). In the effort of proving existence of the geometric variational flow associated with (6), we introduce various classes of "smooth" manifolds where the flow is expected to take place. A first possibility is to introduce the class of Lipschitz ϕ-regular sets. We denote by E a closed set with nonempty interior $\mathrm{int}(E)$ and with *Lipschitz* compact boundary (and the same if $E(t)$ varies with time).

DEFINITION 4.1 (The class $L_\phi R$). We say that E is Lipschitz ϕ-regular, and we write $E \in L_\phi R$, if there exist a neighbourhood U of ∂E and a Lipschitz vector field $\mathbf{N}_\phi$ defined in U, satisfying $\mathbf{N}_\phi \in T_{\phi^o}(\nabla d_\phi)$ almost everywhere in U.

In the crystalline case, several polyhedral sets E are in $L_\phi R$ (while the euclidean unit sphere $\{|x| \leq 1\}$ is not in $L_\phi R$). In addition, there are $L_\phi R$-sets which are not polyhedral, and actually these are of importance, since it may happen that the crystalline mean curvature evolved set starting from a $L_\phi R$-polyhedral set is *not* a $L_\phi R$-polyhedral set at later times. To our best knowledge, a local in time existence result for the geometric flow in the class $L_\phi R$ is known only if $n = 2$, while is not known for $n \geq 3$, even under convexity assumptions. Weaker notions of "smooth" sets will now be given, one of them allowing a local in time existence result in the convex case. Fortunately, a uniqueness result holds in the larger class (Theorem 4.1).

DEFINITION 4.2 (The class ${}_\phi R$). If we require the vector field $\mathbf{N}_\phi$ in Definition 4.1 to satisfy $\mathbf{N}_\phi \in T_{\phi^o}(\nabla d_\phi)$ almost everywhere in U, and to have $\mathrm{div}\mathbf{N}_\phi \in L^\infty(U)$, then we say that E is a ϕ-regular set, and we write $E \in {}_\phi R$.

It is clear that $L_\phi R \subset {}_\phi R$.

The following definition is essentially the one given in [BN].

DEFINITION 4.3 (ϕ-regular flows). Let $a < b$. A ϕ-regular flow in $[a, b]$ is a map $t \in [a, b] \to E(t)$ which admits an open set $A \subseteq \mathbf{R}^n \times [a, b]$ containing $\bigcup_{t\in[a,b]} \partial E(t) \times \{t\}$ with the following properties: the function $d(x, t) := d_\phi(x, E(t)) - d_\phi(x, \mathbf{R}^n \setminus E(t))$ is Lipschitz in A, there exists a vector field $N \in L^\infty(A; \mathbf{R}^n)$ such that $\mathrm{div}N \in L^\infty(A)$ and $N \in T_{\phi^o}(\nabla d)$ almost everywhere in A, and there exists $c > 0$ such that $|\partial_t d - \mathrm{div}N| \leq c|d|$ almost everywhere in A.

The last requirement in Definition 4.3 is a way to impose the manifolds $\partial E(t)$ to flow with velocity equal to a sort of anisotropic mean curvature: Theorem 4.1 below shows that the evolution does not depend on the choice of the vector field N having the required constraints. Note that adopting the evolution law in Definition 4.3 allows to avoid the use of tangential derivatives and, in general, tangential operators on (nonsmooth) manifolds.

The following result, proved in [BN] via a reaction-diffusion approximation, provides uniqueness of a ϕ-regular flow. A different proof based on the properties of the heat semigroup can be found in [CN].

THEOREM 4.1 (Comparison and uniqueness of the flow in the class ${}_\phi R$). *Let $E_1(t)$, $E_2(t)$ be two ϕ-regular flows in $[a, b]$. Then $E_1(a) \subseteq E_2(a) \Rightarrow E_1(t) \subseteq E_2(t)$ for all $t \in [a, b]$.*

In particular if $E_1(a) = E_2(a)$ then $E_1(t) = E_2(t)$ for any $t \in [a,b]$, hence the ϕ-regular flow is unique in $[a,b]$.

We now turn to the problem of the existence of a flow.

Definition 4.4 (The class $\mathrm{I}rB_\phi\mathrm{C}$). Let $r > 0$. We say that E satisfies the interior rB_ϕ-condition, and we write $E \in \mathrm{I}rB_\phi\mathrm{C}$ if, for any $x \in \partial E$, there exists $y \in \mathbf{R}^n$ such that

$$rB_\phi + y \subseteq E \quad \text{and} \quad x \in \partial(rB_\phi + y).$$

It is possible to prove [BNP1], [BCCN] that $E \in L_\phi R \Rightarrow \exists r > 0$ such that $E \in \mathrm{I}rB_\phi\mathrm{C}$ and $\mathbf{R}^n \setminus \mathrm{int}(E) \in \mathrm{I}rB_\phi\mathrm{C}$; moreover $\mathrm{I}rB_\phi\mathrm{C} \subset {}_\phi R$.

Definition 4.5 (rB_ϕ-regular flows). Let $a < b$ and $r > 0$. If in Definition 4.3 we assume also that $E(t)$ and $\mathbf{R}^n \setminus \mathrm{int}(E(t))$ satisfy the interior rB_ϕ-condition for any $t \in [a,b]$, then we say that $t \in [a,b] \to E(t)$ is an rB_ϕ-regular flow in $[a,b]$.

It is interesting to observe that the proof of the local existence result (Theorem 4.2 below) in the class of rB_ϕ-regular flows is based on a notion of weak solution introduced by Almgren-Taylor-Wang [ATW] (the *flat ϕ-curvature flow*, see also [LS] for a similar notion), and on a modification of it introduced in [C].

We now state the local existence result proved in [BCCN], and we refer to that paper for all details. Let $E \in \mathrm{I}rB_\phi\mathrm{C}$ be convex. Let $\mathcal{G} : \ (0,1) \times \big(L^2(\Omega) \cap BV(\Omega)\big) \times \big(L^2(\Omega) \cap BV(\Omega)\big) \to [0,+\infty]$ be the functional

$$\mathcal{G}(h,u,w) := \int_\Omega \phi^o(Du) \ + \frac{1}{2h}\int_\Omega (u-w)^2\,dx. \tag{11}$$

Define in a recursive way the functions d_h^i and the sets E_h^i as follows: for any $h \in (0,1)$ and any $i \in \mathbf{N} \cup \{0\}$, $E_h^0 := E$, $d_h^0(\cdot) := d_\phi(\cdot, E) - d_\phi(\cdot, \mathbf{R}^n \setminus E)$,

$$\mathcal{G}(h,u,d_h^i) = \min\{\mathcal{G}(h,v,d_h^i) : v \in L^2(\Omega) \cap BV(\Omega)\} \tag{12}$$

and

$$E_h^{i+1} := \{u \le 0\}, \qquad d_h^{i+1}(\cdot) := d_\phi(\cdot, E_h^{i+1}) - d_\phi(\cdot, \mathbf{R}^n \setminus E_h^{i+1}). \tag{13}$$

Theorem 4.2 (Local existence of the flow in the class $\mathrm{I}rB_\phi\mathrm{C}$ for convex initial data). *Let $E \subset \mathbf{R}^n$ be a compact convex set satisfying the rB_ϕ-condition for some $r > 0$. Then there is $T > 0$ such that*

$$\lim_{h \to 0} E_h^{[t/h]} =: E(t) \quad \textit{exists for any } t \in [0,T] \textit{ in the Hausdorff distance,}$$

and $E(0) = E$, where $[\cdot]$ denotes the integer part. Each set $E(t)$ is compact, convex and rB_ϕ-regular, and the map $t \in [0,T] \to E(t)$ is the unique local in time ϕ-regular flow starting from E.

Remark 4.1. The main obstacle in the proof of Theorem 4.2 is perhaps represented by the fact that, in general, polyhedral convex initial data may develop, under anisotropic curvature flow, the facet breaking/bending phenomena, see [BNP], [BNP3], [GR1], [GR2]. The time-step minimization procedure is sufficiently general to handle these phenomena.

5. Calibrability, facet breaking phenomenon. Under the assumption of crystalline or mixed anisotropy, in Section 4 we have described uniqueness and existence results for anisotropic mean curvature flows in various classes, adopting a definition of flow based, roughly, on the idea of evolving a tubular neighbourood of the front instead of the front itself. It is however useful to have at our disposal also a definition of anisotropic mean curvature of a single hypersurface (possibly still using differential operators on functions and vector fields extended out of the manifold). This is not immediate; it turns out that the definition of anisotropic mean curvature may become nonlocal. In the crystalline case, for instance, the velocity vector of a facet (a flat portion of ∂E corresponding to a facet of B_ϕ) is, roughly speaking, determined by the *global* shape of the facet itself.

