NONLOCAL AND ABSTRACT PARABOLIC EQUATIONS
AND THEIR APPLICATIONS
BANACH CENTER PUBLICATIONS, VOLUME 86
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2009

WEIGHTED L^2 AND L^q APPROACHES TO FLUID FLOW PAST A ROTATING BODY

R. FARWIG

Department of Mathematics, Darmstadt University of Technology
64289 Darmstadt, Germany
E-mail: farwig@mathematik.tu-darmstadt.de

S. KRAČMAR

Department of Technical Mathematics, Czech Technical University
Karlovo nám. 13, 121 35 Prague 2, Czech Republic
E-mail: kracmar@marian.fsik.cvut.cz

M. KRBEC

Institute of Mathematics, Academy of Sciences of the Czech Republic
Žitná 25, 11567 Prague 1, Czech Republic
E-mail: krbecm@math.cas.cz

Š. NEČASOVÁ

Institute of Mathematics, Academy of Sciences of the Czech Republic
Žitná 25, 11567 Prague 1, Czech Republic
E-mail: matus@math.cas.cz

P. PENEL

Université du Sud Toulon-Var, Mathématiques
B.P. 20132, 83957 La Garde Cedex, France
E-mail: penel@univ-tln.fr

Abstract. Consider the flow of a viscous, incompressible fluid past a rotating obstacle with velocity at infinity parallel to the axis of rotation. After a coordinate transform in order to reduce

2000 *Mathematics Subject Classification*: Primary 76D05; Secondary 35Q30, 35Q35.
Key words and phrases: variational approach, Friedrichs-Poincaré inequality, Littlewood-Paley theory, maximal operators, rotating obstacles, stationary Oseen flow, weak solution, strong solution, weighted estimates, anisotropic weights.
The paper is in final form and no version of it will be published elsewhere.

the problem to a Navier-Stokes system on a fixed exterior domain and a subsequent linearization we are led to a modified Oseen system with two additional terms one of which is not subordinate to the Laplacean.

In this paper we describe two different approaches to this problem in the whole space case. One of them is based on a variational method in L^2-spaces with weights reflecting the anisotropic behaviour of the Oseen fundamental solution. The other approach uses weighted multiplier theory, interpolation and Littlewood-Paley theory to get *a priori* estimates in anisotropically weighted L^q-spaces.

1. Introduction. Consider a three-dimensional rigid body $K \subset\subset \mathbb{R}^3$ rotating with angular velocity $\boldsymbol{\omega} = \omega(0,0,1)^T$, $\omega \neq 0$, and assume that the complement $\mathbb{R}^3 \setminus K$ is filled with a viscous incompressible fluid modelled by the Navier-Stokes equations. We will analyze the viscous flow either past the rotating body K with velocity $\mathbf{u}_\infty = k\mathbf{e}_3 \neq 0$ at infinity or around a rotating body K which is moving in the direction of its axis of rotation with velocity $-\mathbf{u}_\infty$. Given the coefficient of viscosity $\nu > 0$ and an external force $\widetilde{\mathbf{f}} = \widetilde{\mathbf{f}}(\mathbf{y},t)$, we are looking for the velocity $\mathbf{v} = \mathbf{v}(\mathbf{y},t)$ and the pressure $q = q(\mathbf{y},t)$ solving the nonlinear system

$$\begin{aligned} \mathbf{v}_t - \nu\Delta\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla q &= \widetilde{\mathbf{f}} && \text{in } \Omega(t),\ t>0,\\ \operatorname{div}\mathbf{v} &= 0 && \text{in } \Omega(t),\ t>0,\\ \mathbf{v}(\mathbf{y},t) &= \boldsymbol{\omega}\times\mathbf{y} && \text{on } \partial\Omega(t),\ t>0,\\ \mathbf{v}(\mathbf{y},t) &\to \mathbf{u}_\infty \neq 0 && \text{as } |\mathbf{y}|\to\infty. \end{aligned} \tag{1.1}$$

Here the time-dependent exterior domain $\Omega(t)$ is given—due to the rotation with angular velocity $\boldsymbol{\omega}$—by

$$\Omega(t) = O_\omega(t)\Omega$$

where $\Omega \subset \mathbb{R}^3$ is a fixed exterior domain and $O_\omega(t)$ denotes the orthogonal matrix

$$O_\omega(t) = \begin{pmatrix} \cos\omega t & -\sin\omega t & 0\\ \sin\omega t & \cos\omega t & 0\\ 0 & 0 & 1 \end{pmatrix}. \tag{1.2}$$

Introducing the change of variables and the new functions

$$\mathbf{x} = O_\omega^T(t)\mathbf{y} \quad \text{and} \quad \mathbf{u}(\mathbf{x},t) = O_\omega^T(t)(\mathbf{v}(\mathbf{y},t) - \mathbf{u}_\infty), \quad p(\mathbf{x},t) = q(\mathbf{y},t), \tag{1.3}$$

respectively, as well as the force term $\mathbf{f}(\mathbf{x},t) = O_\omega^T(t)\widetilde{\mathbf{f}}(\mathbf{y},t)$ we arrive at the modified Navier-Stokes system

$$\begin{aligned} \mathbf{u}_t - \nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + k\partial_3\mathbf{u} &\\ -(\boldsymbol{\omega}\times\mathbf{x})\cdot\nabla\mathbf{u} + \boldsymbol{\omega}\times\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega\times(0,\infty),\\ \operatorname{div}\mathbf{u} &= 0 && \text{in } \Omega\times(0,\infty),\\ \mathbf{u}(\mathbf{x},t) &\to 0 && \text{as } |\mathbf{x}|\to\infty, \end{aligned} \tag{1.4}$$

with boundary condition $\mathbf{u}(\mathbf{x},t) = \boldsymbol{\omega}\times\mathbf{x} - \mathbf{u}_\infty$ on $\partial\Omega$ in the exterior time-independent domain Ω.

Due to the new coordinate system attached to the rotating body the nonlinear system (1.4) contains two new linear terms, the classical Coriolis force term $\boldsymbol{\omega}\times\mathbf{u}$ (up to a multiplicative constant) and the term $(\boldsymbol{\omega} \times \mathbf{x}) \cdot \nabla \mathbf{u}$ which is *not* subordinate to the Laplacean in unbounded domains. Linearizing (1.4) in $\mathbf{u}$ at $\mathbf{u} \equiv 0$ and considering only the stationary problem we arrive at the modified Oseen system

$$\begin{aligned} -\nu\Delta\mathbf{u} + k\partial_3\mathbf{u} - (\boldsymbol{\omega} \times \mathbf{x}) \cdot \nabla\mathbf{u} + \boldsymbol{\omega} \times \mathbf{u} + \nabla p = \mathbf{f} \quad & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 \quad & \text{in } \Omega, \\ \mathbf{u} \to 0 \quad & \text{at } \infty, \end{aligned} \tag{1.5}$$

together with the boundary condition $\mathbf{u}(\mathbf{x}, t) = \boldsymbol{\omega} \times \mathbf{x} - \mathbf{u}_\infty$ on $\partial\Omega$. Note that there is no boundary condition in the case $\Omega = \mathbb{R}^3$.

In our paper we follow two different ways to handle this problem in $\mathbb{R}^3$. The first approach in an L^2-setting uses variational calculus. This viewpoint has already been applied in [3] by R. Farwig and in [29, 30] by S. Kračmar and P. Penel to solve the scalar model equations

$$-\nu\,\Delta u + k\partial_3 u = f \quad \text{in } \Omega$$

and—with a given non-constant and, in general, non-solenoidal vector function $\mathbf{a}$—

$$-\nu\,\Delta u + k\partial_3 u - \mathbf{a} \cdot \nabla u = f \quad \text{in } \Omega,$$

respectively, in an exterior domain Ω, together with the boundary conditions $u = 0$ on $\partial\Omega$ and $u \to 0$ as $|\mathbf{x}| \to \infty$.

Secondly, to consider more general weights in L^q-spaces, we apply weighted multiplier and Littlewood-Paley theory as well as the theory of one-sided Muckenhoupt weights corresponding to one-sided maximal functions. This approach was introduced firstly by Farwig, Hishida, Müller [7] for the case $\mathbf{u}_\infty = 0$ and in [4], [5] when $\mathbf{u}_\infty \neq 0$ without weights and then extended to the weighted case by Krbec, Farwig, Nečasová [8], [9] and Nečasová, Schumacher [35].

The paper is organized as follows. Section 2 introduces some necessary notation and preliminaries, mainly on weighted function spaces and techniques to deal with operators on them. The main results are presented in Section 3. Next, Section 4 discusses the question of uniqueness. Sections 5 and 6 deal with the (outline of) proofs for the weighted L^2-results and weighted L^q-results, respectively. Finally, we resume the main differences of the two different approaches in Section 7.

2. Notations and preliminaries

2.1. *Anisotropic weights.* To reflect the decay properties near infinity we introduce the following weight functions:

$$w(\mathbf{x}) = \eta_\beta^\alpha = \eta_{\beta,\varepsilon}^{\alpha,\delta}(\mathbf{x}) = (1 + \delta r)^\alpha\,(1 + \varepsilon s)^\beta\,,$$

with $r = r(\mathbf{x}) = |\mathbf{x}| = (\sum_{i=1}^3 x_i^2)^{1/2}$, $s = s(\mathbf{x}) = r - x_3$ where $\mathbf{x} \in \mathbb{R}^3$, $\varepsilon, \delta > 0$, $\alpha, \beta \in \mathbb{R}$. For suitable exponents α and β the corresponding weighted spaces $L^q(\mathbb{R}^3; w)$ will give the appropriate framework to solve (1.5).

We recall the definition of the classical Muckenhoupt class A_2 based on cubes (i.e. open bounded axiparallel cubes Q), which we will use in the L^2 framework. In order to apply estimates for singular integral operators, multiplier operators and maximal operators, the weight function w will be supposed to satisfy the Muckenhoupt A_2-condition. Let us mention that $\eta^{\alpha,\delta}_{\beta,\varepsilon}(\mathbf{x})$ belongs to the class A_2 in $\mathbb{R}^3$ if $-1 < \beta < 1$ and $-3 < \alpha + \beta < 3$.

DEFINITION 2.1. A weight function $0 \le w \in L^1_{\mathrm{loc}}$ belongs to the *Muckenhoupt class A_2* if there exists a constant $C > 0$ such that

$$\sup_Q \left(\frac{1}{|Q|}\int_Q w(\mathbf{x})\,d\mathbf{x}\right)\left(\frac{1}{|Q|}\int_Q w^{-1}\,d\mathbf{x}\right) \le C < +\infty.$$

Concerning the weight functions η^α_β, we will use the two notations $\eta^\alpha_\beta(\mathbf{x})$ and $\eta^{\alpha,\delta}_{\beta,\varepsilon}(\mathbf{x})$ taking advantage of the following estimate: For any $\delta_1, \delta_2, \varepsilon_1, \varepsilon_2 > 0$ one has

$$c_{\min} \cdot \eta^{\alpha,\delta_2}_{\beta,\varepsilon_2} \le \eta^{\alpha,\delta_1}_{\beta,\varepsilon_1} \le c_{\max} \cdot \eta^{\alpha,\delta_2}_{\beta,\varepsilon_2}, \tag{2.1}$$

where $c_{\min} = \min(1, (\frac{\delta_1}{\delta_2})^\alpha) \cdot \min(1, (\frac{\varepsilon_1}{\varepsilon_2})^\beta)$, $c_{\max} = \max(1, (\frac{\delta_1}{\delta_2})^\alpha) \cdot \max(1, (\frac{\varepsilon_1}{\varepsilon_2})^\beta)$. The parameters δ and ε are useful to rescale separately the isotropic and anisotropic parts of the weight function η^α_β.

Now, we define more general Muckenhoupt classes of weights; for further details see e.g. Sawyer [38]. In particular, we will use the weight $w = \eta^{\alpha,1}_{\beta,1}$.

DEFINITION 2.2. Let $\mathcal{R}$ be a collection of bounded sets R in $\mathbb{R}^n$, each of positive Lebesgue measure $|R|$. A weight function $0 \le w \in L^1_{\mathrm{loc}}$ belongs to the *Muckenhoupt class $A_q(\mathcal{R}) = A_q(\mathbb{R}^n, \mathcal{R})$*, $1 \le q < \infty$, if there exists a constant $C > 0$ such that

$$\sup_R \left(\frac{1}{|R|}\int_R w(\mathbf{x})\,d\mathbf{x}\right)\left(\frac{1}{|R|}\int_R w(\mathbf{x})^{-1/(q-1)}\,d\mathbf{x}\right)^{q-1} \le C,$$

if $1 < q < \infty$, and

$$\sup_{R\in\mathcal{R},\, R\ni\mathbf{x}_0} \frac{1}{|R|}\int_R w(\mathbf{x})\,d\mathbf{x} \le C w(\mathbf{x}_0) \quad \text{for a.a. } \mathbf{x}_0 \in \mathbb{R}^n,$$

if $q = 1$, respectively.

Due to the anisotropic nature of our problem the standard Muckenhoupt class $A_q(\mathcal{C}) = A_q(\mathbb{R}^3, \mathcal{C})$, where $\mathcal{C}$ is the set of all cubes $Q \subset \mathbb{R}^3$ with edges parallel to the coordinate axes, is not suitable. Actually, we have to work with a variant where $\mathcal{C}$ is replaced by $\mathcal{J}$, the set of all bounded axiparallel intervals (rectangles) in $\mathbb{R}^3$, leading to the class $A_q(\mathcal{J}) = A_q(\mathbb{R}^3, \mathcal{J})$. Obviously, $A_q(\mathbb{R}^3, \mathcal{J}) \subsetneq A_q(\mathbb{R}^3, \mathcal{C})$.

