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NONLINEAR EVOLUTION EQUATIONS GENERATED BY SUBDIFFERENTIALS WITH NONLOCAL CONSTRAINTS

RISEI KANO, YUSUKE MURASE

Department of Mathematics, Graduate School of Science and Technology
Chiba University, 1-33 Yayoi-cho, Inage-ku, Chiba, 263-8522 Japan
E-mail: risei@graduate.chiba-u.jp, ym@graduate.chiba-u.jp

NOBUYUKI KENMOCHI

Department of Mathematics, Faculty of Education, Chiba University
1-33 Yayoi-cho, Inage-ku, Chiba, 263-8522 Japan
E-mail: kenmochi@faculty.chiba-u.jp

Abstract. We consider an abstract formulation for a class of parabolic quasi-variational inequalities or quasi-linear PDEs, which are generated by subdifferentials of convex functions with various nonlocal constraints depending on the unknown functions. In this paper we specify a class of convex functions $\{\varphi^t(v;\cdot)\}$ on a real Hilbert space H, with parameters $0 \le t \le T$ and v in a set of functions from $[-\delta_0, T]$, $0 < \delta_0 < \infty$, into H, in order to formulate an evolution equation of the form

$$u'(t) + \partial\varphi^t(u; u(t)) \ni f(t), \quad 0 < t < T, \quad \text{in } H.$$

Our objective is to discuss the existence question for the associated Cauchy problem.

1. Introduction. For positive numbers $\delta_0,\ T$, we are given sets $V(-\delta_0, t),\ 0 \le t \le T$, of functions from $(-\delta_0, t)$ into a real Hilbert space H and a family $\{\varphi^s(v;\cdot)\}_{0\le s\le t}$ of proper, lower semicontinuous, convex functions $\varphi^s(v;\cdot)$ with parameters $s \in [0,t]$ and $v \in V(-\delta_0, t)$; here $\varphi^s(v;\cdot)$ continuously depends upon $v \in V(-\delta_0, t)$ in a certain nonlocal way (see section 2 for the detailed definition). We consider a nonlinear evolution equation of the form:

$$u'(t) + \partial\varphi^t(u; u(t)) \ni f(t), \quad 0 < t < T, \quad \text{in } H, \tag{1.1}$$

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subject to the initial condition

$$u(t) = u_0(t), \quad -\delta_0 \leq t \leq 0, \quad \text{in } H, \tag{1.2}$$

where $\partial\varphi^t(u;\cdot)$ is the subdifferential of convex function $\varphi^t(u;\cdot)$ on H, $u' = \frac{du}{dt}$ and $u_0 : [-\delta_0, 0] \to H$ and $f : (0,T) \to H$ are prescribed as the initial and forcing functions, respectively. This is a sort of functional differential equations generated by subdifferentials of $\varphi^t(v;\cdot)$ with a nonlocal dependence upon v. The objective of this paper is to specify a class of convex functions $\{\varphi^s(v;\cdot)\}_{0\leq s\leq t}$ as well as its nonlocal dependence upon $v \in V(-\delta_0, t)$ in order that Cauchy problem $\{(1.1),(1.2)\}$ admits at least one local or global in time solution u.

Variational problems are often called "quasi-variational problems", when the constraints depend upon the unknowns. The stationary cases have been dealt with in many papers, for instance, [2,5,10,13,14], but there are not so many papers dealing with the time-evolution problems, because it is not expected for solutions to have much regularity in time. We recall some papers [11,15,16] for time evolution quasi-variational inequalities. In papers [11,16], the so-called monotonicity property of the mapping $v \to \varphi^s(v;\cdot)$ is used as one of key tools in their treatment. However, the monotonicity property is too restrictive in many important applications, as examples of section 5 suggest. They evolved the theory of quasi-variational evolution equations with a concept of weak solutions. The main theorems (Theorems 3.1, 4.1 and 4.2) of this paper ensure the existence of strong solutions without assuming the monotonicity property of the mapping $v \to \varphi^s(v;\cdot)$. A similar attempt was made in the paper [15] for an evolution problem arising in the theory of semiconductors.

Also, it is important to investige the large time behaviour of solutions to (1.1). However, since the dynamical system associated with (1.1) is multivalued, in general, this is a completely new question and no result has been established yet.

The solvability of evolution equations of the form (1.1) seems delicate, as a simple example shows below. Let us consider a scalar evolution equation

$$u'(t) + \partial I_{[2u(t),\infty)}(u(t)) \ni 1, \quad 0 < t < T, \tag{1.3}$$

where $I_{[2v,\infty)}$ is the indicator function of the real interval $[2v,\infty)$ and $\partial I_{[2v,\infty)}$ is its subdifferential, namely

$$I_{[2v,\infty)}(z) := \begin{cases} 0, & \text{if } z \geq 2v, \\ \infty, & \text{if } z < 2v, \end{cases} \qquad \partial I_{[2v,\infty)}(z) := \begin{cases} 0, & \text{if } z > 2v, \\ (-\infty, 0], & \text{if } z = 2v, \\ \emptyset, & \text{if } z < 2v. \end{cases} \tag{1.4}$$

Any solution u of (1.3) satisfies $u(t) \geq 2u(t)$, hence $u(t) \leq 0$. Also, because of (1.4), $u'(t) \geq 1$, if (1.3) holds at time t. Therefore, when the initial condition is $u(0) = 0$, (1.3) has no solution, since $u(t) > 0$ for any $t > 0$. This is an example which shows that the Cauchy problem has no solution, even if the mapping $v \to \varphi^t(v;\cdot)$ is regular enough.

In this paper we shall specify a class of $\{\varphi^s(v;\cdot)\}$ of convex functions on H and give a nonlocal dependence of $\{\varphi^s(v;\cdot)\}$ upon v in order that the Cauchy problem (1.1)-(1.2) has a local in time solution or more restrictedly a global in time solution.

The solvability of problem (1.1)-(1.2) is based on that of evolution equations generated by subdifferentials of time dependent convex functions $\psi^t(\cdot)$ of the form:

$$u'(t) + \partial\psi^t(u(t)) \ni f(t), \quad 0 < t < T, \text{ in } H \tag{1.5}$$

subject to the initial condition $u(0) = u_0$. Therefore, prior to (1.1)-(1.2) we shall recall the important class of $\psi^t(\cdot)$ which guarantees the well-posedness of Cauchy problems for equation (1.5). The main part of this theory was developed in [3,7,8,17].

As a typical example of equation (1.1), we apply our theorems to the following system of inequalities:

$$\begin{aligned}
&u_t - \Delta u \geq f(x,t) \quad \text{in } Q := \Omega \times (0,T),\\
&u \geq k_c(u;\cdot,\cdot) \quad \text{in } Q,\\
&(u_t - \Delta u - f(x,t))(u - k_c(u;\cdot,\cdot)) = 0 \quad \text{in } Q,\\
&\frac{\partial u}{\partial n} \geq 0, \quad \frac{\partial u}{\partial n}(u - k_c(u;\cdot,\cdot)) = 0 \ \text{ on } \Sigma := \partial\Omega \times (0,T);
\end{aligned} \tag{1.6}$$

here Ω is a bounded smooth domain in $\mathbf{R}^N$, f is a given function on Q, k_c is a integral mapping of the form:

$$k_c(v;x,t) = \int_{-\delta_0}^{t}\int_\Omega \rho(x-y, t-s, v(y,s))dyds, \tag{1.7}$$

where ρ is a smooth function with respect to all the variables on $\mathbf{R}^N \times \mathbf{R} \times \mathbf{R}$. The above system (1.6) is reformulated as a parabolic variational inequality of the form:

$$\begin{aligned}
&u \in W^{1,2}(-\delta_0, T; L^2(\Omega)) \cap L^\infty(-\delta_0, T; H^1(\Omega)) \text{ with } u \geq k_c(u;\cdot,\cdot) \ \text{ a.e. in } Q;\\
&\int_Q \{u_t(u-w) + \nabla u \cdot \nabla(u-w)\}dxdt \leq \int_Q f(x,t)(u-w)dxdt,\\
&\qquad\qquad \forall w \in L^2(0,T;H^1(\Omega)) \text{ with } w \geq k_c(u;\cdot,\cdot) \ \text{ a.e. in } Q.
\end{aligned} \tag{1.8}$$

In the system (1.8) the constraint $k_c = k_c(u;\cdot,\cdot)$ depends upon the unknown u. Moreover, it is easy to check that (1.8) is written in the form (1.1) by using the following convex function $\varphi^s(v;\cdot)$ on $H := L^2(\Omega)$ given by

$$\varphi^s(v;z) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla z|^2 dx, & \text{if } z \in H^1(\Omega), \ z \geq k_c(v;\cdot,s) \text{ a.e. in } \Omega,\\ +\infty, & \text{otherwise.} \end{cases}$$

Thus equation (1.1) is an abstract formulation which includes a class of parabolic quasi-variational inequalities.

Notation and fundamental concepts. In general, for a given real Banach space X we denote by $|\cdot|_X$ the norm in X.

Throughout this paper, let H be a real Hilbert space with inner product $(\cdot,\cdot)_H$ and norm $|\cdot|_H$. Given a proper, lower semi-continuous (l.s.c.) and convex function $\psi(\cdot)$ on H we use the usual notation:

- $D(\psi) := \{z \in H; \psi(z) < \infty\}$ (effective domain).

- $\partial\psi$ is the subdifferential of ψ, which is a (multivalued) mapping in H defined by
$$z^* \in \partial\psi(z) \Leftrightarrow (z^*, v - z)_H \le \psi(v) - \psi(z), \quad \forall v \in H$$
with domain $D(\partial\psi) := \{z \in H; \partial\psi(z) \neq \emptyset\} (\subset D(\psi))$.

