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STABILITY ANALYSIS OF PHASE BOUNDARY MOTION BY SURFACE DIFFUSION WITH TRIPLE JUNCTION

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Abstract. The linearized stability of stationary solutions for the surface diffusion flow with a triple junction is studied. We derive the second variation of the energy functional under the constraint that the enclosed areas are preserved and show a linearized stability criterion with the help of the H^{-1}-gradient flow structure of the evolution problem and the analysis of eigenvalues of a corresponding differential operator.

1. Introduction. The surface diffusion flow

$$V = -\Delta_S H \tag{1}$$

is a geometrical evolution law which describes the surface dynamics for phase interfaces, when mass diffusion only occurs within the interface. Here, V is the normal velocity of the evolving surface, Δ_S is the surface Laplacian, and H is the mean curvature of the surface. The basic property of this flow is that the perimeter of an enclosed volume decreases whereas the volume is conserved.

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In this paper we study the motion by surface diffusion for three curves Γ_t^1, Γ_t^2, and Γ_t^3 which are contained in a bounded domain $\Omega \subset \mathbb{R}^2$ with the conditions that each one of the end points of Γ_t^i $(i = 1, 2, 3)$ is connected at a triple junction $p(t) \in \Omega$ and the other end points intersect with $\partial\Omega$. Then we require for $i = 1, 2, 3$

$$V^i = -m^i \gamma^i \kappa_{ss}^i \quad \text{on} \quad \Gamma_t^i \tag{2}$$

with the boundary conditions at a triple junction $p(t)$

$$\begin{cases} \sphericalangle(\Gamma_t^1, \Gamma_t^2) = \theta^3, \ \ \sphericalangle(\Gamma_t^2, \Gamma_t^3) = \theta^1, \ \ \sphericalangle(\Gamma_t^3, \Gamma_t^1) = \theta^2, \\ \gamma^1\kappa^1 + \gamma^2\kappa^2 + \gamma^3\kappa^3 = 0, \\ m^1\gamma^1\kappa_s^1 = m^2\gamma^2\kappa_s^2 = m^3\gamma^3\kappa_s^3, \end{cases} \tag{3}$$

and at $\Gamma_t^i \cap \partial\Omega$

$$\Gamma^i \perp \partial\Omega, \quad \kappa_s^i = 0. \tag{4}$$

Here, V^i is the normal velocity of Γ_t^i, κ^i is the curvature of Γ_t^i, and s is an arc-length parameter of Γ_t^i. Further, m^i and γ^i are the positive constants concerning the mobility and the surface energy, respectively. In addition, θ^i is the positive constant satisfying

$$\frac{\sin\theta^1}{\gamma^1} = \frac{\sin\theta^2}{\gamma^2} = \frac{\sin\theta^3}{\gamma^3}, \tag{5}$$

which is called *Young's law*. We remark that Young's law is also represented as

$$\gamma^1 T^1 + \gamma^2 T^2 + \gamma^3 T^3 = 0 \ \text{ at } \ p(t),$$

where T^i is the unit tangent to Γ_t^i. In (3) the second and the third condition follow from the continuity of the chemical potentials and the flux balance at the triple junction, respectively. Also, in (4) the second condition is the no-flux condition. The boundary conditions (3) and (4) are the natural boundary conditions when viewing the flow as the H^{-1}-gradient flow of the energy functional

$$E[\Gamma_t] := \sum_{i=1}^{3} \gamma^i L[\Gamma_t^i],$$

where $\Gamma_t = \bigcup_{i=1}^3 \Gamma^i$ and $L[\Gamma_t^i]$ is the length functional of Γ_t^i. It is not difficult to show that under the surface diffusion flow (2) with the boundary conditions (3) and (4) the areas enclosed by Γ_t^i, Γ_t^j, and $\partial\Omega$ for $(i, j) = (1, 2), (2, 3), (3, 1)$ are preserved and the energy $E[\Gamma_t]$ decreases in time. We also find that an arc of a circle or a line segment are stationary under (2)-(4).

The geometric problem (2)-(4) was derived by Garcke and Novick-Cohen [5] as the asymptotic limit of a Cahn-Hilliard system with a degenerate mobility matrix. They also proved the short time existence of a solution for this problem. The stability problem of stationary solutions for (2)-(4) has been addressed by Ito and Kohsaka [7] and by Escher, Garcke and Ito [2] in the case of a geometry with a mirror symmetry and by Ito and Kohsaka [8] in a triangular domain.

Our goal in this paper is to derive the second variation of the energy functional under the constraint that the areas enclosed by Γ_t^i, Γ_t^j, and $\partial\Omega$ for $(i, j) = (1, 2), (2, 3), (3, 1)$ are preserved and also to obtain a linearized stability criterion based on the work of [9]

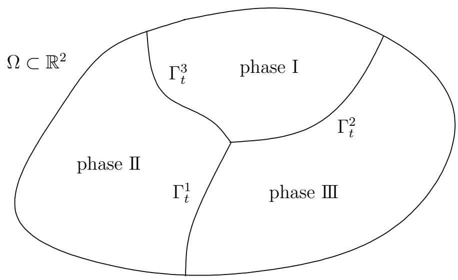

Fig. 1. The phase boundaries with triple junction

and [3]. We remark that [9] is the analysis of three curves with a triple junction for the curvature flow $V^i = \kappa^i$ and [3] is that of one curve for the surface diffusion flow.

This paper proceeds as follows. In Section 2 we give a representation of curves around the stationary solutions by using a modified distance function. It is not possible to use usual distance functions since the triple junction moves with respect to time. Thus we have to introduce a certain tangential adjustment. Then we formulate the evolution problem with the help of this parameterization and give a nonlinear problem. In Section 3 we derive the second variation of the energy functional under the area constraint. In Section 4 we first introduce the linearized system and prove a gradient flow structure with respect to a certain H^{-1} scalar product on networks for the linearized system. Further, we show several properties of the spectrum concerning our system. Finally, we give the stability criterion and analyze the stability for one specific configuration.

2. Parameterization. Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary containing $(0,0)$. We assume that Ω and $\partial\Omega$ are given as

$$\Omega = \{x \in \mathbb{R}^2 \mid \psi(x) < 0\}, \quad \partial\Omega = \{x \in \mathbb{R}^2 \mid \psi(x) = 0\}$$

with a smooth function $\psi : \mathbb{R}^2 \to \mathbb{R}$ with $\nabla\psi(x) \neq 0$ if $x \in \partial\Omega$, i.e. $\psi(x) = 0$. Let Γ_*^i $(i = 1,2,3)$ be straight lines or circular arcs with the constant curvature κ_*^i satisfying

$$\gamma^1\kappa_*^1 + \gamma^2\kappa_*^2 + \gamma^3\kappa_*^3 = 0.$$

Further, Γ_*^i $(i = 1,2,3)$ meet the outer boundary with the angle $\pi/2$ and have $P_* = (0,0)$ (without loss of generality) as a common point (triple junction) with the angle conditions $\sphericalangle(\Gamma_*^i, \Gamma_*^j) = \theta^k$ for $i, j, k \in \{1,2,3\}$ mutually different. Then we define an arc-length parameterization of Γ_*^i as

$$\Gamma_*^i = \{\Phi_*^i(\sigma) \mid \sigma \in [0, l^i]\}$$

with $\Phi_*^i(0) = (0,0)$, $\Phi_*^i(l^i) \in \partial\Omega$. We obtain in particular that l^i is the length of Γ_*^i. Then we will extend Φ_*^i as an arc-length parameterization of the full line or the full circle which contain Γ_*^i. We will now introduce a certain stretched coordinate system in order to allow for parameterizations of curves close to Γ_*^i $(i = 1,2,3)$ over fixed intervals $[0, l^i]$.

Let T_*^i be the unit tangent to Γ_*^i pointing from the triple junction P_* to the outer boundary and let $N_*^i = RT_*^i$, where R is the anti-clockwise rotation by $\pi/2$, be a unit

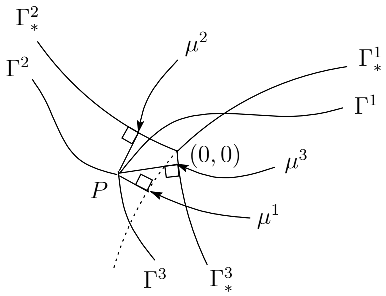

Fig. 2. The positions of μ^i $(i = 1, 2, 3)$

normal. Then we set

$$\mu^i_{\partial\Omega}(q) = \max\{\, \sigma \mid \Phi^i_*(\sigma) + qN^i_*(\sigma) \in \overline{\Omega}\,\}\,.$$

We now define the parameterization of curves $\Gamma = \bigcup_{i=1}^3 \Gamma^i$ close to $\Gamma_* = \bigcup_{i=1}^3 \Gamma^i_*$ having their triple junction at the point P with the help of

$$\rho^i : [0, l^i] \to \mathbb{R}, \quad \mu^i \in \mathbb{R} \quad (i = 1, 2, 3)$$

together with the conditions

$$\Phi^1_*(\mu^1) + \rho^1(0)N^1_*(\mu^1) = \Phi^2_*(\mu^2) + \rho^2(0)N^2_*(\mu^2) = \Phi^3_*(\mu^3) + \rho^3(0)N^3_*(\mu^3). \qquad (6)$$

Here ρ^i are the smooth functions and μ^i are the parameters which allow for a tangential movement of the triple junction along the extended Γ^i_* (see Fig. 2). Set

$$\Psi^i(\sigma, q, \mu^i) = \Phi^i_*(\xi^i(\sigma, q, \mu^i)) + qN^i_*(\xi^i(\sigma, q, \mu^i))\,,$$

where

$$\xi^i(\sigma, q, \mu^i) = \mu^i + \frac{\sigma}{l^i}\{\mu^i_{\partial\Omega}(q) - \mu^i\}\,.$$

Note that $\xi^i(\sigma, 0, 0) = \sigma$ and $\xi^i(0, q, \mu^i) = \mu^i$. Then, if we set

$$\Phi^i(\sigma) = \Psi^i(\sigma, \rho^i(\sigma), \mu^i)\,, \quad \sigma \in [0, l^i]\,, \qquad (7)$$

the functions Φ^i parameterize the curves Γ^i in the neighborhood of Γ_* as $\Gamma^i = \{\Phi^i(\sigma) \mid \sigma \in [0, l^i]\}$. Further, the unit tangent and normal to Γ^i are represented as

$$T^i = \frac{1}{J^i(\boldsymbol{u}^i)}\Phi^i_\sigma, \quad N^i = \frac{1}{J^i(\boldsymbol{u}^i)}R\Phi^i_\sigma,$$

where $\boldsymbol{u}^i = (\rho^i, \mu^i)$ and

$$J^i(\boldsymbol{u}^i) := |\Phi^i_\sigma(\sigma)| = \sqrt{|\Psi^i_\sigma|^2 + 2(\Psi^i_\sigma, \Psi^i_q)_{\mathbb{R}^2}\rho^i_\sigma + |\Psi^i_q|^2|\rho^i_\sigma|^2}.$$