In addition to that, for a more precise description of the anisotropy of the wake region in terms of weights we have to introduce a suitable "hybrid Muckenhoupt class". Roughly speaking, such weights satisfy the Muckenhoupt condition on $\mathbb{R}^3$ and their restrictions to the third variable belong to the *one-sided Muckenhoupt class* corresponding to *one-sided maximal operators* on the real line and are uniform with respect to the radius of the rotation, see Definition 2.3, Theorem 2.7 and Lemma 2.8 below.

DEFINITION 2.3. (i) For every locally integrable function u on the real line let M^+u be defined by

$$M^+u(x) = \sup_{h>0} \frac{1}{h}\int_x^{x+h} |u(t)|\,dt.$$

Analogously,

$$M^{-}u(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |u(t)|\, dt.$$

(ii) A weight function $0 \le w \in L^1_{\mathrm{loc}}(\mathbb{R})$ lies in the *weight class* A_1^- if there exists a constant $c > 0$ such that $M^+w(x) \le cw(x)$ for almost all $x \in \mathbb{R}$. Analogously, $w \in A_1^+$ if and only if $M^-w(x) \le cw(x)$ for almost all $x \in \mathbb{R}$. The smallest constant $c \ge 0$ satisfying $M^\pm w(x) \le cw(x)$ for almost all $x \in \mathbb{R}$ is called the $A_1^\mp$-*constant of* w.

(iii) A weight function $0 \le w \in L^1_{\mathrm{loc}}$ belongs to the *one-sided Muckenhoupt class* A_q^+, $1 < q < \infty$, if there exists a constant $C > 0$ such that for all $x \in \mathbb{R}$

$$\sup_{h>0} \left(\frac{1}{h} \int_{x-h}^{x} w(t)\, dt \right) \left(\frac{1}{h} \int_{x}^{x+h} w(t)^{-1/(q-1)}\, dt \right)^{q-1} \le C.$$

The smallest constant $C \ge 0$ satisfying this estimate is called the A_q^+-*constant* of w. By analogy, we define the set of weights A_q^- and the A_q^--constant of a weight in A_q^-.

Now we are in a position to describe the most general weights considered in this paper. Note that these weights are independent of the angular variable θ in the cylindrical coordinate system $(r, \theta, x_3) \in [0, \infty) \times [0, 2\pi] \times \mathbb{R}$ attached to the axis of revolution $\mathbf{e}_3 = (0, 0, 1)^T$. Hence we will write $w(\mathbf{x}) = w(x_1, x_2, x_3) = w_r(x_3)$ for $r = |(x_1, x_2)|$, $\mathbf{x} = (x_1, x_2, x_3)$.

Definition 2.4. For $1 \le q < \infty$ let

$$\begin{aligned} \widetilde{A}_q^- = \widetilde{A}_q^-(\mathbb{R}^3) = \{ w \in A_q(\mathbb{R}^3) :\ & w \text{ is } \theta\text{-independent for a.a. } r > 0, \\ & w(x_1, x_2, \cdot) = w_r(\cdot) \in A_q^-(\mathbb{R}) \\ & \text{with } A_q^-(\mathbb{R})\text{-constant essentially bounded in } r \}. \end{aligned}$$

2.2. *Function spaces.* Let us outline our notations. Let Ω be an exterior domain with boundary satisfying the cone property or the whole space $\mathbb{R}^3$, and

$$\widehat{W}^{m,q}(\Omega) = \left\{ u \in L^1_{loc}(\Omega) :\ D^l u \in L^q(\Omega),\ \ |l| = m \right\}$$

equipped with the seminorm $|u|_{m,q} = (\sum_{|l|=m} \int_\Omega |u|^q)^{1/q}$. It is known that $\widehat{W}^{m,q}(\Omega)$ is a Banach space (and, if $q = 2$, the space $\widehat{H}^m(\Omega) = \widehat{W}^{m,2}(\Omega)$ is a Hilbert space), provided we identify two functions u_1, u_2 whenever $|u_1 - u_2|_{m,q} = 0$, i.e., u_1, u_2 differ (at most) by a polynomial of degree $m - 1$. As usual, we denote by $\widehat{W}_0^{m,q}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $\widehat{W}^{m,q}(\Omega)$.

Let $(L^2(\Omega;\, w))^3$ be the set of measurable vector functions $\mathbf{f} = (f_1, f_2, f_3)$ in Ω such that

$$\|\mathbf{f}\|^2_{2,\Omega;\,w} = \int_\Omega |\mathbf{f}|^2\, w\, d\mathbf{x} < \infty.$$

We will use the notation $\mathbf{L}^2_{\alpha,\beta}(\Omega)$ instead of $(L^2(\Omega;\, \eta^\alpha_\beta))^3$ and $\|\cdot\|_{2,\alpha,\beta}$ instead of $\|\cdot\|_{(L^2(\Omega;\,\eta^\alpha_\beta))^3}$. Let us define the weighted Sobolev space $\mathbf{H}^1(\Omega; \eta^{\alpha_0}_{\beta_0}, \eta^{\alpha_1}_{\beta_1})$ as the set of functions $\mathbf{u} \in \mathbf{L}^2_{\alpha_0,\beta_0}(\Omega)$ with weak derivatives $\partial_i \mathbf{u} \in \mathbf{L}^2_{\alpha_1,\beta_1}(\Omega)$, $i = 1, 2, 3$. The norm of $\mathbf{u} \in \mathbf{H}^1(\Omega;\, \eta^{\alpha_0}_{\beta_0}, \eta^{\alpha_1}_{\beta_1})$ is given by

$$\|\mathbf{u}\|_{\mathbf{H}^1\left(\Omega;\, \eta^{\alpha_0}_{\beta_0}, \eta^{\alpha_1}_{\beta_1}\right)} = \left(\int_\Omega |\mathbf{u}|^2\, \eta^{\alpha_0}_{\beta_0}\, d\mathbf{x} + \int_\Omega |\nabla \mathbf{u}|^2\, \eta^{\alpha_1}_{\beta_1}\, d\mathbf{x} \right)^{1/2}.$$

As usual, $\mathring{\mathbf{H}}^1(\Omega;\, \eta^{\alpha_0}_{\beta_0}, \eta^{\alpha_1}_{\beta_1})$ will be the closure of $\mathbf{C}^\infty_0(\Omega)$ in $\mathbf{H}^1(\Omega;\, \eta^{\alpha_0}_{\beta_0}, \eta^{\alpha_1}_{\beta_1})$, where $\mathbf{C}^\infty_0(\Omega)$ is $(C^\infty_0(\Omega))^3$.

For simplicity, we shall use the following abbreviations:

$$\begin{array}{lll}\mathbf{L}^2_{\alpha,\beta}(\Omega) & \text{instead of} & (L^2(\Omega;\, \eta^\alpha_\beta))^3, \\ \|\cdot\|_{2,\alpha,\beta} & \text{instead of} & \|\cdot\|_{(L^2(\Omega;\, \eta^\alpha_\beta))^3}, \\ \mathring{\mathbf{H}}^1_{\alpha,\,\beta}(\Omega) & \text{instead of} & \mathring{\mathbf{H}}^1(\Omega;\, \eta^{\alpha-1}_{\beta-1}, \eta^\alpha_\beta), \\ \mathbf{V}_{\alpha,\beta}(\Omega) & \text{instead of} & \mathring{\mathbf{H}}^1(\Omega;\, \eta^{\alpha-1}_{\beta}, \eta^\alpha_\beta).\end{array}$$

We shall use these last two Hilbert spaces for $\alpha \geq 0$, $\beta > 0$, $\alpha + \beta < 3$. As usual, $\mathbf{H}^1(\Omega)$ and $\mathring{\mathbf{H}}^1(\Omega)$ mean the unweighted spaces $(H^1(\Omega;\, 1,1))^3$ and $(\mathring{H}^1(\Omega;\, 1,1))^3$, respectively.

We also use the notation of sets $B_R = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| \leq R\}$, $B^R = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| \geq R\}$, $\Omega_R = B_R \cap \Omega$, $\Omega^R = B^R \cap \Omega$, $B^{R_1}_{R_2} = B^{R_1} \cap B_{R_2}$, $\Omega^{R_1}_{R_2} = B^{R_1}_{R_2} \cap \Omega$, for positive numbers R, R_1, R_2.

Finally, for a nonnegative weight function $w \in L^1_{\mathrm{loc}}$ we introduce the weighted Lebesgue space

$$L^q_w(\mathbb{R}^3) = L^q_w = \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^3) :\ \|u\|_{q,w} = \left(\int_{\mathbb{R}^n} |u(\mathbf{x})|^q w(\mathbf{x})\, d\mathbf{x} \right)^{1/q} < \infty \right\}.$$

2.3. *Properties of Muckenhoupt weights.* To prove Theorem 3.3 below we need several properties of Muckenhoupt weights and of maximal operators. Recall that $\mathcal{J}$ stands for the set of all non-degenerate rectangles in $\mathbb{R}^n$ with edges parallel to the coordinate axes.

PROPOSITION 2.5. *(1) Let μ be a non-negative regular Borel measure such that the strong Hardy-Littlewood maximal operator*

$$\mathcal{M}_{\mathcal{J}}\mu(\mathbf{x}) = \sup_{R \in \mathcal{J},\, R \ni \mathbf{x}} \frac{1}{|R|} \int_R d\mu$$

is finite for almost all $\mathbf{x} \in \mathbb{R}^n$; here R runs through the collection $\mathcal{J}$ of rectangles containing the point $\mathbf{x}$, and $|R|$ denotes the Lebesgue measure of R. Then $(\mathcal{M}_{\mathcal{J}}\mu)^\gamma \in A_1(\mathcal{J})$ for all $\gamma \in [0,1)$.

(2) For all $1 < q < \tau$ we have $A_1(\mathcal{J}) \subset A_q(\mathcal{J}) \subset A_\tau(\mathcal{J})$.

(3) Let $1 < q < \infty$ and $w \in A_q(\mathcal{J})$. Then there are $w_1, w_2 \in A_1(\mathcal{J})$ such that

$$w = \frac{w_1}{w_2^{q-1}}.$$

Conversely, given $w_1, w_2 \in A_1(\mathcal{J})$, the weight $w = w_1 w_2^{1-q}$ belongs to $A_q(\mathcal{J})$.

For the proofs see [15, Chapter IV, § 6]. The claim (3) is a variant of Jones' factorization theorem, see [15, Chapter IV, Theorem 6.8].

For a rapidly decreasing function $u \in \mathcal{S}(\mathbb{R}^n)$ let

$$\mathcal{F}u(\xi) = \hat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\mathbf{x}\cdot\boldsymbol{\xi}} u(\mathbf{x})\, d\mathbf{x}, \quad \xi \in \mathbb{R}^n,$$

be the Fourier transform of u. Its inverse will be denoted by $\mathcal{F}^{-1}$. Moreover, we define the centered Hardy-Littlewood maximal operator

$$\mathcal{M}\mathbf{u}(\mathbf{x}) = \sup_{Q \ni \mathbf{x}} \frac{1}{|Q|} \int_Q |\mathbf{u}(\mathbf{y})| \, d\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^n,$$

for $\mathbf{u} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ where Q runs through the set of all axiparallel cubes centered at $\mathbf{x}$.

THEOREM 2.6. *Let $1 < q < \infty$ and $w \in A_q(\mathcal{C})$.*

(i) The operator $\mathcal{M}$, defined e.g. on $\mathcal{S}(\mathbb{R}^n)$, is a bounded operator from L^q_w to L^q_w.

(ii) Let $m \in C^n(\mathbb{R}^n \setminus \{0\})$ satisfy the pointwise Hörmander-Mikhlin multiplier condition

$$|\xi|^{|\alpha|} \, |D^\alpha m(\xi)| \le c_\alpha \quad \text{for all} \quad \xi \in \mathbb{R}^n \setminus \{0\}$$

and all multiindices $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le n_1 \in \mathbb{N}$, where $n_1 > n/2$. Then the multiplier operator $u \mapsto \mathcal{F}^{-1}(m\hat{u})$, $u \in \mathcal{S}(\mathbb{R}^n)$, can be extended to a bounded linear operator from L^q_w to L^q_w.

(iii) Let m be of class C^n in each "quadrant" of $\mathbb{R}^n$ and let a constant $B \ge 0$ exist such that $\|m\|_\infty \le B$,

$$\sup_{x_{k+1},\dots,x_n} \int_{\mathcal{I}} \left| \frac{\partial^k m(\xi)}{\partial \xi_1 \cdots \partial \xi_k} \right| d\xi_1 \cdots d\xi_k \le B$$

for any dyadic interval $\mathcal{I}$ in $\mathbb{R}^k$, $1 \le k \le n$, and also for any permutation of the variables $\xi_1, \dots, \xi_k$ within $\xi_1, \dots, \xi_n$. If $w \in A_q(\mathbb{R}^n, \mathcal{J})$, then m defines a bounded multiplier operator from $L^q_w(\mathbb{R}^n)$ to $L^q_w(\mathbb{R}^n)$.

For the proof of (i) see [15, Theorem IV 2.8], [32, Theorem 9], for (ii) see [15, Theorem IV 3.9] or [31, Theorem 4]. Note that the pointwise condition on m implies the integral condition in [15], [31]. For the proof of (iii) see [31].

Concerning *one-sided weights* and *one-sided maximal operators* on the real line, see Definition 2.1, we first recall the following duality property: $w \in A_q^+$ if and only if $w^{-q'/q} = w^{-1/(q-1)} \in A_{q'}^-$. Moreover we will need the following results:

THEOREM 2.7 ([38, Theorem 1]). *Let $1 < p < \infty$ and $p' = \frac{p}{p-1}$.*

(i) Let $w_1 \in A_1^+$, $w_2 \in A_1^-$. Then $\frac{w_1}{w_2^{p-1}} \in A_p^+$. Conversely, given $w \in A_p^+$ there exist $w_1 \in A_1^+$, $w_2 \in A_1^-$ such that $w = \frac{w_1}{w_2^{p-1}}$.

(ii) The operator M^+ is continuous from $L^p_w(\mathbb{R})$ to itself if and only if $w \in A_p^+$. Analogously, $M^- : L^p_w(\mathbb{R}) \to L^p_w(\mathbb{R})$ if and only if $w \in A_p^-$.

