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STABILITY AND INSTABILITY OF EQUILIBRIA ON SINGULAR DOMAINS

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Abstract. We show existence of nonconstant stable equilibria for the Neumann reaction-diffusion problem on domains with fractures inside. We also show that the stability properties of all hyperbolic equilibria remain unchanged under domain perturbation in a quite general sense, covered by the theory of Mosco convergence.

1. Introduction. We are going to study the behaviour of solutions to the Neumann boundary problem for the reaction–diffusion equation

$$u_t - \Delta u = g(u) \tag{1}$$

on domains which have a 'splitting' inside: see Figure 1.

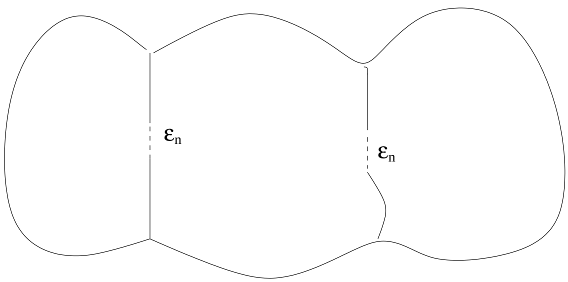

Fig. 1. Our model domain Ω_n

We are particularly interested in the steady (independent of time) solutions of (1) and in their stability, that is, roughly speaking, their being or not 'attractive' for other

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solutions in some neighborhood. This is expressed by the sign of eigenvalues of the linearized problem. As part of the attractor, stable and unstable steady states determine the evolution of other initial data.

On the other hand, it is a well known fact that the domain's shape influences strongly the solutions' properties, and in particular stability. And so, first of all, for Neumann boundary conditions and *convex* regions Ω, the only stable solutions are constants (given by the zeros of g). This was proved in 1978 by Casten and Holland [CH] and independently in 1979 by Matano [M]. It is shown in [CH] that this crucial property holds for a larger class of domains including annuli, and for all domains provided g is convex. On the other hand, Matano [M] constructs an example of a connected region — of quite complex shape — for which there exist non-constant stable equilibria; in other words, a domain for which 'pattern formation' occurs with the reaction–diffusion equation.

Can this be expected for simpler shapes? Many works (Hale and Vegas [HV], Vegas [V], Jimbo [J1, J2], Jimbo and Morita [JM, MJ], de Oliveira et al. [OPP], Arrieta et al. [AHH] and references therein) addressed this question for dumbbell domains: two bigger regions connected by a thin strip. The answer, under various assumptions, is positive: non-constant stable equilibria exist. We want to ask here if the same occurs for regions as in Figure 1, which we refer to as split domains.

The method we propose is different from the approaches of works cited above and has already been used in a special dumbbell type case in our previous work [GV]. The main idea is to consider the problem posed on a disconnected open set Ω as the limit case for a sequence of perturbed problems on $\Omega_n = \Omega(\varepsilon_n)$, and to use the notion of convergence in the sense of Mosco, which is actually an application of Mosco convergence as given in [Mo]; see e.g. Dal Maso et al. [DMa] and references therein for the same approach. Its advantage is to be *equivalent* to convergence of solutions to the stationary Neumann problem on Ω_n to the solution on Ω, and so it seems the most appropriate and general approach to our problem. It also gives dimension independence to our results.

Let us thus list our main assumptions that will allow us to handle split domains. For the perturbation (Ω_n) of Ω we assume that:

(C_1) Ω is an open set, $(\Omega_n)_{n\in\mathbb{N}}$ a sequence of open sets and D a ball in $\mathbb{R}^n$ such that

$$\forall n \in \mathbb{N}, \quad \Omega \subset \Omega_{n+1} \subset \Omega_n \subset D;$$

(C_2) $\forall n \in \mathbb{N}, |\Omega_n| = |\Omega|$;

(C_3) $\partial\Omega$ is Lipschitz;

(C_4) (Ω_n) converge in the sense of Mosco to Ω.