Let us derive the nonlinear problem for ρ^i from the geometric problem (2)-(4). By this parameterization, the surface diffusion flow equation (2) is represented as

$$\rho^i_t = -m^i\gamma^i a^i(\boldsymbol{u}^i)\Delta(\boldsymbol{u}^i)\kappa^i(\boldsymbol{u}^i) + b^i(\boldsymbol{u}^i)\mu^i_t \qquad (8)$$

for $i = 1, 2, 3$, where

$$a^i(\boldsymbol{u}^i) = \frac{J^i(\boldsymbol{u}^i)}{(\Psi^i_q, R\Psi^i_\sigma)_{\mathbb{R}^2}}, \quad b^i(\boldsymbol{u}^i) = -\frac{(\Psi^i_\mu, R\Psi^i_\sigma)_{\mathbb{R}^2} + (\Psi^i_\mu, R\Psi^i_q)_{\mathbb{R}^2}\rho^i_\sigma}{(\Psi^i_q, R\Psi^i_\sigma)_{\mathbb{R}^2}},$$

$$\Delta(\boldsymbol{u}^i) = \frac{1}{\{J^i(\boldsymbol{u}^i)\}^2}\partial^2_\sigma + \frac{1}{J^i(\boldsymbol{u}^i)}\left\{\partial_\sigma \frac{1}{J^i(\boldsymbol{u}^i)}\right\}\partial_\sigma,$$

and the curvature $\kappa^i(\boldsymbol{u}^i)$ is given by

$$\begin{aligned}
\kappa^i(\boldsymbol{u}^i) =& \frac{1}{\{J^i(\boldsymbol{u}^i)\}^3}(\Phi^i_{\sigma\sigma}, R\Phi^i_\sigma)_{\mathbb{R}^2} \\
=& \frac{1}{\{J^i(\boldsymbol{u}^i)\}^3}[(\Psi^i_q, R\Psi^i_\sigma)_{\mathbb{R}^2}\rho^i_{\sigma\sigma} + \big\{2(\Psi^i_{\sigma q}, R\Psi^i_\sigma)_{\mathbb{R}^2} + (\Psi^i_{\sigma\sigma}, R\Psi^i_q)_{\mathbb{R}^2}\big\}\rho^i_\sigma \\
&\qquad + \big\{(\Psi^i_{qq}, R\Psi^i_\sigma)_{\mathbb{R}^2} + 2(\Psi^i_{\sigma q}, R\Psi^i_q)_{\mathbb{R}^2} + (\Psi^i_{qq}, R\Psi^i_q)_{\mathbb{R}^2}\rho^i_\sigma\big\}(\rho^i_\sigma)^2 \\
&\qquad + (\Psi^i_{\sigma\sigma}, R\Psi^i_\sigma)_{\mathbb{R}^2}].
\end{aligned}$$

Further, the boundary conditions (3) are represented as

$$\begin{cases}
(\Phi^1_\sigma, \Phi^2_\sigma)_{\mathbb{R}^2} = |\Phi^1_\sigma||\Phi^2_\sigma|\cos\theta^3, \quad (\Phi^1_\sigma, \Phi^3_\sigma)_{\mathbb{R}^2} = |\Phi^1_\sigma||\Phi^3_\sigma|\cos\theta^2, \\
\gamma^1\kappa^1(\boldsymbol{u}^1) + \gamma^2\kappa^2(\boldsymbol{u}^2) + \gamma^3\kappa^3(\boldsymbol{u}^3) = 0, \\
\dfrac{m^1\gamma^1}{J^1(\boldsymbol{u}^1)}\partial_\sigma\kappa^1(\boldsymbol{u}^1) = \dfrac{m^2\gamma^2}{J^2(\boldsymbol{u}^2)}\partial_\sigma\kappa^2(\boldsymbol{u}^2) = \dfrac{m^3\gamma^3}{J^3(\boldsymbol{u}^3)}\partial_\sigma\kappa^3(\boldsymbol{u}^3)
\end{cases} \tag{9}$$

with the notation

$$(\Phi^i_\sigma, \Phi^j_\sigma)_{\mathbb{R}^2} = (\Psi^i_\sigma, \Psi^j_\sigma)_{\mathbb{R}^2} + (\Psi^i_\sigma, \Psi^j_q)_{\mathbb{R}^2}\rho^j_\sigma + (\Psi^i_q, \Psi^j_\sigma)_{\mathbb{R}^2}\rho^i_\sigma + (\Psi^i_q, \Psi^j_q)_{\mathbb{R}^2}\rho^i_\sigma\rho^j_\sigma,$$

and the boundary conditions (4) are represented as

$$(R\Psi^i_\sigma + R\Psi^i_q\rho^i_\sigma, \nabla\psi(\Psi^i))_{\mathbb{R}^2} = 0, \quad \partial_\sigma\kappa^i(\boldsymbol{u}^i) = 0 \tag{10}$$

for $i = 1, 2, 3$.

3. The variation of the energy functional. The functions Ψ^i have the following properties which we need to derive the variation of the energy.

LEMMA 1. *The parameterizations Ψ^i fulfill the following:*

(i) $\Psi^i(\sigma, 0, 0) = \Phi^i_*(\sigma)$.

(ii) $\Psi^i_\sigma(\sigma, 0, 0) = T^i_*(\sigma)$, $\Psi^i_q(\sigma, 0, 0) = N^i_*(\sigma)$, $\Psi^i_\mu(\sigma, 0, 0) = (1 - \sigma/l^i)T^i_*(\sigma)$.

(iii) $\Psi^i_{\sigma q}(\sigma, 0, 0) = -\kappa^i_* T^i_*(\sigma)$, $\Psi^i_{\sigma\mu}(\sigma, 0, 0) = (-1/l^i)T^i_*(\sigma) + (1 - \sigma/l^i)\kappa^i_* N^i_*(\sigma)$,
$\Psi^i_{qq}(\sigma, 0, 0) = \xi^i_{qq}(\sigma, 0, 0)T^i_*(\sigma)$, $\Psi^i_{q\mu}(\sigma, 0, 0) = -(1 - \sigma/l^i)\kappa^i_* T^i_*(\sigma)$,
$\Psi^i_{\mu\mu}(\sigma, 0, 0) = (1 - \sigma/l^i)^2\kappa^i_* N^i_*(\sigma)$.

(iv) $\Psi^i_{\sigma qq}(\sigma, 0, 0) = \xi^i_{\sigma qq}(\sigma, 0, 0)T^i_*(\sigma) + \xi^i_{qq}(\sigma, 0, 0)\kappa^i_* N^i_*(\sigma)$,
$\Psi^i_{\sigma q\mu}(\sigma, 0, 0) = (\kappa^i_*/l^i)T^i_*(\sigma) - (1 - \sigma/l^i)(\kappa^i_*)^2 N^i_*(\sigma)$,
$\Psi^i_{\sigma\mu\mu}(\sigma, 0, 0) = -(1 - \sigma/l^i)^2(\kappa^i_*)^2 T^i_*(\sigma) - (2/l^i)(1 - \sigma/l^i)\kappa^i_* N^i_*(\sigma)$.

Proof. By the definition of Ψ^i and ξ^i, (i) is obvious. Let us prove (ii). Differentiating $\Psi^i(\sigma, 0, 0) = \Phi^i_*(\sigma)$ with respect to σ, we readily derive $\Psi^i_\sigma(\sigma, 0, 0) = T^i_*(\sigma)$. By the definition of Ψ^i, we have

$$\begin{cases} \Psi^i_q(\sigma,q,\mu^i) = \xi_q(\sigma,q,\mu^i)(1-q\kappa^i_*)T^i_*(\xi(\sigma,q,\mu^i)) + N^i_*(\xi(\sigma,q,\mu^i)), \\ \Psi^i_\mu(\sigma,q,\mu^i) = \xi_\mu(\sigma,q,\mu^i)(1-q\kappa^i_*)T^i_*(\xi(\sigma,q,\mu^i)). \end{cases} \tag{11}$$

According to the definition of ξ^i, we obtain

$$\xi^i_q(\sigma,q,\mu) = (\sigma/l^i)\{\mu^i_{\partial\Omega}(q)\}', \quad \xi^i_\mu(\sigma,q,\mu) = 1-\sigma/l^i. \tag{12}$$

Using $\xi^i(\sigma,0,0)=\sigma$ and $\{\mu^i_{\partial\Omega}(q)\}'|_{q=0}=0$ (see [3, p. 1036]), the second and third of (ii) are derived. Finally, by using $\xi^i(\sigma,0,0)=\sigma$, (11), (12), and Frenet-Serret formulas, we are led to (iii) and (iv). ∎

Also, we derive the following lemma.

LEMMA 2. *Let h^i_* be the curvature of $\partial\Omega$ at $\Gamma^i_*\cap\partial\Omega$. Then*

$$\{\mu^i_{\partial\Omega}(q)\}''|_{q=0} = h^i_*.$$

Proof. Recalling the definition of Ψ^i and ξ, we have

$$\Psi^i(\sigma,q,\mu^i)|_{\sigma=l^i} = \Phi^i_*(\mu^i_{\partial\Omega}(q)) + qN^i_*(\mu^i_{\partial\Omega}(q)).$$

Set $\Psi^i_{\partial\Omega}(q) := \Phi^i_*(\mu^i_{\partial\Omega}(q)) + qN^i_*(\mu^i_{\partial\Omega}(q))$. Then,

$$\{\Psi^i_{\partial\Omega}(q)\}' = (1-q\kappa^i_*)T^i_*(\mu^i_{\partial\Omega}(q))\{\mu^i_{\partial\Omega}(q)\}' + N^i_*(\mu^i_{\partial\Omega}(q)). \tag{13}$$

It follows from $\mu^i_{\partial\Omega}(0)=l^i$ and $\{\mu^i_{\partial\Omega}(q)\}'|_{q=0}=0$ that

$$\{\Psi^i_{\partial\Omega}(q)\}'|_{q=0} = N^i_*(l^i).$$

Further, differentiating (13) and putting $q=0$, we have

$$\{\Psi^i_{\partial\Omega}(q)\}''|_{q=0} = \{\mu^i_{\partial\Omega}(q)\}''|_{q=0}\, T^i_*(l^i).$$