The mean curvature of a hypersurface for a crystalline or mixed anisotropy can be defined via a minimization procedure [GGM], [BNP1], [BNP2]: assume for simplicity that $E \in L_\phi R$ has a facet $F \subset \partial E$ corresponding (in particular parallel) to a facet of B_ϕ, and that E is *convex at* F (i.e., E lies, locally around F, on one side of the hyperplane Π_F containing F). We assume that the facet of B_ϕ corresponding to F is itself the unit ball of an anisotropy in $\mathbf{R}^{n-1}$, denoted by $\widetilde{\phi}$.

In general, we will use the $\widetilde{}$ to indicate objects in reduced dimension; for instance, we denote by $\widetilde{\mathbf{n}}_F$ the unit normal to ∂F (in $\mathbf{R}^{n-1}$) pointing out of F. Finally, we let $\mathbf{n}_F \in \mathbf{S}^{n-1} \subset \mathbf{R}^n$ be the outward unit normal to E at F.

Definition 5.1. The divergence of a vector field minimizing the functional

$$\eta \to \int_F (\mathrm{div}\eta)^2 \, d\mathcal{H}^{n-1} \tag{14}$$

among all $\eta : F \to \Pi_F$ satisfying the constraint $\eta(x) \in T_{\phi^o}(\mathbf{n}_F(x))$ for $\mathcal{H}^{n-1}$-almost every $x \in F$ and having on ∂F maximal normal trace compatible with the constraint, i.e.,

$$\langle \widetilde{\mathbf{n}}_F(x), \eta(x) \rangle = \max\{\langle \widetilde{\mathbf{n}}_F(x), \widetilde{\xi} \rangle : \widetilde{\xi} \in B_{\widetilde{\phi}}\} \qquad \text{for } \mathcal{H}^{n-2}\text{-a.e. } x \in \partial F, \tag{15}$$

is called the anisotropic mean curvature of F.

Remark 5.1. The fact that the normal trace on ∂F must be maximal in the sense of equation (15) is a consequence of the convexity of E at F. It is possible to define the anisotropic mean curvature of the facet also without this convexity assumption; we refer to [BNP1], [BNP2] for the details.

Remark 5.2. Under sufficient "smoothness" assumptions, it is possible to show that the velocity of the flow considered in Section 4 coincides with the anisotropic mean curvature described above, see [BCCN], [BCCN1].

The computation of the anisotropic mean curvature of a facet F of ∂E may be interesting independently of the evolution problem. Moreover, the evolution $F(t)$ of F at (small) later times $t > 0$ could be guessed by looking at the qualitative properties of the anisotropic mean curvature of F at the initial time $t = 0$. For instance, if such a curvature is constant, the facet is expected to translate parallely to itself in normal direction for short times.

Definition 5.2. We say that F is ϕ-calibrable if F has *constant* anisotropic mean curvature, namely if there exists a vector field $\eta : F \to \Pi_F$ satisfying the following elliptic problem with constraints: $\eta(x) \in T_{\phi^o}(\mathbf{n}_F(x))$ for almost every $x \in F$,

$$\text{div}\eta = \text{const} \quad \text{on } F, \tag{16}$$

and η satisfies (15).

The divergence operator in (16) is the divergence on F; note that we could add to η a last component orthogonal to Π_F without changing either (15) (since the new component is orthogonal to $\widetilde{\mathbf{n}}_F$) or (16) (since the new component is constant on F).

It is possible to show that if such a vector field exists, then it minimizes the functional (14).

Remark 5.3. Assume that E is convex at F and that F is convex and ϕ-calibrable. Then the Gauss-Green Theorem yields that the anisotropic mean curvature in (16) is given by

$$\frac{P_{\widetilde{\phi}}(F)}{|F|}, \tag{17}$$

where $P_{\widetilde{\phi}}(F) = \int_{\partial F} \widetilde{\phi}^o(\widetilde{\mathbf{n}}_F)\, d\mathcal{H}^{n-2}$ and $|F|$ is the Lebesgue measure of F.

The following result, proved in [BNP1] in more generality, is a characterization of calibrability.

Theorem 5.1 (Minimality of the quotient perimeter/area). *F is ϕ-calibrable if and only if*

$$\frac{P_{\widetilde{\phi}}(A)}{|A|} \geq \frac{P_{\widetilde{\phi}}(F)}{|F|} \qquad \forall A \subseteq F.$$

The quotient $P_{\widetilde{\phi}}(F)/|F|$ should heuristically be viewed as the mean velocity of the facet F in normal direction. The assertion of Theorem 5.1 can be related to the comparison principle for the associated corresponding flow (Theorem 4.1). Essentially, F is ϕ-calibrable if and only if F does not split for short positive times during the flow. For instance, let us see that if F does not split then it is ϕ-calibrable. Assume that there exists $A \subset F$ such that $P_{\widetilde{\phi}}(A)/|A| < P_{\widetilde{\phi}}(F)/|F|$, and suppose by contradiction that F does not split. Then, identifying this quotient with the normal velocity, we would contradict the comparison principle. On the other hand, the comparison principle holds thanks to Theorem 4.1.

We conclude this section with the following result [BNP3] which gives a verifiable criterion for calibrability.

Theorem 5.2 (Characterization of calibrability in the convex case). *Let $n = 3$. Assume that F is convex and that E is convex at F. Then F is ϕ-calibrable if and only if*

$$\operatorname*{ess\,sup}_{x \in \partial F} \kappa_{\widetilde{\phi}}(x) \leq \frac{P_{\widetilde{\phi}}(F)}{|F|}. \tag{18}$$

6. Applications. We want to show now that the results described in the previous sections, in particular the ones concerning the mixed case, have relations with classical problems in Calculus of Variations.

Example 6.1 (Calibrability for cylindrical anisotropies and the prescribed curvature problem). Assume that $n = 3$ and that B_ϕ is the portion of a cylinder with circular section (mixed anisotropy), i.e.,

$$B_\phi := \widetilde{B} \times [-1, 1], \qquad \widetilde{B} := \{(\xi^1, \xi^2) \in \mathbf{R}^2 : (\xi^1)^2 + (\xi^2)^2 \leq 1\}. \tag{19}$$

Let $E \in L_\phi R$, and $F \subset \partial E$ be a facet of E. Define

$$\lambda := \frac{P(F)}{|F|}, \qquad \mathcal{F}_\lambda(A) := P(A) - \lambda|A| \qquad \forall A \subseteq F.$$

Clearly $\mathcal{F}_\lambda(\emptyset) = \mathcal{F}_\lambda(F) = 0$. The functional $\mathcal{F}_\lambda$ is a particular case of the one in (9), with the choices $n = 2$, $\Omega = F$, $\phi^o(\cdot) = |\cdot|$, $g \equiv \lambda$ and $\mu \equiv 0$. Note that F is ϕ-calibrable if and only if $\min_{A\subseteq F} \mathcal{F}_\lambda(A) = 0$, and non ϕ-calibrability of F means that $\min_{A\subseteq F} \mathcal{F}_\lambda < 0$. It is possible to see that minimizing $\mathcal{F}_\lambda$ among all finite perimeter subsets of F is equivalent to

$$\min_{v\in BV(F;\{\pm 1\})} \mathcal{F}_\lambda(v), \qquad \mathcal{F}_\lambda(v) := \int_F |Dv| - \int_F \lambda v + \int_{\partial F} v, \qquad v \in BV(F; \{\pm 1\}). \tag{20}$$

In [BPV1] it is proved that the minimum problem (20) can be convexified, namely $\mathcal{F}_\lambda$ can be equivalently minimized on the convex set $BV(F; [-1, 1])$. This remark allows to use convex algorithms, for instance based on the approximation $\int_F \sqrt{|Dv|^2 + \epsilon^2}$ [BP] of $\int_F |Dv|$, in order to find numerically the minimizers of $\mathcal{F}_\lambda$.

Example 6.2 (Calibrability for cylindrical anisotropies and the capillarity problem). Let be given a (bounded with nonempty interior) convex set $F \subset \mathbf{R}^2$, and set $\Omega := F \times [0, +\infty)$. The semi-infinite capillary tube Ω is the vessel. Let also $V > 0$ and $\gamma \in \mathbf{R}$ be given. The capillarity problem consists in finding solutions to

$$\inf_{Q\subseteq\Omega} \{P(Q, \Omega) + \cos\gamma \mathcal{H}^{n-2}(\partial Q \cap \partial\Omega) : |Q| = V\},$$

where $P(Q, \Omega)$ is the perimeter of Q in Ω. Take for simplicity $\cos\gamma = 1$. Then a smooth solution of the above minimum problem of the form $Q = \text{subgraph}(u) := \{(x_1, x_2, x_3) \in \Omega : 0 \leq z \leq u(x_1, x_2)\}$, with $|\text{subgraph}(u)| = V$, is such that graph(u) has mean curvature constantly equal to $P(F)/|F|$ in Ω and tangential contact with $\partial\Omega$. The graph of the function u represents a capillary surface defined in F, bounding F with a prescribed finite volume of fluid, and meeting the boundary of the vessel with a prescribed contact angle $\cos\gamma = 1$.