There is an important concept of convergence for convex functions, which was introduced by Mosco [12] in order to characterize the convergence of solutions to variational inequalities. Let $\{\psi_n\}$ be a sequence of proper l.s.c. and convex functions on H. Then it is said that ψ_n converges to a proper, l.s.c. and convex function ψ on H in the sense of Mosco, if the following two conditions (M1) and (M2) are fulfilled:

(M1) $\liminf_{n\to\infty} \psi_n(z) \ge \psi(z)$ for every $z \in H$.
(M2) For each $z \in D(\psi)$ there is a sequence $\{z_n\}$ in H such that $z_n \to z$ in H and $\psi_n(z_n) \to \psi(z)$ as $n \to \infty$.

For basic properties of convex functions we refer to monographs [1,4,9].

2. A class of time-dependent convex functions. Given a family $\{\psi^t\} := \{\psi^t\}_{0\le t\le T}$ of time-dependent proper, l.s.c. and convex functions ψ^t on H for a positive finite number T, let us consider an evolution equation generated by the subdifferential $\partial\psi^t$ in the following form:

$$(CP) \quad \begin{cases} u'(t) + \partial\psi^t(u(t)) \ni f(t), \quad 0 < t < T, \text{ in } H, \\ u(0) = u_0 \text{ in } H, \end{cases} \tag{2.1}$$

where f and u_0 are respectively prescribed in $L^2(0, T; H)$ and H. We say that u is a solution of (CP) on $[0, T]$, if $u \in C([0, T]; H) \cap W^{1,2}_{loc}((0, T]; H)$, $\psi^{(\cdot)}(u) \in L^1(0, T)$, $u(0) = u_0$ and $f(t) - u'(t) \in \partial\psi^t(u(t))$ holds for a.e. $t \in (0, T)$. When the data $\{\psi^t\}$, u_0, f are explicitly indicated, (CP) is denoted by $(CP; \{\psi^t\}, u_0, f)$.

Now, we specify a class of families $\{\psi^t\}$ of time-dependent convex functions on H so that problem (2.1) admits a solution. Let $\{a_r\} := \{a_r; 0 \le r < \infty\}$ and $\{b_r\} := \{b_r; 0 \le r < \infty\}$ be subsets consisting of non-negative functions in $L^2(0, T)$ and $L^1(0, T)$, respectively. Then we define the class $G(\{a_r\}, \{b_r\})$ of $\{\psi^t\}$ as follows.

Definition 2.1. We denote by $G(\{a_r\}, \{b_r\})$ the set of all families $\{\psi^t\} := \{\psi^t\}_{0\le t\le T}$ of proper (i.e. not identically ∞), l.s.c., non-negative and convex function $\psi^t(\cdot)$ on H satisfying that $\forall r > 0$, $\forall s, t \in [0, T]$ with $s \le t$ and $\forall z \in D(\psi^s)$ with $|z|_H \le r$, $\exists \tilde{z} \in D(\psi^t)$ such that

$$|\tilde{z} - z|_H \le \int_s^t a_r(\tau)d\tau(1 + \psi^s(z)^{\frac{1}{2}}), \quad \psi^t(\tilde{z}) - \psi^s(z) \le \int_s^t b_r(\tau)d\tau(1 + \psi^s(z)).$$

We may assume without loss of generality that a_r, b_r are non-decreasing with respect to $r > 0$, namely $a_{r_1} \ge a_{r_2}$, $b_{r_1} \ge b_{r_2}$ a.e. on $[0, T]$, if $r_1 > r_2$. If the time interval $[0, T]$ has to be indicated explicitly, we denote $G(\{a_r\}, \{b_r\})$ by $G_{[0,T]}(\{a_r\}, \{b_r\})$.

Furthermore, let $\{M_r\}_{0\le r<\infty}$ be a family of non-negative numbers. We then put

$$\mathcal{G}(\{M_r\}) = \bigcup_{|a_r|_{L^2(0,T)} \le M_r,\ |b_r|_{L^1(0,T)} \le M_r, 0 \le \forall r < \infty} G(\{a_r\}, \{b_r\}); \tag{2.2}$$

this is denoted by $\mathcal{G}_{[0,T]}(\{M_r\})$, when the interval $[0, T]$ is indicated explicitly.

We recall an existence-uniqueness result on (CP).

Theorem 2.1 (cf. [3,7,8,17]). *Assume that* $\{\psi^t\} \in G(\{a_r\},\{b_r\})$. *Let* $f \in L^2(0,T;H)$ *and* $u_0 \in \overline{D(\psi^0)}$. *Then,* (CP) *has one and only one solution* u *on* $[0,T]$ *such that*

$$\sqrt{t}u' \in L^2(0,T;H), \qquad \sup_{0<t\le T} t\psi^t(u(t)) < \infty.$$

Moreover, if $u_0 \in D(\psi^0)$, *then*

$$u' \in L^2(0,T;H), \qquad \sup_{0\le t\le T} \psi^t(u(t)) < \infty.$$

We recall briefly the proof of the above theorem, since the key ideas for the solvability of our quasi-variational evolution problem (1.1) are found there.

The construction of a solution of (2.1) is made by showing the convergence of the solutions u_λ of the following approximate problems, with real parameters $\lambda \in (0,1]$, as $\lambda \downarrow 0$:

$$u_\lambda'(t) + \partial\psi_\lambda^t(u_\lambda(t)) = f(t) \ \text{ in } H \text{ for a.e. } t \in [0,T], \tag{2.3}$$

with initial condition $u_\lambda(0) = u_0$, where ψ_λ^t is the Moreau-Yosida approximation, i.e.

$$\psi_\lambda^t(v) := \inf_{z\in H}\left\{\frac{1}{2\lambda}|v-z|_H^2 + \psi^t(z)\right\}, \quad \forall v \in H.$$

In order to get the uniform estimates for approximate solutions u_λ with respect to $\lambda \in (0,1]$ we derive the following key inequality from the time-dependence condition on $\psi^t(\cdot)$ mentioned in Definition 2.1:

$$\begin{aligned}&\frac{d}{dt}\psi_\lambda^t(u_\lambda(t)) - (\partial\psi_\lambda^t(u_\lambda(t)), u_\lambda'(t))_H \\ &\le a_r(t)|\partial\psi_\lambda^t(u_\lambda(t))|_H(\psi_\lambda^t(u_\lambda(t))^{\frac12}+1) + b_r(t)(\psi_\lambda^t(u_\lambda(t))+1), \quad \text{a.e. } t\in[0,T],\end{aligned} \tag{2.4}$$

for any $r > |u_\lambda|_{L^\infty(0,T;H)}$.

First of all, taking the inner product of the both sides of (2.3) and $u_\lambda(t) - h(t)$ for any function $h \in W^{1,2}(0,T;H)$ with $\psi^{(\cdot)}(h) \in L^1(0,T)$, we have

$$\begin{aligned}\frac{d}{dt}\left\{\frac12|u_\lambda(t)|_H^2 - (u_\lambda(t),h(t))_H\right\} &+ (u_\lambda(t), h'(t)-f(t))_H + \psi_\lambda^t(u_\lambda(t)) \\ &\le \psi_\lambda^t(h(t)) - (f(t),h(t))_H, \quad \text{for a.e. } t\in[0,T];\end{aligned} \tag{2.5}$$

note here that the existence of such a function h is also shown from our condition on the time-dependence of $\psi^t(\cdot)$, i.e., $\{\psi^t\} \in G(\{a_r\},\{b_r\})$, mentioned in Definition 2.1. Applying the Gronwall's lemma to (2.5) yields an inequality of the form

$$|u_\lambda|_{L^\infty(0,T;H)}^2 + \int_0^T \psi_\lambda^t(u_\lambda(t))dt \le R_1(|u_0|_H + |f|_{L^2(0,T;H)}), \qquad \forall\lambda\in(0,1], \tag{2.6}$$

where $R_1(\cdot)$ is a non-negative and non-decreasing function from $[0,\infty)$ into $[0,\infty)$ which depends only on the class $\mathcal{G}(\{M_r\})$.

Next, taking the inner product of the both sides of (2.3) and u_λ', we obtain that

$$|u_\lambda'(\tau)|_H^2 + (\partial\psi_\lambda^t(u_\lambda(\tau)), u_\lambda'(\tau))_H = (f(\tau), u_\lambda'(\tau))_H, \quad \text{for a.e. } \tau\in(0,T).$$

Using inequality (2.4) in the above relation, we see for any $r > R_1(|u_0|_H + |f|_{L^2(0,T;H)})$ that

$$\begin{aligned}&\frac{d}{d\tau}\psi_\lambda^\tau(u_\lambda(\tau)) + |u_\lambda'(\tau)|_H^2\\ &\qquad \le (f(\tau), u_\lambda'(\tau))_H + a_r(t)|\partial\psi_\lambda^t(u_\lambda(t))|_H(\psi_\lambda^t(u_\lambda(t))^{\frac{1}{2}} + 1) + b_r(t)(\psi_\lambda^t(u_\lambda(t)) + 1),\\ &\qquad\qquad \text{for a.e. } \tau \in (0,T),\ \forall \lambda \in (0,1],\end{aligned}$$

so that

$$\frac{d}{d\tau}\psi_\lambda^\tau(u_\lambda(\tau)) + \frac{1}{2}|u_\lambda'(\tau)|_H^2 \le k_r(\tau)(\psi_\lambda^\tau(u_\lambda(\tau)) + 1), \quad \text{a.e. } \tau \in (0,T),\ \forall \lambda \in (0,1], \tag{2.7}$$

where

$$k_r(\tau) := 6(|a_r(\tau)|^2 + |b_r(\tau)| + |f(\tau)|_H^2), \quad \text{a.e. } \tau \in [0,T]. \tag{2.8}$$

Further, multiplying both sides of (2.7) by τ, we get

$$\begin{gathered}\frac{d}{d\tau}\{\tau\psi_\lambda^\tau(u_\lambda(\tau))\} + \frac{\tau}{2}|u_\lambda'(\tau)|_H^2 \le (\tau k_r(\tau) + 1)(\psi_\lambda^\tau(u_\lambda(\tau)) + 1)\\ \text{a.e. } \tau \in (0,T),\ \forall \lambda \in (0,1],\end{gathered} \tag{2.9}$$