Note that $\psi(\Psi^i_{\partial\Omega}(q))=0$ by the definition of $\mu^i_{\partial\Omega}(q)$. Computing the second derivative of $\psi(\Psi^i_{\partial\Omega}(q))=0$ with respect to q, we are led to

$$([D^2\psi(\Psi^i_{\partial\Omega}(q))]\{\Psi^i_{\partial\Omega}(q)\}', \{\Psi^i_{\partial\Omega}(q)\}')_{\mathbb{R}^2} + (\nabla\psi(\Psi^i_{\partial\Omega}(q)), \{\Psi^i_{\partial\Omega}(q)\}'')_{\mathbb{R}^2} = 0.$$

Thus, putting $q=0$, we obtain

$$([D^2\psi(\Phi^i_*(l^i))]N^i_*(l^i), N^i_*(l^i))_{\mathbb{R}^2} + (\nabla\psi(\Phi^i_*(l^i)), T^i_*(l^i))_{\mathbb{R}^2}\{\mu^i_{\partial\Omega}(q)\}''|_{q=0} = 0.$$

By means of $T^i_*(l^i) = \nabla\psi(\Phi^i_*(l^i))/|\nabla\psi(\Phi^i_*(l^i))|$ and $N^i_*(l^i) = -T_{\partial\Omega}(\Phi^i_*(l^i))$, where $T_{\partial\Omega}$ is the unit tangent vector of $\partial\Omega$, we see

$$\begin{aligned} \{\mu^i_{\partial\Omega}(q)\}''|_{q=0} &= -\frac{([D^2\psi(\Phi^i_*(l^i))]N^i_*(l^i), N^i_*(l^i))_{\mathbb{R}^2}}{(\nabla\psi(\Phi^i_*(l^i)), T^i_*(l^i))_{\mathbb{R}^2}} \\ &= -\frac{([D^2\psi(\Phi^i_*(l^i))]T_{\partial\Omega}(\Phi^i_*(l^i)), T_{\partial\Omega}(\Phi^i_*(l^i)))_{\mathbb{R}^2}}{|\nabla\psi(\Phi^i_*(l^i))|}. \end{aligned}$$

Since the curvature of $\partial\Omega$ is represented as

$$\kappa_{\partial\Omega} = -\frac{([D^2\psi]T_{\partial\Omega}, T_{\partial\Omega})_{\mathbb{R}^2}}{|\nabla\psi|},$$

we are led to the desired result. ∎

For $\boldsymbol{u} = (\boldsymbol{u}^1, \boldsymbol{u}^2, \boldsymbol{u}^3)$ with $\boldsymbol{u}^i = (\rho^i, \mu^i)\,(i = 1, 2, 3)$, we define

$$\mathcal{M} = \{\boldsymbol{u} \,|\, \boldsymbol{u} \text{ satisfies the conditions (6)}\}.$$

Set $\boldsymbol{u}^i(\varepsilon) = (\rho^i(\,\cdot\,;\varepsilon), \mu^i(\varepsilon))\,(i = 1, 2, 3)$ with

$$\rho^i : [0, l^i] \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}, \quad \mu^i : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$$

which fulfill $\rho^i(\,\cdot\,;0) \equiv 0$ and $\mu^i(0) = 0$. Then we choose a variation $\boldsymbol{u}(\varepsilon) = (\boldsymbol{u}^1(\varepsilon), \boldsymbol{u}^2(\varepsilon), \boldsymbol{u}^3(\varepsilon)) \in \mathcal{M}$ with variation vector field

$$\partial_\varepsilon \rho^i(\,\cdot\,;0) = v^i(\cdot), \quad \partial_\varepsilon \mu^i(0) = \tau^i$$

for given $\boldsymbol{\eta} = (\boldsymbol{\eta}^1, \boldsymbol{\eta}^2, \boldsymbol{\eta}^3)$ with $\boldsymbol{\eta}^i = (v^i(\cdot), \tau^i)\,(i = 1, 2, 3)$ satisfying

$$\gamma^1 v^1 + \gamma^2 v^2 + \gamma^3 v^3 = 0 \;\text{ at }\; \sigma = 0, \quad \gamma^1 \tau^1 + \gamma^2 \tau^2 + \gamma^3 \tau^3 = 0. \tag{14}$$

Remark 3. (14) means that $\boldsymbol{\eta} = (\boldsymbol{\eta}^1, \boldsymbol{\eta}^2, \boldsymbol{\eta}^3)$ is in the tangential space to $\mathcal{M}$.

Then we have the following lemma.

Lemma 4. *Let $\boldsymbol{u}(\varepsilon) \in \mathcal{M}$ be a variation with the variation vector field $\boldsymbol{\eta}$ satisfying (14). Then*

$$\tau^i = \frac{1}{s^i}\{c^j v^j(0) - c^k v^k(0)\}$$

for $i, j, k \in \{1, 2, 3\}$ mutually different, where $c^i := \cos\theta^i$ and $s^i := \sin\theta^i$.

Proof. For $\boldsymbol{u}^i = (\rho^i, \mu^i)$ and $\boldsymbol{u}^j = (\rho^j, \mu^j)$, set

$$B^{ij}(\boldsymbol{u}^i, \boldsymbol{u}^j) := \Phi^i_*(\mu^i) + \rho^i(0) N^i_*(\mu^i) - \Phi^j_*(\mu^j) - \rho^j(0) N^j_*(\mu^j).$$

Then it follows from $\boldsymbol{u}(\varepsilon) \in \mathcal{M}$ that $B^{ij}(\boldsymbol{u}^i(\varepsilon), \boldsymbol{u}^j(\varepsilon)) = 0$, so that we have

$$0 = \delta B^{ij}(\mathbf{0}, \mathbf{0})[\boldsymbol{\eta}^i, \boldsymbol{\eta}^j] = \tau^i T^i_*(0) + v^i(0) N^i_*(0) - \tau^j T^j_*(0) - v^j(0) N^j_*(0),$$

where $\delta B^{ij}(\mathbf{0}, \mathbf{0})[\boldsymbol{\eta}^i, \boldsymbol{\eta}^j]$ is the first variation of a functional B^{ij} around a stationary solution in the direction $(\boldsymbol{\eta}^i, \boldsymbol{\eta}^j)$. This implies that

$$\tau^i T^i_*(0) + v^i(0) N^i_*(0) = \tau^j T^j_*(0) + v^j(0) N^j_*(0).$$

Thus we have

$$\tau^1 T^i_*(0) + v^1(0) N^1_*(0) = \tau^2 T^2_*(0) + v^2(0) N^2_*(0) = \tau^3 T^3_*(0) + v^3(0) N^3_*(0), \tag{15}$$

By means of (15), we see

$$\tau^i = \tau^j (T^i_*(0), T^j_*(0))_{\mathbb{R}^2} + v^j(0) (T^i_*(0), N^j_*(0))_{\mathbb{R}^2}.$$

Then it follows from the angle conditions for the stationary solutions Γ^i_* at P_* that

$$(T^i_*(0), T^j_*(0))_{\mathbb{R}^2} = \cos\theta^k, \quad (T^i_*(0), N^j_*(0))_{\mathbb{R}^2} = -\sin\theta^k$$

for $i, j, k \in \{1, 2, 3\}$ mutually different, so that we derive

$$\tau^i = \tau^j \cos\theta^k - v^j(0) \sin\theta^k.$$

Setting $c^i := \cos\theta^i$ and $s^i := \sin\theta^i$, we have

$$(1 - c^i c^j c^k)\tau^i = -\bigl\{c^k c^i s^j v^i(0) + s^k v^j(0) + c^k s^i v^k(0)\bigr\}.$$

Further, (5) and (14) imply

$$(1 - c^i c^j c^k)\tau^i = -\frac{1}{s^i}\bigl[\bigl\{(s^k s^i - c^k c^i (s^j)^2\bigr\} v^j(0) + \bigl\{c^k (s^i)^2 - c^k c^i s^j s^k\bigr\} v^k(0)\bigr]$$

Since we observe

$$s^k s^i - c^k c^i (s^j)^2 = -c^j(1 - c^i c^j c^k), \quad c^k (s^i)^2 - c^k c^i s^j s^k = c^k(1 - c^i c^j c^k),$$

we are led to the desired result. ∎

For $\boldsymbol{u} = (\boldsymbol{u}^1, \boldsymbol{u}^2, \boldsymbol{u}^3)$ with $\boldsymbol{u}^i = (\rho^i, \mu^i)\,(i = 1, 2, 3)$, the energy of $\Gamma = \bigcup_{i=1}^3 \Gamma^i$ is defined as

$$E_\Gamma(\boldsymbol{u}) := \sum_{i=1}^3 \gamma^i L_{\Gamma^i}(\boldsymbol{u}^i) = \sum_{i=1}^3 \gamma^i \int_0^{l^i} J^i(\boldsymbol{u}^i)\, d\sigma, \tag{16}$$

where γ^i is the constant concerning the surface energy and $L_{\Gamma^i}(\boldsymbol{u}^i)$ is the length of Γ^i. Then we have the following lemmas. Here and hereafter, $\delta E(\mathbf{0})[\boldsymbol{\eta}]$ and $\delta^2 E(\mathbf{0})[\boldsymbol{\eta}_1, \boldsymbol{\eta}_2]$ denote the first and second variation of a functional E around a stationary solution, respectively.

Lemma 5 (The first variation of E_Γ). *Let $\boldsymbol{u}(\varepsilon) \in \mathcal{M}$ be a variation with the variation vector field $\boldsymbol{\eta}$ satisfying (14). Then*

$$\delta E_\Gamma(\mathbf{0})[\boldsymbol{\eta}] = -\sum_{i=1}^3 \gamma^i \int_0^{l^i} \kappa_*^i v^i\, d\sigma.$$

Proof. Using Lemma 1, we observe

$$\delta J^i(\mathbf{0})[\boldsymbol{\eta}^i] = -\kappa_*^i v^i - \frac{1}{l^i}\tau^i. \tag{17}$$

By means of (14), we have the desired result. ∎

Lemma 6 (The second variation of E_Γ). *Let $\boldsymbol{u}_j(\varepsilon) = (\boldsymbol{u}_j^1(\varepsilon_j), \boldsymbol{u}_j^2(\varepsilon_j), \boldsymbol{u}_j^3(\varepsilon_j)) \in \mathcal{M}$ $(j = 1, 2)$ with $\boldsymbol{u}_j^i(\varepsilon_j) = (\rho_j^i(\cdot\,; \varepsilon_j), \mu_j^i(\varepsilon_j))$ be a variation which has the variation vector field $\boldsymbol{\eta}_j = (\boldsymbol{\eta}_j^1, \boldsymbol{\eta}_j^2, \boldsymbol{\eta}_j^3)$ $(j = 1, 2)$ with $\boldsymbol{\eta}_j^i = (v_j^i(\cdot), \tau_j^i)$ satisfying (14). Then*

$$\begin{aligned}&\delta^2 E_\Gamma(\mathbf{0})[\boldsymbol{\eta}_1, \boldsymbol{\eta}_2]\\ &= \sum_{i=1}^3 \gamma^i \left\{ \int_0^{l^i} v_{1,\sigma}^i v_{2,\sigma}^i\, d\sigma + h_*^i v_1^i v_2^i\big|_{\sigma = l^i} + \int_0^{l^i} \frac{\kappa_*^i}{l^i}\left(v_1^i \tau_2^i + \tau_1^i v_2^i\right) d\sigma \right\},\end{aligned}$$

where h_*^i is the curvature of $\partial\Omega$ at $\Gamma_*^i \cap \partial\Omega$.