Obviously, $A_p \subset A_p^\pm$ where A_p denotes the usual Muckenhoupt class on the real line. Hence $|\mathbf{x}|^\alpha, (1 + |\mathbf{x}|)^\alpha \in A_p^\pm$ if $-1 < \alpha < p - 1$, $1 < p < \infty$. However, in view of the anisotropic weight $w = \eta_\beta^\alpha$ on $\mathbb{R}^3$, see (3.9), we have to consider also one-dimensional anisotropic weight functions such as

$$\widetilde{w}_{\alpha,\beta}(x) = \widetilde{w}_{\alpha,\beta}(x; r) = (r^2 + x^2)^{\alpha/2} (\sqrt{r^2 + x^2} - x)^\beta, \;\; x \in \mathbb{R}, \, r > 0. \qquad (2.2)$$

LEMMA 2.8 ([9, Lemma 2.4]). *(i) For every $r > 0$ the univariate weight $\widetilde{w}_{\alpha,\beta}(x; r)$ lies in A_1^- if and only if $\beta \ge 0$, $\alpha \le \beta$ and $\alpha + \beta > -1$. Moreover, the A_1^--constant of $\widetilde{w}_{\alpha,\beta}$ is uniformly bounded in r.*

(ii) For every $r > 0$ the univariate weight

$$w_{\alpha,\beta}(x) = w_{\alpha,\beta}(x;r) = (1 + r^2 + x^2)^{\alpha/2}(1 + \sqrt{r^2 + x^2} - x)^{\beta}$$

lies in A_1^- with an A_1^--constant independent of $r > 0$ if and only if

$$\alpha \le 0 \le \beta \text{ and } \alpha + \beta > -1. \tag{2.3}$$

(iii) Let $1 < p < \infty$. Then for every $r > 0$

$$\begin{aligned} w_{\alpha,\beta}(\cdot\,;r) &\in A_p^+ \quad \text{for} \quad \alpha > -1, \quad \beta \le 0, \quad \alpha + \beta < p - 1 \\ w_{\alpha,\beta}(\cdot\,;r) &\in A_p^- \quad \text{for} \quad \alpha < p - 1, \quad \beta \ge 0, \quad \alpha + \beta > -1. \end{aligned} \tag{2.4}$$

Moreover, the $A_p^{\pm}$-constant is uniformly bounded in $r > 0$.

3. Main results. Weighted estimates of the solution to the classical stationary Oseen problem were firstly obtained by R. Finn 1959, see [10], [11], and then improved by R. Farwig [2], [3] in 1992; see [29] for other comments and references.

The linear system (1.5) with $u_\infty = 0$ and the nonlinear system (1.4) in L^2 spaces were investigated by Hishida [17] and by Galdi and Silvestre [13], [14]. In the work of Galdi and Silvestre one can find a combination of L^2-estimates and pointwise anisotropically weighted estimates. For a discussion of weak solutions we refer to [6], [24], [25].

Let us assume for a moment that the pressure p is known. In solving the problem (1.5) with respect to $\mathbf{u}$ and p by means of a purely variational approach, we test (1.5) with $w\,\mathbf{u}$ where w is an appropriate weight function, and get the equation

$$\begin{aligned} \nu \int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2\, w d\mathbf{x} + \nu \int_{\mathbb{R}^3} \mathbf{u} \nabla\, \mathbf{u} \cdot \nabla w\, d\mathbf{x} - \frac{k}{2} \int_{\mathbb{R}^3} |\mathbf{u}|^2\, \partial_1 w d\mathbf{x} \\ -\frac{1}{2} \int_{\mathbb{R}^3} |\mathbf{u}|^2 \operatorname{div}\left(w\, [\boldsymbol{\omega} \times \mathbf{x}]\right) d\mathbf{x} = \int_{\mathbb{R}^3} \mathbf{f} \cdot \mathbf{u}\, w\, d\mathbf{x} - \int_{\mathbb{R}^3} \nabla p \cdot \mathbf{u}\, w\, d\mathbf{x}. \end{aligned} \tag{3.1}$$

First, let us note that $\operatorname{div}(w[\boldsymbol{\omega} \times \mathbf{x}])$ equals zero for $w = \eta_\beta^\alpha$. The left hand side can be estimated from below by

$$\frac{\nu}{2} \int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2 w\, d\mathbf{x} + \frac{1}{2} \int_{\mathbb{R}^3} |\mathbf{u}|^2 (-\nu |\nabla w|^2 / w - k \partial_1 w) d\mathbf{x}. \tag{3.2}$$

Since the term $-\nu|\nabla w|^2/w - k\partial_1 w$ is known explicitly, we will have the possibility to estimate it from below by a small negative quantity in the form $-C\,\eta_{\beta-1}^{\alpha-1}$ without any constraint in $s(\cdot)$, see Lemma 5.3 below.

The first main result of this paper in weighted L^2-spaces, see Theorem 3.1 below, is strongly based on the theorem of Lax-Milgram and an improved weighted Friedrichs-Poincaré type inequality in $\mathring{\mathbf{H}}^1_{\alpha,\,\beta}$. This inequality allows to compensate by the viscous Dirichlet integral the "small" negative contribution in the second integral of (3.2); the parameters α, β, δ, ε will be specified later.

THEOREM 3.1. *Let $\beta > 0$. There are positive constants R_0, c_0, c_1 depending on α, β, δ, ε (essentially, $c_0 = O(\varepsilon^{-2} + \delta^{-2})$ and $c_1 = O(\varepsilon^{-1}\delta^{-1})$ for δ and ε tending to zero, for more details see Lemma 5.3, such that for all $\mathbf{v} \in \mathring{\mathbf{H}}^1_{\alpha,\,\beta}$*

$$\|\mathbf{v}\|^2_{2,\alpha-1,\beta-1} \le c_0 \int_{B_{R_0}} |\nabla \mathbf{v}|^2\, \eta_\beta^\alpha\, d\mathbf{x} + c_1 \int_{B^{R_0}} |\nabla \mathbf{v}|^2\, \eta_\beta^\alpha\, d\mathbf{x}. \tag{3.3}$$

THEOREM 3.2 (Existence and uniqueness). *Let $0 < \beta \le 1$, $0 \le \alpha < y_1\beta$ with y_1 to be given in Subsection 5.4 below, see* (5.12). *Moreover, let $\mathbf{f} \in \mathbf{L}^2_{\alpha+1,\beta}$, $g \in W_0^{1,2}$ with $\operatorname{supp} g = K \subset\subset \mathbb{R}^3$, and $\int_{\mathbb{R}^3} g\,d\mathbf{x} = 0$. Then there exists a unique weak solution $\{\mathbf{u}, p\}$ of the problem* (1.5) *such that $\mathbf{u} \in \mathbf{V}_{\alpha,\beta}$, $p \in L^2_{\alpha,\beta-1}$, $\nabla p \in \mathbf{L}^2_{\alpha+1,\beta}$ and*

$$\|\mathbf{u}\|_{2,\alpha-1,\beta} + \|\nabla\mathbf{u}\|_{2,\alpha,\beta} + \|p\|_{2,\alpha,\beta-1} + \|\nabla p\|_{2,\alpha+1,\beta} \le C(\|\mathbf{f}\|_{2,\alpha+1,\beta} + \|g\|_{1,2}).$$

In an L^q setting the problem is much more difficult and cannot be handled by a variational approach. Moreover, it is necessary to use more advanced harmonic analysis since the integral operators which have to be estimated are not of classical Calderón–Zygmund type. The linear system (1.5) has been analyzed in classical L^q-spaces, $1 < q < \infty$, for the whole space case in [4], [5] proving the *a priori*-estimate

$$\begin{gathered}\|\nu\nabla^2\mathbf{u}\|_q + \|\nabla p\|_q \le c\|\mathbf{f}\|_q,\\ \|k\partial_3\mathbf{u}\|_q + \|(\boldsymbol{\omega}\times\mathbf{x})\cdot\nabla\mathbf{u} + \boldsymbol{\omega}\times\mathbf{u}\|_q \le c\left(1 + \frac{k^4}{\nu^2|\omega|^2}\right)\|\mathbf{f}\|_q\end{gathered} \tag{3.4}$$

with a constant $c > 0$ independent of ν, k and $\boldsymbol{\omega}$. The corresponding case when $\mathbf{u}_\infty = 0$ has recently been analyzed in [6]–[8], [19], [20]. For a more comprehensive introduction including physical considerations and non-Newtonian fluids we refer to [12].

The main result of the second part of the paper can be formulated by the following theorem.

THEOREM 3.3. *Let the weight function $0 \le w \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ be independent of the angular variable θ and satisfy the following condition depending on $q \in (1,\infty)$:*

$$\begin{aligned} 2 \le q < \infty: &\quad w^\tau \in \widetilde{A}^-_{\tau q/2} \quad \text{for some} \quad \tau \in [1,\infty),\\ 1 < q < 2: &\quad w^\tau \in \widetilde{A}^-_{\tau q/2} \quad \text{for some} \quad \tau \in \left(\tfrac{2}{q}, \tfrac{2}{2-q}\right]. \end{aligned} \tag{3.5}$$

(i) Given $\mathbf{f} \in L^q_w(\mathbb{R}^3)^3$ there exists a solution $(\mathbf{u}, p) \in L^1_{\mathrm{loc}}(\mathbb{R}^3)^3 \times L^1_{\mathrm{loc}}(\mathbb{R}^3)$ of (1.5) *satisfying the estimate*

$$\|\nu\nabla^2\mathbf{u}\|_{q,w} + \|\nabla p\|_{q,w} \le c\|\mathbf{f}\|_{q,w}, \tag{3.6}$$

with a constant $c = c(q,w) > 0$ independent of ν, k and $\boldsymbol{\omega}$.

(ii) Let $\mathbf{f} \in L^{q_1}_{w_1}(\mathbb{R}^3)^3 \cap L^{q_2}_{w_2}(\mathbb{R}^3)^3$ such that both (q_1, w_1) and (q_2, w_2) satisfy the conditions (3.5), *and let $\mathbf{u}_1, \mathbf{u}_2 \in L^1_{\mathrm{loc}}(\mathbb{R}^3)^3$ together with corresponding pressure functions $p_1, p_2 \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ be solutions of* (1.5) *satisfying* (3.6) *for (q_1, w_1) and (q_2, w_2), respectively. Then there are $\alpha, \beta \in \mathbb{R}$ such that $\mathbf{u}_1$ coincides with $\mathbf{u}_2$ up to an affine linear field $\alpha\mathbf{e}_3 + \beta\boldsymbol{\omega}\times\mathbf{x}$, $\alpha, \beta \in \mathbb{R}$.*

COROLLARY 3.4. *Let the weight function $0 \le w \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ be independent of the angular variable θ. Moreover, let w satisfy the following condition depending on $q \in (1,\infty)$:*

$$\begin{aligned} 2 \le q < \infty: &\quad w^\tau \in \widetilde{A}^-_{\tau q/2}(\mathcal{J}) \quad \text{for some} \quad \tau \in [1,\infty),\\ 1 < q < 2: &\quad w^\tau \in \widetilde{A}^-_{\tau q/2}(\mathcal{J}) \quad \text{for some} \quad \tau \in \left(\tfrac{2}{q}, \tfrac{2}{2-q}\right], \end{aligned} \tag{3.7}$$

where the weight class $\widetilde{A}^-_\tau(\mathcal{J})$, $1 \le \tau < \infty$, is defined by

$$\widetilde{A}^-_\tau(\mathcal{J}) = \widetilde{A}^-_\tau(\mathbb{R}^3) \cap A_\tau(\mathcal{J}).$$

Given $\mathbf{f} \in L^q_w(\mathbb{R}^3)^3$ *there exists a solution* $(\mathbf{u}, p) \in L^1_{\mathrm{loc}}(\mathbb{R}^3)^3 \times L^1_{\mathrm{loc}}(\mathbb{R}^3)$ *of* (1.5) *satisfying the estimate*

$$\|k\partial_3\mathbf{u}\|_{q,w} + \|(\boldsymbol{\omega}\times\mathbf{x})\cdot\mathbf{u} - \boldsymbol{\omega}\times\mathbf{u}\|_{q,w} \le c\left(1 + \frac{k^5}{\nu^{5/2}|\omega|^{5/2}}\right)\|\mathbf{f}\|_{q,w} \tag{3.8}$$

with a constant $c = c(q,w) > 0$ *independent of* ν, k *and* $\boldsymbol{\omega}$*.*

We remark that the $\boldsymbol{\omega}$-dependent term $1 + \frac{k^5}{\nu^{5/2}|\omega|^{5/2}}$ in (3.8) cannot be avoided in general; see [5] for an example in the space $L^2(\mathbb{R}^3)$.

As an example of anisotropic weight functions we consider as before

$$w(\mathbf{x}) = \eta^\alpha_\beta(\mathbf{x}) = (1+|\mathbf{x}|)^\alpha(1+s(\mathbf{x}))^\beta, \tag{3.9}$$

introduced in [2] to analyze the Oseen equations; see also [9], [24].

Corollary 3.5. *The a priori estimate* (3.6) *holds for the anisotropic weights* $w = \eta^\alpha_\beta$, *see* (3.9), *provided that*

$$\begin{array}{llll} 2 \le q < \infty: & -\frac{q}{2} < \alpha < \frac{q}{2}, & 0 \le \beta < \frac{q}{2} & \text{and} \quad \alpha+\beta > -1, \\ 1 < q < 2: & -\frac{q}{2} < \alpha < q-1, & 0 \le \beta < q-1 & \text{and} \quad \alpha+\beta > -\frac{q}{2}. \end{array}$$

Note that the condition $\beta \ge 0$ will reflect the existence of a wake region in the downstream direction $x_3 > 0$, where the solution of the original nonlinear problem (1.1) will decay slower than in the upstream direction $x_3 < 0$.