From these inequalities (2.6), (2.7) and (2.9) we see that $\{u_\lambda\}$ is bounded in $W^{1,2}(0,T;H)$ and $\{\psi_\lambda^t(u_\lambda)\}$ is bounded in $L^\infty(0,T)$, and moreover by the usual monotonicity argument that u_λ converges to the solution u of (CP) as $\lambda \downarrow 0$ in the sense that

$$\begin{gathered}u_\lambda \to u \ \text{ in } C([0,T];H),\ \text{weakly in } W^{1,2}(0,T;H),\\ \partial\psi_\lambda(u_\lambda) \to \xi \ \text{ in } L^2(0,T;H), \quad \xi(t) := f(t) - u'(t) \in \partial\psi^t(u(t)) \text{ a.e. } t \in (0,T).\end{gathered}$$

Accordingly, integrating energy inequalities (2.7) and (2.9) in time and letting $\lambda \downarrow 0$, we obtain the following estimates for the solution u of (CP):

$$|u|_{L^\infty(0,T;H)}^2 + \int_0^T \psi^t(u(t))dt \le R_1(|u_0|_H + |f|_{L^2(0,T;H)}), \tag{2.10}$$

$$\psi^t(u(t)) - \psi^s(u(s)) + \frac{1}{2}\int_s^t |u'(\tau)|_H^2 d\tau \le \int_s^t k_r(\tau)(\psi^\tau(u(\tau)) + 1)d\tau, \tag{2.11}$$

$$t\psi^t(u(t)) - s\psi^s(u(s)) + \frac{1}{2}\int_s^t \tau|u'(\tau)|_H^2 d\tau \le \int_s^t (\tau k_r(\tau) + 1)(\psi^\tau(u(\tau)) + 1)d\tau, \tag{2.12}$$

$$\text{for all } s,\ t \text{ with } 0 \le s \le t \le T \text{ and all } r > R_1(|u_0|_H + |f|_{L^2(0,T;H)}).$$

The results mentioned above are summarized in the following theorem.

THEOREM 2.2. *Let $\mathcal{G}(\{M_r\})$ be as given by (2.2). Then we have:*

(i) For each positive number p_1 there is a positive constant P_1, depending on $\mathcal{G}(\{M_r\})$, such that

$$|u|_{L^\infty(0,T;H)}^2 + |\sqrt{t}u'|_{L^2(0,T;H)}^2 + \sup_{0\le t\le T} t\psi^t(u(t)) + \int_0^T \psi^t(u(t))dt \le P_1 \tag{2.13}$$

for every solution u of (CP), whenever $|u_0|_H + |f|_{L^2(0,T;H)} \le p_1$.

(ii) For each positive number p_2 there is a positive constant P_2, depending on $\mathcal{G}(\{M_r\})$, such that

$$|u'|^2_{L^2(0,T;H)} + \sup_{0\le t\le T} \psi^t(u(t)) \le P_2 \tag{2.14}$$

for every solution u of (CP), whenever $|u_0|_H + \psi^0(u_0) + |f|_{L^2(0,T;H)} \le p_2$.

In construction of solutions of our quasi-variational evolution equations the next convergence theorem plays an important role together with the above Theorem 2.2. Our result is based on the concept of convergence of convex functions due to Mosco.

THEOREM 2.3. *Let $\{\psi^t_n\}$ be a sequence in $\mathcal{G}(\{M_r\})$ and $\{u_{0n}\}$ and $\{f_n\}$ be sequences in H and $L^2(0,T;H)$, respectively, such that*

(a) ψ^t_n converges to ψ^t on H in the sense of Mosco as $n \to \infty$ for every $t \in [0,T]$, where $\{\psi^t\}$ is a family in $\mathcal{G}(\{M_r\})$.
(b) $u_{0n} \in \overline{D(\psi^0_n)}$, $u_0 \in \overline{D(\psi^0)}$, $u_{n0} \to u_0$ in H, and $f_n \to f$ in $L^2(0,T;H)$ as $n \to \infty$.

Then the solution u_n of $(CP;\{\psi^t_n\}, u_{0n}, f_n)$ tends to the solution u of $(CP;\{\psi^t\}, u_0, f)$ in the sense that

$$u_n \to u \ \text{ in } C([0,T];H), \quad \sqrt{t}u'_n \to \sqrt{t}u' \ \text{ weakly in } L^2([0,T];H), \tag{2.15}$$

$$\int_0^T \psi^t_n(u_n)dt \to \int_0^T \psi^t(u)dt \quad \text{as } n \to \infty. \tag{2.16}$$

Moreover, if $\{\psi^0_n(u_{0n})\}$ is bounded, then

$$u'_n \to u' \ \text{ weakly in } L^2([0,T];H), \quad \psi^t_n(u_n(t)) \to \psi^t(u(t)) \ \text{ for a.e. } t \in [0,T]. \tag{2.17}$$

For the detailed proofs of Theorems 2.1, 2.2 and 2.3, see [8; Chapter 1].

3. Local existence result. In order to formulate functions $\varphi^t(v;\cdot)$ precisely we introduce a time-independent, non-negative, proper, l.s.c. and convex function $\varphi_0(\cdot)$ on H such that

(φ_0) the set $\{z \in H; |z|_H \le r, \ \varphi_0(z) \le r\}$ is compact in H for each $r \ge 0$.

Let δ_0 be a fixed positive number and $T > 0$ be a finite time. For each $t \in [0,T]$ we define a closed convex subset $\mathcal{V}(-\delta_0, t)$ of $W^{1,2}(-\delta_0, t; H)$ by

$$\mathcal{V}(-\delta_0, t) := \{v; V_{[-\delta_0,t]}(v) < \infty\} \tag{3.1}$$

with

$$V_{[-\delta_0,t]}(v) := \sup_{-\delta_0 \le s \le t} \varphi_0(v(s)) + |v(0)|^2_H + |v'|^2_{L^2(-\delta_0,t;H)} \tag{3.2}$$

where $v'(t) = \frac{dv(t)}{dt}$.

Now, to each $v \in \mathcal{V}(-\delta_0, t)$ a family $\{\varphi^s(v;\cdot)\}_{0\le s\le t}$ of functions $\varphi^s(v;\cdot)$ on H is assigned such that

(Φ1) $\varphi^s(v;z)$ is proper, l.s.c., non-negative and convex in $z \in H$, and it is determined by $s \in [0,t]$ and v on $[-\delta_0, s]$; namely, for $v_1, v_2 \in \mathcal{V}(-\delta_0, t)$, we have $\varphi^s(v_1,\cdot) \equiv \varphi^s(v_2,\cdot)$ on H whenever $v_1 \equiv v_2$ on $[-\delta_0, s]$;
(Φ2) $\varphi^s(v;z) \ge \varphi_0(z)$, $\forall v \in \mathcal{V}(-\delta_0, t)$, $\ 0 \le \forall s \le \forall t \le T$;

($\Phi 3$) If $0 \le s_n \le t \le T$, $v_n \in \mathcal{V}(-\delta_0, t)$, $\sup_{n\in\mathbf{N}} V_{[-\delta_0,t]}(v_n) < \infty$, $s_n \to s$ and $v_n \to v$ in $C([-\delta_0, t]; H)$, then $\varphi^{s_n}(v_n; \cdot) \to \varphi^s(v; \cdot)$ on H in the sense of Mosco.

We give the definition of solutions for evolution equation (1.1).

Definition 3.1. Let $u_0 \in C([-\delta_0, 0]; H)$ and $f \in L^2(0, T; H)$. Then we say that u is a solution of the Cauchy problem

$$CP(u_0, f) \quad \begin{cases} u'(t) + \partial\varphi^t(u; u(t)) \ni f(t), \ 0 < t < T, \\ u = u_0 \ \text{ on } [-\delta_0, 0] \end{cases}$$

on $[0, T]$, if u satisfies that $u \in C([-\delta_0, T]; H)$, $u = u_0$ on $[-\delta_0, 0]$, $u \in W^{1,2}(\delta, T; H)$ for every (small) $\delta > 0$, $\varphi^{(\cdot)}(u; u(\cdot)) \in L^1(0, T)$ and $f(t) - u'(t) \in \partial\varphi^t(u; u(t))$ for a.e. $t \in (0, T)$.

Next, in order to formulate our local existence result for $CP(u_0, f)$ we introduce the following function spaces: given any function u_0 in $\mathcal{V}(-\delta_0, 0)$, $0 < R < \infty$ and $t \in [0, T]$, we put

$$\mathcal{V}(u_0; -\delta_0, t) := \{v \in \mathcal{V}(-\delta_0, t); v = u_0 \text{ on } [-\delta_0, 0]\}, \tag{3.3}$$

and

$$\mathcal{V}_R(u_0; -\delta_0, t) := \{v \in \mathcal{V}(u_0; -\delta_0, t); \sup_{0\le s\le t} \{\varphi_0(v(s)) + |v'|^2_{L^2(0,s;H)}\} \le R\}. \tag{3.4}$$

We are in a position to state a local existence result for problem $CP(u_0, f)$.