Proof. Using Lemma 1, we obtain

$$\delta^2 J^i(\mathbf{0})[\boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i] = \xi_{\sigma qq}^i v_1^i v_2^i + \xi_{qq}^i v_1^i v_{2,\sigma}^i + \xi_{qq}^i v_{1,\sigma}^i v_2^i + v_{1,\sigma}^i v_{2,\sigma}^i + \frac{\kappa_*^i}{l^i}\left(v_1^i \tau_2^i + \tau_1^i v_2^i\right). \tag{18}$$

This implies that

$$\begin{aligned}\delta^2 L_{\Gamma^i}(\mathbf{0})[\boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i] &= \int_0^{l^i} \left\{\xi_{\sigma qq}^i v_1^i v_2^i + \xi_{qq}^i v_1^i v_{2,\sigma}^i + \xi_{qq}^i v_{1,\sigma}^i v_2^i + v_{1,\sigma}^i v_{2,\sigma}^i\right\} d\sigma\\ &\quad + \int_0^{l^i} \frac{\kappa_*^i}{l^i}\left(v_1^i \tau_2^i + \tau_1^i v_2^i\right) d\sigma\\ &= \left[\xi_{qq}^i v_1^i v_2^i\right]_{\sigma=0}^{\sigma=l^i} + \int_0^{l^i} v_{1,\sigma}^i v_{2,\sigma}^i\, d\sigma + \int_0^{l^i} \frac{\kappa_*^i}{l^i}\left(v_1^i \tau_2^i + \tau_1^i v_2^i\right) d\sigma.\end{aligned}$$

Then, by means of $\xi^i_{qq}(\sigma,0,0) = (\sigma/l^i)\{\mu^i_{\partial\Omega}(q)\}''|_{q=0}$ and Lemma 2, we have

$$\delta^2 L_{\Gamma^i}(\mathbf{0})[\boldsymbol{\eta}^i_1, \boldsymbol{\eta}^i_2] = \int_0^{l^i} v^i_{1,\sigma} v^i_{2,\sigma}\, d\sigma + h^i_* v^i_1 v^i_2\big|_{\sigma=l^i} + \int_0^{l^i} \frac{\kappa^i_*}{l^i}\big(v^i_1\tau^i_2 + \tau^i_1 v^i_2\big)\, d\sigma.$$

This leads to the desired result. ■

Let D^{ij} be a domain enclosed by Γ^i, Γ^j and $\partial\Omega$. Also, let $Q(s)$ be an arc-length parameterization of $\partial\Omega$ which satisfies

$$Q(S^i(\rho^i)) = \Psi^i_{\partial\Omega}(\rho^i)|_{\sigma=l^i}, \tag{19}$$

where $\Psi^i_{\partial\Omega}(q) := \Phi^i_*(\mu^i_{\partial\Omega}(q)) + qN^i_*(\mu^i_{\partial\Omega}(q))$. Then the area of D_{ij} is represented as

$$\begin{aligned}\mathrm{Area}_{D^{ij}}(\boldsymbol{u}^{ij}) = &-\int_0^{l^i} (\Psi^i, N^i)_{\mathbb{R}^2} J^i\, d\sigma + \int_0^{l^j} (\Psi^j, N^j)_{\mathbb{R}^2} J^j\, d\sigma \\ &+ \int_{\partial\Omega: S^j(\rho^j)\to S^i(\rho^i)} (Q(s), N_{\partial\Omega}(s))_{\mathbb{R}^2}\, ds,\end{aligned}$$

where $\boldsymbol{u}^{ij} = (\boldsymbol{u}^i, \boldsymbol{u}^j)$. Further, let D^{ij}_* be a domain enclosed by Γ^i_*, Γ^j_* and $\partial\Omega$. Then the area of D^{ij}_* is represented as

$$\begin{aligned}\mathrm{Area}_{D^{ij}_*} = &-\int_0^{l^i} (\Phi^i_*, N^i_*)_{\mathbb{R}^2}\, d\sigma + \int_0^{l^j} (\Phi^j_*, N^j_*)_{\mathbb{R}^2}\, d\sigma \\ &+ \int_{\partial\Omega: S^j(0)\to S^i(0)} (Q(s), N_{\partial\Omega}(s))_{\mathbb{R}^2}\, ds.\end{aligned}$$

Thus the area constraint is given by

$$A^{ij}_\Gamma(\boldsymbol{u}^{ij}) := \mathrm{Area}_{D^{ij}}(\boldsymbol{u}^{ij}) - \mathrm{Area}_{D^{ij}_*} = 0.$$

Then we obtain the following lemmas.

LEMMA 7 (The first variation of A^{ij}_Γ). *Let $\boldsymbol{u}^{ij}(\varepsilon) = (\boldsymbol{u}^i(\varepsilon), \boldsymbol{u}^j(\varepsilon))$ with $\boldsymbol{u}^i(\varepsilon), \boldsymbol{u}^j(\varepsilon) \in \mathcal{M}$ be a variation with the variation vector field $\boldsymbol{\eta}^{ij} = (\boldsymbol{\eta}^i, \boldsymbol{\eta}^j)$ where $\boldsymbol{\eta}^i$ and $\boldsymbol{\eta}^j$ satisfy (14). Then*

$$\delta A^{ij}_\Gamma(\mathbf{0})[\boldsymbol{\eta}^{ij}] = -2\int_0^{l^i} v^i\, d\sigma + 2\int_0^{l^j} v^j\, d\sigma.$$

Proof. Set

$$\begin{aligned}F^i(\boldsymbol{u}^i) &:= \int_0^{l^i} (\Psi^i, N^i)_{\mathbb{R}^2} J^i\, d\sigma, \\ G^{ij}(\rho^i, \rho^j) &:= \int_{\partial\Omega: S^j(\rho^j)\to S^i(\rho^i)} (Q(s), N_{\partial\Omega}(s))_{\mathbb{R}^2}\, ds.\end{aligned}$$

Then we obtain that $\mathrm{Area}_{D^{ij}}(\boldsymbol{u}^{ij}) = -F^i(\boldsymbol{u}^i) + F^j(\boldsymbol{u}^j) + G^{ij}(\rho^i, \rho^j)$, so that

$$\delta A^{ij}_\Gamma(\mathbf{0})[\boldsymbol{\eta}^{ij}] = -\delta F^i(\mathbf{0})[\boldsymbol{\eta}^i] + \delta F^j(\mathbf{0})[\boldsymbol{\eta}^j] + \delta G^{ij}(0,0)[v^i, v^j]. \tag{20}$$

Let us first derive $\delta F^i(\mathbf{0})[\boldsymbol{\eta}^i]$. Using Lemma 1, we obtain

$$\delta F^i(\mathbf{0})[\boldsymbol{\eta}^i] = \int_0^{l^i} v^i\,d\sigma - \int_0^{l^i} (\Phi_*^i, T_*^i)_{\mathbb{R}^2}\left\{v_\sigma^i + \left(1-\frac{\sigma}{l^i}\right)\kappa_*^i\tau^i\right\}d\sigma - \int_0^{l^i} (\Phi_*^i, N_*^i)_{\mathbb{R}^2}\left(\kappa_*^i v^i + \frac{1}{l^i}\tau^i\right)d\sigma.$$

By means of the integration by parts and $\Phi_*^i(0) = (0,0)$, we are led to

$$\delta F^i(\mathbf{0})[\boldsymbol{\eta}^i] = 2\int_0^{l^i} v^i\,d\sigma - (\Phi_*^i(l^i), T_*^i(l^i))_{\mathbb{R}^2} v^i(l^i). \tag{21}$$

Let us derive $\delta G^{ij}(0,0)[v^i,v^j]$. Since $Q(S^i(\rho^i(\,\cdot\,;\varepsilon))) = \Psi_{\partial\Omega}^i(\rho^i(\,\cdot\,;\varepsilon))|_{\sigma=l^i}$, $\dot{Q}(S^i(0)) = -N_*^i(l^i)$, and $\{\Psi_{\partial\Omega}^i(q)\}'|_{q=0} = N_*^i(l^i)$, we easily see $(S^i)'(0)v^i = -v^i(l^i)$. Also, note that $Q(S^i(0)) = \Phi_*^i(l^i)$ and $N_{\partial\Omega}(S^i(0)) = T_*^i(l^i)$. Then these imply that

$$\begin{aligned}
&\delta G^{ij}(0,0)[v^i,v^j]\\
&= (Q(S^i(0)), N_{\partial\Omega}(S^i(0)))_{\mathbb{R}^2}(S^i)'(0)v^i - (Q(S^j(0)), N_{\partial\Omega}(S^j(0)))_{\mathbb{R}^2}(S^j)'(0)v^j\\
&= -(\Phi_*^i(l^i), T_*^i(l^i))_{\mathbb{R}^2} v^i(l^i) + (\Phi_*^j(l^j), T_*^j(l^j))_{\mathbb{R}^2} v^j(l^j).
\end{aligned} \tag{22}$$

Thus, by (20), (21), and (22), we have the desired result. ■

Then it follows from Lemma 7 that if the variation preserves areas, they satisfy

$$\int_0^{l^1} v^1\,d\sigma = \int_0^{l^2} v^2\,d\sigma = \int_0^{l^3} v^3\,d\sigma.$$

LEMMA 8 (The second variation of A_Γ^{ij}). *Let $\boldsymbol{u}_k^{ij}(\varepsilon) = (\boldsymbol{u}_k^i(\varepsilon_k), \boldsymbol{u}_k^j(\varepsilon_k))$ ($k=1,2$) with $\boldsymbol{u}_k^i, \boldsymbol{u}_k^j \in \mathcal{M}$ be a variation with the variation vector field $\boldsymbol{\eta}_k^{ij} = (\boldsymbol{\eta}_k^i, \boldsymbol{\eta}_k^j)$ where $\boldsymbol{\eta}_k^i$ and $\boldsymbol{\eta}_k^j$ satisfy (14). Then*

$$\begin{aligned}
&\delta^2 A_\Gamma^{ij}(\mathbf{0})[\boldsymbol{\eta}_1^{ij}, \boldsymbol{\eta}_2^{ij}]\\
&= 2\int_0^{l^i} \kappa_*^i v_1^i v_2^i\,d\sigma + v_1^i\tau_2^i\big|_{\sigma=0} + \tau_1^i v_2^i\big|_{\sigma=0} + 2\int_0^{l^i}\frac{1}{l^i}(v_1^i\tau_2^i + \tau_1^i v_2^i)\,d\sigma\\
&\quad -2\int_0^{l^j} \kappa_*^j v_1^j v_2^j\,d\sigma - v_1^j\tau_2^j\big|_{\sigma=0} - \tau_1^j v_2^j\big|_{\sigma=0} - 2\int_0^{l^j}\frac{1}{l^j}(v_1^j\tau_2^j + \tau_1^j v_2^j)\,d\sigma.
\end{aligned}$$

