NONLOCAL AND ABSTRACT PARABOLIC EQUATIONS
AND THEIR APPLICATIONS
BANACH CENTER PUBLICATIONS, VOLUME 86
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2009

ABSTRACT QUASI-VARIATIONAL INEQUALITIES OF ELLIPTIC TYPE AND APPLICATIONS

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Abstract. A class of quasi-variational inequalities (QVI) of elliptic type is studied in reflexive Banach spaces. The concept of QVI was earlier introduced by A. Bensoussan and J.-L. Lions [2] and its general theory has been developed by many mathematicians, for instance, see [6, 7, 9, 13] and a monograph [1]. In this paper we give a generalization of the existence theorem established in [14]. In our treatment we employ the compactness method along with a concept of convergence of nonlinear multivalued operators of monotone type (cf. [11]). We shall prove an abstract existence result for our class of QVI's, and moreover, give some applications to QVI's for elliptic partial differential operators.

1. Introduction. Let X be a real reflexive Banach space and X^* be its dual. We assume that X and X^* are strictly convex and denote by $\langle \cdot, \cdot \rangle$ the duality pairing between X^* and X. Given a nonlinear operator A from X into X^*, an element $g^* \in X^*$ and a closed convex subset K of X, the variational inequality is formulated as a problem to find u in X such that

$$\begin{cases} u \in K, \\ \langle Au - g^*, u - w \rangle \leq 0, \quad \forall w \in K. \end{cases} \tag{1}$$

This has been studied by many mathematicians, for instance see [4, 5, 10] and their references.

The concept of quasi-variational inequality was introduced by A. Bensoussan and J. L. Lions [2] in order to solve some problems in the control theory. Given an operator $A : X \to X^*$, an element $g^* \in X^*$ and a family $\{K(v); v \in X\}$ of closed convex subsets

2000 *Mathematics Subject Classification*: Primary 35K45; Secondary 35K50.
Key words and phrases: elliptic quasi-variational inequality, semi-monotone, PDE.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc86-0-15

of X, the quasi-variational inequality is a problem to find u in X such that

$$\begin{cases} u \in K(u), \\ \langle Au - g^*, u - w \rangle \leq 0, \quad \forall w \in K(u). \end{cases} \tag{2}$$

The constraint $K(u)$ for the quasi-variational inequality depends upon the unknown u, which causes one of main difficulties in the mathematical treatment of quasi-variational inequalities.

The theory of quasi-variational inequalities has been developed for various classes of mappings $v \to K(v)$ and linear or nonlinear operators $A : X \to X^*$; see for instance [6,7,13], in which two approaches to quasi-variational inequalities were proposed. One of them is the so-called monotonicity method in Banach lattices X (cf. [13]), and for the mapping $v \to K(v)$ the monotonicity condition

$$\min\{w_1, w_2\} \in K(v_1), \quad \max\{w_1, w_2\} \in K(v_2), \quad \text{if } v_1, v_2 \in X \text{ with } v_1 \leq v_2, \tag{3}$$

is required, and an existence result for (2) is proved with the help of a fixed point theorem in Banach lattices. Another is the so-called compactness method in which some compactness properties are required for the mapping $v \to K(v)$ such as $K(v_n)$ converges to $K(v)$ in the Mosco sense, if $v_n \to v$ weakly in X as $n \to \infty$. In the latter framework, an existence result for (2) was shown in [7].

However, these results seem to be insufficient for application from some point of wiew. Therefore their generalizations were established in [14]. In that paper, it is assumed that $A : X \to X^*$ is a pseudo-monotone operator, $Au = \tilde{A}(u, u)$, generated by a semi-monotone operator $\tilde{A} : X \times X \to X^*$. In such a case our quasi-variational inequality is of the form: Find $u \in X$ and

$$\begin{cases} u \in K(u), & u^* \in Au, \\ \langle u^* - g^*, u - w \rangle \leq 0, & \forall w \in K(u). \end{cases} \tag{4}$$

In this paper, we discuss the following quasi-variational inequality which is a further generalization of the case treated in [14]. For a given function $\varphi : X \times X \to \mathbb{R}$, our quasi-variational inequality is written as

$$\begin{cases} \varphi(u, u) < \infty, \quad u^* \in Au; \\ \langle u^* - g^*, u - v \rangle + \varphi(u, u) \leq \varphi(u, v), \ \forall v \in X. \end{cases} \tag{5}$$