Theorem 3.1. *Let $0 < T < \infty$ and $u_0 \in \mathcal{V}(-\delta_0, 0)$ with $\varphi^0(u_0; u_0(0)) < \infty$. Assume that there are positive numbers $T_0 \le T$ and $R > \varphi^0(u_0; u_0(0))$, a family $\{M_r\}_{0\le r<\infty}$ of positive numbers M_r and a set $\{\{\varphi^t(v; \cdot)\}; v \in \mathcal{V}_R(u_0; -\delta_0, T_0)\}$ of families $\{\varphi^t(v; \cdot)\}_{0\le t\le T_0}$ of convex functions satisfying the following condition:*

($*$) There are two families $\{a_r^v; v \in \mathcal{V}_R(u_0; -\delta_0, T_0), 0 \le r < \infty\}$ of non-negative functions in $L^2(0, T_0)$ and $\{b_r^v; v \in \mathcal{V}_R(u_0; -\delta_0, T_0), 0 \le r < \infty\}$ of non-negative functions in $L^1(0, T_0)$ such that

(H1) $|a_r^v|_{L^2(0,T_0)} \le M_r$ and $|b_r^v|_{L^1(0,T_0)} \le M_r$ for all $r > 0$ and all $v \in \mathcal{V}_R(u_0; -\delta_0, T_0)$, and $\{\varphi^t(v; \cdot)\} \in G(\{a_r^v\}, \{b_r^v\})$ for all $v \in \mathcal{V}_R(u_0; -\delta_0, T_0)$;
(H2) for each finite $r > 0$ and $\varepsilon > 0$ there is a positive number $\delta_{r\varepsilon} > 0$ such that

$$\int_0^{\delta_{r\varepsilon}} (a_r^v(\tau)^2 + b_r^v(\tau))d\tau < \varepsilon, \quad \forall v \in \mathcal{V}_R(u_0; -\delta_0, T_0).$$

Then, for each $f \in L^2(0, T_0; H)$, problem $CP(u_0, f)$ has at least one solution u on an interval $[0, T']$ with $0 < T' \le T_0$ such that $u \in \mathcal{V}(-\delta_0, T'; H)$ and $\sup_{0\le t\le T'} \varphi^t(u; u(t)) < \infty$.

In the rest of this section we give a proof of Theorem 3.1. Let v be any element in $\mathcal{V}_R(u_0; -\delta_0, T_0)$. Then, by (H1), $\{\varphi^t(v; \cdot)\} \in G(\{a_r^v\}, \{b_r^v\})$ with $|a_r^v|_{L^2(0,T_0)} \le M_r$ and $|b_r^v|_{L^1(0,T_0)} \le M_r$ for all $r \ge 0$. Now, consider the problem

$$\begin{cases} u'(t) + \partial\varphi^t(v; u(t)) \ni f(t), \quad \text{a.e. } t \in [0, T_0], \\ u(0) = u_0(0). \end{cases} \tag{3.5}$$

By virtue of Theorem 2.1, this problem has a unique solution $u \in W^{1,2}(0, T_0; H)$ such that $\sup_{0\le t\le T_0} \varphi^t(v; u(t)) < \infty$. On account of (ii) of Theorem 2.2 we have the following uniform estimates with respect to $v \in \mathcal{V}_R(u_0; -\delta_0, T_0)$:

$$|u'|^2_{L^2(0,T_0;H)} + \sup_{0\le t\le T_0} \varphi^t(v; u(t)) \le P_2, \text{ hence } |u|_{L^\infty(0,T_0;H)} \le |u_0|_H + \sqrt{T_0 P_2} =: r_0, \quad (3.6)$$

where P_2 is a positive constant independent of $v \in \mathcal{V}_R(u_0; -\delta_0, T_0)$

Moreover, we have:

LEMMA 3.1. *Let τ_0 be a positive number such that $R > \varphi^0(u_0; u_0(0)) + \tau_0$. Then there exists a positive number T_1 such that*

$$\varphi^t(v; u(t)) + |u'|^2_{L^2(0,T_1;H)} \le \varphi^0(u_0; u_0(0)) + \tau_0, \quad \forall t \in [0, T_1], \quad (3.7)$$

for any $v \in \mathcal{V}_R(u_0; -\delta_0, T_0)$ and the solution u of (3.5).

Proof. By taking the inner product between the both sides of the equation in (3.5) and $u'(t)$, we have

$$|u'(t)|^2_H + (\xi(t), u'(t))_H = (f(t), u'(t))_H, \quad \text{a.e. } t \in [0, T_0], \quad (3.8)$$

where $\xi(t) := f(t) - u'(t) \in \partial\varphi^t(v; u(t))$ for a.e. $t \in [0, T_0]$. By (3.6), we have $|u|_{L^\infty(0,T_0;H)} < r := r_0 + 1$ and hence (cf. (2.4))

$$\frac{d}{dt}\varphi^t(v; u(t)) - (\xi(t), u'(t))_H \quad (3.9)$$
$$\le a^v_r(t)|u'(t) - f(t)|_H(\varphi^t(v; u(t))^{\frac{1}{2}} + 1) + b^v_r(t)(\varphi^t(v; u(t)) + 1), \quad \text{a.e. } t \in [0, T_0].$$

Using this inequality, we see from (3.8) that

$$|u'(t)|^2_H + \frac{d}{dt}\varphi^t(v; u(t))$$
$$\le |f(t)|_H|u'(t)|_H + (P_2^{\frac{1}{2}} + 1)a^v_r(t)(|u'(t)|_H + |f(t)|_H) + (P_2 + 1)b^v_r(t)$$

for a.e. $t \in [0, T_0]$. Hence, for any $t \in [0, T_0]$,

$$\int_0^t |u'(\tau)|^2_H d\tau + \varphi^t(v; u(t))$$
$$\le \varphi^0(u_0; u_0(0)) + P_2^{\frac{1}{2}}\left(\int_0^t |f(\tau)|^2_H \, d\tau\right)^{\frac{1}{2}}$$
$$+ \left(\int_0^t a^v_r(\tau)^2 \, d\tau\right)^{\frac{1}{2}} \cdot 2(P_2 + 1)(P_2^{\frac{1}{2}} + |f|_{L^2(0,T_0;H)}) + (P_2 + 1)\int_0^t b^v_r(\tau) \, d\tau.$$

Therefore, by condition (H2) there exists a small positive number T_1, independent of $v \in \mathcal{V}_R(u_0; -\delta_0, T_0)$, such that

$$P_2^{\frac{1}{2}}\left(\int_0^{T_1} |f(\tau)|^2_H \, d\tau\right)^{\frac{1}{2}} + \left(\int_0^{T_1} a^v_r(\tau)^2 \, d\tau\right)^{\frac{1}{2}} \cdot 2(P_2 + 1)(P_2^{\frac{1}{2}} + |f|_{L^2(0,T_0;H)})$$
$$+ (P_2 + 1)\int_0^{T_1} b^v_r(\tau) d\tau \le \tau_0.$$

Thus we have (3.7). ■

Proof of Theorem 3.1. By (3.4) and assumption (φ_0) about the level set compactness of φ_0, $\mathcal{V}_R(u_0; -\delta_0, T_0)$ is non-empty, compact and convex in $C([-\delta_0, T_0]; H)$. Now, consider a mapping $S : \mathcal{V}_R(u_0; -\delta_0, T_0) \to \mathcal{V}(-\delta_0, T)$ which is defined as follows:

$$[Sv](t) = \begin{cases} u_0(t), & \text{for } t \in [-\delta_0, 0], \\ u(t), & \text{for } t \in (0, T_1], \\ u(T_1), & \text{for } t \in (T_1, T_0], \end{cases} \tag{3.10}$$

where u is the solution of (3.5) associated with $v \in \mathcal{V}_R(u_0; -\delta_0, T_0)$ and T_1 is the same number as in Lemma 3.1. Then, for every $v \in \mathcal{V}_R(u_0; -\delta_0, T_0)$, it follows from the definition of S and Lemma 3.1 that $Sv \in \mathcal{V}(u_0; -\delta_0, T_0)$ and

$$\begin{aligned} \sup_{0 \le s \le T_0} \{\varphi_0([Sv](s)) + |[Sv]'|^2_{L^2(0,s;H)}\} &= \sup_{0 \le s \le T_1} \{\varphi_0(u(s)) + |u'|^2_{L^2(0,s;H)}\} \\ &\le \sup_{0 \le s \le T_1} \{\varphi^s(v; u(s)) + |u'|^2_{L^2(0,s;H)}\} \\ &\le \varphi^0(u_0; u_0(0)) + \tau_0 \le R, \end{aligned}$$

where u is the solution of (3.5). Thus S maps $\mathcal{V}_R(u_0; -\delta_0, T_0)$ into itself.

Next, we show the continuity of S in $\mathcal{V}_R(u_0; -\delta_0, T_0)$ with respect to the topology of $C([-\delta_0, T_0]; H)$. Let $\{v_n\}$ be any sequence in $\mathcal{V}_R(u_0; -\delta_0, T_0)$ and suppose that $v_n \to v$ in $C([-\delta_0, T_0]; H)$ (as $n \to \infty$). It is clear that $v \in \mathcal{V}_R(u_0; -\delta_0, T_0)$ and $\{V_{[-\delta_0, T_0]}(v_n)\}$ is bounded. By assumption (Φ3), we see that $\varphi^t(v_n; \cdot) \to \varphi^t(v; \cdot)$ on H in the sense of Mosco for every $t \in [0, T_0]$. Therefore, according to Theorem 2.3, the solution u_n of (3.5) corresponding to $v = v_n$ converges to the solution u of (3.5) in the sense that

$$u_n \to u \ \text{ in } C([0, T_0]; H), \quad u_n' \to u' \ \text{ weakly in } L^2(0, T_0; H),$$

This means that $Sv_n \to Sv$ in $C([-\delta_0, T_0]; H)$.

We are in a position to apply the fixed-point theorem for continuous mappings in compact and convex sets. Applying it to the mapping S we see that S has at least one fixed-point u_* in $\mathcal{V}_R(u_0; -\delta_0, T_0)$, i.e. $Su_* = u_*$. Denoting by u the restriction of u_* on $[-\delta_0, T_1]$, we easily check from the definition (3.10) that u is a solution of $CP(u_0, f)$ on the time interval $[0, T_1]$. ∎

4. Global existence result. Let φ_0 be the same as in the previous section as well as $\delta_0 > 0$ and $T > 0$. In this section, we consider a closed convex subset $\tilde{\mathcal{V}}(-\delta_0, t)$ of $L^2(-\delta_0, t; H)$ for each $t \in [0, T]$, as is defined below, in place of $\mathcal{V}(-\delta_0, t)$.