We also assume that g is a $C^1(\mathbb{R})$ function satisfying

(G) $\displaystyle\limsup_{|x|\to+\infty} \frac{g(x)}{x} < 0.$

The geometrical sense of (C_1)–(C_2) is clear. (C_3) ensures compactness of the injection from $H^1(\Omega)$ into $L^2(\Omega)$, see for example [Ne, Chapter 1, Theorem 1.4]; this is essential in the proofs. Let us comment on (C_4). The definition of Mosco convergence is introduced rigorously in Section 2. It is important to note here, see for instance Henrot [H1], that it occurs in particular if the capacity — by which we mean the 2-capacity — of $\Omega_n \setminus \Omega$ converges to 0. (As for the notion and properties of capacity, as a tool for measuring very fine sets, we refer to Evans and Gariepy [EG, Chapter 4] and note here only that all sets in $\mathbb{R}^N$ of Hausdorff dimension greater than $N-2$ are of non-zero capacity). In our case, one can see that (C_4) is satisfied if Ω is composed of some number of connected components, each at zero distance from the next one, and Ω_n is obtained by 'making holes' in the joining parts of the boundary, under the condition that the number of holes does not grow too rapidly and if their size decreases, see also Damlamian [Dam]. For most applications it is sufficient to assume that the number of holes remains constant.

With these assumptions, we show that that any hyperbolic, i.e. linearly stable or unstable steady state on Ω is a limit of a sequence $\{u_n\}$ of hyperbolic steady states on Ω_n, in the sense of $L^2(D)$ convergence (see (C_1)). Moreover, for n big enough, u_n has the same stability as u: the eigenvalues of the linearized operator $-\Delta - g'(u_n)$ converge to eigenvalues of the operator of $-\Delta - g'(u)$ (where Δ states for the Neumann–laplacian and the second term is the multiplication operator). We also have convergence of all respective eigenspaces, still in in the sense of $L^2(D)$ distance. This is proved in Theorems 3.1 and 3.2, by methods of the degree and operators' perturbation theories. In Theorem 3.1 we use the Leray–Schauder fixed point index and in Theorem 3.2, we infer the result from uniform convergence for the resolvent operators of the linearized equations.

Existence of non-constant stable equilibria on Ω_n, for big n, is an immediate consequence of Theorems 3.1 and 3.2. Indeed, take g with two stable zeros, like $g(u) = u(1-u^2)$ and put u to be equal to $+1$ and -1 on each connected component of Ω. This u forms a stable equilibrium which by our result has to be approached, in $L^2(D)$, by a sequence u_n of *stable* equilibria on Ω_n. What remains an open question, is the rate of this convergence: from what order of $\varepsilon(n)$ equilibria actually become stable.

In case when the system admits only hyperbolic equilibria, we show that their number is equal on Ω and on Ω_n. Theorem 3.6 states that in this case, the Hausdorff distance in $L^2(D)$ between the sets of stable steady points on Ω_n and the set of stable steady points on Ω is going to zero. The same is true for unstable equilibria. As a consequence, we get the equality between cardinalities of these sets.

Let us stress that (C_2) is important to our method of proving stability; we actually use it for getting convergence of resolvent operators in Theorem 3.2. Of course, this is not a necessary condition. However, the results of Arrieta et al. [AHH] let us think that this main theorem is false in general when $|\Omega_n| > |\Omega|$, for example for dumbbell domains, where we would need additional hypotheses. This will be the subject of our forthcoming paper.

2. Functional framework

2.1. *Main operator and the semigroup.* Let D and Ω be given by (C_1). For all $f \in L^2(\Omega)$, the linear equation

$$\begin{cases} -\Delta u + u = f & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2}$$

has a unique variational solution in $H^1(\Omega)$. Let us denote by A_Ω the Neumann–Laplacian operator equal to $-\Delta + I$ in its domain

$$D(A_\Omega) = \{u \in H^1(\Omega) : \ \exists f \in L^2(\Omega) \text{ such that } u \text{ is a solution of (2)}\}. \tag{3}$$

A_Ω has compact resolvent and its first eigenvalue is equal to 1. It follows that A_Ω is sectorial in the sense of Henry [H2], satisfying

$$\|(\lambda - A_\Omega)^{-1}\|_{\mathcal{L}(L^2(\Omega))} \leq \frac{2}{|\lambda - 1|}, \tag{4}$$

for all λ in the sector

$$\mathcal{S} = \{\lambda : \pi/3 \leq |arg(\lambda - 1)| \leq \pi, \lambda \neq 1\}. \tag{5}$$

The operator $-A_\Omega$ is then the infinitesimal generator of an analytic linear semigroup $\{S_\Omega(t)\}_{t\geq 0}$, where

$$S_\Omega(t) = \frac{1}{2i\pi} \int_\Gamma e^{\lambda t}(\lambda + A_\Omega)^{-1} d\lambda, \tag{6}$$

and Γ is a contour in the resolvent set $\rho(-A_\Omega)$, with $\arg\lambda \to \pm\theta$ as $|\lambda| \to \infty$ for some $\theta \in (\frac{\pi}{2}, \pi)$.