The above abstract result is applied to a quasi-variational inequality arising in the elastic-plastic torsion problem for visco-elastic materials: Find $u \in H_0^1(\Omega)$ and $\tilde{u} \in L^2(\Omega)$ satisfying

$$\begin{cases} |\nabla u| \leq k_c(u) \text{ a.e. on } \Omega, \quad \tilde{u} \in \beta(u) \text{ a.e. on } \Omega, \\ \displaystyle\sum_{i,j=1}^{N} \int_\Omega a_{ij}(x, u) \frac{\partial u}{\partial x_i} \frac{\partial (u - w)}{\partial x_j} dx + \int_\Omega \tilde{u}(u - w) dx \leq \int_\Omega f(u - w) dx, \\ \qquad\qquad \forall w \in H_0^1(\Omega) \quad \text{with} \quad |\nabla w| \leq k_c(u) \quad \text{a.e. on } \Omega, \end{cases}$$

where Ω is a bounded smooth domain in $\mathbf{R}^N$, f is given in $L^2(\Omega)$, $k_c(\cdot)$ is a positive, smooth and bounded function on $\mathbf{R}$ and $\beta(\cdot)$ is a maximal monotone graph in $\mathbf{R} \times \mathbf{R}$. In this case our abstract result is applied to

$$\varphi(v, u) = \int_\Omega I_{K(v)}(u)dx + \hat{\beta}(u)$$

with $K(v) := \{w \in H_0^1(\Omega); |\nabla w| \le k_c(u) \text{ a.e. on } \Omega\}$ and indicator function $I_{K(v)}$ of $K(v)$, where $\hat{\beta}$ is the primitive of β, i.e. $\partial\hat{\beta} = \beta$, and to

$$Au := -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(x, u)\frac{\partial u}{\partial x_i}\right) + \beta(u).$$

It should be noted that the family $\{K(v); v \in H_0^1(\Omega)\}$ does not satisfy the monotonicity condition (3), and that the term $\beta(u)$ is in general multivalued.

2. Main results. Let X be a real Banach space and X^* be its dual space, and assume that X and X^* are strictly convex. We denote by $\langle\cdot,\cdot\rangle$ the duality pairing between X^* and X, and by $|\cdot|_X$ and $|\cdot|_{X^*}$ the norms of X and X^*, respectively. For various general concepts on nonlinear multivalued operators from X into X^*, for instance, monotonicity and maximal monotonicity of operators, we refer to the monograph [1,4]. In this paper, operators are multivalued, in general. Given a general nonlinear operator A from X into X^*, we use the notations $D(A)$, $R(A)$ and $G(A)$ to denote its domain, range and graph. Now we formulate quasi-variational inequalities for a class of nonlinear operators, called semi-monotone, from $X \times X$ into X^*.

DEFINITION 2.1. An operator $\tilde{A}(\cdot,\cdot) : X \times X \to X^*$ is called *semi-monotone* if $D(\tilde{A}) = X \times X$ and the following conditions (SM1) and (SM2) are satisfied:

(SM1) $\widetilde{A}(v,\cdot) : X \to X^*$ is maximal monotone, and $D(\widetilde{A}(v,\cdot)) = X$ for every $v \in X$.
(SM2) $\{v_n\} \subset X$ and $v_n \to v$ weakly in X as $n \to \infty$

$$\Rightarrow \quad \forall u^* \in \widetilde{A}(v, u),\ \exists\{{u_n}^*\} \subset X^* \text{ such that } \begin{cases} {u_n}^* \in \widetilde{A}(v_n, u),\ \forall n \in \mathbb{N}, \\ {u_n}^* \to u^* \text{ in } X^*. \end{cases}$$

DEFINITION 2.2. If a sequence $\{\varphi_n\}$ of proper l.s.c. convex functions on X satisfies the following (MC1) and (MC2), then we say that φ_n converges to a proper l.s.c. convex function φ on X in the sense of Mosco [11].

(MC1) $\forall z \in D(\varphi),\ \exists\{z_n\} \subset X$ such that $z_n \to z$ in X and $\varphi_n(z_n) \to \varphi(z)$.
(MC2) If $\{\varphi_{n_k}\} \subset \{\varphi_n\}$ and $\{z_k\} \subset X$ such that $z_k \to z$ weakly in X, then $\liminf_{k\to\infty} \varphi_{n_k}(z_k) \ge \varphi(z)$.