For each $t \in [0, T]$ we define

$$\tilde{\mathcal{V}}(-\delta_0, t) := \{v; \tilde{V}_{[-\delta_0, t]}(v) < \infty\}, \tag{4.1}$$

where

$$\tilde{V}_{[-\delta_0, t]}(v) := |v|^2_{L^\infty(-\delta_0, t; H)} + \int_{-\delta_0}^t \varphi_0(v(s)) ds. \tag{4.2}$$

Now, we suppose that to each $v \in \tilde{\mathcal{V}}(-\delta_0, t)$ a family $\{\varphi^s(v; \cdot)\}_{0 \le s \le t}$ of functions $\varphi^s(v; \cdot)$ on H is assigned such that

($\tilde{\Phi}$1) $\varphi^s(v; z)$ is proper, l.s.c., non-negative and convex in $z \in H$, and it is determined by $s \in [0, t]$ and v on $[-\delta_0, s]$; namely, for $v_1, v_2 \in \tilde{\mathcal{V}}(-\delta_0, t)$, we have $\varphi^s(v_1, \cdot) \equiv \varphi^s(v_2, \cdot)$ on H whenever $v_1 = v_2$ a.e. on $(-\delta_0, s)$;

($\tilde{\Phi}2$) $\varphi^s(v;z) \ge \varphi_0(z)$, $\forall v \in \tilde{\mathcal{V}}(-\delta_0,t)$, $\forall 0 \le s \le t \le T$;

($\tilde{\Phi}3$) If $0 \le s_n \le t \le T$, $v_n \in \tilde{\mathcal{V}}(-\delta_0,t)$, $\sup_{n\in\mathbf{N}} \tilde{V}_{[-\delta_0,t]}(v_n) < \infty$, $s_n \to s$ and $v_n \to v$ in $L^2(-\delta_0,t;H)$, then $\varphi^{s_n}(v_n;\cdot) \to \varphi^s(v;\cdot)$ on H in the sense of Mosco.

Next, we define a function space $\tilde{\mathcal{V}}_M(-\delta_0,t)$ for each $M > 0$ and $t \in [0,T]$ by

$$\tilde{\mathcal{V}}_M(-\delta_0,t) := \{v \in \tilde{\mathcal{V}}(-\delta_0,t);\ \tilde{V}_{[-\delta_0,t]}(v) \le M\}.$$

In order to show the existence of a solution of $CP(u_0,f)$ on the whole interval $[0,T]$ we relax assumptions (H1) and (H2) as follows: For each $M > 0$ there is a family $\{M_r\}_{0\le r<\infty}$ of positive numbers M_r and a set $\{\{\varphi^t(v;\cdot)\}; v \in \tilde{\mathcal{V}}_M(-\delta_0,T)\}$ of families $\{\varphi^t(v;\cdot)\}_{0\le t\le T}$ of convex functions satisfying the following condition:

(**) There are two families $\{a_r^v; v \in \tilde{\mathcal{V}}_M(-\delta_0,T), 0 \le r < \infty\}$ of non-negative functions in $L^2(0,T)$ and $\{b_r^v; v \in \tilde{\mathcal{V}}_M(-\delta_0,T), 0 \le r < \infty\}$ of non-negative functions in $L^1(0,T)$ such that

($\tilde{H}1$) $|a_r^v|_{L^2(0,T)} \le M_r$ and $|b_r^v|_{L^1(0,T)} \le M_r$ for all $r > 0$ and all $v \in \tilde{\mathcal{V}}_M(-\delta_0,T)$, and $\{\varphi^t(v;\cdot)\} \in G(\{a_r^v\},\{b_r^v\})$ for all $v \in \tilde{\mathcal{V}}_M(-\delta_0,T)$;

($\tilde{H}2$) for each finite $r > 0$ and $\varepsilon > 0$ there is a positive number $\delta_{r\varepsilon} > 0$ such that

$$\int_t^{t+\delta_{r\varepsilon}} (a_r^v(\tau)^2 + b_r^v(\tau))d\tau < \varepsilon, \quad \forall t \in [0,T-\delta_{r\varepsilon}],\ \forall v \in \tilde{\mathcal{V}}_M(-\delta_0,T).$$

It should be noted that these conditions are independent of initial data. Moreover we require the following assumption ($\tilde{H}3$):

($\tilde{H}3$) there are a positive number R_0 and a family $\{h_v\} := \{h_v; v \in \tilde{\mathcal{V}}(-\delta_0,T)\}$ of functions in $W^{1,2}(0,T;H)$ such that

$$|h_v|_{W^{1,2}(0,T;H)} \le R_0, \quad \int_0^T \varphi^t(v;h_v(t))dt \le R_0, \quad \forall v \in \tilde{\mathcal{V}}(-\delta_0,T).$$

We first show the existence of a solution $CP(u_0,f)$ on the whole interval $[0,T]$ for good initial values u_0.

THEOREM 4.1. *Suppose that ($\tilde{H}1$) and ($\tilde{H}2$) hold for every $M > 0$ as well as ($\tilde{H}3$). Let $u_0 \in \mathcal{V}(-\delta_0,0)$ with $\varphi^0(u_0;u_0(0)) < \infty$ and f be any function in $L^2(0,T;H)$. Then $CP(u_0,f)$ has at least one solution u on $[0,T]$ such that*

$$u \in W^{1,2}(0,T;H), \qquad \sup_{0\le t\le T} \varphi^t(u;u(t)) < \infty.$$

Proof. It is clear that ($\Phi1$) – ($\Phi3$) automatically satisfied, if ($\tilde{\Phi}1$) – ($\tilde{\Phi}3$) hold, and that ($H1$) and ($H2$) follow immediately from ($\tilde{H}1$) and ($\tilde{H}2$). Therefore, according to Theorem 3.1, $CP(u_0,f)$ has a solution u on a certain time interval $[0,\tau](\subset [0,T])$ such that

$$u \in W^{1,2}(0,\tau;H), \qquad \sup_{0\le t\le \tau} \varphi^t(u;u(t)) < \infty.$$

Consider an ordered set Z given by

$$Z := \{(u,\tau); 0 < \tau \le T,\ u \text{ is a solution of } CP(u_0,f) \text{ on } [0,\tau]\}$$

with an order $\prec$ defined by

$$(u_1,\tau_1) \prec (u_2,\tau_2) \Leftrightarrow \tau_1 \le \tau_2,\ u_1 = u_2 \text{ on } [-\delta_0,\tau_1].$$

Then, by the local existence result mentioned above, Z is non-empty. Now, let Y be any totally ordered set in Z with respect to the above order $\prec$. Then, putting $\hat{u}(t) = u(t)$ if $(u,\tau) \in Y$ and $0 \le t \le \tau$, we see that $\hat{u}$ is well defined on the interval $[0,\hat{\tau})$ with $\hat{\tau} := \sup_{(u,\tau)\in Y} \tau$. Moreover, we obtain that $\hat{u}_0 := \lim_{t\uparrow\hat{\tau}} \hat{u}(t)$ exists in H. In fact, since $\hat{u}$ is a solution of $CP(u_0, f)$ on any compact interval $[0,\tau]$ with $0 < \tau < \hat{\tau}$, it follows (cf. (2.5)) that

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{1}{2}|\hat{u}(t)|_H^2 - (\hat{u}(t), h(t))_H\right\} + (\hat{u}(t), h'(t) - f(t))_H + \varphi^t(\hat{u}; \hat{u}(t)) \qquad (4.3)\\ &\le \varphi^t(\hat{u}; h(t)) - (f(t), h(t))_H, \quad \text{for a.e. } t \in [0,\tau], \end{aligned}$$

if $0 < \tau < \hat{\tau}$, $h \in W^{1,2}(0,\tau;H)$ and $\varphi^{(\cdot)}(\hat{u}; h(\cdot))$ is integrable on $[0,\tau]$. Here, we use assumption $(\tilde{H}3)$ as follows. Take an increasing sequence $\{\tau_n\}$ with $\tau_n \uparrow \hat{\tau}$ and define a sequence $\{u_n\}$ of functions by

$$u_n(t) = \begin{cases} \hat{u}(t) & \text{for } t \in [-\delta_0, \tau_n], \\ \hat{u}(\tau_n) & \text{for } t \in [\tau_n, T]. \end{cases}$$

Since $\{u_n\} \subset \tilde{\mathcal{V}}(-\delta_0, T)$, we see from $(\tilde{H}3)$ that there are functions h_n for all n such that

$$|h_n|_{W^{1,2}(0,T;H)} \le R_0, \quad \sup_{n\in\mathbf{N}} \int_0^T \varphi^t(u_n; h_n(t))dt \le R_0.$$

Noting that $\varphi^t(u_n; h_n(t)) = \varphi^t(\hat{u}; h_n(t))$ a.e. on $[0,\tau_n]$, we infer from (4.3) with $h = h_n$ that

$$\sup_{n\in\mathbf{N}} |\hat{u}|_{L^\infty(0,\tau_n;H)} < \infty, \quad \sup_{n\in\mathbf{N}} \int_0^{\tau_n} \varphi^t(\hat{u}; \hat{u}(t))dt < \infty. \qquad (4.4)$$

Hence, $\hat{u} \in L^\infty(0,\hat{\tau};H)$ and $\varphi^{(\cdot)}(\hat{u}; \hat{u}(\cdot))$ is integrable on $[0,\hat{\tau}]$, namely $\hat{u} \in \tilde{\mathcal{V}}(-\delta_0, \hat{\tau})$. This shows by $(\tilde{H}1)$ that $\{\varphi^t(\hat{u}; \cdot)\} \in \mathcal{G}_{[0,\hat{\tau}]}(\{M_r\})$ for some family $\{M_r\} := \{M_r\}_{0\le r<\infty}$ of positive numbers. By Theorem 2.1, the Cauchy problem

$$w'(t) + \partial\varphi^t(\hat{u}; w(t)) \ni f(t), \quad 0 < t < \hat{\tau}, \quad w(0) = u_0(0).$$

has a unique solution w on the interval $[0,\hat{\tau}]$ such that

$$w \in W^{1,2}(0,\hat{\tau};H), \qquad \sup_{0\le t\le\hat{\tau}} \varphi^t(\hat{u}; w(t)) < \infty.$$

Since $w = \hat{u}$ on $[0,\hat{\tau})$, it follows that $\hat{u}(\tau_n) = w(\tau_n) \to w(\hat{\tau})$. If w is denoted by $\hat{u}$, the element $(\hat{u},\hat{\tau})$ is an upper bound of Y. Therefore, by virtue of Zorn's lemma, we conclude that Z has at least one maximal element (u^*, τ^*).