If $\mathbf{n}_u = (\eta, n_3)$ is the unit normal to graph(u) pointing toward the epigraph of u, $\eta = -\nabla u/\sqrt{1 + |\nabla u|^2}$, then $\eta(x) \in \widetilde{B}$ for all $x \in F$ (namely $|\eta(x)| \leq 1$), div$\eta = P(F)/|F|$, and $\eta = \widetilde{\mathbf{n}}_F$ on ∂F. Therefore, the existence of η is equivalent to saying that F is ϕ-calibrable, when ϕ as in (19).

Example 6.3 (Cylindrical anisotropies and the total variation flow). Let $n = 3$. The total variation flow is the formal gradient flow of the total variation functional $\int_{\mathbf{R}^n} |Du|$, in which the vertical velocity of the graph of the solution is given by the local curvature of the (horizontal) level curves. Let B_ϕ be as in (19). In view of the previous discussions, we expect that F evolves with vertical velocity equal to $P(F)/|F|$, as long as it remains

ϕ-calibrable. If F is convex, the facet is expected to bend close to regions where the curvature of ∂F is large, see Theorem 5.2.

7. Weak solutions: minimal barriers. It is well known that mean curvature flow develops singularities at finite time, starting from a smooth compact initial datum (the simplest singularity being the extinction of the flowing manifold). It is therefore meaningful to look for weak solutions, which are well defined for all times; there has been a lot of recent research in this direction, see for instance [Br], [CGG], [ES], [Am], [Il1]-[Il4], [ESS].

Concerning anisotropic mean curvature flow for mixed or crystalline anisotropies, one possible way to obtain a weak solution is to use the minimizing movement method [ATW] already recalled in Section 4. In this section we want to propose a notion of weak solution to crystalline motion by mean curvature for convex sets, based on Theorem 4.2 and on the notion of minimal barrier of De Giorgi [DeG], [BP2], [BNb]. Such a method seems to be applicable to a rather large class of problems: we mention in this direction the two barriers approach to mean curvature flow in higher codimension [BNc] and the barrier approach to systems of ordinary differential equations [BG].

Assume that ϕ is a crystalline anisotropy. Let us denote by $\mathcal{P}(\mathbf{R}^n)$ the family of all subsets of $\mathbf{R}^n$.

Definition 7.1 (The class of tests). Let $a, b \in \mathbf{R}$, $a < b$. A function $f : [a,b] \to \mathcal{P}(\mathbf{R}^n)$ belongs to $\mathcal{T}$ if and only if the map $t \to f(t)$ is an rB_ϕ-regular flow of compact convex sets in the sense of Definition 4.5.

Definition 7.2 (Barriers with respect to $\mathcal{T}$). We say that a function ϕ is a barrier with respect to $\mathcal{T}$, and we write $\phi \in \mathrm{Bar}(\mathcal{T})$, if and only if $\phi : [0, +\infty) \to \mathcal{P}(\mathbf{R}^n)$ and the following condition holds: if $f : [a,b] \subset [0,+\infty) \to \mathcal{P}(\mathbf{R}^n)$ belongs to $\mathcal{T}$ and $f(a) \subseteq \phi(a)$ then $f(b) \subseteq \phi(b)$.

Definition 7.3 (Minimal barrier starting from E). Let $E \subset \mathbf{R}^n$ be a compact convex set. For any $t \in [0, +\infty[$ the minimal barrier $M(\mathcal{T}, E)(t)$ starting from E at time 0 is defined as follows:

$$M(\mathcal{T}, E)(t) = \bigcap \left\{\phi(t) : \ \phi : [0, +\infty[\to \mathcal{P}(\mathbf{R}^n), \ \phi \in \mathrm{Bar}(\mathcal{T}), \ \phi(0) \supseteq E\right\}.$$

Definition 7.4 (Upper and lower regularizations). Let $E \subset \mathbf{R}^n$ be a compact convex set. For any $\rho > 0$ define $E^+_\rho := \{x \in \mathbf{R}^n : \mathrm{dist}(x, E) < \rho\}$, $E^-_\rho := \{x \in \mathbf{R}^n : \mathrm{dist}(x, \mathbf{R}^n \setminus E) > \rho\}$. For any $t \in [0, +\infty[$ the upper and lower regularizations starting from E at time 0 are defined as follows:

$$M^*(\mathcal{T}, E)(t) = \bigcap_{\rho > 0} M(\mathcal{T}, E^+_\rho)(t), \qquad M_*(\mathcal{T}, E)(t) = \bigcup_{\rho > 0} M(\mathcal{T}, E^-_\rho)(t).$$

In view of Theorem 4.2, the class $\mathcal{T}$ is *not empty*, and any convex $\mathrm{I}r'B_\phi\mathrm{C}$-set can be flowed (for short times) remaining inside the class $\mathcal{T}$ of rB_ϕ-regular flows ($r' > r$). Therefore Definitions 7.3, 7.4 are meaningful, and provide weak notions of crystalline mean curvature flow of convex sets defined for all times. The properties of $M(\mathcal{T}, E)$,

$M^*(\mathcal{T}, E)$, and $M_*(\mathcal{T}, E)$ deserve further investigation. One property that we may expect is that convexity of E is enough to ensure that fattening does not develop, i.e., $|M^*(\mathcal{T}, E)(t) \setminus M_*(\mathcal{T}, E)(t)| = 0$ for any $t \in [0, +\infty)$.

8. Higher order flows. Anisotropic surface diffusion. We have already mentioned in the introduction that higher order flows may be of interest in differential geometry and materials science. They may also furnish a way to regularize lower order flows, such as for instance mean curvature flow. We begin with the instructive example of the Willmore functional in the case $n = 2$ (called elastica functional, possibly with the addition of the length of the curve). Note that, in general, in presence of geometric evolution equations of order (at least) four, self-intersections of the flowing manifold are often unavoidable. We therefore sometimes switch our viewpoint to a parametric one. The Willmore flow corresponds to the choice $\Phi(\mathbf{n}, B) = (\mathrm{tr}(B))^2$ in formula (3).

EXAMPLE 8.1 (Elastica flow). Let $\gamma : \mathbf{S}^1 \to \mathbf{R}^2$ be a smooth immersed plane curve, and consider the elastica functional

$$\int_\gamma \left(1 + \epsilon\kappa^2\right) ds \tag{21}$$

where κ is the curvature of γ, s is the arclength parameter, and $\varepsilon > 0$. Then the gradient flow [GH] of this functional reads as

$$\frac{\partial\gamma}{\partial t} = \left(\kappa - 2\epsilon\partial_s^2\kappa - \epsilon\kappa^3\right)\mathbf{n}. \tag{22}$$

Note the presence of the higher order term $\partial_s^2\kappa$ ("Laplace-Beltrami" operator on γ of the curvature). The problem of taking the limit as $\varepsilon \to 0^+$ in (21) and its relations with the curvature flow of γ has been deepened in [BMN], as well as similar questions in the case of flowing manifolds of arbitrary codimension in $\mathbf{R}^n$.

We now want to investigate a possible way of defining the *ϕ-tangential laplacian* of κ_ϕ on ∂E, when ϕ^o is a regular anisotropy, ∂E is smooth, and κ_ϕ is defined in Section 3. We begin with some observations. When ϕ^o is Riemannian, namely $\phi^o(\xi^*) = \sqrt{g^{ij}\xi_i^*\xi_j^*}$, the Laplace-Beltrami operator on ∂E of a smooth function $f : \partial E \to \mathbf{R}$ is defined as $\Delta_{\partial E} f = \mathrm{div}_\tau(\mathrm{grad}_\tau f)$, where div_τ and grad_τ denote the tangential divergence and the tangential gradient respectively. We recall that, if for instance $g^{ij} = \delta^{ij}$, and if f^{ext} denotes a smooth extension of f on a suitable neighbourhood of ∂E, then $\mathrm{grad}_\tau f = (\mathrm{Id} - \mathbf{n} \otimes \mathbf{n})\mathrm{grad}(f^{\mathrm{ext}})$. The Laplace-Beltrami operator on ∂E can be derived by looking at the operator associated with the Euler-Lagrange equation of the g^{ij}-Dirichlet energy of f on ∂E, or by inspecting the second variation of the perimeter P_ϕ, and finally also by looking at the leading order term in the Euler-Lagrange equation of geometric functionals depending on the mean curvature. In the Finsler case these various approaches could lead, in principle, to different operators (see [BCS], [Sh] for general results on Finsler geometry). Here we derive a notion of ϕ-tangential laplacian of κ_ϕ using the latter possibility, and we show that the leading term of the Euler-Lagrange equation for the anisotropic Willmore functional coincides with the leading term of the second variation of P_ϕ [BF]. Let us introduce some notation. Let U be a sufficiently small tubular neighborhood of ∂E and

$u \in \mathcal{C}^\infty(U)$ be such that $\{u \le 0\} = E$, $\{u > 0\} = U \setminus E$, $\{u = 0\} = \partial E$ and $u^2 + |\mathrm{grad} u|^2 > 0$ in U (we call such a u a representing function of ∂E). The symbol ∇u denotes the differential of u, and is a one-covector; if necessary, it will be considered as a row. The symbol $\mathrm{grad} u$ denotes the gradient of u, and is a one-vector; if necessary, it will be considered as a column. If $X = (X^1, \ldots, X^n)$ is a smooth vector field, the symbol JX denotes the Jacobian of X, that is the matrix representing the differential of X. We recall that $(JX)_{ij} = \frac{\partial X^i}{\partial x^j}$.