Let $\tilde{A} : D(\tilde{A}) := X \times X \to X^*$ be a semi-monotone operator. Then we define $A : D(A) = X \to X^*$ by putting $Au := \tilde{A}(u, u)$ for all $u \in X$, which is called the operator generated by $\tilde{A}$.

Now, for an operator A generated by a semi-monotone operator $\widetilde{A}$, any $g^* \in X^*$ and a mapping $v \to K(v)$ we consider a quasi-variational inequality, denoted by P(g^*,φ), to find $u \in X$ and $u^* \in X^*$ such that

$$\mathrm{P}(g^*,\varphi) \qquad \begin{cases} \varphi(u, u) < \infty, \quad u^* \in Au; \\ \langle u^* - g^*, u - v\rangle + \varphi(u, u) \le \varphi(u, v),\ \forall v \in X. \end{cases} \tag{6}$$

Our main results are as follows.

Theorem 2.1. *Let* $\widetilde{A} : D(\widetilde{A}) = X \times X \to X^*$ *be a bounded semi-monotone operator,* $A : X \to X^*$ *be the operator generated by* $\widetilde{A}$*,* K_0 *be a bounded closed convex subset of* X*, and* $g^* \in X^*$*. Assume that* $\varphi : X \times X \to \mathbb{R} \cup \{\infty\}$ *is such that* $\varphi(v, \cdot)$ *is proper l.s.c. convex on* X *for each* $v \in K_0$*, and* K_0 *contains effective domains* $D(\varphi(v, \cdot))$ *for all* $v \in K_0$*. Moreover, assume the following condition (K):*

$$\text{(K)} \quad \{v_n\} \subset K_0 \ \text{ and } \ v_n \to v \text{ weakly in } X \implies \varphi(v_n, \cdot) \to \varphi(v, \cdot) \ \text{ on } X \text{ in the sense of Mosco.}$$

Then, the quasi-variational inequality $P(g^*,\varphi)$ *has at least one solution* u*.*

The following theorem is a slightly more general version of Theorem 2.1.

Theorem 2.2. *Let* $\widetilde{A} : D(\widetilde{A}) = X \times X \to X^*$ *be bounded and semi-monotone,* $A : X \to X^*$ *be the operator generated by* $\widetilde{A}$*. Assume that* $\varphi : X \times X \to \mathbb{R} \cup \{\infty\}$ *is a function such that* $\varphi(v, \cdot)$ *is a proper l.s.c. convex function for each* $v \in X$*, and there exists a bounded set* $G_0 \subset X$ *such that* $D(\varphi(v, \cdot)) \cap G_0 \neq \emptyset$ *for all* $v \in X$*, and the following boundedness and coerciveness conditions are satisfied:*

$$\exists R > 0 \ \text{ such that } \ \inf_{z \in G_0} \varphi(v, z) \leq R(|v|_X + 1), \quad \forall v \in X, \tag{7}$$

$$\inf_{w^* \in Aw} \left(\frac{\langle w^*, w - v \rangle + \varphi(w, w)}{|w|_X} \right) \to \infty \ \text{ as } \ |w|_X \to \infty \ \text{ uniformly in } v \in G_0. \tag{8}$$

Moreover, assume the following condition (K'):

$$\text{(K')} \quad \{v_n\} \subset X, \ \ v_n \to v \text{ weakly in } X \implies \varphi(v_n, \cdot) \to \varphi(v, \cdot) \ \text{ on } X \text{ in the sense of Mosco.}$$

Then, the problem $P(g^*,\varphi)$ *has at least one solution* u*.*

In our proof of Theorems 2.1 and 2.2 we use some results on nonlinear operators of monotone type, which are mentioned below.

Proposition 2.1. *Let* $\tilde{A} : D(\tilde{A}) = X \times X \to X^*$ *be a semi-monotone operator and let* $A : X \to X^*$ *be the operator generated by* $\widetilde{A}$*. Then, the following two properties* (a) *and* (b) *hold:*

(a) *For any* $v, u \in X, A(v, u)$ *is a non-empty, closed, bounded and convex subset of* X^**.*
(b) *Let* $\{u_n\}$ *and* $\{v_n\}$ *be sequences in* X *such that* $u_n \to u$ *weakly in* X *and* $v_n \to v$ *weakly in* X *(as* $n \to \infty$*). If* $u_n^* \in \tilde{A}(v_n, u_n)$*,* $u_n^* \to g$ *weakly in* X^* *and* $\limsup_{n\to\infty} \langle u_n^*, u_n \rangle \leq \langle g, u \rangle$*, then* $g \in \tilde{A}(v, u)$ *and* $\lim_{n\to\infty} \langle u_n^*, u_n \rangle = \langle g, u \rangle$*.*

For a proof of Proposition 2.1, see [14].