We set now

$$f = g + Id \tag{7}$$

and consider the original nonlinear equation (1), written as

$$\begin{cases} u_t + A_\Omega u &= f(u), \quad t > 0 \\ u(0) &= u_0. \end{cases} \tag{8}$$

The assumption (G) takes now the form

$$\limsup_{|x| \to +\infty} \frac{f(x)}{x} < 1. \tag{9}$$

By a solution of the problem (8) we understand a continuous function from $[0, +\infty)$ into $L^2(\Omega)$, satisfying on $(0, +\infty)$ the following integral equation

$$u(t) = S_\Omega(t)u_0 + \int_0^t S_\Omega(t-s) f(u(s)) ds. \tag{10}$$

Under conditions (9) and $f \in C^1(\mathbb{R})$, we know that for all $u_0 \in L^2(\Omega)$, there exists a unique solution of (8). Let $T_\Omega : \mathbb{R}^+ \times L^2(\Omega) \mapsto L^2(\Omega)$ be the nonlinear semigroup which to (t, u_0) associates the solution of the equation (8) at time t with the initial datum u_0.

The condition (9) implies that the equilibria are uniformly bounded in $L^\infty(D)$ and that solutions of the parabolic equation with initial conditions in $L^\infty(D)$ are also bounded in $L^\infty(D)$. This is summed up in the next proposition, which can be proved by standard methods:

PROPOSITION 2.1. *Let $SP(\Omega)$ be the set of stationary (time-independent) solutions of (1). There exists $K > 0$ such that*

$$\forall u \in SP(\Omega) \qquad \|u\|_{L^\infty(\Omega)} \leq K,$$
$$\forall t > 0, \qquad \|u(t)\|_{L^\infty(\Omega)} \leq \max\{K, \|u_0\|_{L^\infty(\Omega)}\}$$

where $u(t)$ is the solution of (8) with the initial condition u_0.

Proposition (2.1) implies in particular that if we consider uniformly bounded in $L^\infty(D)$ initial data only, we can assume that f is Lipschitz continuous.

Let K be the constant given in Proposition 2.1, we denote by C_f^K the smallest constant such that

$$|f(x)-f(y)| \le C_f^K |x-y| \ \ \forall x,y \in \overline{B}(0,K).$$

2.2. *Restriction and extension between D and $\Omega \subset D$.* As we are interested in perturbing the domain, we will need one large space in which solutions can be considered and compared. We choose the space $L^2(D)$ as the reference, D being given by (C_1). Let us denote by $\|.\|$ the norm in $\mathcal{L}(L^2(D))$ and fix the following operators of restriction and extension:

$$r_\Omega \in \mathcal{L}(L^2(D), L^2(\Omega)) \quad \text{and} \quad p_\Omega \in \mathcal{L}(L^2(\Omega)), L^2(D))$$

defined by

$$\begin{array}{ll} \forall u \in L^2(D), & r_\Omega(u) = u \quad \text{in} \quad \Omega, \\ \forall u \in L^2(\Omega), & p_\Omega(u) = u \quad \text{in} \quad \Omega, \quad p_\Omega(u) = 0 \in \Omega^c. \end{array} \tag{11}$$

Of course p_Ω is not continuous in $H^1(D)$. We note now that the operators $p_\Omega \circ (\lambda - A_\Omega)^{-1} \circ r_\Omega$ and $p_\Omega \circ S_\Omega(t) \circ r_\Omega$ belong to $\mathcal{L}(L^2(D))$ and the values of the norms remain unchanged. So, we will consider the resolvent operator $(\lambda - A_\Omega)^{-1}$ as an operator from $L^2(D)$ to $L^2(D)$. In what follows, we will also often omit the operators p_Ω and r_Ω. It is clear that the formula (6) remains valid when we consider it in the sense of composition with p_Ω and r_Ω.

Also, for simplicity, let us write A_n, S_n, T_n (respectively A, S) instead of A_{Ω_n}, S_{Ω_n} (resp. A_Ω, S_Ω); p_n instead of p_{Ω_n} and p instead of p_Ω.