4. Uniqueness in $\mathbb{R}^3$.

In this section, we will start with the question of uniqueness of weak solutions to problem (1.5) in $\Omega = \mathbb{R}^3$. The approach will also be used in Section 5 in the proof of existence of solenoidal solutions.

Theorem 4.1 (Uniqueness in $\mathbb{R}^3$). *Let* $\{\mathbf{u}, p\}$ *be a solution in* $\mathcal{S}'$ *of the problem* (1.5) *with* $\mathbf{f} = \mathbf{0}$, $g = 0$.

(i) If $\mathbf{u} \in \widehat{\mathbf{H}}^{1,2}_0$ *and* $p \in L^2_{loc}$, *then* $\mathbf{u} = \mathbf{0}$ *and* $p = const.$

(ii) If $\nabla^2\mathbf{u} \in L^q_w(\mathbb{R}^3)^3$ *where* $1 < q < \infty$ *and* $w \in A_q(\mathcal{C})$, *then* $\mathbf{u} = \alpha\mathbf{e}_3 + \beta\boldsymbol{\omega}\times\mathbf{x}$ *with constants* α, $\beta \in \mathbb{R}$ *and* $p = const.$

Proof. Because $\operatorname{div}((\boldsymbol{\omega}\times\mathbf{x})\cdot\nabla\mathbf{u} - \boldsymbol{\omega}\times\mathbf{u}) = (\boldsymbol{\omega}\times\mathbf{x})\cdot\nabla\operatorname{div}\mathbf{u} = 0$, we have $\triangle p = 0$. Hence, applying the Laplacean and the Fourier transform we get

$$\triangle(-\nu\,\Delta\mathbf{u} + k\partial_3\mathbf{u} - (\boldsymbol{\omega}\times\mathbf{x})\cdot\nabla\mathbf{u} + \boldsymbol{\omega}\times\mathbf{u}) = \mathbf{0},$$
$$|\xi|^2(\nu|\xi|^2\widehat{\mathbf{u}} + i\,k\,\xi_3\widehat{\mathbf{u}} - (\boldsymbol{\omega}\times\xi)\cdot\nabla_{\boldsymbol{\xi}}\widehat{\mathbf{u}} + \boldsymbol{\omega}\times\widehat{\mathbf{u}}) = \mathbf{0} \qquad \text{in } \mathcal{S}'.$$

Consider the latter equation in cylindrical coordinates (ξ_3, ρ, φ) such that $(\boldsymbol{\omega}\times\xi)\cdot\nabla_{\boldsymbol{\xi}}\widehat{\mathbf{u}} = \omega\,\partial_\varphi\widehat{\mathbf{u}}$ and let $\widehat{\mathbf{v}} = T(\varphi)^{-1}\widehat{\mathbf{u}}(\xi_3,\rho,\varphi)$, where

$$T(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\widehat{\mathbf{v}}$ satisfies the equation

$$|\xi|^2\{-\partial_\varphi\widehat{\mathbf{v}} + [(\nu/\omega)|\xi|^2 + i(k/\omega)\xi_3]\widehat{\mathbf{v}}\} = \mathbf{0} \quad \text{in } \mathcal{S}'. \tag{4.1}$$

We will show that this equation implies $\operatorname{supp}\widehat{\mathbf{v}} \subset \{0\}$; moreover, due to the definition of $\widehat{\mathbf{v}}$ we will also have $\operatorname{supp}\widehat{\mathbf{u}} \subset \{\mathbf{0}\}$. Hence $\mathbf{u}$ is a polynomial of x_1, x_2, x_3. In (i) we have $\nabla\mathbf{u} \in \mathbf{L}^2$, so that the polynomial $\mathbf{u}$ belongs to $\mathbf{L}^6$. Consequently, $\mathbf{u} = 0$. In (ii) the polynomial $\nabla^2\mathbf{u} \in L^q_w(\mathbb{R}^3)^{27}$ must vanish so that $\nabla\mathbf{u}$ is linear. Now it is easily seen that there exists constants α, $\beta \in \mathbb{R}$ such that $\mathbf{u} = \alpha\mathbf{e}_3 + \boldsymbol{\omega}\times\mathbf{x}$, for the complete proof of part (ii) see [4, Theorem 1.1 (2)]. Moreover, in both cases, (1.5) implies that $\nabla p = 0$ and $p = const.$

So, we have to prove that an arbitrary vector function $\Psi \in \mathbf{C}_0^\infty(\mathbb{R}^3 \setminus \{\mathbf{0}\})$ satisfies $\langle\widehat{\mathbf{v}}, \Psi\rangle = 0$. If for each $\Psi \in \mathbf{C}_0^\infty(\mathbb{R}^3 \setminus \{\mathbf{0}\})$ there is a function $\Phi \in \mathbf{C}_0^\infty(\mathbb{R}^3 \setminus \{\mathbf{0}\})$ such that

$$\partial_\varphi(|\xi|^2\Phi) + [(\nu/\omega)|\xi|^2 + i(k/\omega)\xi_3](|\xi|^2\Phi) = \Psi, \tag{4.2}$$

then (4.1) implies that

$$\begin{aligned}0 &= \langle|\xi|^2\{-\partial_\varphi\widehat{\mathbf{v}} + [(\nu/\omega)|\xi|^2 + i(k/\omega)\xi_3]\widehat{\mathbf{v}}\}, \Phi\rangle\\ &= \langle\widehat{\mathbf{v}}, \partial_\varphi(|\xi|^2\Phi) + [(\nu/\omega)|\xi|^2 + i(k/\omega)\xi_3](|\xi|^2\Phi)\rangle = \langle\widehat{\mathbf{v}}, \Psi\rangle.\end{aligned}$$

Hence, the proof of $\operatorname{supp}\widehat{\mathbf{v}} \subset \{0\}$ is reduced to the solvability of (4.2).

First we note that it is sufficient to solve the equation

$$\partial_\varphi\zeta + ((\nu/\omega)|\xi|^2 + i(k/\omega)\xi_3)\zeta = \Psi, \tag{4.3}$$

for the division by the expression $|\xi|^2$ defines a one-to-one correspondence of the space $\mathbf{C}_0^\infty(\mathbb{R}^3 \setminus \{\mathbf{0}\})$ onto itself. Finally, the solvability of (4.3) in $\mathbf{C}_0^\infty(\mathbb{R}^3 \setminus \{\mathbf{0}\})$ follows from standard arguments on ordinary differential equations. ∎

5. Existence of a solution in $\mathbb{R}^3$. In this section we will construct a weak solution of the problem (1.5) assuming that $g = 0$.

5.1. *Existence of the pressure in $\mathbb{R}^3$ for a solenoidal solution.* If there exist distributions $\mathbf{u}, p$ satisfying

$$\begin{aligned}-\nu\,\Delta\mathbf{u} + k\,\partial_3\mathbf{u} - (\boldsymbol{\omega}\times\mathbf{x})\cdot\nabla\mathbf{u} + \boldsymbol{\omega}\times\mathbf{u} + \nabla p &= \mathbf{f} \qquad \text{in } \mathbb{R}^3,\\ \operatorname{div}\mathbf{u} &= 0 \qquad \text{in } \mathbb{R}^3,\end{aligned}$$

then the pressure p satisfies the equation

$$\triangle p = \operatorname{div}\mathbf{f}, \tag{5.1}$$

for $\operatorname{div}((\boldsymbol{\omega}\times\mathbf{x})\cdot\nabla\mathbf{u} - \boldsymbol{\omega}\times\mathbf{u}) = (\boldsymbol{\omega}\times\mathbf{x})\cdot\nabla\operatorname{div}\mathbf{u} = 0$, and $\operatorname{div}(\Delta\mathbf{u} + k\,\partial_3\mathbf{u}) = 0$.

Let $\mathcal{E}$ be the fundamental solution of the Laplace equation, i.e. $\mathcal{E} = -1/(4\pi r)$. Assuming firstly $\mathbf{f} \in \mathbf{C}_0^\infty$ we have $p = \mathcal{E} \star \operatorname{div}\mathbf{f}$ and $\nabla p = \nabla\mathcal{E} \star \operatorname{div}\mathbf{f}$ and so, $p = \nabla\mathcal{E} \star \mathbf{f}$ and $\nabla p = \nabla^2\mathcal{E} \star \mathbf{f}$. It is well known that both formulas can be extended to $\mathbf{f} \in \mathbf{L}^2_{\alpha+1,\,\beta}$ with $0 < \beta < 1$ and $-2 < \alpha + \beta < 2$ (concerning the term $\nabla p = \nabla^2\mathcal{E} \star \mathbf{f}$ note that $\nabla^2\mathcal{E}$ is a singular kernel of Calderón-Zygmund type and that $\eta^{\alpha+1}_\beta$ belongs to the Muckenhoupt class of weights A_2), see [2, Thm. 3.2, Thm. 5.5], [28, Thm. 4.4, Thm. 5.4], where the theorems are formulated for the pressure part $\mathcal{P}$ of the fundamental solution of the classical Oseen problem. For $\mathbf{f} \in \mathbf{L}^2_{\alpha+1,\,\beta}$ we get $p \in L^2_{\alpha,\,\beta-1}$ and $\nabla p \in \mathbf{L}^2_{\alpha+1,\,\beta}$, and there are positive constants C_1, C_2 such that the following estimates are satisfied:

$$\|p\|_{2,\alpha,\beta-1} \le C_1\|\mathbf{f}\|_{2,\alpha+1,\beta}, \quad \|\nabla p\|_{2,\alpha+1,\beta} \le C_2\|\mathbf{f}\|_{2,\alpha+1,\beta}. \tag{5.2}$$

5.2. *Friedrichs-Poincaré inequality.* In this subsection we formulate an inequality of Friedrichs-Poincaré type in the weighted Sobolev space $\mathring{\mathbf{H}}^1_{\alpha,\,\beta}$. It will be necessary for our aims to estimate constants in this inequality carefully. We also recall some technical assertion; for more details see Kračmar and Penel [29].

LEMMA 5.1. *Let $\alpha \geq 0$, $\beta > 0$, $\alpha + \beta < 3$, $\kappa > 1$. Let δ and ε be arbitrary positive constants, such that $(\beta - \alpha)(2\varepsilon - \delta) \geq 0$. Then for all $\mathbf{u} \in \mathring{\mathbf{H}}^1_{\alpha,\,\beta}$*

$$\|\mathbf{u}\|^2_{2,\alpha-1,\beta-1} \leq c_0 \,\|\, \nabla\mathbf{u}|\, B_{R_0}\|^2_{2,\alpha,\beta} + c_1 \left\|\, \nabla\mathbf{u}|\, B^{R_0}\right\|^2_{2,\alpha,\beta}, \tag{5.3}$$

where $c_0 = [(\alpha\delta + 2\beta\varepsilon)/(\beta\beta^*\delta\varepsilon)]^2$, $c_1 = [(2\kappa)/(\delta\varepsilon)] \cdot [(\alpha+\beta)/(\beta\beta^*)]^2$ and $R_0 \geq |\delta^{-1} - (2\varepsilon)^{-1}|(\kappa-1)^{-1}$.

REMARK 5.2. Observe that if additionally $\delta < 2\varepsilon$ and $1 < \kappa \leq 2\varepsilon/\delta + \delta/(2\varepsilon) - 1$ then $c_0 \geq c_1$.

LEMMA 5.3. *Let $0 \leq \alpha < \beta$, $\kappa > 1$, $0 < \varepsilon \leq (1/(2\kappa)) \cdot (k/\nu) \cdot ((\beta-\alpha)/\beta^2)$ and δ, ν, $k > 0$. Then the function*

$$F_{\alpha,\beta}(s,r;\,\nu) \cdot \eta^{\alpha-1}_{\beta-1} \equiv -\nu|\nabla\eta^{\alpha}_{\beta}|^2/\eta^{\alpha}_{\beta} - k\,\partial_3\,\eta^{\alpha}_{\beta} \tag{5.4}$$

satisfies the estimate

$$F_{\alpha,\beta}\,(s,r;\nu) - (1-\kappa^{-1})k\delta\varepsilon(\beta-\alpha)s \geq -\alpha\delta k(1+\nu k^{-1}\alpha\delta) \tag{5.5}$$

for all $r > 0$ and $s \in [0, 2r]$.