Proposition 2.2. *Let* $A_1 : D(A_1) \subset X \to X^*$ *be a maximal monotone operator and* $A_2 : D(A_2) = X \to X^*$ *be a maximal monotone operator. Suppose that*

$$\inf_{v_1^* \in A_1 v, \ v_2^* \in A_2 v} \frac{\langle v_1^* + v_2^*, v - v_0 \rangle}{|v|_X} \to \infty \ \text{ as } \ |v|_X \to \infty, \ v \in D(A_1).$$

Then $R(A_1 + A_2) = X^*$*.*

For a proof of Proposition 2.2, see [4, 5, 8].

3. Proof of main theorems. We begin with the proof of Theorem 2.1.

Proof of Theorem 2.1. The theorem is proved in the following two steps. In the first step (A), we prove the case when $\tilde{A}(v,\cdot)$ is strictly monotone from X into X^* for every $v \in X$, and the second step (B) is the general case as in the statement of Theorem 2.1.

Case (A). First, we solve the following problem for each $v \in K_0$.

$$\begin{cases} \varphi(v,u) < \infty, \quad u^* \in \widetilde{A}(v,u); \\ \langle u^* - g^*, u - w\rangle + \varphi(v,u) \le \varphi(v,w), \ \forall w \in X. \end{cases} \tag{9}$$

Now $\widetilde{A}(v,\cdot)$ is maximal monotone with $D(\widetilde{A}(v,\cdot)) = X$, and the operator $\partial\varphi(v,\cdot)$, which is the subdifferential of $\varphi(v,\cdot)$, is maximal monotone. Furthermore, for each $v \in D(\varphi(v,\cdot))$,

$$\inf_{w_1^* \in \partial\varphi(v,w),\ \ w_2^* \in \widetilde{A}(v,w)} \frac{\langle w_1^* + w_2^*, w - w_0\rangle}{|w|_X} \to \infty \ \text{ as } \ |w|_X \to \infty,\ w \in D(\partial\varphi(v,\cdot)) \tag{10}$$

is trivially satisfied because $D(\partial\varphi(v,\cdot)) \subset K_0$. From Proposition 2.2, $R(\partial\varphi(v,\cdot) + \widetilde{A}(v,\cdot)) = X^*$, so that

$$\exists u \in D(\varphi(v,\cdot)), \quad \exists u^* \in \widetilde{A}(v,u) \ \text{ s.t. } \ g^* - u^* \in \partial\varphi(v,u). \tag{11}$$

Therefore,

$$\langle u^* - g^*, u - w\rangle + \varphi(v,u) \le \varphi(v,w), \ \forall w \in X, \tag{12}$$

and the problem (9) has a solution u. By the strict monotonicity, this solution is unique.

Now we consider the operator $S : K_0 \to K_0$ which assigns to each $v \in K_0$ the solution $u \in K_0$ of (9). We shall show that S is weakly continuous in K_0. Assume that $\{v_n\} \subset K_0$, $v_n \to v$ weakly in X, and $Sv_n = u_n$. Then, since K_0 is weakly compact in X, there exist a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $u \in K_0$ such that $u_{n_k} \to u$ weakly in X $(k \to \infty)$. For each k, we have

$$\begin{cases} \varphi(v_{n_k}, u_{n_k}) < \infty, \quad u^*_{n_k} \in \widetilde{A}(v_{n_k}, u_{n_k}); \\ \langle u^*_{n_k} - g^*, u_{n_k} - w\rangle + \varphi(v_{n_k}, u_{n_k}) \le \varphi(v_{n_k}, w), \ \forall w \in X. \end{cases} \tag{13}$$