2.3. *Mosco convergence.* We introduce now briefly the notion of Mosco convergence which is our main assumption (C_4). We mean by this (see also e.g. [DMa] for the same approach to Neumann perturbation problems) convergence in the sense of Mosco [Mo, Definition 1.1], of the following linear subspaces of $L^2(D)^{N+1}$

$$X_{\Omega_n} = \{(p_{\Omega_n}(u), p_{\Omega_n}(\nabla u)) \ : \ u \in L^2(\Omega_n)\}$$

to $X_\Omega = \{(p_\Omega(u), p_\Omega(\nabla u)) \ : \ u \in L^2(\Omega)\}$. (The operator of extension by zero p_Ω applied to a vector acts on each of its components). We specify this in the following

Definition 2.2. Let Ω be an open set and $(\Omega_n)_{n\in\mathbb{N}}$ a sequence of open sets. We say that $(\Omega_n)_{n\in\mathbb{N}}$ *converges in the sense of Mosco* to Ω if the following conditions (M_1) and (M_2) hold:

(M_1) if $u_n \in H^1(\Omega_n)$ are such that

$$p_{\Omega_n}(u_n) \xrightarrow{L^2(D)} v, \quad p_{\Omega_n}(\nabla u_n) \xrightarrow{[L^2(D)]^N} b,$$

then there exists $u \in H^1(\Omega)$ such that $v = p_\Omega(u)$ and $b = p_\Omega(\nabla u)$.

(M_2) for all $u \in H^1(\Omega)$, there exists $u_n \in H^1(\Omega_n)$ such that

$$p_{\Omega_n}(u_n) \xrightarrow{L^2(D)} p_\Omega(u), \quad p_{\Omega_n}(\nabla u_n) \xrightarrow{[L^2(D)]^N} p_\Omega(\nabla u).$$

It is known that under conditions (M_1) and (M_2), the solution to the linear problem (2) is continuous with respect to domain perturbations; the same was shown for semilinear case in [DMa].

3. Stability. In all what follows in this part, we consider a domain Ω and a sequence of domains $(\Omega_n)_{n\in\mathbb{N}}$ satisfying conditions (C_1, C_2, C_3, C_4). The point of our interest here will be the set, denoted by $SP(\Omega)$, of stationary points for the equation (10), i.e. the solutions of:

$$A_\Omega u = f(u). \tag{12}$$

3.1. *Notation.* For all $u \in SP(\Omega)$ and $k \in \mathbb{N}$ let $\lambda_k(A_\Omega - f'(u))$ be the k^{th} eigenvalue of the operator $A_\Omega - f'(u)$. We denote by e^k the eigenvector corresponding to the eigenvalue $\lambda_k(A_\Omega - f'(u))$ and W^k the subspace generated by the first k eigenvectors

$$W^k = span[e^1; \dots ; e^k].$$

In the same way, we denote by $W_n^k = span[e_n^1; \dots ; e_n^k]$ where e_n^k is the eigenvector corresponding to the eigenvalue $\lambda_k(A_{\Omega_n} - f'(u_n))$.

The following notation refers to the stability of steady states in the sense of linearization:

$$SP^+(\Omega) = \{u \in SP(\Omega) : \lambda_1(A_\Omega - f'(u)) > 0\},$$
$$SP^0(\Omega) = \{u \in SP(\Omega) : \ \exists k \in \mathbb{N},\ \lambda_k(A_\Omega - f'(u)) = 0\},$$
$$SP^-(\Omega) = \{u \in SP(\Omega) : \lambda_1(A_\Omega - f'(u)) < 0, \quad \forall k \in \mathbb{N},\ \lambda_k(A_\Omega - f'(u)) \neq 0\}.$$

The set $SP^+(\Omega) \cup SP^-(\Omega)$ is the subset of hyperbolic equilibria. It is known that each of them is isolated in $L^2(\Omega)$. It is known also that stable steady states are attractive in the sense of Liapunov, and the unstable ones are repulsive for almost all data in their neighborhood.

3.2. *Main theorems.* We state now continuity of the hyperbolic equilibrium point with respect to our domain perturbation.