Given a representing function u of ∂E, we define

$$\mathbf{N}_\phi^{*,u} := \frac{\nabla u}{\phi^o(\nabla u)}, \qquad \mathbf{N}_\phi^u := T_{\phi^o}(\nabla u) \qquad \text{in } U. \tag{23}$$

It can be checked that on $\{u = 0\}$ these definitions do not depend on the choice of u. Therefore if we choose $u = d_\phi$ then, recalling the notation of Section 3.1, where now $\mathbf{N}_\phi^{*,u} = \mathbf{N}_\phi^*$ and $\mathbf{N}_\phi^u = \mathbf{N}_\phi$, on ∂E we have

$$\mathbf{N}_\phi^* = \mathbf{n}_\phi^* = \nabla d_\phi = (\mathbf{n}_{\phi 1}^*, \ldots, \mathbf{n}_{\phi n}^*), \qquad \mathbf{N}_\phi = \mathbf{n}_\phi = T^o(\nabla d_\phi) = (\mathbf{n}_\phi^1, \ldots, \mathbf{n}_\phi^n). \tag{24}$$

We recall also that

$$(\phi^o(\mathbf{N}_\phi^*))^2 = (\phi(\mathbf{N}_\phi))^2 = \langle \mathbf{N}_\phi^*, \mathbf{N}_\phi \rangle = 1 \qquad \text{in } U. \tag{25}$$

Remark 8.1. The ϕ-divergence in Ω of a vector field $X = (X^1, \ldots, X^n)$ defined in Ω turns out to be equal to $\mathrm{div} X = \partial_{x^i} X^i$, and the ϕ-Laplace-Beltrami operator in Ω of $u : \Omega \to \mathbf{R}$ turns out to be defined as $\Delta_\phi u = \mathrm{div}(\mathrm{grad}_\phi(u))$, $\mathrm{grad}_\phi(u) := T_{\phi^o}(\nabla u)$, see [BP1]. We want to extend these definitions to tangential operators on ∂E.

The symbol $\mathbf{N}_\phi^* \otimes \mathbf{N}_\phi$ denotes the $(1,1)$ tensor formed as the tensor product of the covector $\mathbf{N}_\phi^*$ with the vector $\mathbf{N}_\phi$, and can be identified with a rank-one matrix, $(\mathbf{N}_\phi^* \otimes \mathbf{N}_\phi)_{ij} = \mathbf{N}_{\phi i}^* \mathbf{N}_\phi^j$. The symbol $\mathbf{N}_\phi \otimes \mathbf{N}_\phi^*$ stands for the transposed matrix.

Remark 8.2. On ∂E we have

$$\ker(\mathrm{Id} - \mathbf{n}_\phi^* \otimes \mathbf{n}_\phi) = \mathrm{span}\{\mathbf{n}_\phi^*\}, \qquad \mathrm{Im}(\mathrm{Id} - \mathbf{n}_\phi^* \otimes \mathbf{n}_\phi) = (\mathbf{n}_\phi)^\perp;$$
$$(\mathrm{Id} - \mathbf{n}_\phi^* \otimes \mathbf{n}_\phi)^2 = \mathrm{Id} - \mathbf{n}_\phi^* \otimes \mathbf{n}_\phi,$$
$$\ker(\mathrm{Id} - \mathbf{n}_\phi \otimes \mathbf{n}_\phi^*) = \mathrm{span}\{\mathbf{n}_\phi\}, \qquad \mathrm{Im}(\mathrm{Id} - \mathbf{n}_\phi \otimes \mathbf{n}_\phi^*) = (\mathbf{n}_\phi^*)^\perp;$$
$$(\mathrm{Id} - \mathbf{n}_\phi \otimes \mathbf{n}_\phi^*)^2 = \mathrm{Id} - \mathbf{n}_\phi \otimes \mathbf{n}_\phi^*.$$

Definition 8.1. Let $X = (X^1, \ldots, X^n)$ be a smooth vector field defined on ∂E. We define the ϕ-tangential divergence of X on ∂E as

$$\mathrm{div}_{\tau,\phi} X := \mathrm{tr}\left[\left(\mathrm{Id} - \mathbf{n}_\phi \otimes \mathbf{n}_\phi^*\right) JX^{\mathrm{ext}}\right] = \mathrm{div}(X^{\mathrm{ext}}) - \mathbf{n}_\phi^j \mathbf{n}_{\phi i}^* \frac{\partial}{\partial x^j} {X^{\mathrm{ext}}}^i, \tag{26}$$

where X^{ext} is any smooth extension of X in U.

Recall that $\mathrm{tr}(\mathbf{N}_\phi \otimes \mathbf{N}_\phi^* JX) = \langle \mathbf{N}_\phi^* JX, \mathbf{N}_\phi \rangle = \mathbf{N}_\phi^j (\mathbf{N}_\phi^* JX^{\mathrm{ext}})_j = \mathbf{N}_\phi^j \mathbf{N}_{\phi i}^* (JX^{\mathrm{ext}})_{ij} = \mathbf{N}_\phi^j \mathbf{N}_{\phi i}^* \frac{\partial}{\partial x^j} {X^{\mathrm{ext}}}^i$.

Since if Y is a vector field defined in U which vanishes on ∂E then $\nabla Y^i = \langle \nabla Y^i, \mathbf{n}_\phi \rangle \mathbf{n}_\phi^*$ on ∂E for every $i \in \{1, \ldots, n\}$, it can be checked that $\mathrm{div}_{\tau,\phi}$ does not depend on the extension of X on U.

Observe that

$$\mathrm{div}_{\tau,\phi}\mathbf{n}_\phi = \mathrm{div}\mathbf{N}_\phi = \mathrm{div}_\tau \mathbf{n}_\phi \qquad \text{on } \partial E.$$

Indeed the first equality follows by differentiating $\phi(\mathbf{N}_\phi) = 1$ in U, and the second equality follows from the first one also recalling that $\mathbf{n}^*_\phi$ and $\mathbf{n}$ are parallel.

We are interested in computing the first variation of the anisotropic Willmore functional, defined as

$$\mathcal{W}_\phi(E) := \int_{\partial E} (\mathrm{div}_{\tau,\phi}\mathbf{n}_\phi)^2 \phi^o(\mathbf{n})\, d\mathcal{H}^{n-1} =: \int_{\partial E} (\mathrm{div}_{\tau,\phi}\mathbf{n}_\phi)^2\, dP_\phi.$$

We give the definiton of ϕ-tangential gradient on ∂E.

In order to extend on the whole of U a smooth scalar field f defined on ∂E we consider $p_\phi : U \to \partial E$ to be the projection along geodesics of U on ∂E, i.e.,

$$p_\phi(y) = y - d_\phi(y)\mathbf{N}_\phi(y) \qquad y \in U.$$

Then we define f in the whole of U, and we denote this (canonical) extension by $\overline{f}$, as

$$\overline{f}(y) = f(p_\phi(y)), \qquad y \in U.$$

Definition 8.2. Let f be a smooth scalar field defined on ∂E. We define

$$\mathrm{grad}_{\partial E,\phi} f := \mathrm{grad}\overline{f} \qquad \text{on } \partial E. \tag{27}$$

Note that $\mathrm{grad}_{\partial E,\phi} f$ can be identified with $\nabla f(\mathrm{Id} - \mathbf{n}_\phi \otimes \mathbf{n}^*_\phi) = \nabla f - \langle \nabla f, \mathbf{n}_\phi\rangle \mathbf{n}^*_\phi = \nabla_\tau f(\mathrm{Id} - \mathbf{n}_\phi \otimes \mathbf{n}^*_\phi)$.

Observe also that we are not considering ∂E endowed with the Finsler metric on the tangent bundle $T\partial E$ obtained as the restriction $r(x,\xi)$ of ϕ to $T\partial E$, and therefore we expect our operators (in Definitions 8.1, 8.2) not to coincide with the ones corresponding to looking at the Finsler manifold $(\partial E, r)$.

Let ψ be a smooth scalar function defined on ∂E and X a smooth vector field defined on ∂E. Then from (26) we obtain

$$\mathrm{div}_{\tau,\phi}(\psi X) = \psi\ \mathrm{div}_{\tau,\phi} X + (\mathrm{grad}\overline{\psi}, (\mathrm{Id} - \mathbf{n}_\phi \otimes \mathbf{n}^*_\phi)X) = \psi\ \mathrm{div}_{\tau,\phi} X + (\mathrm{grad}_{\partial E,\phi}\psi, X).$$

We are now in a position to introduce an operator defined on the space $\mathcal{C}^2(\partial E)$ with values in $\mathcal{C}^0(\partial E)$, which will turn out to be the leading term in the expression of the first variation of $\mathcal{W}$.