This means $u \in D(\varphi(v,\cdot))$ by (MC2). By (MC1), we can see that there exists a sequence $\{\widetilde{u}_k\} \subset K_0$ such that $\varphi(v_{n_k}, \widetilde{u}_k) \to \varphi(v,u)$ and $\widetilde{u}_k \to u$ in X. Moreover, $\langle u^*_{n_k}, u_{n_k} - \widetilde{u}_k\rangle \le \langle g^*, u_{n_k} - \widetilde{u}_k\rangle - \varphi(v_{n_k}, u_{n_k}) + \varphi(v_{n_k}, \widetilde{u}_k)$ for all $k \in \mathbb{N}$ by (13), and $\limsup_{k\to\infty}(-\varphi(v_{n_k}, u_{n_k})) \le -\varphi(v,u)$ by (MC2), so that

$$\begin{aligned} \limsup_{k\to\infty} \langle u^*_{n_k}, u_{n_k}\rangle &= \limsup_{k\to\infty} (\langle u^*_{n_k}, u_{n_k} - \widetilde{u}_k\rangle + \langle u^*_{n_k}, \widetilde{u}_k\rangle) \\ &\le \limsup_{k\to\infty} (\langle g^*, u_{n_k} - \widetilde{u}_k\rangle + \langle u^*_{n_k}, \widetilde{u}_k\rangle \\ &\qquad\qquad - \varphi(v_{n_k}, u_{n_k}) + \varphi(v_{n_k}, \widetilde{u}_k)) \\ &\le \langle u^*, u\rangle. \end{aligned} \tag{14}$$

Next, by (SM2), for any $w \in X$ and any $w^* \in \widetilde{A}(v,w)$, there exists some sequence $\{w_k^*\}$ in X^*, such that $w_k^* \in \widetilde{A}(v_{n_k}, w)$ and $w_k^* \to w^*$ in X^*. In addition, $\langle u^*_{n_k} - w_k^*, u_{n_k} - w\rangle \ge 0$ for all $k \in \mathbb{N}$, because $\widetilde{A}(v_{n_k}, \cdot)$ is monotone. Hence

$$0 \le \limsup_{k\to\infty} \langle u^*_{n_k} - w_k^*, u_{n_k} - w\rangle \le \langle u^* - w^*, u - w\rangle \tag{15}$$

and $\widetilde{A}(v,\cdot)$ is maximal monotone, which implies that $u^* \in \widetilde{A}(v,u)$. For these u and u^*, we use (SM2) to get a sequence $\{\widetilde{w}_k^*\} \subset X^*$ which satisfies that $\widetilde{w}_k^* \in \widetilde{A}(v_{n_k},u)$ and $\widetilde{w}_k^* \to u^*$ in X^*. Furthermore, since $\langle u_{n_k}^*, u_{n_k}\rangle \geq \langle u_{n_k}^*, u\rangle + \langle \widetilde{w}_k^*, u_{n_k}-u\rangle$ by the monotonicity of $\widetilde{A}(v_{n_k},\cdot)$, we see that

$$\begin{aligned}\liminf_{k\to\infty}\langle u_{n_k}^*, u_{n_k}\rangle &\geq \liminf_{k\to\infty}(\langle u_{n_k}^*, u\rangle + \langle \widetilde{w}_k^*, u_{n_k}-u\rangle)\\ &= \langle u^*, u\rangle. \end{aligned} \tag{16}$$

Combining this with (14), we get $\lim_{k\to\infty}\langle u_{n_k}^*, u_{n_k}\rangle = \langle u^*, u\rangle$.

For any $w \in D(\varphi(v,\cdot))$, by (MC1) we are able to find a sequence $\{w_k\}$ such that $w_k \to w$ in X and $\varphi(v_{n_k}, w_k) \to \varphi(v,w)$. Moreover, note from (13) that $\langle u_{n_k}^* - g^*, u_{n_k} - w_k\rangle + \varphi(v_{n_k}, u_{n_k}) \leq \varphi(v_{n_k}, w_k)$. Therefore,

$$\begin{aligned}\varphi(v,w) &= \liminf_{k\to\infty}\varphi(v_{n_k}, w_k)\\ &\geq \liminf_{k\to\infty}(\langle u_{n_k}^*, u_{n_k}\rangle - \langle u_{n_k}^*, w_k\rangle - \langle g^*, u_{n_k}-w_k\rangle + \varphi(v_{n_k}, u_{n_k}))\\ &= \langle u^*-g^*, u-w\rangle + \varphi(v,u). \end{aligned} \tag{17}$$

This shows that $Sv = u$ and S is weakly continuous. By the fixed point theorem for compact operators (Leray-Schauder type), S has at least one fixed point u in K_0, namely $Su = u$, which is a solution of P(g^*,φ).