THEOREM 3.1. *For all $u \in SP^+(\Omega) \cup SP^-(\Omega)$, there exists $(u_n)_{n\in\mathbb{N}}$ which converges to u in $L^2(D)$ and such that $u_n \in SP(\Omega_n)$ for all n.*

THEOREM 3.2. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence such that $u_n \in SP(\Omega_n)$ and which converges to u in $L^2(D)$. Then $u \in SP(\Omega)$ and for every $k \in \mathbb{N}$,*

$$\lambda_k(A_{\Omega_n} - f'(u_n)) \to \lambda_k(A_\Omega - f'(u)) \qquad \text{as } n \to \infty,$$
$$d_{L^2(D)}(W_n^k; W^k) \to 0 \qquad \text{as } n \to \infty.$$

Without loss of generality, with Proposition 2.1, we can suppose that f is Lipschitz continuous with $\operatorname{Lip} f = C_f^K$. Here again, we consider that the functions are extended by zero outside the open set in which they are naturally defined. To prove this result, we will use the Leray-Schauder Fixed-Point Index (see e.g. [Z, volume I, chapter 12]). We denote by R_A the resolvent operator which to v associates $A^{-1}(f(v))$, extended by zero outside Ω:

$$R_A \,:\, L^2(D) \to L^2(D), \qquad v \mapsto p_\Omega(A^{-1}(f(v))). \tag{13}$$

We define R_{A_n} in the same way. Note that only $v|_\Omega$ enter into the definition of $R_A(v)$, and so we can consider $L^2(\Omega)$ as the effective domain of R_A. We also have

$$u \in SP(\Omega) \Leftrightarrow R_A(u) = u.$$

It is clear that R_A is a compact operator such that for all $v \in L^2(D)$

$$\|R_A(v)\|_{H^1(\Omega)} \leq C_f^K \|v\|_{L^2(D)}.$$

The following result will be essential for the proof of Theorem 3.1.

LEMMA 3.3. *Suppose that (w_n) is a sequence in $L^2(D)$ such that $(f(w_n))$ converges weakly to h in $L^2(D)$. Then*

$$\lim_{n\to\infty} \|R_{A_n}(w_n) - A^{-1}(h)\|_{L^2(D)} = 0,$$
$$\lim_{n\to\infty} \|R_A(w_n) - A^{-1}(h)\|_{L^2(D)} = 0.$$

Proof. Let us prove the first claim. Let $u_n = R_{A_n}(w_n)$:

$$\int_{\Omega_n} \nabla u_n \nabla \phi_n + u_n \phi_n = \int_{\Omega_n} f(w_n)\phi_n \tag{14}$$

for all $\phi_n \in H^1(\Omega_n)$. It is easy to see that $(p_n(u_n), p_n(\nabla u_n))$ is bounded in $L^2(D)^{N+1}$. So, by (M_1),

$$(p_n(u_n), p_n(\nabla u_n)) \xrightarrow{[L^2(D)]^{N+1}} (p(u), p(\nabla u)).$$

Let (φ_n) and φ be given by (M_2) and take them as test functions in (14). We can now pass to the limit, obtaining $Au = h$ on Ω. So, (u_n) converges to $A^{-1}(h)$ weakly in $L^2(D)$. By (C_3), we obtain that (u_n) converges to $A^{-1}(h)$ strongly in $L^2(\Omega)$. With (C_2), this gives

$$\int_{\Omega_n} |u_n - A^{-1}(h)|^2 dx = \int_\Omega |u_n - A^{-1}(h)|^2 dx \to 0 \text{ as } n \to \infty. \tag{15}$$

This ends the proof of the first statement. The second one comes from the weak continuity of the operator A^{-1} and the compact injection of $H^1(\Omega)$ into $L^2(D)$, (C_3). ∎

REMARK 3.4. Lemma 3.3 remains true for domains which do not satisfy (C_2), but a more general condition $|\Omega_n \setminus \Omega| \to 0$. One should just replace (15) by

$$\lim_{n\to\infty} \int_{D\setminus\Omega} |u_n - A^{-1}(h)|^2 dx = 0.$$

This follows from boundedness of f, which gives uniform boundedness of $(u_n) = R_{A_n}(w_n)$ in $L^\infty(D)$.