5.3. *The problem in B_R—solenoidal solutions.* In this subsection we will study the existence of a weak solution of the following problem in a bounded domain B_R, where the pressure p is assumed to be known so that the right hand side equals $\mathbf{f} - \nabla p = \widetilde{\mathbf{f}} \in \mathbf{L}^2_{\alpha+1,\,\beta}$:

$$-\nu\Delta\mathbf{u} + k\,\partial_3\mathbf{u} - (\boldsymbol{\omega}\times\mathbf{x})\cdot\nabla\mathbf{u} + \boldsymbol{\omega}\times\mathbf{u} = \widetilde{\mathbf{f}} \qquad \text{in } B_R, \tag{5.6}$$

$$\mathbf{u} = 0 \qquad \text{on } \partial B_R. \tag{5.7}$$

We show the existence of a weak solution $\mathbf{u}_R \in \mathring{\mathbf{H}}^1(B_R)$. Following (3.1), (3.2) again with $w = \eta^0_{\beta_0}$, $\beta_0 \in (0,1]$, let us introduce the continuous bilinear form $\widetilde{Q}(\cdot,\cdot)$ on $\mathring{\mathbf{H}}^1(B_R) \times \mathring{\mathbf{H}}^1(B_R)$ by

$$\widetilde{Q}(\mathbf{u},\mathbf{v}) = \int_{B_R} \nu\nabla\mathbf{u}\cdot\nabla(\mathbf{v}\cdot\eta^0_{\beta_0})\,d\mathbf{x} + k\int_{B_R} \partial_3\mathbf{u}\cdot(\mathbf{v}\eta^0_{\beta_0})\,d\mathbf{x}$$

$$+ \int_{B_R} (\boldsymbol{\omega}\times\mathbf{x})\cdot\nabla\mathbf{u}(\mathbf{v}\eta^0_{\beta_0})\,d\mathbf{x} + \int_{B_R} (\boldsymbol{\omega}\times\mathbf{u})\cdot(\mathbf{v}\eta^0_{\beta_0})\,d\mathbf{x},$$

so that, using (5.4),

$$\widetilde{Q}(\mathbf{v},\mathbf{v}) \geq \frac{\nu}{2}\int_{B_R} |\nabla\mathbf{v}|^2\eta^0_{\beta_0}d\mathbf{x} + \frac{1}{2}\int_{B_R} |\mathbf{v}|^2 F_{0,\beta_0}(s,r;\,\nu)\eta^{-1}_{\beta_0-1}d\mathbf{x}. \tag{5.8}$$

LEMMA 5.4. *Let $0 < \beta_0 \leq 1$. Then for all $\widetilde{\mathbf{f}} \in \mathbf{L}^2_{1,\beta_0}(B_R)$, where $\eta^{\alpha}_{\beta_0} \equiv \eta^{\alpha,\varepsilon_0}_{\beta_0,\varepsilon_0}$, $\varepsilon_0 < \frac{1}{2}\frac{k}{\nu}\frac{1}{\beta_0}$, there exists a unique $\mathbf{u}_R \in \mathring{\mathbf{H}}^1(B_R)$ such that for all $\mathbf{v} \in \mathring{\mathbf{H}}^1(B_R)$*

$$\widetilde{Q}\,(\mathbf{u}_R,\mathbf{v}) = \int_{B_R} \widetilde{\mathbf{f}}\cdot\mathbf{v}\eta^0_{\beta_0}\,d\mathbf{x}. \tag{5.9}$$

Proof. The bilinear form $\widetilde{Q}$ is coercive, i.e., there exists a constant $C_R > 0$ such that $\widetilde{Q}(\mathbf{v},\mathbf{v}) \geq C_R\|\mathbf{v}\|^2$, where $\|\cdot\|$ is the norm in the space $\overset{\circ}{\mathbf{H}}{}^1(B_R)$. Indeed, we get

$$\widetilde{Q}(\mathbf{v},\mathbf{v}) \geq \frac{\nu}{2}\int_{B_R} |\nabla \mathbf{v}|^2 \eta^0_{\beta_0} d\mathbf{x} + \frac{1}{2}\int_{B_R} |\mathbf{v}|^2 F_{0,\beta_0}(s,r;\,\nu)\eta^{-1}_{\beta_0-1} d\mathbf{x}\,.$$

Since $\varepsilon_0 < \frac{1}{2}\frac{k}{\nu}\frac{1}{\beta_0}$, there is a constant $\kappa > 1$ satisfying all previous conditions and additionally $\varepsilon_0 \leq \frac{1}{2\kappa}\frac{k}{\nu}\frac{1}{\beta_0}$. Because $\alpha = 0$, we get from Lemma 5.3

$$\int_{B_R} |\mathbf{v}|^2 F_{0,\beta_0}(s,r;\,\nu)\eta^{-1}_{\beta_0-1} d\mathbf{x} \geq \left(1-\frac{1}{\kappa}\right) k\varepsilon_0^2\beta_0 \int_{B_R} |\mathbf{v}|^2\eta^{-1}_{\beta_0-1} s\, d\mathbf{x},$$

$$\widetilde{Q}(\mathbf{v},\mathbf{v}) \geq \frac{\nu}{2}\int_{B_R} |\nabla \mathbf{v}|^2 \eta^0_{\beta_0} d\mathbf{x} + \frac{1}{2}\left(1-\frac{1}{\kappa}\right) k\varepsilon_0\beta_0 \int_{B_R} |\mathbf{v}|^2\eta^{-1}_{\beta_0-1} (\varepsilon_0 s)\, d\mathbf{x}.$$

Using Lemma 5.1 and Remark 5.2 we derive that

$$\widetilde{Q}(\mathbf{v},\mathbf{v}) \geq C_R\left(\int_{B_R} |\nabla \mathbf{v}|^2 d\mathbf{x} + \int_{B_R} |\mathbf{v}|^2 d\mathbf{x}\right) = C_R\|\mathbf{v}\|^2, \tag{5.10}$$

where $C_R = (\nu/4)\cdot(1-\kappa^{-1})\cdot\min\{1,\, \varepsilon_0^2\beta_0^2/4,\, 2(k/\nu)\beta\varepsilon_0\}\cdot(1+\varepsilon_0\, R)$. Using the Lax-Milgram theorem we get that there is a unique $\mathbf{u}_R \in \overset{\circ}{\mathbf{H}}{}^1(B_R)$ such that (5.9) is satisfied. ∎

Remark 5.5. An arbitrary function $\mathbf{\Phi} \in \overset{\circ}{\mathbf{H}}{}^1(B_R)$ can be expressed in the form $\mathbf{\Phi} = \phi\,\eta^0_{\beta_0}$, where ϕ is a function from $\overset{\circ}{\mathbf{H}}{}^1(B_R)$. Therefore, we have

$$Q(\mathbf{u}_R,\mathbf{\Phi}) = \int_{B_R} \widetilde{\mathbf{f}}\cdot\mathbf{\Phi}\, d\mathbf{x}, \tag{5.11}$$

for all $\mathbf{\Phi} \in \overset{\circ}{\mathbf{H}}{}^1(B_R)$ where by definition $Q(\mathbf{u}_R,\mathbf{\Phi}) \equiv Q(\mathbf{u}_R,\phi\cdot\eta^0_{\beta_0}) \equiv \widetilde{Q}(\mathbf{u}_R,\phi)$.

5.4. *Uniform estimates of* $\mathbf{u}_R$ *in* $\mathbb{R}^3$*—solenoidal solutions.* Our next aim is to prove that the weak solutions $\mathbf{u}_R$ of (5.9) are uniformly bounded in $\mathbf{V}_{\alpha,\beta}$ as $R \to +\infty$.

Let y_1 be the unique real solution of the algebraic equation

$$4y^3 + 8y^2 + 5y - 1 = 0. \tag{5.12}$$

It is easy to verify that $y_1 \in (0,1)$. We will explain later why the control of α/β by y_1 is necessary.

Lemma 5.6. *Let* $0 < \beta \leq 1$, $0 \leq \alpha < y_1\beta$, *and* $\widetilde{\mathbf{f}} \in \mathbf{L}^2_{\alpha+1,\beta}$. *Then, as* $R \to +\infty$, *the weak solutions* $\mathbf{u}_R$ *of (5.9) given by Lemma 5.4 are uniformly bounded in* $\mathbf{V}_{\alpha,\beta}$. *There is a constant* $c > 0$ *independent of* R *such that*

$$\int_{\mathbb{R}^3} |\tilde{\mathbf{u}}_R|^2\eta^{\alpha-1}_\beta d\mathbf{x} + \int_{\mathbb{R}^3} |\nabla\tilde{\mathbf{u}}_R|^2\eta^{\alpha}_\beta d\mathbf{x} \leq c\int_{\mathbb{R}^3} |\widetilde{\mathbf{f}}|^2\eta^{\alpha+1}_\beta d\mathbf{x} \tag{5.13}$$

for all R *larger than some* $R_0 > 0$*; here* $\tilde{\mathbf{u}}_R$ *is the extension by zero of* $\mathbf{u}_R$ *on* $\mathbb{R}^3\setminus B_R$.

Proof. Our aim is to get an estimate of $\mathbf{u}_R$ with a constant not depending on R. First, we consider $\mathbf{u}_R$ on a bounded subdomain $B_{R_0} \subset B_R$; the choice of R_0 will be given in the next part of the proof. Let us substitute $\mathbf{v} = \mathbf{u}_R$ into (5.9). Hence we get

$$\widetilde{Q}(\mathbf{u}_R,\mathbf{u}_R) = \int_{B_R} \widetilde{\mathbf{f}}\cdot\mathbf{u}_R\eta^0_{\beta_0} d\mathbf{x} \geq C_1\left(\int_{B_R} |\nabla\mathbf{u}_R|^2\eta^0_{\beta_0} d\mathbf{x} + \int_{B_R} |\mathbf{u}_R|^2\eta^{-1}_{\beta_0} d\mathbf{x}\right),$$

with a constant $C_1 > 0$. Let R_0 be some fixed number such that $0 < R_0 < R$. We get

$$\int_{B_{R_0}} |\nabla \mathbf{u}_R|^2 \eta_\beta^\alpha d\mathbf{x} + \int_{B_{R_0}} |\mathbf{u}_R|^2 \eta_\beta^{\alpha-1} d\mathbf{x} \le C_2 \int_{B_R} |\widetilde{\mathbf{f}}|\, |\mathbf{u}_R| \eta_\beta^\alpha d\mathbf{x}, \tag{5.14}$$

where the constant $C_2 = C_1^{-1}(1+\varepsilon_0 R_0)^\alpha (1+\varepsilon_0 2R_0)^{|\beta-\beta_0|}$ depends on k, ν, α, β, β_0, ε_0, R_0, κ, but not depend on R.

Now, we are going to derive an estimate of $\mathbf{u}_R$ on B_R. Using the test function $\mathbf{\Phi} = \mathbf{u}_R \eta_\beta^\alpha = \mathbf{u}_R (1+\delta r)^\alpha (1+\varepsilon s)^\beta \in \mathring{\mathbf{H}}^1(B_R)$ in (5.11) we get after integration by parts that

$$\nu \int_{B_R} |\nabla \mathbf{u}_R|^2 \eta_\beta^\alpha d\mathbf{x} + \nu \int_{B_R} \mathbf{u}_R \nabla \mathbf{u}_R \cdot \nabla \eta_\beta^\alpha \, d\mathbf{x} - \frac{k}{2} \int_{B_R} |\mathbf{u}_R|^2 \, \partial_3 \eta_\beta^\alpha d\mathbf{x}$$
$$= \int_{B_R} \widetilde{\mathbf{f}} \cdot \mathbf{u}_R \eta_\beta^\alpha d\mathbf{x}\,.$$

So, for some $\kappa > 1$

$$\frac{\nu}{2} \int_{B_R} |\nabla \mathbf{u}_R|^2 \eta_\beta^\alpha d\mathbf{x} + \frac{1}{2} \int_{B_R} |\mathbf{u}_R|^2 F_{\alpha,\beta}(s,r;\,\nu) \eta_{\beta-1}^{\alpha-1} d\mathbf{x} \le \int_{B_R} |\widetilde{\mathbf{f}}|\, |\mathbf{u}_R| \eta_\beta^\alpha d\mathbf{x}\,.$$

Let $R_0 \ge |\frac{1}{\delta} - \frac{1}{2\varepsilon}|(\kappa-1)^{-1}$. Using Lemma 5.3 with $0 \le \alpha < \beta$, $\varepsilon \le \frac{1}{2\kappa}\frac{k}{\nu}\frac{\beta-\alpha}{\beta^2}$ and Lemma 5.1 (with $\delta < 2\varepsilon$), the second term in the previous estimate can be evaluated from below as follows:

$$\int_{B_R} |\mathbf{u}_R|^2 \, F_{\alpha,\beta}(s,r;\,\nu)) \eta_{\beta-1}^{\alpha-1} \, d\mathbf{x}$$
$$\ge -\alpha\delta k \left(1 + \frac{\nu\kappa}{k}\alpha\delta\right) \frac{2\kappa}{\delta\varepsilon} \left(\frac{\alpha+\beta}{\beta\beta^*}\right)^2 \int\limits_{B_R^{R_0}} |\nabla \mathbf{u}_R|^2 \eta_\beta^\alpha d\mathbf{x}$$
$$+ (1-\kappa^{-1}) k\delta\varepsilon(\beta-\alpha) \int_{B_R^{R_0}} |\mathbf{u}_R^2| \eta_{\beta-1}^{\alpha-1} s \, d\mathbf{x} - 2C_4 \int_{B_{R_0}} |\nabla \mathbf{u}_R|^2 \eta_\beta^\alpha d\mathbf{x}\,.$$

Let

$$C_5 = \alpha\delta k \left(1 + \frac{\kappa\nu}{k}\alpha\delta\right) \frac{\kappa}{\delta\,\varepsilon} \left(\frac{\alpha+\beta}{\beta\beta^*}\right)^2.$$

It is clear that $C_5 \le \frac{\nu}{2\,\kappa^2} < \frac{\nu}{2\,\kappa}$ if $1 + \nu\kappa\alpha\delta/k \le \kappa$, i.e. $\delta \le \frac{k}{\nu} \cdot \frac{\kappa-1}{\kappa\beta}$, and $\alpha \le \frac{1}{2\kappa^4} \cdot \frac{\nu}{k} \cdot l(\frac{\beta\beta^*}{\alpha+\beta} r)^2 \varepsilon$. Using Lemma 5.1 and Remark 5.2 we get, if $\delta < 2\varepsilon$ and $1 < \kappa \le \frac{2\varepsilon}{\delta} + \frac{\delta}{2\varepsilon} - 1$, that

$$\int_{B_R} |\nabla \mathbf{u}_R|^2 \eta_\beta^\alpha d\mathbf{x} + \int_{B_R} |\mathbf{u}_R|^2 \, \eta_\beta^{\alpha-1} d\mathbf{x} \le c \int_{\mathbb{R}^3} |\widetilde{\mathbf{f}}|^2 \eta_\beta^{\alpha+1} d\mathbf{x}.$$

It can be easily shown that all conditions on α, β, δ, ε, κ used in the proof are compatible if $0 \le \alpha < y_1\beta$. ■

5.5. *The problem in $\mathbb{R}^3$—solenoidal solutions.* Let y_1 be the same constant as in (5.12).