Definition 8.3. Let $f \in \mathcal{C}^2(\partial E)$. We set

$$\Delta_{\partial E,\phi} f := \mathrm{div}_{\tau,\phi}\left(\phi^o_{\xi^*\xi^*}(\mathbf{n}^*_\phi)\mathrm{grad}_{\partial E,\phi} f\right).$$

In the euclidean case, that is, when $\phi^o(\xi^*) = |\xi^*|$, $\Delta_{\partial E,\phi}$ coincides with the usual tangential laplacian operator on ∂E. Finally, we recall the following result [BF].

Lemma 8.2. *Let $f, g \in \mathcal{C}^2(\partial E)$. Then*

$$-\int_{\partial E} g\Delta_{\partial E,\phi} f\, dP_\phi = \int_{\partial E} (\mathrm{grad}_{\partial E,\phi} f, \phi^o_{\xi^*\xi^*}(\mathbf{n}^*_\phi)\mathrm{grad}_{\partial E,\phi} f)\, dP_\phi. \tag{28}$$

Let $\psi \in \mathcal{C}^2(\partial E)$. Define

$$\alpha : \mathbf{R} \times U \to U, \qquad (\lambda, y) \mapsto \alpha(\lambda, y) = \alpha_\lambda(y) := y + \lambda\overline{\psi}(y)\mathbf{N}_\phi(y) + o(\lambda),$$

and set

$$E_\lambda := \alpha_\lambda(E).$$

THEOREM 8.3. *We have*

$$\frac{d}{d\lambda}\mathcal{W}_\phi(E_\lambda)|_{\lambda=0} = \int_{\partial E} [2\ \Delta_{\partial E,\phi}\kappa_\phi + 2\kappa_\phi \mathrm{tr}[(J\mathbf{N}_\phi)^2] - (\kappa_\phi)^3]\psi\ dP_\phi. \tag{29}$$

In the euclidean case (29) agrees with the first variation of the Willmore functional. Let us also notice that a result similar to Theorem 8.3 has been proved in [Cl].

Proof. If $|\lambda|$ is small enough, say $|\lambda| < \varepsilon$, we define the smooth function v^λ in U through the formula

$$v^\lambda(y) := d_\phi(\alpha_\lambda^{-1}(y)).$$

Then v^λ is a defining function for ∂E_λ for every $\varepsilon > 0$ small enough. Furthermore

$$\mathbf{n}_{\partial E_\lambda}(y) = \frac{\nabla v^\lambda(y)}{|\nabla v^\lambda(y)|} =: \mathbf{n}_\lambda(y), \qquad \mathbf{n}_\phi^{\partial E_\lambda}(y) = \phi_\xi^o(\nabla v^\lambda(y)) =: \mathbf{n}_\phi^\lambda(y). \tag{30}$$

Let $\mathbf{g}(y) := \overline{\psi}(y)\mathbf{N}_\phi(y)$. Then

$$\nabla v^\lambda(y) = \nabla d_\phi(\alpha_\lambda^{-1}(y))\left[\mathrm{Id} - \lambda J\mathbf{g}(y) + o(\lambda)\right],$$

and

$$J\mathbf{g}(y) = \overline{\psi}(y) J\mathbf{N}_\phi(y) + \mathbf{n}_\phi(y) \otimes \nabla\overline{\psi}(y).$$

Therefore

$$\dot{\mathbf{n}}_0 := \frac{d}{d\lambda}\mathbf{n}^\lambda|_{\lambda=0} = -\mathbf{n}J\mathbf{g} + \langle \mathbf{n}, \mathbf{n}J\mathbf{g}\rangle\mathbf{n}. \tag{31}$$

Let us rewrite the expression of $\mathcal{W}_\phi$ in a more convenient way. We have

$$\mathcal{W}_\phi(E) = \int_{\partial E} (\mathrm{tr}[(\mathrm{Id} - \mathbf{n}_\phi \otimes \mathbf{n}_\phi^*)J\mathbf{N}_\phi])^2\ dP_\phi. \tag{32}$$

Hence

$$\begin{aligned}
\frac{d}{d\lambda}\mathcal{W}_\phi(E_\lambda)|_{\lambda=0} &= \frac{d}{d\lambda}\int_{\partial E} ((\mathrm{div}_{\tau,\phi}\mathbf{n}_\phi^\lambda)(\alpha_\lambda))^2\ dP_\phi(\alpha_\lambda)|_{\lambda=0} \\
&= \int_{\partial E} (\mathrm{tr}[(\mathrm{Id} - \mathbf{n}_\phi \otimes \mathbf{n}_\phi^*)J\mathbf{N}_\phi])^2 \frac{d}{d\lambda}\ dP_\phi(\alpha_\lambda)|_{\lambda=0} \\
&+ \int_{\partial E} \frac{d}{d\lambda}[(\mathrm{Id} - \mathbf{n}_\phi^\lambda(\alpha_\lambda) \otimes \mathbf{n}_\phi^{\lambda *}(\alpha_\lambda))(J\mathbf{n}_\phi^\lambda)(\alpha_\lambda)]^2|_{\lambda=0}\ dP_\phi =: I + II.
\end{aligned} \tag{33}$$

We start by computing I. Recall that

$$\frac{d}{d\lambda} d\mathcal{H}^{n-1}(\alpha_\lambda(x))|_{\lambda=0} = \mathrm{div}_\tau \mathbf{g}(x) d\mathcal{H}^{n-1}(x). \tag{34}$$

Using (31), (25), [BF, Lemma 3.2] and the relation $\phi^o_\xi(\mathbf{n}) = \mathbf{n}_\phi$ we have

$$
\begin{aligned}
(35) \qquad \frac{d}{d\lambda}\phi^o(\mathbf{n}^\lambda)|_{\lambda=0} &= (\phi^o_\xi(\mathbf{n}), \dot{\mathbf{n}}_0) = \phi^o(\mathbf{n})(\phi^o_\xi(\mathbf{n}), \left[-\mathbf{n}^*_\phi J\mathbf{g} + (\mathbf{n}, \mathbf{n}J\mathbf{g})\mathbf{n}^*_\phi\right]) \\
&= \phi^o(\mathbf{n})(\mathbf{n}_\phi, [-\nabla\overline{\psi}\langle\mathbf{n}^*_\phi, \mathbf{n}_\phi\rangle - \psi\ \mathbf{n}^*_\phi J\mathbf{N}_\phi \\
&\quad + (\mathbf{n}, \mathbf{n}\ \mathbf{n}_\phi \otimes \nabla\overline{\psi} + \phi^o(\mathbf{n})\psi\ \mathbf{n}^*_\phi J\mathbf{N}_\phi)\mathbf{n}^*_\phi]) \\
&= \phi^o(\mathbf{n})[-(\nabla\overline{\psi}, \mathbf{n}_\phi\ ) + (\mathbf{n}, \mathbf{n}\ \mathbf{n}_\phi \otimes \nabla\overline{\psi})] \\
&= \phi^o(\mathbf{n})[-\langle\nabla\overline{\psi}, \mathbf{n}_\phi\rangle + \langle(\mathbf{n}, \nabla\overline{\psi}\ )\mathbf{n}, \mathbf{n}_\phi\rangle].
\end{aligned}
$$

Therefore we obtain

$$
\begin{aligned}
\frac{d}{d\lambda}\ dP_\phi(\alpha_\lambda)|_{\lambda=0} &= \phi^o(\mathbf{n})(-\langle\nabla\overline{\psi}, \mathbf{n}_\phi\rangle + \langle\ (\mathbf{n}, \nabla\overline{\psi}\ )\mathbf{n}, \mathbf{n}_\phi\rangle) + \phi^o(\mathbf{n})\mathrm{div}_\tau\mathbf{g}\ d\mathcal{H}^{n-1} \\
&= [-\langle\nabla_\tau\psi, \mathbf{n}_\phi\rangle - \langle\ \langle\nabla\overline{\psi}, \mathbf{n}\ \rangle\mathbf{n}, \mathbf{n}_\phi\rangle + \langle\ \langle\mathbf{n}, \nabla\overline{\psi}\ \rangle\mathbf{n}, \mathbf{n}_\phi\rangle + \langle\nabla_\tau\psi, \mathbf{n}_\phi\rangle \\
&\quad + \psi\,\mathrm{div}_\tau\mathbf{n}_\phi]\phi^o(\mathbf{n})\ d\mathcal{H}^{n-1} = \psi\ \mathrm{div}_\tau\mathbf{n}_\phi\ dP_\phi.
\end{aligned}
$$