Proof of Theorem 3.1. Let $u \in SP^+(\Omega) \cup SP^-(\Omega)$. Being hyperbolic, u is isolated, i.e. for ε small enough, u is the unique fixed point of R_A in $\overline{B}(u,\varepsilon)$, where $B(u,\varepsilon)$ denotes the ball in $L^2(D)$ of center u and radius ε. Let $i(.,.)$ be the Leray-Schauder Fixed-Point

Index, we have then $i(R_A, B(u,\varepsilon)) \neq 0$. Let $H_n : \overline{B}(u,\varepsilon) \times [0,1] \to L^2(D)$ be the map defined by

$$H_n(x,t) = tR_{A_n}(x) + (1-t)R_A(x). \tag{16}$$

Suppose that, for n large enough, H_n is a compact homotopy, then

$$i(R_{A_n}, B(u,\varepsilon)) = i(R_A, B(u,\varepsilon)) \neq 0.$$

This implies that there exists $u_n \in SP(\Omega_n) \cap B(u,\varepsilon)$. And by a direct application of Lemma 3.3 we get that the sequence (u_n) converges strongly to u in $L^2(D)$: this would end the proof.

In order to prove that H_n is a compact homotopy, we have to verify that H_n is compact and that $H_n(x,t) \neq x$ for all $(x,t) \in \partial B(u,\varepsilon) \times [0,1]$. Compactness of H_n follows from compactness of A_n^{-1} and A^{-1}. Suppose that there exists a sequence $(v_k, t_k) \in \partial B(u,\varepsilon) \times [0,1]$ such that $H_n(v_k,t_k) = v_k$. Note that for a subsequence, after renumbering, we can assume that $H_n(v_n,t_n) = v_n$. Let v be the weak limit in $L^2(D)$ of v_n, and h the weak limit of $f(v_n)$. Since $H_n(v_n,t_n) = v_n$, the sequence (v_n) converges strongly to $v \in \partial B(u,\varepsilon)$ in $L^2(D)$ and as f is Lipschitz continuous, $h = f(v)$. On the other hand, by Lemma 3.3, $(R_{A_n}(v_n))_n$ and $(R_A(v_n))_n$ converge strongly in $L^2(D)$ to $A^{-1}(h) = A^{-1}(f(v))$. So, by (16), $v = R_A(v)$. This contradicts the fact that u is the unique fixed point in $\overline{B}(u,\varepsilon)$. ■

Proof of Theorem 3.2. The fact that $u \in SP(\Omega)$ is obvious. The two convergences come directly (see [K, IV, 3.5]) from the convergence of the resolvent operators $((A_n + \lambda - f'(u_n))^{-1})$ to the resolvent operator $(A + \lambda - f'(u))^{-1}$ in $\mathcal{L}(L^2(D))$, for some real λ. Note that these operators have of course the same eigenvalues and eigenspaces as $((A_n - f'(u_n))^{-1})$, $(A - f'(u))^{-1}$. We will prove now convergence of these resolvents. We recall that f can be considered as being Lipschitz continuous with the Lipschitz constant C_f^K and take $\lambda > C_f^K - 1$. Let (h_n) be a sequence in $L^2(D)$ such that $\|h_n\|_{L^2(D)} \le 1$. We aim at showing that

$$\lim_{n\to\infty} \|(A_n + \lambda - f'(u_n))^{-1}(h_n) - (A + \lambda - f'(u))^{-1}(h_n)\|_{L^2(D)} = 0. \tag{17}$$

Up to a subsequence, h_n converges weakly in $L^2(D)$ to h. Let

$$v_n = ((A_n + \lambda - f'(u_n))^{-1})(h_n),$$

then

$$\int_{\Omega_n} |\nabla v_n|^2 + \int_{\Omega_n} v_n^2 + \lambda \int_{\Omega_n} v_n^2 + \int_{\Omega_n} f'(u_n) v_n^2 = \int_{\Omega_n} h_n v_n,$$

which gives

$$\|v_n\|^2_{H^1(\Omega_n)} \le 1 + \frac{1}{1 + \lambda - C_f^K}.$$

So, by (M_1) there exists $v \in H^1(\Omega)$ such that, up to a subsequence,

$$(p_n(\nabla v_n), p_n(v_n)) \xrightarrow{[L^2(D)]^{N+1}} (p(\nabla v), p(v)).$$

Also, by (C_3), v_n converge strongly to v in $L^2(\Omega)$. Let $\varphi \in H^1(\Omega)$ and let $\varphi_n \in H^1(\Omega_n)$ be given by (M_2). Take them as test functions in the equation defining v_n:

$$\int_{\Omega_n} \{\nabla v_n \nabla \varphi_n + (1+\lambda) v_n \varphi_n - f'(u_n) v_n \varphi_n\} dx = \int_{\Omega_n} h_n \varphi_n dx,$$

for all $n \in \mathbb{N}$. Passing to the limit, with boundedness of f', we obtain that $v = (A + \lambda - f'(u))^{-1}(h)$. We use now v_n as test function in the above equation. Note that, by (C_2) and (C_3),

$$\int_{\Omega_n} h_n v_n \, dx = \int_{\Omega} h_n v_n \, dx \to \int_{\Omega} hv \, dx.$$