If $\tau^* = T$ is shown, then u^* is a solution of $CP(u_0, f)$ on $[0,T]$, namely it is enough to prove $\tau^* = T$ to complete the proof. Assume that $\tau^* < T$. Then, since $\varphi^{\tau^*}(u^*; u^*(\tau^*)) < \infty$, it follows that u^* is extended beyond time τ^* as a solution of $CP(u_0, f)$. In fact, we consider the problem

$$\begin{cases} \tilde{u}'(t) + \partial\tilde{\varphi}^t(\tilde{u}; \tilde{u}(t)) \ni \tilde{f}(t), \quad 0 < t < \tilde{T}, \\ \tilde{u} = \tilde{u}_0^* \quad \text{on } [-\tilde{\delta}_0, 0], \end{cases} \qquad (4.5)$$

where $\tilde{T} := T - \tau^*$, $\tilde{\delta}_0 := \delta_0 + \tau^*$, $\tilde{u}_0^*(t) = u^*(t + \tau^*)$ for $t \in [-\tilde{\delta}_0, 0]$, $\tilde{f}(t) := f(t + \tau^*)$ for $t \in (0, \tilde{T})$ and

$$\tilde{\varphi}^t(v; \cdot) := \varphi^{t+\tau^*}(v(\cdot + \tau^*); \cdot), \quad \forall v \in \tilde{\mathcal{V}}(-\tilde{\delta}_0, t),\ 0 < t \le \tilde{T}.$$

It is easy to see from $(\tilde{H}1)$ and $(\tilde{H}2)$ that assumptions $(H1)$ and $(H2)$ of Theorem 3.1 are satisfied for the family $\{\tilde{\varphi}^t(v; \cdot)\}_{0 \le t \le \tilde{T}}$, initial datum $\tilde{u}_0^*$ and any $R > \tilde{\varphi}^0(\tilde{u}_0^*; \tilde{u}_0^*(0))$. Therefore, problem (4.5) has a solution $\tilde{u}$ on a certain interval $[0, \tilde{T}']$ such that

$$\tilde{u} \in W^{1,2}(0, \tilde{T}'; H), \qquad \sup_{0 \le t \le \tilde{T}'} \tilde{\varphi}^t(\tilde{u}; \tilde{u}(t)) < \infty.$$

Putting

$$u(t) := \begin{cases} u^*(t) & \text{for } t \in [-\delta_0, \tau^*), \\ \tilde{u}(t - \tau^*) & \text{for } t \in [\tau^*, \tau^* + \tilde{T}'], \end{cases}$$

we observe that $u \in W^{1,2}(0, \tau^* + \tilde{T}'; H)$, $\sup_{0 \le t \le \tau^* + \tilde{T}'} \varphi^t(u; u(t)) < \infty$ and u is a solution of $CP(u_0, f)$ on $[0, \tau^* + \tilde{T}']$. This contradicts the fact that (u^*, τ^*) is maximal in Z. Consequently, $\tau^* = T$ must be true. ∎

Finally we show the existence of a solution of $CP(u_0, f)$ for a slightly more general class of initial data.

THEOREM 4.2. *Suppose that* $(\tilde{\Phi}1)$, $(\tilde{\Phi}2)$ *and* $(\tilde{\Phi}3)$ *hold and that* $(\tilde{H}1)$ *and* $(\tilde{H}2)$ *hold for every* $M > 0$ *as well as* $(\tilde{H}3)$. *Let* $u_0 \in \tilde{\mathcal{V}}(-\delta_0, 0) \cap C([-\delta_0, 0]; H)$ *such that there is a sequence* $\{u_{0n}\}$ *in* $\mathcal{V}(-\delta, 0)$ *with* $\varphi^0(u_{0n}; u_{0n}(0)) < \infty$ *satisfying that*

$$\sup_{n \in \mathbf{N}} \tilde{V}_{[-\delta_0, 0]}(u_{n0}) < \infty, \quad u_{n0} \to u_0 \quad \text{in } C([-\delta_0, 0]; H). \tag{4.6}$$

Then $CP(u_0, f)$ *has at least one solution* u *on* $[0, T]$ *such that*

$$u \in C([0, T]; H), \quad \sqrt{t} u' \in L^2(0, T; H), \qquad \sup_{0 < t \le T} t\varphi^t(u; u(t)) < \infty. \tag{4.7}$$

Proof. Since $u_{n0} \in \mathcal{V}(-\delta_0, 0)$ and $\varphi^0(u_{n0}; u_{n0}(0)) < \infty$, by virtue of Theorem 4.1 problem $CP(u_{n0}, f)$ has at least one solution u_n on $[0, T]$, i.e.

$$u_n'(t) + \xi_n(t) = f(t), \quad \xi_n(t) \in \partial\varphi^t(u_n; u_n(t)), \quad \text{a.e. } t \in (0, T), \tag{4.8}$$

and

$$u_n = u_{n0} \text{ on } [-\delta_0, 0],$$

such that $u_n \in W^{1,2}(0, T; H)$ and $\sup_{0 \le t \le T} \varphi^t(u_n; u_n(t)) < \infty$. Also, note from $(\tilde{H}3)$ that there is a sequence $\{h_n\}$ such that

$$\sup_{n \in \mathbf{N}} \left\{ |h_n|^2_{W^{1,2}(0,T;H)} + \int_0^T \varphi^t(u_n; h_n(t))\, dt \right\} < \infty. \tag{4.9}$$

Taking the inner product of both sides of (4.8) and $u_n(t) - h_n(t)$, we get (cf. (2.5))

$$\frac{d}{dt}\left\{ \frac{1}{2}|u_n(t)|_H^2 - (u_n(t), h_n(t))_H \right\} + (u_n(t), h_n'(t) - f(t))_H + \varphi^t(u_n; u_n(t))$$
$$\le \varphi^t(u_n; h_n(t)) - (f(t), h_n(t))_H$$

for a.e. $t \in (0,T)$. Integrating this inequality in time, we obtain with the help of Gronwall's lemma and (4.9) that

$$\sup_{n\in\mathbf{N}} \left\{ |u_n|^2_{L^\infty(0,T;H)} + \int_0^T \varphi^t(u_n;u_n(t))\,dt \right\} < \infty.$$

Therefore we have for some constant $M > 0$

$$\tilde{V}_{[-\delta_0,T]}(u_n) \le M, \quad \forall n = 1,2,\cdots.$$

Hence, by condition $(\tilde{H}1)$, there is a family $\{M_r\}$ of positive numbers M_r such that

$$\{\varphi^t(u_n;\cdot)\} \in \mathcal{G}(\{M_r\}), \quad \forall n = 1,2,\cdots.$$

Furthermore, from (i) of Theorem 2.2 and our assumption (4.6) it follows that there is a positive constant P_1 satisfying

$$|u_n|^2_{L^\infty(0,T;H)} + |\sqrt{t}u_n'|^2_{L^2(0,T;H)} + \sup_{0<t\le T} t\varphi^t(u_n;u_n(t)) + \int_0^T \varphi^t(u_n;u_n(t))dt \le P_1, \quad (4.10)$$

for all $n = 1,2,\cdots$, and by $(\tilde{\Phi}2)$,

$$\varphi_0(u_n(t)) \le \frac{P_1}{t}, \quad \forall t \in (0,t], \quad \forall n = 1,2,\cdots. \quad (4.11)$$

Since the level set of φ_0 is compact in H, by (4.10) and (4.11) it is easy to extract a subsequence $\{u_{n_k}\}$ from $\{u_n\}$ such that $u_{n_k} \to u$ in $C_{loc}((0,T];H)$ and hence in $L^2(0,T;H)$ (as $k \to \infty$) for a certain $u \in \tilde{\mathcal{V}}(u_0;-\delta_0;T)$. This shows by our assumption $(\tilde{\Phi}3)$ that $\varphi^t(u_{n_k};\cdot) \to \varphi^t(u;\cdot)$ on H in the sense of Mosco for every $t \in [0,T]$. Here, apply Theorem 2.3 to the sequence of problems

$$u_{n_k}'(t) + \partial\varphi^t(u_{n_k};u_{n_k}(t)) \ni f(t), \quad 0 < t < T, \quad u_{n_k}(0) = u_{n_k0}(0),$$

to see that u_{n_k} converges in $C([0,T];H)$ to the solution w of

$$w'(t) + \partial\varphi^t(u;w(t)) \ni f(t), \quad 0 < t < T, \quad w(0) = u_0(0).$$

Then, clearly, $w = u$ on $(0,T]$ and thus u must be a solution of $CP(u_0,f)$ on $[0,T]$ and satisfies (4.7). ∎

5. Obstacle problems. We begin this section with some artificial examples in order to explore our assumptions $(H1) - (H2)$ for local existence in time or $(\tilde{H}1) - (\tilde{H}3)$ for global existence in time.

EXAMPLE 5.1. Let $H := \mathbf{R}$, δ_0 and T be fixed positive numbers. We consider a scalar quasi-variational inequality, choosing $\varphi_0 \equiv 0$ on $\mathbf{R}$ and

$$\varphi^s(v;z) := \begin{cases} 0, & \text{if } z \in [k_c(v;s),\infty), \\ \infty, & \text{otherwise}, \end{cases} \qquad \forall v \in W^{1,2}(-\delta_0,t),\ \forall 0 \le s \le t \le T, \quad (5.1)$$

where

$$k_c(v;s) := 2v(0) + 2\int_0^s |v'(\tau)|^p d\tau, \quad \forall v \in W^{1,2}(-\delta_0,t),\ \forall 0 \le s \le t \le T, \quad (5.2)$$

for a fixed number p with $0 < p \le 1$. It is easy to check conditions (φ_0) and $(\Phi1) - (\Phi3)$ for φ_0 and $\varphi^t(v,\cdot)$, respectively.