Hence, using the relations $\mathrm{div}_{\tau,\phi}\mathbf{n}_\phi = \mathrm{div}\mathbf{N}_\phi = \mathrm{div}_\tau\mathbf{n}_\phi$ on ∂E, we have

$$
(36) \qquad I = \int_{\partial E} \psi(\mathrm{div}_{\tau,\phi}\mathbf{n}_\phi)^2\mathrm{div}_\tau\mathbf{n}_\phi\ dP_\phi = \int_{\partial E} \psi(\mathrm{div}_{\tau,\phi}\mathbf{n}_\phi)^3\ dP_\phi.
$$

We now prove that

$$
(37) \qquad II = \int_{\partial E} 2\ \mathrm{div}_{\tau,\phi}\mathbf{n}_\phi[\mathrm{div}_{\tau,\phi}(\phi^o_{\xi^*\xi^*}(\mathbf{n}^*_\phi)\nabla_{\tau,\phi}\psi) + \mathrm{tr}[(J\mathbf{N}_\phi)^2]]\ dP_\phi.
$$

We start by noticing that for every $y \in U$ we have

$$
(38) \qquad \mathbf{n}^\lambda_\phi(y) = \mathbf{N}_\phi(y) + \lambda\frac{d}{d\lambda}\mathbf{n}^\lambda_\phi(y)|_{\lambda=0} + o(\lambda).
$$

Now we have

$$
\begin{aligned}
\frac{d}{d\lambda}\mathbf{n}^\lambda_\phi(y)|_{\lambda=0} &= \frac{d}{d\lambda}\phi^o_\xi(\nabla v^\lambda(y))|_{\lambda=0} = \frac{d}{d\lambda}\phi^o_\xi(\nabla d_\phi(\alpha^{-1}_\lambda(y))|_{\lambda=0} \\
&= \phi^o_{\xi^*\xi^*}(\nabla d_\phi(y))\frac{d}{d\lambda}(\nabla d_\phi(\alpha^{-1}_\lambda(y))[\mathrm{Id} - \lambda J\mathbf{g}(y) + o(\lambda)])|_{\lambda=0} \\
&= \phi^o_{\xi^*\xi^*}(\nabla d_\phi(y))[-\nabla^2 d_\phi(y)\mathbf{g}(y) - \nabla d_\phi(y)J\mathbf{g}(y)] \\
&= -\ \overline{\psi}(y)\phi^o_{\xi^*\xi^*}(\nabla d_\phi(y))[\nabla^2 d_\phi(y)\mathbf{N}_\phi(y)] \\
&\quad - \phi^o(\nabla d_\phi(y))\phi^o_{\xi^*\xi^*}(\nabla d_\phi(y))\left[\overline{\psi}(y)\frac{\nabla d_\phi(y)}{\phi^o(\nabla d_\phi(y))}J\mathbf{N}_\phi(y)\right. \\
&\quad \left. + \nabla\overline{\psi}(y)\left\langle\frac{\nabla d_\phi(y)}{\phi^o(\nabla d_\phi(y))}, \mathbf{N}_\phi(y)\right\rangle\right] \\
&= -\ \phi^o_{\xi^*\xi^*}(\mathbf{N}^*_\phi(y))[\overline{\psi}(y)\mathbf{N}^*_\phi(y)J\mathbf{N}_\phi(y) + \nabla\overline{\psi}(y)\langle\mathbf{N}^*_\phi(y), \mathbf{N}_\phi(y)\rangle] \\
&= -\ \phi^o_{\xi^*\xi^*}(\mathbf{N}^*_\phi(y))\nabla\overline{\psi}(y),
\end{aligned}
$$

where in the last two equalities we used [BF, Lemma 3.2], (25), and the following relations, which involve also [BF, Remark 3.3]:

$$\begin{aligned}\mathbf{N}_\phi(y) = \phi^o_\xi(\nabla d_\phi) \Rightarrow J\mathbf{N}_\phi &= \phi^o_{\xi^*\xi^*}(\nabla d_\phi)\nabla^2 d_\phi\\ &\Rightarrow 0 = J\mathbf{N}_\phi\mathbf{N}_\phi = \phi^o_{\xi^*\xi^*}(\nabla d_\phi)[\nabla^2 d_\phi\ \mathbf{N}_\phi]\end{aligned}$$

($J\mathbf{N}_\phi\mathbf{N}_\phi = 0$ on ∂E is obtained by differentiating $\mathbf{N}_\phi$ along $\mathbf{N}_\phi$ itself).

Therefore, from (38) and the expression we obtained above for $\frac{d}{d\lambda}\mathbf{n}^\lambda_\phi(y)|_{\lambda=0}$, it follows

$$\begin{aligned}\frac{d}{d\lambda}(\mathrm{div}_{\tau,\phi}\mathbf{n}^\lambda_\phi(\alpha_\lambda))^2|_{\lambda=0} &= 2\ \mathrm{div}_{\tau,\phi}\mathbf{n}_\phi \frac{d}{d\lambda}\mathrm{tr}[(\mathrm{Id} - \mathbf{n}^\lambda_\phi(\alpha_\lambda)\otimes\mathbf{n}^{\lambda*}_\phi(\alpha_\lambda))J\mathbf{n}^\lambda_\phi(\alpha_\lambda)]|_{\lambda=0}\\
&= 2\ \mathrm{div}_{\tau,\phi}\mathbf{n}_\phi\Big\{\frac{d}{d\lambda}[\mathrm{tr}\left(\mathrm{Id} - \mathbf{n}^\lambda_\phi(\alpha_\lambda)\otimes\mathbf{n}^{\lambda*}_\phi(\alpha_\lambda)\right)J\mathbf{N}_\phi(\alpha_\lambda)|_{\lambda=0}\\
&\quad - \frac{d}{d\lambda}\ \mathrm{tr}(\mathrm{Id} - \mathbf{n}^\lambda_\phi(\alpha_\lambda)\otimes\mathbf{n}^{\lambda*}_\phi(\alpha_\lambda))\nabla(\phi^o_{\xi^*\xi^*}(\mathbf{N}^*_\phi)\nabla\overline{\psi})\Big\}\\
&= 2\ \mathrm{div}_{\tau,\phi}\mathbf{n}_\phi\{\mathrm{tr}\left[\left(\mathrm{Id} - \mathbf{n}_\phi\otimes\mathbf{n}^*_\phi\right)\frac{d}{d\lambda}J\mathbf{N}_\phi(\alpha_\lambda)|_{\lambda=0}\right]\\
&\quad - \mathrm{tr}\left[\frac{d}{d\lambda}\left(\mathbf{n}^\lambda_\phi(\alpha_\lambda)\otimes\mathbf{n}^{\lambda*}_\phi(\alpha_\lambda)\right)|_{\lambda=0}J\mathbf{N}_\phi\right]\\
&\quad - \mathrm{tr}[(\mathrm{Id} - \mathbf{n}_\phi\otimes\mathbf{n}^*_\phi)\nabla(\phi^o_{\xi^*\xi^*}(\mathbf{N}^*_\phi)\nabla\overline{\psi})]\}\\
&=: 2\ \mathrm{div}_{\tau,\phi}\mathbf{n}_\phi\{(i) - (ii) - \mathrm{div}_{\tau,\phi}(\phi^o_{\xi^*\xi^*}(\mathbf{n}^*_\phi)\nabla_{\partial E,\phi}\psi)\}.\end{aligned}$$

We claim that $(i) = \psi\ \mathrm{tr}[(J\mathbf{N}_\phi)^2]$ and $(ii) = 0$.

Firstly, abbreviating $\frac{\partial}{\partial x^h}$ to ∂_h and similarly for derivatives of higher order, we notice that

$$\begin{aligned}\frac{d}{d\lambda}(J\mathbf{N}_\phi)(\alpha_\lambda(x))|_{\lambda=0} &= \Big(\sum_j \partial_{ij}\mathbf{N}^k_\phi(x)\dot\alpha^j_0(x)\Big)_{1\le i,k\le n}\\ &= \Big(\psi(x)\sum_j \partial_{ij}\mathbf{N}^k_\phi(x)\mathbf{n}^j_\phi(x)\Big)_{1\le i,k\le n}.\end{aligned}$$

Since $J\mathbf{N}_\phi\mathbf{N}_\phi = 0$ on ∂E we have

$$\partial_i\Big(\sum_j \partial_j\mathbf{N}^k_\phi(x)\mathbf{n}^j_\phi(x)\Big) = 0 \qquad \forall i,k\in\{1,\dots,n\},\ \forall x\in\partial E,$$

hence

$$\sum_j \partial_{ij}\mathbf{N}^k_\phi(x)\mathbf{n}^j_\phi(x) = -\sum_j \partial_j\mathbf{N}^k_\phi(x)\partial_i\mathbf{N}^j_\phi(x) \qquad \forall i,k\in\{1,\dots,n\},\ \forall x\in\partial E.$$