THEOREM 5.7 (Existence and uniqueness in $\mathbb{R}^3$). *Let $0 < \beta \le 1$, $0 \le \alpha < y_1\beta$, and $\mathbf{f} \in \mathbf{L}^2_{\alpha+1,\beta}$. Then there exists a unique weak solution $\{\mathbf{u}, p\}$ of the problem*

$$-\nu\,\Delta\mathbf{u} + k\partial_3\mathbf{u} - (\boldsymbol{\omega}\times\mathbf{x})\cdot\nabla\mathbf{u} + \boldsymbol{\omega}\times\mathbf{u} + \nabla p = \mathbf{f} \qquad \textit{in } \mathbb{R}^3, \tag{5.15}$$

$$\operatorname{div}\mathbf{u} = 0 \qquad \textit{in } \mathbb{R}^3 \tag{5.16}$$

such that $\mathbf{u} \in \mathbf{V}_{\alpha,\beta}$, $p \in L^2_{\alpha,\beta-1}$, $\nabla p \in \mathbf{L}^2_{\alpha+1,\beta}$ *and*

$$\|\mathbf{u}\|_{2,\alpha-1,\beta} + \|\nabla\mathbf{u}\|_{2,\alpha,\beta} + \|p\|_{2,\alpha,\beta-1} + \|\nabla p\|_{2,\alpha+1,\beta} \le C\,\|\mathbf{f}\|_{2,\alpha+1,\beta}\,. \tag{5.17}$$

Proof. Existence. Let p be the function constructed in Subsection 5.1 and satisfying the estimate (5.2). Choose a sequence $\{R_n\}$ of positive real numbers converging to $+\infty$. Let $\mathbf{u}_{R_n}$ be the weak solution of (5.6), (5.7) on B_{R_n}. Extending $\mathbf{u}_{R_n}$ by zero on $\mathbb{R}^3 \setminus B_{R_n}$ to a function $\tilde{\mathbf{u}}_n \in \mathbf{V}_{\alpha,\beta}$ we get a bounded sequence $\{\tilde{\mathbf{u}}_n\}$ in $\mathbf{V}_{\alpha,\beta}$. Thus, there is a subsequence $\tilde{\mathbf{u}}_{n_k}$ of $\tilde{\mathbf{u}}_n$ with a weak limit $\mathbf{u}$ in $\mathbf{V}_{\alpha,\beta}$. Obviously, $\mathbf{u}$ is a weak solution of (5.15) and

$$\begin{aligned}\|\mathbf{u}\|^2_{2,\alpha-1,\beta} + \|\nabla\mathbf{u}\|^2_{2,\alpha,\beta} &\le \liminf_{k\in\mathbb{N}} \left(\int_{\mathbb{R}^3} |\tilde{\mathbf{u}}_{n_k}|^2 \eta_\beta^{\alpha-1}\,d\mathbf{x} + \int_{\mathbb{R}^3} |\nabla\tilde{\mathbf{u}}_{n_k}|^2 \eta_\beta^{\alpha}\,d\mathbf{x}\right)\\ &\le c|\widetilde{\mathbf{f}}|^2\eta_\beta^{\alpha+1}\,d\mathbf{x} = c\int_{\mathbb{R}^3} |\mathbf{f}-\nabla p|^2\eta_\beta^{\alpha+1}\,d\mathbf{x}.\end{aligned}$$

Taking into account (5.2) we get (5.17).

Let us also check that $\mathbf{u}$ satisfies also the equation (5.16). Note that $\mathbf{u} \in \mathbf{H}^2_{\mathrm{loc}}$ because $\mathbf{f} - \nabla p \in \mathbf{L}^2_{\alpha+1,\,\beta}$. So, computing the divergence of (5.15) we get

$$-\nu\,\Delta(\operatorname{div}\mathbf{u}) + k\partial_3(\operatorname{div}\mathbf{u}) - (\boldsymbol{\omega}\times\mathbf{x})\cdot\nabla(\operatorname{div}\mathbf{u}) = \operatorname{div}\mathbf{f} - \triangle\mathbf{p} \tag{5.18}$$

in the sense of distributions. From (5.1) and (5.17) we have

$$-\nu\,\Delta\gamma + k\partial_3\gamma - (\boldsymbol{\omega}\times\mathbf{x})\cdot\nabla\gamma = 0$$

for $\gamma = \operatorname{div}\mathbf{u} \in L^2_{\alpha,\,\beta} \subset L^2$. Using Fourier transform we get

$$(\nu|\xi|^2 + ik\xi_3)\widehat{\gamma} - (\boldsymbol{\omega}\times\xi)\cdot\nabla_{\boldsymbol{\xi}}\widehat{\gamma} = 0 \quad \text{in } \mathcal{S}'.$$

Assuming $\widehat{\gamma}$ in cylindrical coordinates $[\xi_3,\rho,\varphi]$, $\rho = (\xi_1^2+\xi_2^2)^{1/2}$, we can rewrite the equation in the form

$$-\partial_\varphi\widehat{\gamma} + [(\nu/\omega)|\xi|^2 + i(k/\omega)\xi_3]\widehat{\gamma} = 0.$$

Using the same approach as in the proof of the uniqueness, see Theorem 4.1, we prove that $\operatorname{supp}\widehat{\gamma} \subset \{0\}$. The proof of this fact is reduced to the solvability of the equation (4.3) which was proved for arbitrary $\Psi \in C_0^\infty(\mathbb{R}^3\setminus\{0\})$ in the proof of Theorem 4.1. So, by the same procedure we derive that γ is a polynomial in $\mathbb{R}^3$ and, since $\gamma \in L^2$, we get $\gamma \equiv 0$, i.e. (5.16). The uniqueness of the solution follows from Theorem 4.1. ∎

5.6. *The problem in* $\mathbb{R}^3$ *with non-zero divergence.* First of all let us formulate a lemma which will be used for the extension of our results to the case with non-zero divergence.

Lemma 5.8 (Bogovski, Galdi and Sohr). *Let* $\Omega \subseteq \mathbb{R}^n$, $n \ge 2$, *be a bounded Lipschitz domain, and* $1 < q < \infty$. *Then for each* $g \in W_0^{k,q}(\Omega)$ *with* $\int_\Omega g\,dx = 0$, *there exists* $\mathbf{G} \in (W_0^{k+1,q}(\Omega))^n$ *satisfying*

$$\operatorname{div}\mathbf{G} = g, \qquad \|\mathbf{G}\|_{(W_0^{k+1,\,q}(\Omega))^n} \le C\|g\|_{W_0^{k,\,q}(\Omega)}$$

with some constant $C = C(q,k,\Omega) > 0$.

For the proof and further references see e.g. [39, Lemma 2.3.1]. Next we will prove the following theorem:

THEOREM 5.9 (Existence and uniqueness in $\mathbb{R}^3$). *Let* $0 < \beta \le 1$, $0 \le \alpha < y_1\beta$, $\mathbf{f} \in \mathbf{L}^2_{\alpha+1,\beta}$, $g \in W_0^{1,2}$ *with* $\operatorname{supp} g = K \subset\subset \mathbb{R}^3$, *and* $\int_{\mathbb{R}^3} g\,dx = 0$. *Then there exists a unique weak solution* $\{\mathbf{u}, p\}$ *of the problem*

$$-\nu\,\Delta\mathbf{u} + k\partial_3\mathbf{u} - (\boldsymbol{\omega}\times\mathbf{x})\cdot\nabla\mathbf{u} + \boldsymbol{\omega}\times\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \mathbb{R}^3,$$
$$\operatorname{div}\mathbf{u} = g \quad \text{in } \mathbb{R}^3$$

such that $\mathbf{u} \in \mathbf{V}_{\alpha,\beta}$, $p \in L^2_{\alpha,\beta-1}$, $\nabla p \in \mathbf{L}^2_{\alpha+1,\beta}$ *and*

$$\|\mathbf{u}\|_{2,\alpha-1,\beta} + \|\nabla\mathbf{u}\|_{2,\alpha,\beta} + \|p\|_{2,\alpha,\beta-1} + \|\nabla p\|_{2,\alpha+1,\beta} \le C(\|\mathbf{f}\|_{2,\alpha+1,\beta} + \|g\|_{1,2}).$$

Proof. Using Lemma 5.8 we find $\mathbf{G} \in \mathbf{W}_0^{2,2}$, $\operatorname{supp}\mathbf{G} \subset \mathcal{K}$, where $\mathcal{K}$ is a bounded Lipschitz domain containing an ε-neighbourhood $\mathcal{K}_\varepsilon$ of the compact set K for an arbitrary $\varepsilon > 0$, such that $\operatorname{div}\mathbf{G} = g$, $\|\mathbf{G}\|_{2,2} \le C\|g\|_{1,2}$. Let us consider the problem

$$-\nu\,\Delta\mathbf{U} + k\,\partial_3\mathbf{U} - (\boldsymbol{\omega}\times\mathbf{x})\cdot\nabla\mathbf{U} + \boldsymbol{\omega}\times\mathbf{U} + \nabla p = \mathbf{F} \quad \text{in } \mathbb{R}^3,$$
$$\operatorname{div}\mathbf{U} = 0 \quad \text{in } \mathbb{R}^3$$

where $\mathbf{U} = \mathbf{u} - \mathbf{G}$ and $\mathbf{F} = \mathbf{f} + \nu\,\Delta\mathbf{G} - k\,\partial_1\mathbf{G} + (\boldsymbol{\omega}\times\mathbf{x})\cdot\nabla\mathbf{G} - \omega\times\mathbf{G}$. Now the assertion of Theorem 5.9 follows from Theorem 5.7. ■

6. Weighted L^q approach. Working first of all formally or in the space $\mathcal{S}'(\mathbb{R}^3)$ of tempered distributions we apply the Fourier transform $\mathcal{F} = \widehat{\ }$ to (1.5). With the Fourier variable $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ and $s = |\xi|$ we get from (1.5)

$$(\nu s^2 + ik\xi_3)\widehat{\mathbf{u}} - \omega(\partial_\varphi\widehat{\mathbf{u}} - \mathbf{e}_3\times\widehat{\mathbf{u}}) + i\xi\widehat{p} = \widehat{\mathbf{f}}, \quad i\xi\cdot\widehat{\mathbf{u}} = 0. \tag{6.1}$$

The unknown pressure p is given by $-|\xi|^2\widehat{p} = i\xi\cdot\widehat{\mathbf{f}}$, i.e.,

$$\widehat{\nabla p}(\xi) = i\xi\cdot\widehat{p} = \frac{(\xi\cdot\widehat{\mathbf{f}})\widehat{\mathbf{f}}}{|\xi|^2}\,.$$

Then the Hörmander-Mikhlin multiplier theorem on weighted L^q-spaces (Theorem 2.7 (ii)) yields for every weight $w \in A_q(\mathbb{R}^3, \mathcal{C})$ the estimate

$$\|\nabla p\|_{q,w} \le c\|\mathbf{f}\|_{q,w} \tag{6.2}$$

where $c = c(q, w) > 0$; in particular $\nabla p \in L^q_w$.

Hence $\mathbf{u}$ is a (solenoidal) solution of the reduced problem

$$(\nu s^2 + ik\xi_3)\widehat{\mathbf{u}} - \omega(\partial_\varphi\widehat{\mathbf{u}} - \mathbf{e}_3\times\widehat{\mathbf{u}}) = \widehat{\mathbf{F}}, \tag{6.3}$$

where $\mathbf{F} = \mathbf{f} - \nabla p$ or, equivalently, $\widehat{\mathbf{F}} = \widehat{\mathbf{f}} - \widehat{\nabla p}$. Equation (6.3) may be considered as a first order ordinary differential equation with respect to $\varphi \in (0, 2\pi)$ with periodic boundary conditions. As shown in [4] (6.3) has a unique solution

$$\widehat{\mathbf{u}}(\xi) = \int_0^\infty e^{-\nu|\xi|^2 t} O_\omega^T(t)\mathcal{F}\mathbf{F}(O_\omega(t)\cdot - kt\mathbf{e}_3)(\mathbf{x})\,dt, \tag{6.4}$$

or, in $\mathbf{x}$-space, using the heat kernel $E_t(\mathbf{x}) = (4\pi\nu t)^{-3/2}e^{-|\mathbf{x}|^2/4\nu t}$,

$$\mathbf{u}(\mathbf{x}) = \int_0^\infty E_t * O_\omega^T(t)\mathbf{F}(O_\omega(t)\cdot - kt\mathbf{e}_3)(\mathbf{x})\,dt. \tag{6.5}$$

The main ingredients of the proof of Theorem 3.3 are a weighted version of Littlewood-Paley theory and a decomposition of the integral operator

$$
\begin{aligned}
T\mathbf{F}(\mathbf{x}) &= \int_0^\infty \widehat{\psi}_{\nu t}(\xi) O_\omega^T(t) \mathcal{F}\mathbf{F}(O_\omega(t)\cdot -kt\mathbf{e}_3)(\xi)\,\frac{dt}{t} \\
&= \int_0^\infty \widehat{\psi}_t(\xi) O_{\omega/\nu}^T(t) \mathcal{F}\mathbf{F}\Big(O_{\omega/\nu}(t)\cdot -\frac{k}{\nu}t\mathbf{e}_3\Big)(\xi)\,\frac{dt}{t},
\end{aligned} \tag{6.6}
$$

where

$$
\widehat{\psi}(\xi) = \frac{1}{(2\pi)^{3/2}}|\xi|^2 e^{-|\xi|^2} \quad \text{and} \quad \widehat{\psi}_t(\xi) = \widehat{\psi}(\sqrt{t}\xi), \ \ t>0, \tag{6.7}
$$

are the Fourier transforms of the function $\psi = -\Delta E_1 \in \mathcal{S}(\mathbb{R}^3)$ and of $\psi_t(\mathbf{x}) = t^{-3/2}\psi(\mathbf{x}/\sqrt{t})$, $t>0$, respectively. Note that due to [40, III.1., Proposition 3] and Theorem 2.6 (ii) it suffices to find an estimate of $\|\Delta\mathbf{u}\|_{q,w}$ in order to estimate all second order derivatives $\partial_j\partial_k\mathbf{u}$ of $\mathbf{u}$.