Case (B). We approximate $\widetilde{A}$ by $\widetilde{A}_\varepsilon(v,u) = \widetilde{A}(v,u) + \varepsilon J(u)$ for every $u, v \in X$, $\varepsilon \in (0,1]$, and the duality mapping J from X into X^*. Then $\widetilde{A}_\varepsilon$ is semi-monotone and $\widetilde{A}_\varepsilon(v,\cdot)$ is strictly monotone. Applying the case (A) for the operator A_ε generated by $\widetilde{A_\varepsilon}$, we see that

$$\exists u_\varepsilon \in K_0 \ \text{ s.t. } \ \begin{cases} u_\varepsilon \in D(\varphi(u_\varepsilon,\cdot)), \quad u_\varepsilon^* \in A_\varepsilon(u_\varepsilon);\\ \langle u_\varepsilon + \varepsilon\cdot Ju_\varepsilon - g^*, u_\varepsilon - w\rangle + \varphi(u_\varepsilon,u_\varepsilon) \leq \varphi(u_\varepsilon, w), \ \forall w \in X. \end{cases} \tag{18}$$

Now choose a sequence $\{\varepsilon_n\}$ and a subsequence $\{u_{\varepsilon_n}\}$ of $\{u_\varepsilon\}$ such that ε_n converges to 0, and u_{ε_n} converges to some u weakly in X. Then $\varphi(u,u) < \infty$ by condition (K'). According to (MC1), there exists a sequence $\{\widetilde{u}_n\}$ which satisfies $\varphi(u_{\varepsilon_n}, \widetilde{u}_n) \to \varphi(u,u)$ and $\widetilde{u}_n \to u$ in X. For this sequence, we can choose a subsequence $\{u_{\varepsilon_{n_k}}^*\}$ of $\{u_{\varepsilon_n}^*\}$ such that $\{u_{\varepsilon_{n_k}}^*\}$ converges to some u^* weakly in X, because $\widetilde{A}$ is bounded. We now observe by using (MC2) and (18) that

$$\begin{aligned}&\limsup_{k\to\infty}\langle u_{\varepsilon_{n_k}}^*, u_{\varepsilon_{n_k}} - u\rangle\\ &= \limsup_{k\to\infty}(\langle u_{\varepsilon_{n_k}}^* + \varepsilon_{n_k}\cdot Ju_{\varepsilon_{n_k}}, u_{\varepsilon_{n_k}} - \widetilde{u}_{n_k}\rangle + \langle u_{\varepsilon_{n_k}}^* + \varepsilon_{n_k}\cdot Ju_{\varepsilon_{n_k}}, \widetilde{u}_{n_k} - u\rangle)\\ &\leq \limsup_{k\to\infty}(\langle g^*, u_{\varepsilon_{n_k}} - \widetilde{u}_{n_k}\rangle - \varphi(u_{\varepsilon_{n_k}}, u_{\varepsilon_{n_k}}) + \varphi(u_{\varepsilon_{n_k}}, \widetilde{u}_{n_k})\\ &\qquad\qquad + \langle u_{\varepsilon_{n_k}}^* + \varepsilon_{n_k}\cdot Ju_{\varepsilon_{n_k}}, \widetilde{u}_{n_k} - u\rangle)\\ &\leq 0. \end{aligned} \tag{19}$$

From Proposition 2.1-(b) and this inequality, we get $\lim_{k\to\infty}\langle u_{\varepsilon_{n_k}}^*, u_{\varepsilon_{n_k}}\rangle = \langle u^*, u\rangle$. Moreover, as is seen from (MC1), for each $w \in X$

$$\exists \{w_k\} \subset X \ \text{ s.t. } \ \begin{cases} \varphi(u_{\varepsilon_{n_k}}, w_k) \to \varphi(u, w), \\ w_k \to w \ \text{ in } \ X. \end{cases}$$

With these sequences and (18), we have

$$\begin{aligned} \varphi(u,w) &= \lim_{k\to\infty} \varphi(u_{\varepsilon_{n_k}}, w_k) \\ &\geq \lim_{k\to\infty} (\langle u^*_{\varepsilon_{n_k}} + \varepsilon_{n_k} \cdot Ju_{\varepsilon_{n_k}} - g^*, u_{\varepsilon_{n_k}} - w_k\rangle + \varphi(u_{\varepsilon_{n_k}}, u_{\varepsilon_{n_k}})) \\ &\geq \langle u^* - g^*, u - w\rangle + \varphi(u,u) \end{aligned}$$

for all $w \in X$, so u is a solution of P(g^*,φ). ■

Next, we show Theorem 2.2 using the result of Theorem 2.1.