Therefore for every $x\in\partial E$ we have

$$\frac{d}{d\lambda}J\mathbf{N}_\phi(\alpha_\lambda(x))|_{\lambda=0} = \Big(\psi(x)\Big(\sum_j \partial_{ij}\mathbf{N}^k_\phi(x)\mathbf{n}^j_\phi(x)\Big)\Big)_{1\le i,k\le n} = \psi(x)(J\mathbf{N}_\phi(x))^2,$$

and finally using [BF, Lemma 3.2] we obtain

$$(i) = -\psi\ \mathrm{tr}[(J\mathbf{N}_\phi)^2] + \psi\mathrm{tr}(\mathbf{n}_\phi\otimes\mathbf{n}^*_\phi)(J\mathbf{N}_\phi)^2 = -\psi\ \mathrm{tr}[(J\mathbf{N}_\phi)^2]. \tag{39}$$

Let us prove that $(ii) = 0$. We observe that, using (31) and (35), we obtain

$$\begin{aligned}\frac{d}{d\lambda}\mathbf{N}_\phi^{\lambda *}(y)|_{\lambda=0} &= \frac{\dot{\mathbf{n}}_0(y)}{\phi^o(\mathbf{n})} - \frac{\mathbf{n}(y)}{\phi^o(\mathbf{n}(y))^2}\frac{d}{d\lambda}\phi^o(\mathbf{n}^\lambda(y))|_{\lambda=0} \\ &= -\mathbf{N}_\phi^*(y)J\mathbf{g}(y) + (\mathbf{n}(y), \mathbf{n}(y)J\mathbf{g}(y))\mathbf{N}_\phi^*(y) \\ &\quad - \frac{1}{\phi^o(\mathbf{n}(y))}(\dot{\mathbf{n}}_0(y), \mathbf{N}_\phi(y))\mathbf{N}_\phi^*(y) \\ &= -\mathbf{n}_\phi^*(y)J\mathbf{g}(y) + (\mathbf{n}(y), \mathbf{n}(y)J\mathbf{g}(y))\mathbf{N}_\phi^*(y) \\ &\quad + (\mathbf{N}_\phi^*(y)J\mathbf{g}(y), \mathbf{N}_\phi(y)\rangle\mathbf{N}_\phi^*(y) - \langle\mathbf{n}(y), \mathbf{n}(y)J\mathbf{g}(y))\mathbf{N}_\phi^*(y) \\ &= -\nabla\overline{\psi}(y) + \langle\nabla\overline{\psi}(y), \mathbf{N}_\phi(y)\rangle\mathbf{N}_\phi^*(y) = -\nabla\overline{\psi}(y)(\mathrm{Id} - \mathbf{N}_\phi^*(y)\otimes\mathbf{N}_\phi(y))\end{aligned}$$

(notice that the last quantity in the above equality is exactly $\nabla_{\partial E,\phi}\psi(x)$ when $x \in \partial E$). Hence using the last calculation, [BF, Lemma 3.2] and [BF, Remark 3.3] we have

$$\begin{aligned}(ii) &= \mathrm{tr}\Big[\frac{d}{d\lambda}(\mathbf{n}_\phi^\lambda(\alpha_\lambda)\otimes\mathbf{n}_\phi^{\lambda *}(\alpha_\lambda)|_{\lambda=0}J\mathbf{N}_\phi\Big] \\ &= \mathrm{tr}\Big[\frac{d}{d\lambda}\mathbf{n}_\phi^\lambda(\alpha_\lambda)|_{\lambda=0}\otimes(\mathbf{n}_\phi^*J\mathbf{N}_\phi) + \mathbf{n}_\phi\otimes\frac{d}{d\lambda}\mathbf{n}_\phi^{*\,\lambda}|_{\lambda=0}J\mathbf{N}_\phi\Big] \\ &= 0 + \mathrm{tr}[\mathbf{n}_\phi\otimes(\nabla\overline{\psi}(\mathrm{Id} - \mathbf{n}_\phi\otimes\mathbf{n}_\phi^*)J\mathbf{N}_\phi)] \\ &= \mathrm{tr}[\mathbf{n}_\phi\otimes(\nabla\overline{\psi}\ J\mathbf{N}_\phi)] = \sum_j \mathbf{n}_\phi^j\Big(\sum_i \partial_i\overline{\psi}\ \partial_j\mathbf{N}_\phi^i\Big) \\ &= (\nabla\overline{\psi}, J\mathbf{N}_\phi\mathbf{n}_\phi) = 0.\end{aligned} \tag{40}$$

We conclude that

$$\begin{aligned}II &= \int_{\partial E}\frac{d}{d\lambda}(\mathrm{div}_{\tau,\phi}\mathbf{n}_\phi^\lambda(\alpha_\lambda))^2|_{\lambda=0}\ dP_\phi \\ &= \int_{\partial E} -2\mathrm{div}_{\tau,\phi}\mathbf{n}_\phi\mathrm{div}_{\tau,\phi}(\phi^o_{\xi^*\xi^*}(\mathbf{n}_\phi^*)\nabla_{\partial E,\phi}\psi) - 2\mathrm{div}_{\tau,\phi}\mathbf{n}_\phi\mathrm{tr}((J\mathbf{N}_\phi)^2)\ dP_\phi \\ &= \int_{\partial E}\big\{-2\mathrm{div}_{\tau,\phi}[\phi^o_{\xi^*\xi^*}(\mathbf{n}_\phi^*)\nabla_{\partial E,\phi}(\mathrm{div}_{\tau,\phi}\mathbf{n}_\phi)] - 2\mathrm{div}_{\tau,\phi}\mathbf{n}_\phi\mathrm{tr}[(J\mathbf{N}_\phi)^2]\big\}\ \psi\ dP_\phi.\end{aligned} \tag{41}$$

Summing up I and II we obtain (29). ∎

The expression of the first variation of $\mathcal{W}_\phi$ obtained in Theorem 8.3 is a suggestion for considering as the anisotropic surface diffusion flow the one where the normal velocity is given by minus the operator given in Definition 8.3 applied to κ_ϕ.

Acknowledgements. We acknowledge the De Giorgi Center of Mathematical Research for partial support.

References

[ABD] G. Alberti, G. Bouchittè and G. Dal Maso, *The calibration method for the Mumford-Shah functional and free-discontinuity problems*, Calc. Var. Partial Differential Equations, 16 (2003), 299–333.

[ATW] F. J. Almgren, J. E. Taylor and L.-H. Wang, *Curvature-driven flows: a variational approach*, SIAM J. Control Optim. 31 (1993), 387–438.

[Am] L. Ambrosio, *Lecture notes on geometric evolution problems, distance function and viscosity solutions*, in: Calculus of Variations and Partial Differential Equations. Topics on Geometrical Evolution Problems and Degree Theory, G. Buttazzo, A. Marino and M. K. V. Murthy (eds.), Springer, Berlin, 1999, 5–94.

[AFP] L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Mathematical Monographs, 2000.

[AT] L. Ambrosio and V. M. Tortorelli, *On the approximation of free discontinuity problems*, Boll. Unione Mat. Ital. 6-B (1992), 105–123.

[BCS] D. Bao, S. S. Chern, and Z. Shen, *An Introduction to Riemann-Finsler Geometry*, Graduate Texts in Mathematics, Springer, New York, 2000.

[BGN] J. W. Barrett, H. Garcke and R. Nürnberg, *On the variational approximation of combined second and fourth order geometric evolution equations*, SIAM J. Sci. Comput. 29 (2007), 1006–1041.

[BGN1] J. W. Barrett, H. Garcke and R. Nürnberg, *Finite element approximation of a phase field model for surface diffusion of voids in a stressed solid*, Math. Comput. 75 (2006), 7–41.

[BBR] G. Bellettini, P. Buttà and E. Presutti, *Sharp interface limits of non local anisotropic interactions*, Arch. Ration. Mech. Anal. 159 (2001), 109–135.

[BCCN] G. Bellettini, V. Caselles, A. Chambolle and M. Novaga, *Crystalline mean curvature flow of convex sets*, Arch. Ration. Mech. Anal. 179 (2006), 109–152.

[BCCN1] G. Bellettini, V. Caselles, A. Chambolle and M. Novaga, *The volume preserving crystalline mean curvature flow of convex sets in R^N*, J. Math. Pures Appl., to appear.

[BCN] G. Bellettini, M. Chermisi and M. Novaga, *Crystalline curvature flow of networks*, Interfaces Free Bound. 8 (2006), 481–521.

[BCN1] G. Bellettini, M. Chermisi and M. Novaga, *The level set method for systems of PDEs*, Comm. Partial Differential Equations 32 (2007), 1043–1064.

[BF] G. Bellettini and I. Fragalà, *Elliptic approximations to prescribed mean curvature surfaces in Finsler geometry*, Asymp. Anal. 22 (2000), 87–111.

[BG] G. Bellettini and G. F. Gronchi, *Barriers for systems of ODEs: an application to the two-body problem*, Rend. Atti Acc. Naz. Sci. XL Mem. Mat. 26 (2002), 145–160.

[BMN] G. Bellettini, C. Mantegazza and M. Novaga, *Singular perturbations of mean curvature flow*, J. Differential Geom. 57 (2007), 403–431.