To decompose $\widehat{\psi}_t$ choose $\widetilde{\chi} \in C_0^\infty\left(\frac{1}{2},2\right)$ satisfying $0 \le \widetilde{\chi} \le 1$ and $\sum_{j=-\infty}^{\infty}\widetilde{\chi}(2^{-j}s) = 1$ for all $s>0$. Then define χ_j, $j\in\mathbb{Z}$, by its Fourier transform $\widehat{\chi}_j(\xi) = \widetilde{\chi}(2^{-j}|\xi|)$, $\xi\in\mathbb{R}^n$, yielding $\sum_{j=-\infty}^{\infty}\widehat{\chi}_j = 1$ on $\mathbb{R}^n\setminus\{0\}$ and

$$
\operatorname{supp}\widehat{\chi}_j \subset A(2^{j-1},2^{j+1}) := \{\xi\in\mathbb{R}^3 : 2^{j-1} \le |\xi| \le 2^{j+1}\}. \tag{6.8}
$$

Using χ_j, we define for $j\in\mathbb{Z}$

$$
\psi^j = \frac{1}{(2\pi)^{3/2}}\chi_j * \psi \quad (\widehat{\psi} = \widehat{\chi}_j\cdot\widehat{\psi}). \tag{6.9}
$$

Obviously, $\sum_{j=-\infty}^{\infty}\psi^j = \psi$ on $\mathbb{R}^3$. Finally, in view of (6.6), (6.9), we define the linear operators

$$
\begin{aligned}
T_j\mathbf{F}(\mathbf{x}) &= \int_0^\infty \widehat{\psi}^j_{\nu t}(\xi) O_\omega^T(t) \mathcal{F}\mathbf{F}(O_\omega(t)\cdot -kt\mathbf{e}_3)(\xi)\frac{dt}{t} \\
&= \int_0^\infty \widehat{\psi}^j_t(\xi) O_{\omega/\nu}^T(t) \mathcal{F}\mathbf{F}\Big(O_{\omega/\nu}(t)\cdot -\frac{k}{\nu}t\mathbf{e}_3\Big)(\xi)\frac{dt}{t}.
\end{aligned} \tag{6.10}
$$

Since formally $T = \sum_{j=-\infty}^{\infty}T_j$, we have to prove that this infinite series converges even in the operator norm on L^q_w.

For later use we cite the following lemma, see [7].

Lemma 6.1. *The functions ψ^j, ψ^j_t, $j\in\mathbb{Z}$, $t>0$, have the following properties:*

(i) $\operatorname{supp}\widehat{\psi}^j_t \subset A(2^{j-1}/\sqrt{t}, 2^{j+1}/\sqrt{t})$.

(ii) For $m>\frac{3}{2}$ let $h(\mathbf{x}) = (1+|\mathbf{x}|^2)^{-m}$ and $h_t(\mathbf{x}) = t^{-3/2}h(\mathbf{x}/\sqrt{t})$, $t>0$. Then there exists a constant $c>0$ independent of $j\in\mathbb{Z}$ such that

$$
\begin{aligned}
&|\psi^j(\mathbf{x})| \le c2^{-2|j|}\,h_{2^{-2j}}(\mathbf{x}), \quad \mathbf{x}\in\mathbb{R}^3, \\
&\|\psi^j\|_1 \le c2^{-2|j|}\,.
\end{aligned} \tag{6.11}
$$

To introduce a weighted Littlewood-Paley decomposition of L^q_w choose $\widetilde{\varphi}\in C_0^\infty(\frac{1}{2},2)$ such that $0\le\widetilde{\varphi}\le 1$ and $\int_0^\infty \widetilde{\varphi}(s)^2\,\frac{ds}{s} = \frac{1}{2}$. Then define $\varphi\in\mathcal{S}(\mathbb{R}^3)$ by its Fourier transform $\widehat{\varphi}(\xi) = \widetilde{\varphi}(|\xi|)$ yielding for every $s>0$

$$
\widehat{\varphi}_s(\xi) = \widetilde{\varphi}(\sqrt{s}|\xi|), \quad \operatorname{supp}\widehat{\varphi}_s \subset A\Big(\frac{1}{2\sqrt{2}}, \frac{2}{\sqrt{2}}\Big) \tag{6.12}
$$

and the normalization $\int_0^\infty \widehat{\varphi}_s(\xi)^2\,\frac{ds}{s} = 1$ for all $\xi\in\mathbb{R}^n\setminus\{0\}$.

THEOREM 6.2. *Let $1 < q < \infty$ and $w \in A_q(\mathbb{R}^3)$. Then there are constants $c_1, c_2 > 0$ depending on q, w and φ such that for all $\mathbf{f} \in L^q_w$*

$$c_1\|\mathbf{f}\|_{q,w} \le \left\| \left(\int_0^\infty |\varphi_s * \mathbf{f}(\cdot)|^2 \, \frac{ds}{s} \right)^{1/2} \right\|_{q,w} \le c_2\|\mathbf{f}\|_{q,w} \tag{6.13}$$

where $\varphi_s \in \mathcal{S}(\mathbb{R}^n)$ is defined by (6.12).

For the proof see [37, Proposition 1.9, Theorem 1.10], and also [31], [42].

6.1. *Outline of the proof.* As a preliminary version of Theorem 3.3 we prove the following proposition. The extension to more general weights based on complex interpolation of L^q_w-spaces and a factorization theorem is postponed to the end of this section. On the one hand can hardly proceed directly: the reason is the difficulty to verify directly that a given weight function belongs to a particular Muckenhoupt class. On the other hand, there will be no loss when constructing a weight function by abstract tools because factorization gives necessary and sufficient condition for a function to be in a Muckenhoupt class in terms of the A_1 weights. The latter class consists of functions majoring their maximal functions (up to a multiplicative constant) and can effectively be handled.

PROPOSITION 6.3. *Let the weight $w \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ be independent of the angle θ and define $w_r(x_3) := w(x_1, x_2, x_3)$ for fixed $r = |(x_1, x_2)| > 0$. Assume that*

$$\begin{array}{lll} w \in \widetilde{A}^-_{q/2} & & \text{if} \quad q > 2, \\ w \in \widetilde{A}^-_1 \quad \text{or} \quad \frac{1}{w} \in \widetilde{A}^+_1 & & \text{if} \quad q = 2, \\ w^{2/(2-q)} \in \widetilde{A}^-_{q/(2-q)} & & \text{if} \quad 1 < q < 2. \end{array} \tag{6.14}$$

Then the linear operator T defined by (6.6) *satisfies the estimate*

$$\|T\mathbf{F}\|_{q,w} \le c\|\mathbf{F}\|_{q,w} \quad \text{for all} \quad \mathbf{F} \in L^q_w \tag{6.15}$$

with a constant $c = c(q, w) > 0$ independent of $\mathbf{F}$.

Proof. Step 1. First we consider the case $q > 2$, $w \in \widetilde{A}^-_{q/2} \subset A_q$, and define the sublinear operator $\mathcal{M}^j$, a modified maximal operator, by

$$\mathcal{M}^j g(\mathbf{x}) = \sup_{s>0} \int_{A_s} (|\psi^j_t| * |g|)\Big(O^T_{\omega/\nu}(t)\mathbf{x} + \frac{k}{\nu} t\mathbf{e}_3\Big) \, \frac{dt}{t}\,, \tag{6.16}$$

where $A_s = [\frac{s}{16}, 16s]$. Then $T_j\mathbf{F}$ satisfies the estimate

$$\|T_j\mathbf{F}\|_{q,w} \le c\|\psi^j\|_1^{1/2} \|\mathcal{M}^j\|^{1/2}_{L_v^{(q/2)'}} \|\mathbf{F}\|_{q,w}\,, \quad j \in \mathbb{Z}, \tag{6.17}$$

where v denotes the θ-independent weight

$$v = w^{-(\frac{q}{2})'/(\frac{q}{2})} = w^{-\frac{2}{q-2}} \in \widetilde{A}^+_{(q/2)'} = \widetilde{A}^+_{q/(q-2)}, \tag{6.18}$$

see [9].

Step 2. We estimate $\|\mathcal{M}^j g\|_{(q/2)',v}$. For functions γ depending on θ, x_3 only let $\mathcal{M}_{\mathrm{hel}}$ denote the "helical" maximal operator

$$\mathcal{M}_{\mathrm{hel}}\gamma(\theta, x_3) = \sup_{s>0} \frac{1}{s} \int_{A_s} |\gamma|\Big(\theta - \frac{\omega}{\nu} t, x_3 + \frac{k}{\nu} t\Big) \, dt,$$

where $A_s = \left[\frac{s}{16}, 16s\right]$. Then, writing $p := (\frac{q}{2})'$, the following estimates

$$\mathcal{M}^j g(\mathbf{x}) \le c2^{-2|j|}\mathcal{M}(\mathcal{M}_{\mathrm{hel}} g)(\mathbf{x}) \quad \text{for a.a. } \mathbf{x} \in \mathbb{R}^n, \tag{6.19}$$

$$\|\mathcal{M}^j g\|_{p,v} \le c2^{-2|j|}\|\mathcal{M}_{\mathrm{hel}} g\|_{p,v}, \tag{6.20}$$

are satisfied, where in (6.19) $\mathcal{M}_{\mathrm{hel}} g$ is considered as $\mathcal{M}_{\mathrm{hel}} g(r,\cdot,\cdot)$ for almost all $r > 0$ and where $\mathcal{M}$, the centered Hardy-Littlewood maximal operator, is bounded from $L^p_v(\mathbb{R}^3)$ to itself by Theorem 2.8 (ii). For more details see [9].

Step 3. Note that up to now we have not yet used any specific properties of the weight $v \in A_p$. To estimate $\mathcal{M}_{\mathrm{hel}} g$ in (6.20) we shall work with a suitable one-sided maximal operator since our weight belongs to a Muckenhoupt class in $\mathbb{R}^3$, but a problem occurs when the weight is considered with respect to x_3 only. This naturally corresponds to the physical circumstances of the problem, where in the Oseen case the wake should appear. Let us write $g_r(\theta, x_3) = g(r, \theta, x_3) = g(\mathbf{x})$ and $v_r(x_3) = v(\mathbf{x})$, $r = |(x_1, x_2)| > 0$, for the θ-independent weight v. Then by the 2π-periodicity of g_r and v_r with respect to θ we get for almost all $r > 0$

$$\begin{aligned}
&\int_{\mathbb{R}}\int_0^{2\pi} \mathcal{M}_{\mathrm{hel}} g_r(\theta, x_3)^p v_r(x_3)\, d\theta\, dx_3 \\
&\le \int_{\mathbb{R}}\int_0^{2\pi}\left|\sup_{s>0}\frac{1}{s}\int_0^{16s} |g_r|\left(\theta - \frac{\omega}{k}\left(x_3 + \frac{k}{\nu}t\right), x_3 + \frac{k}{\nu}t\right) dt\right|^p v_r(x_3)\, d\theta\, dx_3 \\
&= \int_{\mathbb{R}}\int_0^{2\pi}\left|\sup_{s>0}\frac{1}{s}\int_0^{16s} \gamma_{r,\theta}\left(x_3 + \frac{k}{\nu}t\right) dt\right|^p d\theta\, v_r(x_3)\, dx_3 \\
&= 16\int_0^{2\pi}\int_{\mathbb{R}} |M^+\gamma_{r,\theta}(x_3)|^p v_r(x_3)\, dx_3\, d\theta
\end{aligned}$$

where $\gamma_{r,\theta}(y_3) = |g_r|(\theta - \frac{\omega}{k}y_3, y_3)$ and M^+ denotes the one-sided maximal operator, see Definition 2.3. Since $w_r \in A^-_{q/2}$, by (6.18) and Theorem 2.7 (i) $v_r = w_r^{-(q/2)'/(q/2)} \in A^+_{(q/2)'} = A^+_p$ with an A^+_p-constant uniformly bounded in $r > 0$. Then Theorem 2.7 (ii) yields the estimate

$$\begin{aligned}
&\int_{\mathbb{R}}\int_0^{2\pi} \mathcal{M}_{\mathrm{hel}} g_r(\theta, x_3)^p v_r(x_3)\, d\theta\, dx_3 \\
&\quad \le c\int_0^{2\pi}\int_{\mathbb{R}} |\gamma_{r,\theta}(x_3)|^p v_r(x_3)\, dx_3\, d\theta = c\|g_r\|^p_{L^p(\mathbb{R}\times(0,2\pi),\, v_r(x_3))},
\end{aligned}$$

where $c > 0$ is independent of k, ν. Integrating with respect to $r\,dr$, $r \in (0,\infty)$, Fubini's theorem allows to consider an extension of $\mathcal{M}_{\mathrm{hel}}$ to a bounded operator from $L^p_v(\mathbb{R}^3)$ to itself with an operator norm bounded uniformly in k, ν. Hence, (6.20) implies the estimate

$$\|\mathcal{M}_{\mathrm{hel}} g\|_{p,v} \le c2^{-2|j|}\|g\|_{p,v},$$

and (6.17) as well as Lemma 6.1 (ii) show that

$$\|T_j\mathbf{F}\|_{q,w} \le c2^{-2|j|}\|\mathbf{F}\|_{q,w}$$

for all $\mathbf{F} \in L^q_w(\mathbb{R}^3)$ with a constant $c > 0$ independent of $j \in \mathbb{Z}$. Summarizing the previous inequalities we obtain (6.15) for $q > 2$.

Step 4. We apply the Littlewood-Paley theory in the case $q = 2$, $w \in \widetilde{A}_1^-$.

Step 5. Applying a duality argument we get the estimate for $1 < q < 2$.

Step 6. Finally, we prove the claim for $q = 2$ and $w = \frac{1}{w} \in \widetilde{A}_+^1$ by a duality argument.

Further important ingredients needed to proceed with the general case are complex interpolation of weighted Lebesgue spaces, see, e.g. [1], and a suitable factorization theorem. Actually, we need an anisotropic variant of Jones' factorization theorem tailored to our situation, where we have to cope with one-sided Muckenhoupt weights with respect to x_3, which, at the same time, satisfy the standard Muckenhoupt condition in three dimensions. We include the proof since its idea might be useful to tackle similar anisotropic situations.