Proof of Theorem 2.2. First, we put

$$d_1 := \sup_{w\in G_0} |w|_X,$$

$$d_2 := \sup\left\{ |w|_X \middle| w \in X, \inf_{w^*\in Aw} \left(\frac{\langle w^*, w - v\rangle + \varphi(w,w)}{|w|_X} \right) \leq |g^*|_{X^*}(1 + d_1), \forall v \in G_0 \right\},$$

and $M_0 = d_1 + d_2 + 1$, and $B_M = \{w \in X \mid |w|_X \leq M\}$ for $M \geq M_0$. Now we define the function φ_M on $X \times X$ by

$$\varphi_M(v,u) = \begin{cases} \varphi(v,u) & \text{if } \ |u|_X \leq M, \\ \infty & \text{otherwise.} \end{cases}$$

Then $\varphi_M(v,\cdot)$ is a proper l.s.c. convex function on X for each $v \in X$.

Next we show that $\{\varphi_M(v_n,\cdot)\}$ converges to $\varphi_M(v,\cdot)$ on X in the sense of Mosco. In fact, for any $w \in D(\varphi_M(v,\cdot))$, we use (MC1) to find a sequence $\{w_n\}$ such that $w_n \to w$ in X and $\varphi(v_n, w_n) \to \varphi(v,w)$. If $|w|_X < M$, then $|w_n|_X < M$ is satisfied for all large $n \geq N_0$ for a certain $N_0 \in \mathbb{N}$. Now putting

$$\widetilde{w}_n = \begin{cases} \text{any } \ w \in D(\varphi_M(v_n,\cdot)) & (n < N_0), \\ w_n & (n \geq N_0), \end{cases}$$

we can see that (MC1) holds for $\{\widetilde{w}_n\}$. If $|w|_X = M$, we approximate w by

$$w^{(m)} = \left(1 - \frac{1}{m}\right) w + \frac{1}{m} v_0 \qquad (m = 1, 2, \cdots), \ \ v_0 \in D(\varphi(v,\cdot)) \cap G_0.$$

It is clear that $w^{(m)} \to w$ as $m \to \infty$, $w^{(m)} \in D(\varphi(v,\cdot))$, and $|w^{(m)}|_X < M$ for all $m \in \mathbb{N}$. By the above result with $|w|_X < M$, we see that

$$\forall m \in \mathbb{N}, \ \ \exists \{w_n^{(m)}\} \ \text{ s.t. } \ \begin{cases} w_n^{(m)} \to w^{(m)} \ \text{ in } \ X, \\ \varphi_M(v_n, w_n^{(m)}) \to \varphi_M(v, w^{(m)}). \end{cases} \tag{20}$$

Furthermore, for each $m \in \mathbb{N}$, there exists $n(m) \in \mathbb{N}$ such that

$$|w_n^{(m)} - w^{(m)}|_X < \frac{1}{2m} \ \text{ and } \ |\varphi_M(v_n, w_n^{(m)}) - \varphi_M(v, w^{(m)})|_X \leq \frac{1}{2m}, \ \forall n \geq n(m).$$

We choose a sequence $\{n'(m)\}$ which satisfies $n'(m) < n'(m+1)$, $n(m) < n'(m)$, and $m < n'(m)$ for every $m \in \mathbb{N}$. Then for all $n \in \mathbb{N}$ there exists $i \in \mathbb{N}$ such that $n'(i) \leq n <$

$n'(i+1)$. We choose $w_n^{(i)}$ as $\widetilde{w}_n$. Then,

$$|\widetilde{w}_n - w|_X \le |w_n^{(i)} - w^{(i)}|_X + |w^{(i)} - w|_X \to 0.$$

Moreover, we easily see $\varphi_M(v, w^{(m)}) \to \varphi_M(v, w)$ and $\varphi_M(v_n, w_n^{(m)}) \to \varphi_M(v, w^{(m)})$, so that

$$|\varphi_M(v_n, \widetilde{w}_n) - \varphi_M(v, w)| \le |\varphi_M(v_n, w_n^{(i)}) - \varphi_M(v, w^{(i)})| + |\varphi_M(v, w^{(i)}) - \varphi_M(v, w)| \to 0$$