Thus

$$\lim_{n\to\infty} \int_{\Omega_n} \{|\nabla v_n|^2 + (1+\lambda)|v_n|^2 - f'(u_n)|v_n|^2\}dx = \int_{\Omega} \{|\nabla v|^2 + (1+\lambda)|v|^2 - f'(u)|v|^2\}dx.$$

It follows that

$$\lim_{n\to\infty} \int_D \{|p_n(\nabla v_n) - p(\nabla v)|^2 + |p_n(v_n) - p(v)|^2\}dx = 0.$$

So,

$$\lim_{n\to\infty} \int_D |p_n(v_n) - p(v)|^2 dx = 0.$$

This means that $(A_n + \lambda - f'(u_n))^{-1}(h_n)$ converges to $(A + \lambda - f'(u))^{-1}(h)$ in $L^2(D)$. On the other hand, it is easy to see that $(A+\lambda-f'(u))^{-1}(h_n)$ converges to $(A+\lambda-f'(u))^{-1}(h)$ in $L^2(D)$ so that (17) is proved. This ends the proof. ■

Remark 3.5. We don't need (C_2) for Theorem 3.1, but it is crucial for the proof of Theorem 3.2.

In the final part of this section, we suppose that the flux in the domain Ω has no non-hyperbolic equilibrium. The structure of the set of equilibria in the network domain is then very similar to the one in the disconnected domain.

Theorem 3.6. *Suppose that $SP(\Omega)^0 = \emptyset$, then*

$$\lim_{n\to\infty} d^H(SP^+(\Omega_n), SP^+(\Omega)) = 0,$$
$$\lim_{n\to\infty} d^H(SP^-(\Omega_n), SP^-(\Omega)) = 0.$$

Here, d^H denotes the Hausdorff distance between sets in $L^2(D)$. The functions are considered, as usual, in $L^2(D)$ by extension by zero. As an immediate consequence of Theorem 3.6, we have the following corollary.

Corollary 3.7. *Suppose that $SP(\Omega)^0 = \emptyset$, then for n large enough*

$$\mathrm{card} SP^+(\Omega_n) = \mathrm{card} SP^+(\Omega),$$
$$\mathrm{card} SP^-(\Omega_n) = \mathrm{card} SP^-(\Omega).$$

Proof. The assumption (G), or (9) on f, implies that $SP(\Omega)$ is bounded in $H^1(\Omega)$ and then compact in $L^2(\Omega)$. So, if $SP(\Omega)$ is hyperbolic, the subset $SP^+(\Omega) \cup SP^-(\Omega)$ is finite. It is then clear, by Theorems 3.1 and 3.2, that

$$\lim_{n\to\infty} \sup_{u\in SP^+(\Omega)} \inf_{v\in SP^+(\Omega_n)} \|u - v\|_{L^2(D)} = 0.$$

On the other hand, let $u_n \in SP^+(\Omega_n)$. As $SP(\Omega_n)$ is uniformly bounded with respect to n in $H^1(\Omega_n)$, using the Mosco conditions, it is easy to see that, up to a subsequence, u_n

converge strongly to $u \in SP(\Omega)$ in $L^2(\Omega)$. Since $u_n \in L^\infty(\Omega_n)$, the convergence holds in $L^2(D)$. Using Theorem 3.2 again, we conclude that $u \in SP^+(\Omega)$ and then

$$\lim_{n\to\infty} \sup_{v\in SP^+(\Omega_n)} \inf_{u\in SP^+(\Omega)} \|u-v\|_{L^2(D)} = 0.$$

This ends the proof of the first claim. For the second, we use the same argument. Moreover, with the first Mosco condition, we have $\limsup \lambda(\Omega_n) \le \lambda(\Omega)$ so if (u_n) is a sequence in $SP^-(\Omega_n)$, each limit of a subsequence is in $SP^-(\Omega_n) \cup SP^0(\Omega_n)$, then in $SP(\Omega_n)$. ∎

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