Lemma 6.4 (Anisotropic version of Jones' Factorization Theorem). *Suppose that $w \in \widetilde{A}_q^-$, $1 < q < \infty <$. Then there exist weights $w_1 \in \widetilde{A}_1^-$ and $w_2 \in \widetilde{A}_1^+$ such that*

$$w = \frac{w_1}{w_2^{q-1}}.$$

Here $\widetilde{A}_1^+$ is defined by analogy with $\widetilde{A}_1^-$, cf. Definition 2.5, by assuming for $w_2 \in \widetilde{A}_1^+$ that $(w_2)_r \in A_1^+$ with A_1^+-constants uniformly bounded in $r > 0$. An analogous result holds for $w \in \widetilde{A}_q^+$.

Proof. Let $q \geq 2$. Given $w \in \widetilde{A}_q^-$ we consider the operator T defined by

$$\begin{aligned} Tf &= (w^{-1/q}\mathcal{M}(f^{q/q'}w^{1/q}))^{q'/q} + w^{1/q}\mathcal{M}(fw^{-1/q}) \\ &\quad + (w^{-1/q}M_1^+(f_r^{q/q'}w_r^{1/q}))^{q'/q} + w^{1/q}M_1^-(f_r w_r^{-1/q}) \end{aligned}$$

where $r = |(x_1, x_2)|$. Then for all $f \in L^q(\mathbb{R}^3)$

$$\begin{aligned} \|Tf\|_q^q &\leq c\Big\{\int_{\mathbb{R}^3} w^{-q'/q}(\mathcal{M}(f^{q/q'}w^{1/q}))^{q'}\,d\mathbf{x} + \int_{\mathbb{R}^3} w(\mathcal{M}(fw^{-1/q}))^q\,d\mathbf{x} \\ &\quad + \int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}} w_r^{-q'/q}(M_1^+(f_r^{q/q'}w_r^{1/q}))^{q'}\,dx_3\Big)d(x_1,x_2) \\ &\quad + \int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}} w_r(M_1^+(f_r w_r^{-1/q}))^q\,dx_3\Big)d(x_1,x_2)\Big\} \\ &\leq A^q\|f\|_q^q, \end{aligned}$$

with a constant $A = A(q, w) > 0$.

Let us fix a non-negative θ-independent function $f \in L^q(\mathbb{R}^3)$ with $\|f\|_q = 1$ and define

$$\eta = \sum_{k=1}^{\infty}(2A)^{-k}T^k(f),$$

where $T^k(f) = T(T^{k-1}(f))$. Obviously Tf and therefore also η are θ-independent. Moreover, $\eta \in L^q(\mathbb{R}^3)$ and $\|\eta\|_q \leq \sum_{k=1}^{\infty} 2^{-k} = 1$. In particular, $\eta(\mathbf{x}) < \infty$ for a.a. $\mathbf{x} \in \mathbb{R}^3$,

$\eta_r(\cdot) \in L^q(\mathbb{R})$ for a.a. $(x_1, x_2) \in \mathbb{R}^2$ and $\eta_r(x_3) < \infty$ for a.a. $x_3 \in \mathbb{R}$. Since T is subadditive and positivity-preserving, we get the pointwise inequality

$$T\eta \le \sum_{k=1}^{\infty}(2A)^{-k}T^{k+1}(f) = \sum_{k=2}^{\infty}(2A)^{1-k}T^{k}(f) \le (2A)\eta.$$

Now let $w_1 := w^{1/q}\eta^{q/q'}$ and $w_2 := w^{-1/q}\eta$ such that $w = w_1/w_2^{q-1}$. Then

$$\begin{aligned}
\mathcal{M}(w_1) &\le w^{1/q}(T\eta)^{q/q'} \le w^{1/q}\eta^{q/q'}(2A)^{q/q'} = (2A)^{q/q'}w_1\\
M_1^{+}((w_1)_r) &\le w^{1/q}(T\eta)^{q/q'} \le w^{1/q}\eta^{q/q'}(2A)^{q/q'} = (2A)^{q/q'}(w_1)_r\\
\mathcal{M}(w_2) &\le w^{-1/q}T(\eta) \quad \le w^{-1/q}\eta\, 2A = 2Aw_2\\
M_1^{-}((w_2)_r) &\le w^{-1/q}T(\eta) \quad \le w^{-1/q}\eta\, 2A = 2A(w_2)_r
\end{aligned}$$

proving that $w_1 \in \widetilde{A}_1^-$, $w_2 \in \widetilde{A}_1^+$.

The case $1 < q < 2$ follows by a simple duality argument, since $w \in \widetilde{A}_q^-$ is equivalent to $w^{-q'/q} \in \widetilde{A}_{q'}^+$. ∎

7. Concluding remarks. In the first part the variational approach in an L^2-setting with anisotropic weights was applied under assumptions on the weights, which follow from the Friedrichs-Poincaré inequality and the A_2-condition. In the second case, to construct strong solutions in L^q-spaces Littlewood-Paley decomposition was used to estimate second order derivatives of the solution. This naturally leads to the Muckenhoupt class of $A_{q/2}$ weights. Since it turned out to be necessary that the weights are also of Muckenhoupt type in the direction of the axis of rotation, we had to use the theory of (one-dimensional) one-sided weights instead of the standard weighted theory, since the natural anisotropic weights for the Oseen operator do not belong to the class $A_p(\cdot,\cdot,x_3)$. In the L^2-framework the pressure p was obtained by the fundamental solution of the Laplace equation and the weighted estimate of ∇p follows from the Calderón-Zygmund theory. On the other hand, to be consistent with the analysis of the velocity field, in the L^q-framework we applied multiplier theory. Both techniques can be used in L^2 as well as in L^q to get an estimate of ∇p. Also we would like to point out that we are looking for weak solutions in L^2, but for strong solutions in L^q. Anyway, in the L^2-setting without weights we can apply the Plancherel theorem which gives the desired results immediately, see remarks in [7] and work of Hishida [18].

Acknowledgements. Š. N. and M. K. were supported by the Grant Agency of the Academy of Sciences No. IAA100190505 and IAA100190804, further their research was supported by the Academy of Sciences of the Czech Republic, Institutional Research Plan No. AV0Z10190503. S. K. was supported by the Grant Agency of the Czech Republic n. 201050005, and n.201080012 and his research was supported by the Research Plan of the Ministry of Education of the Czech Republic No. 6840770010. M. K., Š. N. and R. F. were partially supported by Joint Research Project D/04/25763 of DAAD and D-CZ 3/05-06 of the Academy of Sciences.

References

[1] J. Bergh and J. Löfström, *Interpolation Spaces*, Springer-Verlag, New York, 1976.

[2] R. Farwig, *The stationary exterior 3D-problem of Oseen and Navier-Stokes equations in anisotropically weighted Sobolev spaces*, Math. Z. 211 (1992), 409–447.

[3] R. Farwig, *A variational approach in weighted Sobolev spaces to the operator* $-\triangle + \partial/\partial_1$ *in exterior domains of* $\mathbb{R}^3$, Math. Z. 210 (1992), 449–464.

[4] R. Farwig, *An* L^q*-analysis of viscous fluid flow past a rotating obstacle*, Tôhoku Math. J. 58 (2005), 129–147.

[5] R. Farwig, *Estimates of lower order derivatives of viscous fluid flow past a rotating obstacle*, in: Banach Center Publications 70, 2005, 73–84.

[6] R. Farwig and T. Hishida, *Stationary Navier-Stokes flow around a rotating obstacle*, Funkcial. Ekvac. 50 (2007), 371–403.

[7] R. Farwig, T. Hishida and D. Müller, L^q*-theory of a singular "winding" integral operator arising from fluid dynamics*, Pacific J. Math. 215 (2004), 297–312.

[8] R. Farwig, M. Krbec and Š. Nečasová, *A weighted* L^q *approach to Stokes flow around a rotating body*, Ann. Univ. Ferrara Sez. VII 54 (2008), 61–84.

[9] R. Farwig, M. Krbec and Š. Nečasová, *A weighted* L^q*-approach to Oseen flow around a rotating body*, Math. Methods in the Appl. Sci. 31 (2008), 551–574.

[10] R. Finn, *Estimates at infinity for stationary solution of Navier-Stokes equations*, Bull. Math. Soc. Sci. Math. Phys. R.P.R., N. Ser. 3(51) (1960), 387–418.

[11] R. Finn, *On the exterior stationary problem for the Navier-Stokes equations, and associated perturbation problems*, Arch. Rational Mech. Anal. 19 (1965), 363–406.

[12] G. P. Galdi, *On the motion of a rigid body in a viscous liquid: A mathematical analysis with applications*, in: Handbook of Mathematical Fluid Dynamics, Vol. 1, S. Friedlander and D. Serre (eds.), Elsevier, 2002.

[13] G. P. Galdi, *Steady flow of a Navier-Stokes fluid around a rotating obstacle*, J. Elasticity 71 (2003), 1–31.

[14] G. P. Galdi and A. L. Silvestre, *On the steady motion of a Navier-Stokes liquid around a rigid body*, Arch. Rational Mech. Anal. 184 (2007), 371–400.

[15] J. García-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland, Amsterdam, 1985.

[16] M. Geissert, H. Heck and M. Hieber, L^p *theory of the Navier-Stokes flow in the exterior of a moving or rotating obstacle*, J. Reine Angew. Math. 596 (2006), 45–62.

[17] T. Hishida, *The Stokes operator with rotation effect in exterior domains*, Analysis 19 (1999), 51–67.

[18] T. Hishida, *An existence theorem for the Navier-Stokes flow in the exterior of a rotating obstacle*, Arch. Rational Mech. Anal. 150 (1999), 307–348.

[19] T. Hishida, L^q *estimates of weak solutions to the stationary Stokes equations around a rotating body*, J. Math. Soc. Japan 58 (2006), 743–767.

[20] T. Hishida and Y. Shibata, L_p-L_q *estimate of the Stokes operator and Navier-Stokes flows in the exterior of a rotating obstacle*, preprint, 2006.

[21] J.-L. Impagliazzo, *Résolution des équations de Navier-Stokes compressibles à l'aide de la méthode de décomposition*, Thèse de doctorat, Université du Sud Toulon-Var, 1997.

[22] T. Kobayashi and Y. Shibata, *On the Oseen equation in the three-dimensional exterior domains*, Math. Ann. 310 (1998), 1–45.

[23] S. Kračmar, Š. Nečasová and P. Penel, *Remarks on the non-homogeneous Oseen problem arising from modeling of the fluid around a rotating body*, in: Hyperbolic Problems: Theory, Numerics, Applications (Lyon, 2006), S. Benzoni-Gavage and D. Serre (eds.), Springer, 2008, 775–782.

[24] S. Kračmar, Š. Nečasová and P. Penel, *Anisotropic L^2 estimates of weak solutions to the stationary Oseen-type equations in $\mathbb{R}^3$ for a rotating body*, RIMS Kokyuroku Bessatsu B1 (2007), 219–235.

[25] S. Kračmar, Š. Nečasová and P. Penel, *L^q approach of weak solutions of Oseen flow around a rotating body*, in: Parabolic and Navier-Stokes Equations, Banach Center Publications 81 (2008), 259–276.

[26] H. Kozono and H. Sohr, *New apriori estimates for the Stokes equations in exterior domains*, Indiana Univ. Math. J. 41 (1991), 1–27.

[27] S. Kračmar, A. Novotný and M. Pokorný, *Estimates of three dimensional Oseen kernels in weighted L^p spaces*, in: Applied Nonlinear Analysis, Kluwer, 1999, 281–316.

[28] S. Kračmar, A. Novotný and M. Pokorný, *Estimates of Oseen kernels in weighted L^p spaces*, J. Math. Soc. Japan 53 (2001), 59–111.

[29] S. Kračmar and P. Penel, *Variational properties of a generic model equation in exterior 3D domains*, Funkcial. Ekv. 47 (2004), 499–523.

[30] S. Kračmar and P. Penel, *New regularity results for a generic model equation in exterior 3D domains*, in: Banach Center Publications 70, 2005, 139–155.

[31] D. S. Kurtz, *Littlewood-Paley and multiplier theorems on weighted L^p spaces*, Trans. Amer. Math. Soc. 259 (1980), 235–254.

[32] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. 165 (1972), 207–226.

[33] Š. Nečasová, *Asymptotic properties of the steady fall of a body in a viscous fluid*, Math. Meth. Appl. Sci. 27 (2004), 1969–1995.

[34] Š. Nečasová, *On the problem of the Stokes flow and Oseen flow in $\mathbb{R}^3$ with Coriolis force arising from fluid dynamics*, IASME Trans. 2 (2005), 1262–1270.

[35] Š. Nečasová and K. Schumacher, *Strong solution to the Stokes equations of a flow around a rotating body in weighted L^q spaces*, submitted.

[36] C. W. Oseen, *Neuere Methoden und Ergebnisse in der Hydrodynamik*, Akad. Verlagsgesellschaft, Leipzig, 1927.

[37] V. Rychkov, *Littlewood-Paley theory and function spaces with A_p^{loc} weights*, Math. Nachr. 224 (2001), 145–180.

[38] E. Sawyer, *Weighted inequalities for the one-sided Hardy-Littlewood maximal function*, Trans. Amer. Math. Soc. 297 (1986), 53–61.

[39] H. Sohr, *The Navier-Stokes Equations: An Elementary Functional Analytic Approach*, Birkhäuser, Basel 2001.

[40] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, NJ, 1970.

[41] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, NJ, 1993.

[42] J.-O. Strömberg and A. Torchinsky, *Weighted Hardy Spaces*, Lecture Notes in Math. 1381, Springer, Berlin, 1989.

[43] E. A. Thomann and R. B. Guenther, *The fundamental solution of the linearized Navier-Stokes equations for spinning bodies in three spatial dimensions—time dependent case*, J. Math. Fluid Mech. 8 (2006), 77–98.