Proof. Let us first derive $\delta^2 F^i(\mathbf{0})[\boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i]$. Using Lemma 1, (17), and

$$\delta N^i(\mathbf{0})[\boldsymbol{\eta}_k^i] = -\left\{v_{k,\sigma}^i + \left(1-\frac{\sigma}{l^i}\right)\kappa_*^i\tau_k^i\right\}T_*^i \quad (k=1,2), \tag{23}$$

we obtain

$$\begin{aligned}
&\delta^2 F^i(\mathbf{0})[\boldsymbol{\eta}_1^i, \boldsymbol{\eta}_2^i]\\
&= -2\int_0^{l^i} \kappa_*^i v_1^i v_2^i\,d\sigma - \int_0^{l^i}\frac{1}{l^i}(v_1^i\tau_2^i + \tau_1^i v_2^i)\,d\sigma - \int_0^{l^i}\left(1-\frac{\sigma}{l^i}\right)^2\kappa_*^i\tau_1^i\tau_2^i\,d\sigma
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^{l^i}\left(1-\frac{\sigma}{l^i}\right)\tau_1^i v_{2,\sigma}^i\,d\sigma-\int_0^{l^i}\left(1-\frac{\sigma}{l^i}\right)v_{1,\sigma}^i\tau_2^i\,d\sigma\\
&+\int_0^{l^i}(\Phi_*^i,\delta N^i(\mathbf{0})[\boldsymbol{\eta}_1^i])_{\mathbb{R}^2}\,\delta J^i(\mathbf{0})[\boldsymbol{\eta}_2^i]\,d\sigma+\int_0^{l^i}(\Phi_*^i,\delta N^i(\mathbf{0})[\boldsymbol{\eta}_2^i])_{\mathbb{R}^2}\,\delta J^i(\mathbf{0})[\boldsymbol{\eta}_1^i]\,d\sigma\\
&+\int_0^{l^i}(\Phi_*^i,\delta^2 N^i(\mathbf{0})[\boldsymbol{\eta}_1^i,\boldsymbol{\eta}_2^i])_{\mathbb{R}^2}\,d\sigma+\int_0^{l^i}(\Phi_*^i,N_*^i)_{\mathbb{R}^2}\,\delta^2 J^i(\mathbf{0})[\boldsymbol{\eta}_1^i,\boldsymbol{\eta}_2^i]\,d\sigma.
\end{aligned}
$$

Then, by integration by parts, $\Phi_*^i(0)=(0,0)$, $\xi_{qq}^i(l^i,0,0)=h_*^i$, (17), (18), (23), and

$$
\begin{aligned}
&\delta^2 N^i(\mathbf{0})[\boldsymbol{\eta}_1^i,\boldsymbol{\eta}_2^i]\\
&=-\left\{v_{1,\sigma}^i v_{2,\sigma}^i+\left(1-\frac{\sigma}{l^i}\right)^2(\kappa_*^i)^2\tau_1^i\tau_2^i+\left(1-\frac{\sigma}{l^i}\right)\kappa_*^i(v_{1,\sigma}^i\tau_2^i+\tau_1^i v_{2,\sigma}^i)\right\}N_*^i\\
&\quad-\left\{\xi_{qq}^i\kappa_*^i v_1^i v_2^i+\left(\kappa_*^i v_1^i+\frac{1}{l^i}\tau_1^i\right)v_{2,\sigma}^i+\left(\kappa_*^i v_2^i+\frac{1}{l^i}\tau_2^i\right)v_{1,\sigma}^i\right\}T_*^i,
\end{aligned}
$$

we are led to

$$
\begin{aligned}
&\delta^2 F^i(\mathbf{0})[\boldsymbol{\eta}_1^i,\boldsymbol{\eta}_2^i]\\
&=-2\int_0^{l^i}\kappa_*^i v_1^i v_2^i\,d\sigma-2\int_0^{l^i}\frac{1}{l^i}(v_1^i\tau_2^i+\tau_1^i v_2^i)\,d\sigma-v_1^i\tau_2^i\big|_{\sigma=0}-\tau_1^i v_2^i\big|_{\sigma=0}\\
&\quad+h_*^i(\Phi_*^i(l^i),N_*^i(l^i))_{\mathbb{R}^2}v_1^i(l^i)v_2^i(l^i).
\end{aligned}\tag{24}
$$

Let us derive $\delta^2 G^i(0,0,0,0)[v_1^i,v_1^j,v_2^i,v_2^j]$. Recalling $Q(S^i(\rho^i(\,\cdot\,;\varepsilon)))=\Psi_{\partial\Omega}^i(\rho^i(\,\cdot\,;\varepsilon))|_{\sigma=l^i}$ and computing the second variation of it, we have

$$
\begin{aligned}
&\ddot{Q}(S^i(0))\{(S^i)'(0)v_1^i\}\{(S^i)'(0)v_2^i\}+\dot{Q}(S^i(0))\{(S^i)''(0)v_1^i v_2^i\}\\
&=\{\Psi_{\partial\Omega}^i(q)\}''|_{q=0}\,v_1^i(l^i)v_2^i(l^i).
\end{aligned}
$$

Since $(S^i)'(0)v_k^i=-v_k^i(l^i)$, $\dot{Q}(S^i(0))=-N_*^i(l^i)$, and

$$
\ddot{Q}(S^i(0))=\kappa_{\partial\Omega}(S^i(0))N_{\partial\Omega}(S^i(0))=h_*^i T_*^i(l^i)=\{\Psi_{\partial\Omega}^i(q)\}''|_{q=0},
$$

we obtain $(S^i)''(0)v_1^i v_2^i=0$. Then it follows that

$$
\begin{aligned}
&\delta^2 G^{ij}(0,0,0,0)[v_1^i,v_1^j,v_2^i,v_2^j]\\
&=-\kappa_{\partial\Omega}(S^i(0))(Q(S^i(0)),T_{\partial\Omega}(S^i(0)))_{\mathbb{R}^2}\{(S^i)'(0)v_1^i\}\{(S^i)'(0)v_2^i\}\\
&\quad+(Q(S^i(0)),N_{\partial\Omega}(S^i(0)))_{\mathbb{R}^2}(S^i)''(0)v_1^i v_2^i\\
&\quad+\kappa_{\partial\Omega}(S^j(0))(Q(S^j(0)),T_{\partial\Omega}(S^j(0)))_{\mathbb{R}^2}\{(S^j)'(0)v_1^j\}\{(S^j)'(0)v_2^j\}\\
&\quad-(Q(S^j(0)),N_{\partial\Omega}(S^j(0)))_{\mathbb{R}^2}(S^j)''(0)v_1^j v_2^j\\
&=h_*^i(\Phi_*^i(l^i),N_*^i(l^i))_{\mathbb{R}^2}v_1^i(l^i)v_2^i(l^j)-h_*^j(\Phi_*^j(l^j),N_*^j(l^j))_{\mathbb{R}^2}v_1^j(l^j)v_2^j(l^j).
\end{aligned}\tag{25}
$$

Thus, by means of (24), (25), and $\mathrm{Area}_{D^{ij}}(\boldsymbol{u}^{ij})=-F^i(\boldsymbol{u}^i)+F^j(\boldsymbol{u}^j)+G^{ij}(\rho^i,\rho^j)$, we are led to the desired result. ■

If $\Gamma_*=\bigcup_{i=1}^3\Gamma_*^i$ is a extremal value of the energy functional under the area constraint, we have

$$
\delta E_\Gamma(\mathbf{0})[\boldsymbol{\eta}]+\lambda_1\delta A_\Gamma^{12}(\mathbf{0})[\boldsymbol{\eta}^{12}]+\lambda_2\delta A_\Gamma^{23}(\mathbf{0})[\boldsymbol{\eta}^{23}]=0,\tag{26}
$$

where $\lambda_1,\ \lambda_2 \in \mathbb{R}$. Then, by means of Lemma 5 and Lemma 7, we obtain

$$-\sum_{i=1}^{3} \gamma^i \int_0^{l^i} \kappa_*^i v^i \, d\sigma + \lambda_1 \Big\{ -2 \int_0^{l^1} v^1 \, d\sigma + 2 \int_0^{l^2} v^2 \, d\sigma \Big\}$$
$$+\lambda_2 \Big\{ -2 \int_0^{l^2} v^2 \, d\sigma + 2 \int_0^{l^3} v^3 \, d\sigma \Big\} = 0.$$

That is, it follows that

$$\int_0^{l^1} (-\gamma^1 \kappa_*^1 - 2\lambda_1) v^1 \, d\sigma + \int_0^{l^2} (-\gamma^2 \kappa_*^2 + 2\lambda_1 - 2\lambda_2) v^2 \, d\sigma$$
$$+ \int_0^{l^3} (-\gamma^3 \kappa_*^3 + 2\lambda_2) v^3 \, d\sigma = 0.$$

Since v^i $(i = 1, 2, 3)$ are arbitrary functions, we obtain

$$-\gamma^1 \kappa_*^1 - 2\lambda_1 = 0, \quad -\gamma^2 \kappa_*^2 + 2\lambda_1 - 2\lambda_2 = 0, \quad -\gamma^3 \kappa_*^3 + 2\lambda_2 = 0.$$

Recalling $\gamma^1 \kappa_*^1 + \gamma^2 \kappa_*^2 + \gamma^3 \kappa_*^3 = 0$, we see $\lambda_1 = -\gamma^1 \kappa_*^1 / 2$ and $\lambda_2 = \gamma^3 \kappa_*^3 / 2$.