Now, we consider

$$u'(t) + \partial\varphi^t(u; u(t)) \ni f(t), \quad 0 < t < T, \tag{5.3}$$

with initial condition

$$u(t) = u_0(t) \text{ for all } t \in [-\delta_0, 0], \tag{5.4}$$

where f is given in $L^2(0,T)$ and u_0 in $W^{1,2}(-\delta_0,0)$ with $u_0(0) \le 0$ (hence $\varphi^0(u_0; u_0(0)) < \infty$). For such an initial datum u_0 and any number $M > 0$, we choose

$$a_r^v(\tau) := 2|v'(\tau)|^p, \quad b_r^v(\tau) := 0 \text{ for a.e. } \tau \in (0,T),$$
$$\forall v \in W^{1,2}(-\delta_0,T) \text{ with } v = u_0 \text{ on } [-\delta_0,0] \text{ and } |v'|^2_{L^2(0,T)} \le M, \ \forall r \ge 0.$$

Then, condition $(*) = \{(H1)-(H2)\}$ with $T_0 := T$ is satisfied; in fact, given $z \ge k_c(v;s)$, $\tilde{z} = z - k_c(v;s) + k_c(v;t)$ satisfies that

$$|\tilde{z} - z| \le 2\int_s^t |v'(\tau)|^p d\tau, \quad \varphi^t(v;\tilde{z}) - \varphi^s(v;z) = 0.$$

Since $a_r^v := 2|v'|^p \in L^{\frac{2}{p}}(0,T)$, $(H1)$ holds. Also, if $0 < p < 1$, then $(H2)$ holds, too. When $p = 1$, as is easily checked, $(H2)$ does not hold. If $0 < p < 1$ and $u_0(0) \le 0$, our Theorem 3.1 says that the scalar problem (5.3)-(5.4) has a local in time solution. However, in case $p = 1$ and $u_0(0) = 0$, the problem has no solution. By the way, when $p = \frac{1}{2}$, $f \equiv 0$ and $u_0 \equiv 0$ on $[-\delta_0, 0]$, it is easy to see that the function $u(t) = 4t$ is a solution of (5.3)-(5.4) on $[0,T]$.

Example 5.2. Let ρ_0 be a smooth function on $\mathbf{R}$ and define $k_c(v;\cdot)$ by

$$k_c(v;s) := \int_{-\delta_0}^s \rho_0(s-\tau)v(\tau)d\tau, \quad \forall v \in L^2(-\delta_0,t), \ 0 \le \forall s \le \forall t \le T.$$

We define $\varphi^s(v;\cdot)$ by (5.1) for this obstacle function $k_c(v;\cdot)$. It is easy to see that conditions $(\tilde{\Phi}1) - (\tilde{\Phi}3)$ are fulfilled. Also, for any $M > 0$, conditions $(\tilde{H}1)$ and $(\tilde{H}2)$ hold for $a_r^v \equiv M \max|\rho_0|$ and $b_r^v \equiv 0$. Therefore, by Theorem 3.1, problem (5.3)-(5.4) has a local in time solution, if $f \in L^2(0,T)$ and the initial datum u_0 is given so that $u_0(0) \ge k_c(u_0;0)$. Furthermore, if the obstacle function is replaced by

$$k_c(v;s) := \int_{-\delta_0}^s \rho_0(s-\tau)\min\{v(\tau), m_0\}\, d\tau$$

with a positive constant m_0, then k_c is bounded by $m_1 := m_0(\delta_0+T)\max|\rho_0|$ from above and hence condition $(\tilde{H}3)$ is satisfied; in fact, we can choose as h_v in $(\tilde{H}3)$ the constant function m_1. Accordingly, by Theorem 4.1, problem (5.3) and (5.4) has a solution on the whole interval $[0,T]$.

Next, we give two typical applications to parabolic partial differential inequalities with the unkonwn dependent obstacles; one of them is the one mentioned in the introduction.

Example 5.3. Let Ω be a bounded domain in $\mathbf{R}^N$ with smooth boundary $\Gamma := \partial\Omega$. We put $Q := \Omega \times (0,T)$, $0 < T < \infty$, and $H := L^2(\Omega)$. As the proper, l.s.c. convex function

φ_0 on $L^2(\Omega)$ we take

$$\varphi_0(z) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla z|^2 dx, & \forall z \in H^1(\Omega), \\ \infty, & \text{otherwise.} \end{cases} \tag{5.5}$$

Also, let $\rho(\cdot,\cdot,\cdot)$ be a smooth function on $\mathbf{R}^N \times \mathbf{R} \times \mathbf{R}$. Assume that ρ and its partial derivatives $\rho_i(x,t,v) := \frac{\partial}{\partial x_i}\rho(x,t,v)$, $i = 1, 2, \cdots, N$, satisfy

$$|\rho(x,s,v_1) - \rho(x,t,v_2)| + \sum_{i=1}^N |\rho_i(x,s,v_1) - \rho_i(x,t,v_2)| \le c_0(|t-s| + |v_1 - v_2|), \tag{5.6}$$

$$\forall (x,s,v_1), (x,t,v_2) \in \mathbf{R}^N \times \mathbf{R} \times \mathbf{R},$$

where c_0 is a positive constant. We define

$$\varphi^s(v;z) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla z|^2 dx, & \text{if } z \in H^1(\Omega) \text{ and } z \ge k_c(v;\cdot,s) \text{ a.e. on } \Omega, \\ \infty, & \text{otherwise,} \end{cases} \tag{5.7}$$

$$\forall v \in L^2(-\delta_0, t; H^1(\Omega)),\ \forall 0 \le s \le t \le T,$$

where δ_0 is a fixed positive number and

$$k_c(v;x,s) = \int_{-\delta_0}^s \int_\Omega \rho(x-y, s-\tau, v(y,\tau)) dy d\tau, \quad \forall (x,s) \in \Omega \times [-\delta_0, t].$$

It is easy to see that conditions (φ_0) and $(\tilde{\Phi}1) - (\tilde{\Phi}3)$ are fulfilled by $\varphi_0(\cdot)$ and $\varphi^s(\cdot;\cdot)$ given by (5.5) and (5.7), respectively. Next, we check condition $(*)$ or $(**)$. Let $M > 0$ be any number and v be any function in

$$\tilde{\mathcal{V}}_M(-\delta_0, T) := \left\{ v;\ |v|^2_{L^\infty(-\delta_0,T;L^2(\Omega))} + \frac{1}{2}|\nabla v|^2_{L^2(-\delta_0,T;L^2(\Omega))} \le M \right\}. \tag{5.8}$$

Then, for each function $z \in H^1(\Omega)$ with $z \ge k_c(v;\cdot,s)$ a.e. on Ω and $0 \le s \le t \le T$, the function $\tilde{z} := z - k_c(v;\cdot,s) + k_c(v;\cdot,t)$ satisfies that $\tilde{z} \in H^1(\Omega)$ and $\tilde{z} \ge k_c(v;\cdot,t)$ a.e. on Ω. From (5.6) it follows that

$$\begin{aligned} &|\tilde{z}(x) - z(x)| \\ &\quad \le |k_c(v;x,t) - k_c(v;x,s)| \\ &\quad \le c_0(\delta_0 + s)|\Omega||s-t| + \int_s^t \int_\Omega (|\rho(x-y, t-\tau, 0)| + c_0|v(y,\tau)|) dy d\tau \\ &\quad \le c_0(\delta_0 + s)|\Omega||s-t| + \{|\Omega||\rho(\cdot,\cdot,0)|_{L^\infty(Q)} + c_0|\Omega|^{\frac{1}{2}}|v|_{L^\infty(-\delta_0,T;L^2(\Omega))}\}|t-s| \\ &\quad \le c_1(M)|t-s| \end{aligned}$$

for all $v \in \tilde{\mathcal{V}}_M(-\delta_0, T)$, where $|\Omega|$ denotes the volume of Ω and

$$c_1(M) := c_0(\delta_0 + T)|\Omega| + |\Omega||\rho(\cdot,\cdot,0)|_{L^\infty(Q)} + c_0|\Omega|^{\frac{1}{2}}\sqrt{M}.$$

Similarly,

$$\begin{aligned}&\left|\frac{\partial}{\partial x_i}(k_c(v;x,s)-k_c(v;x,t))\right|\\&\quad\le c_0(\delta_0+s)|\Omega||s-t|+\int_s^t\int_\Omega(|\rho_i(x-y,t-\tau,0)|+c_0|v(y,\tau)|)dyd\tau,\\&\quad\le c_1(M)|t-s|\end{aligned}$$

for $i=1,2,\cdots,N$ and all $v\in\tilde{\mathcal{V}}_M(-\delta_0,T)$. Therefore,

$$|\tilde z-z|_{L^2(\Omega)}\le c_1(M)|\Omega|^{\frac12}|t-s|$$

and

$$\begin{aligned}\frac12|\nabla\tilde z|^2_{L^2(\Omega)}-\frac12|\nabla z|^2_{L^2(\Omega)}&\le|\nabla z|_{L^2(\Omega)}\cdot c_1(M)\sqrt{N|\Omega|}|t-s|+c_1(M)^2NT|\Omega||t-s|^2\\&\le c_2(M)|t-s|\left(\frac{1}{\sqrt2}|\nabla z|_{L^2(\Omega)}+1\right),\end{aligned}$$

where $c_2(M):=\sqrt2c_1(M)\sqrt{N|\Omega|}+c_1(M)^2NT|\Omega|$; namely,

$$\varphi^t(v;\tilde z)-\varphi^s(v;z)\le c_2(M)|t-s|(\varphi^s(v;z)^{\frac12}+1).$$