[BNb] G. Bellettini and M. Novaga, *Some aspects of De Giorgi's barriers for geometric evolutions*, in: Calculus of Variations and Partial Differential Equations. Topics on Geometrical Evolution Problems and Degree Theory, Springer-Verlag, 1999, 115–151.

[BNc] G. Bellettini and M. Novaga, *A result on motion by mean curvature in arbitrary codimension*, Diff. Geom. Appl. 11 (1999), 205–220.

[BN] G. Bellettini and M. Novaga, *Approximation and comparison for non-smooth anisotropic motion by mean curvature in $\mathbf{R}^N$*, Math. Mod. Meth. Appl. Sc. 10 (2000), 1–10.

[BNP] G. Bellettini, M. Novaga and M. Paolini, *Facet-breaking for three-dimensional crystals evolving by mean curvature*, Interfaces Free Bound. 1 (1999), 39–55.

[BNP1] G. Bellettini, M. Novaga and M. Paolini, *On a crystalline variational problem. I. First variation and global L^∞ regularity*, Arch. Ration. Mech. Anal. 157 (2001), 165–191.

[BNP2] G. Bellettini, M. Novaga and M. Paolini, *On a crystalline variational problem. II. BV regularity and structure of minimizers on facets*, Arch. Ration. Mech. Anal. 157 (2001), 193–217.

[BNP3] G. Bellettini, M. Novaga and M. Paolini, *Characterization of facet-breaking for non-smooth mean curvature flow in the convex case*, Interfaces Free Bound. 3 (2001), 415–446.

[BP] G. Bellettini and M. Paolini, *Convex approximations of an inhomogeneous anisotropic functional*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. (9) Mat. Appl. 5 (1994), 177–187.

[BP1] G. Bellettini and M. Paolini, *Anisotropic motion by mean curvature in the context of Finsler geometry*, Hokkaido Math. J. 25 (1996), 537–566.

[BP2] G. Bellettini and M. Paolini, *Some results on minimal barriers in the sense of De Giorgi applied to driven motion by mean curvature*, Rend. Atti Acc. Naz. Sci. XL Mem. Mat. 19 (1995), 43–67; Errata Corrige, 26 (2002), 161–165.

[BPV1] G. Bellettini, M. Paolini and C. Verdi, *Convex approximations of functionals with curvature*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. (9) Mat. Appl. 2 (1991), 297–306.

[BPV2] G. Bellettini, M. Paolini and C. Verdi, *Numerical minimization of functionals with curvature by convex approximations*, in: Progress in Partial Differential Equations: Calculus of Variations, Applications, Pitman Research Notes in Mathematics Series 267, C. Bandle, J. Bemelmans, M. Chipot, M. Grüter and J. Saint Jean Paulin (eds.), Longman Scientific & Technical, Harlow, 1992, 124–138.

[Br] K. A. Brakke, *The Motion of a Surface by its Mean Curvature*, Princeton Univ. Press, Princeton, 1978.

[CHT] J. W. Cahn, C. A. Handwerker and J. E. Taylor, *Geometric models of crystal growth*, Acta Metall. Mater 40 (1992), 1443–1474.

[Cl] U. Clarenz, *The Wulff shape minimizes an anisotropic Willmore functional*, Interfaces Free Bound. 6 (2004), 351–359.

[CRCT] W. C. Carter, A. R. Roosen, J. W. Cahn, and J. E. Taylor, *Shape evolution by surface diffusion and surface attachment limited kinetics*, Acta Metall. Mater. 43 (1995), 4309–4323.

[C] A. Chambolle, *An algorithm for mean curvature motion*, Interfaces Free Bound. 6 (2004), 195–218.

[CN] A. Chambolle and M. Novaga, *Convergence of an algorithm for the anisotropic and crystalline mean curvature flow*, SIAM J. Math. Anal. 37 (2006), 1978–1987.

[CGG] Y.-G. Chen, Y. Giga and S. Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, J. Differential Geom. 33 (1991), 749–786.

[Ch] M. Chermisi, *Crystalline flow of planar partitions and a geometric apporach for systems of PDEs*, Ph.D. Thesis, Univ. Roma Tor Vergata, 2006.

[DG] F. Davi and M. E. Gurtin, *On the motion of a phase interface by surface diffusion*, Z. Angew. Math. Phys. 41 (1990), 782–811.

[DeG] E. De Giorgi, *Barriers, boundaries, motion of manifolds*, Conference held at the Department of Mathematics of Pavia, 1994.

[D] G. Dziuk, *Computational parametric Willmore flow*, Numer. Math. 111 (2008), 55–80.

[Ec] K. Ecker, *Regularity Theory for Mean Curvature Flow*, Birkhäuser, Boston, 2004.

[EH] K. Ecker and G. Huisken, *Mean curvature flow of entire graphs*, Ann. of Math. (2) 130 (1989), 453–471.

[Ev] L. C. Evans, *A geometric interpretation of the heat equation with multivalued initial data*, SIAM J. Math. Anal. 27 (1996), 932–958.

[ESS] L. C. Evans, H.-M. Soner and P. E. Souganidis, *Phase transitions and generalized motion by mean curvature*, Comm. Pure Appl. Math. 45 (1992), 1097–1123.

[ES] L. C. Evans and J. Spruck, *Motion of level sets by mean curvature I*, J. Differential Geom. 33 (1991), 635–681.

[FGP] F. Fierro, R. Goglione and M. Paolini, *Numerical simulations of mean curvature flow in presence of a nonconvex anisotropy*, Math. Models Methods Appl. Sci. 8 (1998), 573–601.

[GN] H. Garcke and B. Nestler, *A mathematical model for grain growth in thin metallic films*, Math. Models Methods Appl. Sci. 10 (2000), 895–921.

[GH] M. Giaquinta and S. Hildebrandt, *Calculus of Variations I*, Grundlehren der mathematischen Wissenschaften 310, Springer 1996.

[Gi] Y. Giga, *Singular diffusivity-facets, shocks and more*, in: Applied Mathematics Entering the 21st Century, SIAM, 2004, 121–138.

[GG] Y. Giga and S. Goto, *Geometric evolutions of phase-boundaries*, in: On the Evolution of Phase Boundaries, M. Gurtin and G. McFadden (eds.), IMA Vols. Math. Appl. 43, Springer, Berlin, 1992, 51–66.

[GGIS] Y. Giga, S. Goto, H. Ishii and M.-H. Sato, *Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains*, Indiana Univ. Math. J. 40 (1991), 443-470.

[GGM] Y. Giga, M. E. Gurtin and J. Matias, *On the dynamics of crystalline motion*, Japan J. Industrial Appl. Math. 15 (1998), 7–50.

[GR1] Y. Giga and P. Rybka, *Stability of facets of crystals growing from vapor*, Discrete Contin. Dyn. Syst. 14 (2006), 689–706.

[GR2] Y. Giga and P. Rybka, *Stability of facets of self-similar motion of a crystal*, Adv. Differential Equations 10 (2005), 601–634.

[Gr] M. A. Grayson, *The heat equation shrinks embedded plane curves to round points*, J. Differential Geom. 26 (1987), 285–314.

[Hu] G. Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Differential Geom. 20 (1984), 237–266.

[Hu1] G. Huisken, *Nonparametric mean curvature evolution with boundary conditions*, J. Differential Equations 77 (1989), 369–378.

[KS] E. Kuwert and R. Schaetzle, *Gradient flow for the Willmore functional*, Comm. Anal. Geom. 10 (2001), 409–441.

[Il1] T. Ilmanen, *Generalized flow of sets by mean curvature on a manifold*, Indiana Univ. Math. J. 41 (1992), 671–705.

[Il2] T. Ilmanen, *The level set flow on a manifold*, in: Differential Geometry: Partial Differential Equations on Manifolds (Los Angeles, CA 1990), Proc. Sympos. Pure Math. 54, Part 1, Amer. Math. Soc., Providence, RI, 1993, 193–204.

[Il3] T. Ilmanen, *Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature*, J. Differential Geom. 38 (1993) 417–461.

[Il4] T. Ilmanen, *Elliptic regularization and partial regularity for motion by mean curvature*, Mem. Amer. Math. Soc. 108 (1994).

[LS] S. Luckhaus and T. Sturzenhecker, *Implicit time discretization for the mean curvature flow equation*, Calc. Var. Partial Differential Equations 3 (1995), 253-271.

[MT] U. Massari and N. Taddia, *A variational approach to mean curvature flow with Dirichlet conditions*, Ann. Univ. Ferrara Sez. VII Sc. Mat. 46 (2000), 81–100.

[Sc] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Encyclopedia of Mathematics and its Applications 44, Cambridge University Press, Cambridge, 1993.

[Sp] H. Spohn, *Interface motion in models with stochastic dynamics*, J. Stat. Phys. 71 (1993), 1081–1132.

[Sh] Z. Shen, *Lectures on Finsler Geometry*, World Scientific, 2001.

[TC] J. E. Taylor and J. W. Cahn, *Linking anisotropic sharp and diffuse surface motion laws via gradient flows*, J. Stat. Phys. 77 (1994), 183–197.