Let us consider the second variation under (26). Set

$$\Xi_\Gamma(\boldsymbol{u}) := E_\Gamma(\boldsymbol{u}) - \frac{1}{2} \gamma^1 \kappa_*^1 A_\Gamma^{12}(\boldsymbol{u}^{12}) + \frac{1}{2} \gamma^3 \kappa_*^3 A_\Gamma^{23}(\boldsymbol{u}^{23}).$$

Then $\delta \Xi_\Gamma(\mathbf{0})[\boldsymbol{\eta}] = 0$. By means of Lemma 6, Lemma 8, and $\gamma^1 \kappa_*^1 + \gamma^2 \kappa_*^2 + \gamma^3 \kappa_*^3 = 0$, we have

$$\begin{aligned}
\delta^2 \Xi_\Gamma(\mathbf{0})[\boldsymbol{\eta}_1, \boldsymbol{\eta}_2] = & \sum_{i=1}^{3} \gamma^i \Big\{ \int_0^{l^i} v_{1,\sigma}^i v_{2,\sigma}^i \, d\sigma - (\kappa_*^i)^2 \int_0^{l^i} v_1^i v_2^i \, d\sigma + h_*^i v_1^i v_2^i \big|_{\sigma = l^i} \Big\} \\
& - \frac{1}{2} \gamma^1 \kappa_*^1 \big(v_1^1 \tau_2^1 \big|_{\sigma=0} + \tau_1^1 v_2^1 \big|_{\sigma=0} \big) - \frac{1}{2} \gamma^2 \kappa_*^2 \big(v_1^2 \tau_2^2 \big|_{\sigma=0} + \tau_1^2 v_2^2 \big|_{\sigma=0} \big) \\
& - \frac{1}{2} \gamma^3 \kappa_*^3 \big(v_1^3 \tau_2^3 \big|_{\sigma=0} + \tau_1^3 v_2^3 \big|_{\sigma=0} \big).
\end{aligned}$$

Using Lemma 4, we obtain

$$\begin{aligned}
\gamma^1 \kappa_*^1 \big\{ v_1^1(0) \tau_2^1 + \tau_1^1 v_2^1(0) \big\} = & \frac{\kappa_*^1}{s^1} \big\{ -2\gamma^2 c^2 v_1^2(0) v_2^2(0) + 2\gamma^3 c^3 v_1^3(0) v_2^3(0) \\
& + (\gamma^2 c^3 - \gamma^3 c^2)(v_1^2(0) v_2^3(0) + v_1^3(0) v_2^2(0)) \big\},
\end{aligned}$$

where $s^i = \sin \theta^i$ and $c^i = \cos \theta^i$. Here we see

$$\begin{aligned}
& v_1^2(0) v_2^3(0) + v_1^3(0) v_2^2(0) \\
& = \frac{1}{\gamma^2 \gamma^3} \big\{ (\gamma^1)^2 v_1^1(0) v_2^1(0) - (\gamma^2)^2 v_1^2(0) v_2^2(0) - (\gamma^3)^2 v_1^3(0) v_2^3(0) \big\}.
\end{aligned}$$

This implies that

$$\begin{aligned}
& \gamma^1 \kappa_*^1 \big\{ v_1^1(0) \tau_2^1 + \tau_1^1 v_2^1(0) \big\} \\
& = \kappa_*^1 \Big(\frac{c^3}{s^3} - \frac{c^2}{s^2} \Big) \gamma^1 v_1^1(0) v_2^1(0) + \frac{\kappa_*^1}{s^3} \gamma^2 v_1^2(0) v_2^2(0) - \frac{\kappa_*^1}{s^2} \gamma^3 v_1^3(0) v_2^3(0)
\end{aligned}$$

Applying a similar argument, we have

$$\begin{aligned}
&\gamma^2\kappa_*^2\{v_1^2(0)\tau_2^2+\tau_1^2v_2^2(0)\}\\
&=-\frac{\kappa_*^2}{s^3}\gamma^1v_1^1(0)v_2^1(0)+\kappa_*^2\left(\frac{c^1}{s^1}-\frac{c^3}{s^3}\right)\gamma^2v_2^2(0)v_2^2(0)+\frac{\kappa_*^2}{s^1}\gamma^3v_1^3(0)v_2^3(0),\\
&\gamma^3\kappa_*^3\{v_1^3(0)\tau_2^3+\tau_1^3v_2^3(0)\}\\
&=\frac{\kappa_*^3}{s^2}\gamma^1v_1^1(0)v_2^1(0)-\frac{\kappa_*^3}{s^1}\gamma^2v_1^2(0)v_2^2(0)+\kappa_*^3\left(\frac{c^2}{s^2}-\frac{c^1}{s^1}\right)\gamma^3v_2^3(0)v_2^3(0).
\end{aligned}$$

Then, using $\gamma^1\kappa_*^1+\gamma^2\kappa_*^2+\gamma^3\kappa_*^3=0$ and (5), we are led to

$$\kappa_*^i\left(\frac{c^k}{s^k}-\frac{c^j}{s^j}\right)-\frac{\kappa_*^j}{s^k}+\frac{\kappa_*^k}{s^j}=\frac{2}{s^i}(c^j\kappa_*^j-c^k\kappa_*^k)$$

for $(i,j,k)=(1,2,3)$, $(2,3,1)$, $(3,1,2)$. This leads to

$$\begin{aligned}
\delta^2\Xi_\Gamma(\mathbf{0})[\boldsymbol{\eta}_1,\boldsymbol{\eta}_2]=&\sum_{i=1}^3\gamma^i\left\{\int_0^{l^i}v_{1,\sigma}^iv_{2,\sigma}^i\,d\sigma-(\kappa_*^i)^2\int_0^{l^i}v_1^iv_2^i\,d\sigma+h_*^iv_1^iv_2^i\big|_{\sigma=l^i}\right\}\\
&-\frac{\gamma^1}{s^1}(c^2\kappa_*^2-c^3\kappa_*^3)v_1^1v_2^1\big|_{\sigma=0}-\frac{\gamma^2}{s^2}(c^3\kappa_*^3-c^1\kappa_*^1)v_1^2v_2^2\big|_{\sigma=0}\\
&-\frac{\gamma^3}{s^3}(c^1\kappa_*^1-c^2\kappa_*^2)v_1^3v_2^3\big|_{\sigma=0}.
\end{aligned}$$

Remark 9. We remark that this kind of bilinear form also appears in the analysis of the double bubble, see [6] and [10].

4. Gradient flow structure and stability analysis.

This section is a survey of [4]. The details will appear in [4].

4.1. *Gradient flow structure.* Let us first introduce the linearized system for the nonlinear problem (8)-(10), which is the first variation of (8)-(10) around a stationary solution. Using Lemma 1 and the fact that

$$\begin{aligned}
&\Psi_{\sigma\sigma}^i(\sigma,0,0)=\kappa_*^iN_*^i(\sigma),\quad \Psi_{\sigma\sigma q}^i(\sigma,0,0)=-(\kappa_*^i)^2N_*^i(\sigma),\\
&\Psi_{\sigma\sigma\mu}^i(\sigma,0,0)=-\frac{2\kappa_*^i}{l^i}N_*^i(\sigma)-\left(1-\frac{\sigma}{l^i}\right)(\kappa_*^i)^2T_*^i(\sigma),
\end{aligned}$$

the linearization of (8) is represented as

$$v_t^i=-m^i\gamma^i\{v_{\sigma\sigma}^i+(\kappa_*^i)^2v^i\}_{\sigma\sigma} \tag{27}$$

for $\sigma\in(0,l^i)$ and $i=1,2,3$. To get (27), we apply a similar argument to [3, Lemma 3.2]. Further, we have

$$\gamma^1v^1+\gamma^2v^2+\gamma^3v^3=0, \tag{28}$$

and the linearizations of (9) are given by

$$\frac{1}{s^1}(c^2\kappa_*^2-c^3\kappa_*^3)v^1+v_\sigma^1=\frac{1}{s^2}(c^3\kappa_*^3-c^1\kappa_*^1)v^2+v_\sigma^2=\frac{1}{s^3}(c^1\kappa_*^1-c^2\kappa_*^2)v^3+v_\sigma^3, \tag{29}$$

$$\gamma^1\{v_{\sigma\sigma}^1+(\kappa_*^1)^2v^1\}+\gamma^2\{v_{\sigma\sigma}^2+(\kappa_*^2)^2v^2\}+\gamma^3\{v_{\sigma\sigma}^3+(\kappa_*^3)^2v^3\}=0, \tag{30}$$

$$m^1\gamma^1\{v_{\sigma\sigma}^1+(\kappa_*^1)^2v^1\}_\sigma=m^2\gamma^2\{v_{\sigma\sigma}^2+(\kappa_*^2)^2v^2\}_\sigma=m^3\gamma^3\{v_{\sigma\sigma}^3+(\kappa_*^3)^2v^3\}_\sigma \tag{31}$$

at $\sigma = 0$, and those of (10) are given by

$$v^i_\sigma + h^i_* v^i = 0, \tag{32}$$

$$m^i \gamma^i \{v^i_{\sigma\sigma} + (\kappa^i_*)^2 v^i\}_\sigma = 0 \tag{33}$$

at $\sigma = l^i$ for $i = 1, 2, 3$. (29) are derived from the angle conditions in (9) by applying the same argument as for (17) and also using Lemma 4 and (5). As to (30), (31), and (33), see [3, Lemma 3.2 and 3.3]. To get (32), we apply a similar argument to [3, Lemma 3.3].

Set $I[\boldsymbol{v}_1, \boldsymbol{v}_2] := \delta^2 \Xi_\Gamma(\mathbf{0})[\boldsymbol{\eta}_1, \boldsymbol{\eta}_2]$ where $\boldsymbol{v}_j = (v^1_j, v^2_j, v^3_j)$ $(j = 1, 2)$. Also, for $k \in \mathbb{N}$, set

$$\begin{aligned}
&\mathcal{H}^k := H^k(0, l^1) \times H^k(0, l^2) \times H^k(0, l^3),\\
&(\mathcal{H}^k)' := (H^k(0, l^1))' \times (H^k(0, l^2))' \times (H^k(0, l^3))',\\
&\mathcal{E} := \Big\{(v^1, v^2, v^3) \in \mathcal{H}^1 \,\Big|\, \gamma^1 v^1 + \gamma^2 v^2 + \gamma^3 v^3 = 0 \ \text{at} \ \sigma = 0,\\
&\qquad\qquad\qquad \int_0^{l^1} v^1\, d\sigma = \int_0^{l^2} v^2\, d\sigma = \int_0^{l^3} v^3\, d\sigma\Big\},\\
&\mathcal{X} := \big\{(w^1, w^2, w^3) \in (\mathcal{H}^1)' \,|\, \langle w^1, 1\rangle = \langle w^2, 1\rangle = \langle w^3, 1\rangle\big\},
\end{aligned}$$

where $H^k(0, l^i)$ is Sobolev space and $\langle \cdot, \cdot \rangle$ is the duality pairing between $(H^1(0, l^i))'$ and $H^1(0, l^i)$. Note that we need $\langle w^1, 1\rangle = \langle w^2, 1\rangle = \langle w^3, 1\rangle$ in $\mathcal{X}$ to analyze the linearized system (27)-(33) since the original geometric problem (2)-(4) has area-preserving property. In addition, we define the inner product as

$$(\boldsymbol{v}_1, \boldsymbol{v}_2)_{-1} := \sum_{i=1}^{3} (v^i_1, v^i_2)_{-1} = \sum_{i=1}^{3} m^i \int_0^{l^i} \partial_\sigma u_{v^i_1} \partial_\sigma u_{v^i_2}\, d\sigma, \tag{34}$$

where $(u_{v^1_j}, u_{v^2_j}, u_{v^3_j})$ for a given $\boldsymbol{v}_j = (v^1_j, v^2_j, v^3_j) \in \mathcal{X}$ is a weak solution of

$$\begin{cases}
-m^i \partial^2_\sigma u_{v^i_j} = v^i_j \ \text{for} \ \sigma \in (0, l^i),\\
u_{v^1_j} + u_{v^2_j} + u_{v^3_j} = 0 \ \text{at} \ \sigma = 0,\\
m^1 \partial_\sigma u_{v^1_j} = m^2 \partial_\sigma u_{v^2_j} = m^3 \partial_\sigma u_{v^3_j} \ \text{at} \ \sigma = 0,\\
\partial_\sigma u_{v^i_j} = 0 \ \text{at} \ \sigma = l^i.
\end{cases}$$

Then we obtain the following proposition which ensures that the linearized system has the gradient flow structure.