Thus, putting $a_r^v:=c_1(M)|\Omega|^{\frac12}$ and $b_r^v:=c_2(M)$ for all $v\in\tilde{\mathcal{V}}(-\delta_0,T)$ and all $r\ge0$, we see that $(\tilde H1)$ and $(\tilde H2)$ hold. Therefore, by virtue of Theorem 3.1, for given $f\in L^2(Q)$ and $u_0\in W^{1,2}(-\delta_0,0;L^2(\Omega))\cap L^\infty(-\delta_0,0;H^1(\Omega))$ with $u_0(\cdot,0)\ge k_c(u_0;\cdot,0)$ a.e. on Ω, the quasi-variational problem, denoted by $(QV1)$, formulated on $Q':=\Omega\times(0,T')$, $0<T'\le T$,

$$u\in W^{1,2}(-\delta_0,T';L^2(\Omega))\cap L^\infty(-\delta_0,T';H^1(\Omega))\text{ with }u\ge k_c(u;\cdot,\cdot)\ \text{ a.e. on }Q';$$

$$\int_Q\{u_t(u-w)+\nabla u\cdot\nabla(u-w)\}dxdt\le\int_Q f(x,t)(u-w)dxdt,$$
$$\forall w\in L^2(0,T';H^1(\Omega))\text{ with }w\ge k_c(u;\cdot,\cdot)\ \text{ a.e. on }Q',$$

$$u=u_0\ \text{ a.e. on }\Omega\times[-\delta_0,0],$$

has at least one solution u on a certain interval $[0,T']\subset[0,T]$. Further suppose that ρ is bounded from above on $\mathbf{R}^N\times\mathbf{R}\times\mathbf{R}$. Then, so is k_c on $\tilde{\mathcal{V}}(-\delta_0,T)\times\mathbf{R}^N\times\mathbf{R}$, that is, $k_c\le k^*$ for a certain positive constant k^*. In this case, $\varphi^t(v;k^*)=0$ for all $v\in\tilde{\mathcal{V}}(-\delta_0,T)$ and $t\in[0,T]$, which shows that $(\tilde H3)$ holds. In such a case, our Theorem 4.1 says that problem $(QV1)$ has a solution u on the whole interval $[0,T]$.

EXAMPLE 5.4. Let Ω, Γ, Q and Σ be as in Example 5.3, as well as $0<\delta_0<\infty$ and $0<T<\infty$. Also, we take as φ_0 the same function as in Example 5.3, too. Let Ω_i, $i=1,2,\cdots,n$, be a finite number of smooth subdomains of Ω such that $\overline{\Omega}_i\subset\Omega$, $i=1,2,\cdots,n$, and $\overline{\Omega}_k\cap\overline{\Omega}_j=\emptyset$ if $k\ne j$. We define a mapping $\Lambda_i:L^2(-\delta_0,t;H_0^1(\Omega))\to C([0,t])$, $0\le s\le t\le T$, by

$$[\Lambda_iv](s):=\int_{-\delta_0}^t\int_{\Omega_i}\gamma_i(s-\tau,v(x,\tau))dxds,\quad\forall v\in L^2(-\delta_0,t;H_0^1(\Omega)),\quad\forall s\in[0,t],$$

for each $i = 1, 2, \cdots, n$, where $\gamma_i(\cdot,\cdot)$ are Lipschitz continuous functions on $\mathbf{R} \times \mathbf{R}$, $i = 1, 2, \cdots, n$, i.e.

$$|\gamma_i(\tau_1, v_1) - \gamma_i(\tau_2, v_2)| \le c_0(|\tau_1 - \tau_2| + |v_1 - v_2|), \quad \forall(\tau_1, v_1), (\tau_2, v_2) \in \mathbf{R} \times \mathbf{R}, \qquad (5.9)$$

for a positive constant c_0. Now, for each $v \in L^2(-\delta_0, t; H_0^1(\Omega))$ and $s \in [0, T]$, consider

$$K(v; s) := \{z \in H_0^1(\Omega); |\nabla z| \le k_c([\Lambda_i v](s)) \text{ a.e. on } \Omega_i,\ i = 1, 2, \cdots, n\},$$

where $k_c(\cdot)$ is a smooth and strictly positive function on $\mathbf{R}$. Clearly $K(v; s)$ is a closed convex subset of $H_0^1(\Omega)$ and non-empty, since $0 \in K(v; s)$.

We consider a quasi-variational problem, denoted by $(QV2)$, to find a function u such that

$$u \in L^2(-\delta_0, T; H_0^1(\Omega)) \cap W^{1,2}(0, T; L^2(\Omega)) \text{ with } u(t) \in K(u; t) \text{ for all } t \in [0, T],$$
$$\int_0^T \int_\Omega u_t(u - w)dxdt + \int_0^T \int_\Omega \nabla u \cdot \nabla(u - w)dxdt \le \int_0^T \int_\Omega f(u - w)dxdt,$$
$$\forall w \in L^2(-\tau_0, T; H_0^1(\Omega)) \text{ with } w(t) \in K(u; t) \text{ for a.e. } t \in (0, T),$$
$$u(t) = u_0(t) \quad \text{for all } t \in [-\delta_0, 0],$$

where f is given in $L^2(0, T; L^2(\Omega))$ and u_0 in $W^{1,2}(-\delta_0, 0; L^2(\Omega)) \cap L^\infty(-\delta_0, 0; H^1(\Omega))$ with $u_0(0) \in K(u_0; 0)$. Defining a proper, l.s.c. and convex function $\varphi^s(v; \cdot)$ on $L^2(\Omega)$ for each $v \in \tilde{\mathcal{V}}(-\delta_0, t); L^2(\Omega))$ (cf.(5.9)), $0 \le s \le t \le T$, by

$$\varphi^s(v; z) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla z|^2 dx, & \text{if } z \in K(v; s), \\ \infty, & \text{otherwise}, \end{cases}$$

we see easily that problem $(QV2)$ can be described as a Cauchy problem of the form $CP(u_0, f)$ on $[0, T]$. It is easy to check that our convex functions φ_0 and $\varphi^s(v; \cdot)$ satisfy conditions (φ_0) and $(\tilde{\Phi}1) - (\tilde{\Phi}3)$.

Let us verify assumptions in Theorems 4.1 (and hence Theorem 3.1). Choose a collection $\{\eta_k\}_{0 \le k \le n}$ of smooth non-negative functions on $\mathbf{R}^N$ corresponding to the family $\{\Omega_i\}$ of subdomains of Ω such that

$$\eta_k = 1 \text{ on } \Omega_k,\ \forall 1 \le k \le n, \text{ and } \sum_{k=0}^n \eta_k = 1 \text{ on } \Omega. \qquad (5.10)$$

Now, given $M > 0$ and any function v with $|v|^2_{L^\infty(-\delta_0, T; L^2(\Omega))} + \frac{1}{2}|\nabla v|^2_{L^2(-\delta_0, T; L^2(\Omega))} \le M$. For any $0 \le s \le t \le T$ and $z \in K(v; s)$, we put

$$\tilde{z} = \eta_0 z + \sum_{i=1}^n \eta_i \cdot \frac{k_i(v; s)}{k_i(v; t)} z, \quad k_i(v; \cdot) := k_c([\Lambda_i v](\cdot)),\ i = 1, 2, \cdots, n.$$

Then we see from (5.10) that

$$\nabla \tilde{z} = \frac{k_i(v; t)}{k_i(v; s)} \nabla z \ \text{ on } \Omega_i,\ i = 1, 2, \cdots, n, \text{ namely, } \tilde{z} \in K(v; t).$$

Moreover, we have

$$|\tilde{z}(x) - z(x)| \le \sum_{i=1}^n \eta_i(x) \frac{|k_i(v; t) - k_i(v; s)|}{k_i(v; s)} \cdot |z(x)|$$

and

$$\begin{aligned}\nabla \tilde{z}(x) = &\left\{\eta_0(x) + \sum_{i=1}^{n} \frac{k_i(v;t)}{k_i(v;s)}\eta_i(x)\right\}\nabla z(x) \\ &+ \left\{\nabla\eta_0(x) + \sum_{i=1}^{n} \frac{k_i(v;t)}{k_i(v;s)}\nabla\eta_i(x)\right\}z(x) \\ = &\nabla z(x) + \left\{\sum_{i=1}^{n} \frac{k_i(v;t)-k_i(v;s)}{k_i(v;s)}\eta_i(x)\right\}\nabla z(x) \\ &+ \left\{\sum_{i=1}^{n} \frac{k_i(v;t)-k_i(v;s)}{k_i(v;s)}\nabla\eta_i(x)\right\}z(x).\end{aligned}$$

Since $k_i(v;\tau)$ is Lipschitz continuous in τ, i.e. $|k_i(v;t) - k_i(v;s)| \le c_i(M)|t-s|$, where

$$c_i(M) := c_0(T+\delta_0)|\Omega_i| + |\Omega_i| \max|\gamma_i(\cdot,0)| + c_0\{|\Omega_i|M\}^{\frac{1}{2}}, \quad i = 1,2,\cdots,n,$$

we obtain from the above relations that

$$|\tilde{z} - z|_{L^2(\Omega)} \le c_*(M)|t-s|, \quad \varphi^t(v;t) - \varphi^s(v;z) \le c_*(M)|t-s|(\varphi^s(v;s)^{\frac{1}{2}} + 1),$$

where $c_*(M)$ is a positive constant depending only on M. Thus conditions $(\tilde{H}1)$ and $(\tilde{H}2)$ are satisfied by the families of functions $a_r^v \equiv b_r^v \equiv c_*(M)$, and $(\tilde{H}3)$ is trivially satisfied by $h_v \equiv 0$. Accordingly, by Theorem 4.1, our quasi-variational problem $(QV2)$ has at least one solution u on the whole interval $[0,T]$.

Some other applications are found in a recent paper [5] treating one dimensional gradient obstacle problems of parabolic type.

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