Proposition 10. *Let $\boldsymbol{w} = (w^1, w^2, w^3) \in \mathcal{X}$ be given. Then $\boldsymbol{v} = (v^1, v^2, v^3) \in \mathcal{H}^3$ with*

$$\int_0^{l^1} v^1\, d\sigma = \int_0^{l^2} v^2\, d\sigma = \int_0^{l^3} v^3\, d\sigma$$

is a weak solution of

$$w^i = -m^i \gamma^i \{v^i_{\sigma\sigma} + (\kappa^i_*)^2 v^i\}_{\sigma\sigma}$$

with the boundary conditions (28)-(33) if and only if

$$(\boldsymbol{w}, \boldsymbol{\varphi})_{-1} = -I[\boldsymbol{v}, \boldsymbol{\varphi}]$$

holds for all $\boldsymbol{\varphi} = (\varphi^1, \varphi^2, \varphi^3) \in \mathcal{E}$.

Proof. Let $\boldsymbol{v}$ be a weak solution of the linearized system. Setting

$$\zeta^i = \gamma^i\{v^i_{\sigma\sigma} + (\kappa^i_*)^2 v^i\},$$

we derive

$$\begin{aligned}
\sum_{i=1}^3 (w^i,\varphi^i)_{-1} &= \sum_{i=1}^3 m^i \int_0^{l^i} \partial_\sigma u_{w^i}\partial_\sigma u_{\varphi^i}\,d\sigma = \sum_{i=1}^3 \langle w^i, u_{\varphi^i}\rangle = \sum_{i=1}^3 m^i \int_0^{l^i} \partial_\sigma \zeta^i \partial_\sigma u_{\varphi^i}\,d\sigma \\
&= \sum_{i=1}^3 \int_0^{l^i} \zeta^i\varphi^i\,d\sigma = \sum_{i=1}^3 \gamma^i \int_0^{l^i} \{v^i_{\sigma\sigma} + (\kappa^i_*)^2 v^i\}\varphi^i\,d\sigma \\
&= \sum_{i=1}^3 \gamma^i \int_0^{l^i} v^i_{\sigma\sigma}\varphi^i\,d\sigma + \sum_{i=1}^3 \gamma^i(\kappa^i_*)^2 \int_0^{l^i} v^i\varphi^i\,d\sigma \\
&= \sum_{i=1}^3 \gamma^i \Big[v^i_\sigma\varphi^i\Big]_{\sigma=0}^{\sigma=l^i} - \sum_{i=1}^3 \gamma^i \int_0^{l^i} v^i_\sigma\varphi^i_\sigma + \sum_{i=1}^3 \gamma^i(\kappa^i_*)^2 \int_0^{l^i} v^i\varphi^i\,d\sigma.
\end{aligned}$$

Using $\gamma^1\varphi^1 + \gamma^2\varphi^2 + \gamma^3\varphi^3 = 0$ at $\sigma = 0$, (29), and (32), we are led to the desired result. ∎

4.2. *Stability analysis.* Let us study the spectrum for the linearized system (27)-(33). Set

$$\begin{aligned}
\mathcal{D}(\mathcal{A}) =& \big\{(v^1,v^2,v^3) \in \mathcal{H}^3 \,\big|\, (v^1,v^2,v^3) \text{ satisfy (28)-(30), (32), and} \\
& \int_0^{l^1} v^1\,d\sigma = \int_0^{l^2} v^2\,d\sigma = \int_0^{l^3} v^3\,d\sigma\big\}.
\end{aligned}$$

Then the linearized operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{X}$ is given by

$$\langle \mathcal{A}\boldsymbol{v}, \boldsymbol{\xi}\rangle = \sum_{i=1}^3 m^i \int_0^{l^i} \big[\gamma^i\big\{v^i_{\sigma\sigma} + (\kappa^i_*)^2 v^i\big\}\big]_\sigma \xi^i_\sigma\,d\sigma$$

for all $\boldsymbol{\xi} \in \{(\xi^1,\xi^2,\xi^3) \in \mathcal{H}^1 \,|\, \xi^1+\xi^2+\xi^3 = 0\}$. Then, using Proposition 10, we obtain for all $\boldsymbol{\varphi} \in \mathcal{E}$

$$(\mathcal{A}\boldsymbol{v}, \boldsymbol{\varphi})_{-1} = -I[\boldsymbol{v},\boldsymbol{\varphi}].$$

For this operator $\mathcal{A}$, we have the following proposition.

Proposition 11. *The operator $\mathcal{A}$ satisfies the following:*

(i) The operator $\mathcal{A}$ is self-adjoint with respect to the inner product $(\cdot,\cdot)_{-1}$.
(ii) The spectrum of $\mathcal{A}$ contains a countable system of eigenvalues.
(iii) The initial value problem (27)-(33) is solvable for a initial data in $\mathcal{X}$.
(iv) The zero solution is an asymptotically stable solution of (27)-(33) if and only if the largest eigenvalue of $\mathcal{A}$ is negative.

Remark 12. The proof of Proposition 11 will appear in [4].

To establish the linearized stability, the following lemma is helpful.

Lemma 13. *Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$ be the eigenvalues of $\mathcal{A}$ (taking the multiplicity into account).*

(i) *For all* $n \in \mathbb{N}$

$$\lambda_n = -\inf_{\mathcal{W}\in\Sigma_{n-1}} \sup_{\boldsymbol{v}\in\mathcal{W}^\perp\setminus\{\mathbf{0}\}} \frac{I[\boldsymbol{v},\boldsymbol{v}]}{(\boldsymbol{v},\boldsymbol{v})_{-1}},$$

$$\lambda_n = -\sup_{\mathcal{W}\in\Sigma_{n-1}} \inf_{\boldsymbol{v}\in\mathcal{W}^\perp\setminus\{\mathbf{0}\}} \frac{I[\boldsymbol{v},\boldsymbol{v}]}{(\boldsymbol{v},\boldsymbol{v})_{-1}}.$$

Here Σ_n *is the collection of* n*-dimensional subspaces of* $\mathcal{E}$ *and* $\mathcal{W}^\perp$ *is the orthogonal complement of* $\mathcal{W}$ *with respect to the* H^{-1}*-inner product.*

(ii) *The eigenvalues depend continuously on* h_*^i, l^i, *and* κ_*^i. *Further, the eigenvalues are monotone decreasing in each of the parameters* h_*^i $(i = 1,2,3)$.

Proof. The lemma follows with the help of Courant's maximum-minimum principle together with the fact that I depends continuously on h_*^i, l^i, and κ_*^i, and is monotone with respect to h_*^i. The proof follows the lines of Courant and Hilbert [1, Chapter VI]. ∎

By means of Proposition 11 and Lemma 13, we have the following theorem.

THEOREM 14. *Let* $\Gamma_* = \bigcup_{i=1}^3 \Gamma_*^i$ *be the stationary solution of (2)-(4). Then, if there exists a constant* $c > 0$ *such that*

$$I[\boldsymbol{v},\boldsymbol{v}] \geq c\|\boldsymbol{v}\|_{-1}^2 \quad \text{for all } \boldsymbol{v} \in \mathcal{E}\setminus\{\mathbf{0}\},$$

the stationary solution Γ_* *is linearly stable.*

4.3. *Example.* Let us consider the stability of the stationary solution for one specific configuration. Assume that

$$\gamma^1 = \gamma^2 = \gamma^3 = 1, \quad l^1 = l^2 = l^3 = 1, \quad \kappa_*^1 = \kappa_*^2 = \kappa_*^3 = 0. \tag{35}$$

Then it follows from the first assumption of (35) and (5) that

$$\theta^1 = \theta^2 = \theta^3 = 120^\circ.$$

Also, the third assumption of (35) implies that all of Γ_*^i $(i = 1,2,3)$ are the line segments. Further, the assumptions (35) give the linearized system

$$v_t^i = -m^i v_{\sigma\sigma\sigma\sigma}^i \quad \text{for } \sigma \in (0,1),$$

$$v^1 + v^2 + v^3 = 0 \quad \text{at } \sigma = 0, \tag{36}$$

$$v_\sigma^1 = v_\sigma^2 = v_\sigma^3 \quad \text{at } \sigma = 0, \tag{37}$$

$$v_{\sigma\sigma}^1 + v_{\sigma\sigma}^2 + v_{\sigma\sigma}^3 = 0 \quad \text{at } \sigma = 0, \tag{38}$$

$$m^1 v_{\sigma\sigma\sigma}^1 = m^2 v_{\sigma\sigma\sigma}^2 = m^3 v_{\sigma\sigma\sigma}^3 \quad \text{at } \sigma = 0, \tag{39}$$

$$v_\sigma^i + h_*^i v^i = 0 \quad \text{at } \sigma = 1, \tag{40}$$

$$v_{\sigma\sigma\sigma}^i = 0 \quad \text{at } \sigma = 1, \tag{41}$$

and the bilinear form

$$I[\boldsymbol{v},\boldsymbol{v}] = \sum_{i=1}^3 \left\{ \int_0^1 (v_\sigma^i)^2\, d\sigma + h_*^i (v^i)^2\big|_{\sigma=1} \right\}.$$

The following lemma is needed in order to analyze the stability of $\Gamma_* = \bigcup_{i=1}^3 \Gamma_*^i$.

Lemma 15. *Assume (35).*

(i) The operator $\mathcal{A}$ has zero eigenvalues if and only if $\Lambda(h^1_*, h^2_*, h^3_*) = 0$, where

$$\Lambda(h^1_*, h^2_*, h^3_*) = 3h^1_*h^2_*h^3_* + 7(h^1_*h^2_* + h^2_*h^3_* + h^3_*h^1_*) + 15(h^1_* + h^2_* + h^3_*) + 27.$$

(ii) Set $\mathcal{S} = \{(h^1_*, h^2_*, h^3_*) \,|\, \Lambda(h^1_*, h^2_*, h^3_*) = 0\}$. The multiplicity of possible zero eigenvalues is equal to two if $(h^1_*, h^2_*, h^3_*) = (-3, -3, -3) \in \mathcal{S}$. Further, it is equal to one if $(h^1_*, h^2_*, h^3_*) \in \mathcal{S} \setminus \{(-3, -3, -3)\}$

Proof. Let us first prove (i). Assume that $-m^i v^i_{\sigma\sigma\sigma\sigma} = 0$. Then the functions v^i $(i = 1, 2, 3)$ can be denoted by $v^i(\sigma) = \alpha^i_3\sigma^3 + \alpha^i_2\sigma^2 + \alpha^i_1\sigma + \alpha^i_0$, where α^i_k are constants. The simple computation gives

$$v^i_\sigma(\sigma) = 3\alpha^i_3\sigma^2 + 2\alpha^i_2\sigma + \alpha^i_1, \quad v^i_{\sigma\sigma}(\sigma) = 6\alpha^i_3\sigma + 2\alpha^i_2, \quad v^i_{\sigma\sigma\sigma}(\sigma) = 6\alpha^i_3.$$

By means of (41), we have $\alpha^i_3 = 0$. This implies that

$$v^i(\sigma) = \alpha^i_2\sigma^2 + \alpha^i_1\sigma + \alpha^i_0, \quad v^i_\sigma(\sigma) = 2\alpha^i_2\sigma + \alpha^i_1, \quad v^i_{\sigma\sigma}(\sigma) = 2\alpha^i_2.$$

Using (36), (37), and (38), we are led to

$$\alpha^1_0 + \alpha^2_0 + \alpha^3_0 = 0, \quad \alpha^1_1 = \alpha^2_1 = \alpha^3_1, \quad \alpha^1_2 + \alpha^2_2 + \alpha^3_2 = 0. \tag{42}$$

Also, (40) gives

$$(2\alpha^i_2 + \alpha^i_1) + h^i_*(\alpha^i_2 + \alpha^i_1 + \alpha^i_0) = 0. \tag{43}$$

Further, by means of

$$\int_0^1 v^1\,d\sigma = \int_0^1 v^2\,d\sigma = \int_0^1 v^3\,d\sigma,$$

we obtain

$$\frac{1}{3}\alpha^1_2 + \frac{1}{2}\alpha^1_1 + \alpha^1_0 = \frac{1}{3}\alpha^2_2 + \frac{1}{2}\alpha^2_1 + \alpha^2_0 = \frac{1}{3}\alpha^3_2 + \frac{1}{2}\alpha^3_1 + \alpha^3_0. \tag{44}$$

Then the eigenvalue $\lambda = 0$ if and only if the equations (42)-(44) have a nontrivial solution $(\alpha^1_0, \alpha^2_0, \alpha^3_0, \alpha^1_1, \alpha^2_1, \alpha^3_1, \alpha^1_2, \alpha^2_2, \alpha^3_2) \neq \mathbf{0}$, which is equivalent to $\det[M(h^1_*, h^2_*, h^3_*)] = 0$, where $M(h^1_*, h^2_*, h^3_*)$ is the 9×9-matrix

$$\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
h^1_* & 0 & 0 & 1 + h^1_* & 0 & 0 & 2 + h^1_* & 0 & 0 \\
0 & h^2_* & 0 & 0 & 1 + h^2_* & 0 & 0 & 2 + h^2_* & 0 \\
0 & 0 & h^3_* & 0 & 0 & 1 + h^3_* & 0 & 0 & 2 + h^3_* \\
1 & -1 & 0 & \dfrac{1}{2} & -\dfrac{1}{2} & 0 & \dfrac{1}{3} & -\dfrac{1}{3} & 0 \\
1 & 0 & -1 & \dfrac{1}{2} & 0 & -\dfrac{1}{2} & \dfrac{1}{3} & 0 & -\dfrac{1}{3}
\end{bmatrix}.$$

Setting $\Lambda(h^1_*, h^2_*, h^3_*) := (-3/4) \cdot \det[M(h^1_*, h^2_*, h^3_*)]$, we are led to (i).

Let us prove (ii). By using MAPLE, we can derive

$$\operatorname{rank}[M(-3, -3, -3)] = 7.$$

This implies that the multiplicity of zero eigenvalues is equal to two, provided that $(h^1_*, h^2_*, h^3_*) = (-3,-3,-3) \in \mathcal{S}$. Also, we see that for $(h^1_*, h^2_*, h^3_*) \in \mathcal{S} \setminus \{(-3,-3,-3)\}$

$$\operatorname{rank}[M(h^1_*, h^2_*, h^3_*)] = 8,$$

which means that the multiplicity of zero eigenvalues is equal to one, provided that $(h^1_*, h^2_*, h^3_*) \in \mathcal{S} \setminus \{(-3,-3,-3)\}$. ■

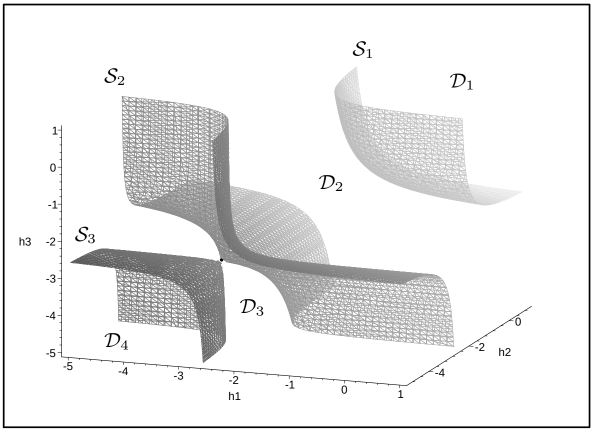

Fig. 3. $\mathcal{S} = \{(h^1_*, h^2_*, h^3_*) \mid \Lambda(h^1_*, h^2_*, h^3_*) = 0\} = \mathcal{S}^1 \cup \mathcal{S}^2 \cup \mathcal{S}^3$

Let us analyze the stability of Γ_*. Assume that $(h^1_*, h^2_*, h^3_*) = (0,0,0)$. Then

$$I[\boldsymbol{v}, \boldsymbol{v}] = \sum_{i=1}^{3} \int_0^1 (v^i_\sigma)^2 \, d\sigma \geq 0.$$

Since the maximal eigenvalue λ_1 allows the characterization

$$\lambda_1 = -\inf_{\boldsymbol{v} \in \mathcal{E}\setminus\{\boldsymbol{0}\}} \frac{I[\boldsymbol{v}, \boldsymbol{v}]}{(\boldsymbol{v}, \boldsymbol{v})_{-1}},$$

we have $\lambda_1 \leq 0$. On the other hand, it follows from Lemma 15(i) and $\Lambda(0,0,0) = 27 > 0$ that all of eigenvalues are not zero for $(h^1_*, h^2_*, h^3_*) = (0,0,0)$. Thus, in this case, we see $\lambda_1 < 0$. Hence, we have $I[\boldsymbol{v}, \boldsymbol{v}] \geq (-\lambda_1)\|\boldsymbol{v}\|^2_{-1}$ with $\lambda_1 < 0$ for $(h^1_*, h^2_*, h^3_*) = (0,0,0)$. That is, Γ_* is linearly stable. Further, by means of $(h^1_*, h^2_*, h^3_*) = (0,0,0) \in \mathcal{D}_1$ (see Fig. 3), Lemma 13, and Lemma 15, we are led to $\lambda_1 < 0$ as long as $(h^1_*, h^2_*, h^3_*) \in \mathcal{D}_1$. Thus Γ_* is linearly stable, provided that $(h^1_*, h^2_*, h^3_*) \in \mathcal{D}_1$. In addition, using Lemma 13 and Lemma 15, we obtain

$$\begin{aligned}
&N_U = 0,\ N_N = 0 \quad \text{if} \quad (h^1_*, h^2_*, h^3_*) \in \mathcal{D}_1,\\
&N_U = 0,\ N_N = 1 \quad \text{if} \quad (h^1_*, h^2_*, h^3_*) \in \mathcal{S}_1,\\
&N_U = 1,\ N_N = 0 \quad \text{if} \quad (h^1_*, h^2_*, h^3_*) \in \mathcal{D}_2,\\
&N_U = 1,\ N_N = 1 \quad \text{if} \quad (h^1_*, h^2_*, h^3_*) \in \mathcal{S}_2 \setminus \{(-3,-3,-3)\},\\
&N_U = 1,\ N_N = 2 \quad \text{if} \quad (h^1_*, h^2_*, h^3_*) = (-3,-3,-3) \in \mathcal{S}_2 \cap \mathcal{S}_3,\\
&N_U = 2,\ N_N = 0 \quad \text{if} \quad (h^1_*, h^2_*, h^3_*) \in \mathcal{D}_3,\\
&N_U = 2,\ N_N = 1 \quad \text{if} \quad (h^1_*, h^2_*, h^3_*) \in \mathcal{S}_3 \setminus \{(-3,-3,-3)\},\\
&N_U = 3,\ N_N = 0 \quad \text{if} \quad (h^1_*, h^2_*, h^3_*) \in \mathcal{D}_4,
\end{aligned}$$

where N_U is the number of the positive eigenvalues and N_N is the number of the zero eigenvalues. Consequently, we see that $\mathcal{S}_1$ is a criterion of the stability under the assumption (35).

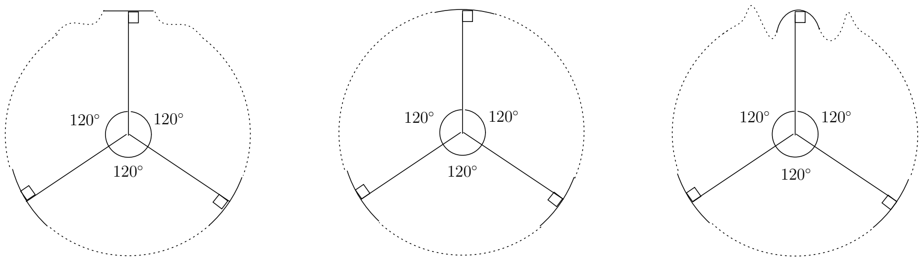

Fig. 4. [left] Stable. $(h^1_*, h^2_*, h^3_*) = (0, -1, -1) \in \mathcal{D}_1$. [middle] Neutral. $(h^1_*, h^2_*, h^3_*) = (-1, -1, -1) \in \mathcal{S}_1$. [right] Unstable. $h^1_* < -1$, $h^2_* = h^3_* = -1$.

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