From these results, we can see that $\{\varphi_M(v_n, \cdot)\}$ has property (MC1). The verification of (MC2) is easy. We are now in a position to apply Theorem 2.1 for B_M and φ_M for all M, and we see that

$$\exists u_M \in X \ \text{ s.t. } \begin{cases} \varphi_M(u_M, u_M) < \infty, \quad u_M^* \in A u_M, \\ \langle u_M^* - g^*, u_M - w \rangle + \varphi_M(u_M, u_M) \le \varphi(u_M, w), \ \forall w \in X; \end{cases} \tag{21}$$

note that condition (K) for $K_0 = B_M$ and $\varphi = \varphi_M$ follows from (K'). From (7), it follows that

$$\forall M \ge M_0, \ \ \exists w_M \in D(\varphi_M(u_M, \cdot)) \cap G_0 \ \text{ s.t. } \ \varphi(u_M, w_M) \le R(|u_M|_X + 2).$$

Now, we obtain from (21) that

$$\frac{\langle u_M^*, u_M - w_M \rangle + \varphi(u_M, u_M)}{|u_M|_X} \le |g^*|_{X^*} + R + \frac{|g^*|_{X^*} + 2R}{|u_M|_X}.$$

Hence, our coerciveness assumption (8) implies that $\{u_M\}_{M \ge M_0}$ is bounded in X. Therefore,

$$\exists \{M_n\} \subset \{M\}_{M \ge M_0} \ \text{ s.t. } \begin{cases} u_{M_n} \to u \ \text{ weakly in } X, \\ u_{M_n}^* \to u^* \ \text{ weakly in } X^*, \end{cases} \tag{22}$$

and $u \in D(\varphi(u, \cdot))$ by (MC2). On account of (MC1), for the weak limit u and any element w of $D(\varphi(u, \cdot))$,

$$\exists \{\widehat{u}_n\}, \ \exists \{\widehat{w}_n\} \ \text{ s.t. } \begin{cases} \widehat{u}_n \to u, \quad \widehat{w}_n \to w \ \text{ in } X, \\ \varphi(u_{M_n}, \widehat{u}_n) \to \varphi(u, u), \quad \varphi(u_{M_n}, \widehat{w}_n) \to \varphi(u, w). \end{cases} \tag{23}$$

Using (21), we see that $\limsup_{n\to\infty} \langle u_{M_n}^*, u_{M_n} \rangle \le \langle u^*, u \rangle$ in the same way as in Theorem 2.1. Now by Proposition 2.1-(b), we have $\langle u_{M_n}^*, u_{M_n} \rangle \to \langle u^*, u \rangle$, and hence

$$\begin{aligned} \varphi(u, w) &= \liminf_{n\to\infty} \varphi(u_{M_n}, \widehat{w}_n) \\ &\ge \liminf_{n\to\infty} (\langle u_{M_n}^*, u_{M_n} \rangle - \langle u_{M_n}^*, \widehat{w}_n \rangle - \langle g^*, u_{M_n} - \widehat{w}_n \rangle + \varphi(u_{M_n}, u_{M_n})) \\ &\ge \langle u^* - g^*, u - w \rangle + \varphi(u, u). \end{aligned}$$

This means that u is a solution of P(g^*,φ). ■

4. Applications. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, $a_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ and $a_i(\cdot,\cdot,\cdot) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ $(i = 1, 2, \cdots, N)$. We assume the following conditions hold.

(a1) $a_0(\cdot, \eta)$, $a_i(\cdot, \eta, \xi)$ are measurable on Ω, $\forall \eta \in \mathbb{R}$, $\forall \xi \in \mathbb{R}^N$, $i = 1, 2, \cdots, N$.

(a2) $a_0(x, \cdot)$ is continuous on $\mathbb{R}$ for a.e. $x \in \Omega$ and $a_i(x, \cdot, \cdot)$ is continuous on $\mathbb{R} \times \mathbb{R}^N$ for a.e. $x \in \Omega$, $i = 1, 2, \cdots, N$.

(a3) There exist positive constants c_0 and c_1 such that
$$\begin{cases} c_0(|\xi|^{p-1}-1) \le a_i(x,\eta,\xi) \le c_1(|\xi|^{p-1}+1), \\ c_0(|\eta|^{p-1}-1) \le a_0(x,\eta) \le c_1(|\eta|^{p-1}+1), \end{cases}$$
for $i = 1, 2, \cdots, N,\ \eta \in \mathbb{R},\ \xi \in \mathbb{R}^N,$ and a.e. $x \in \Omega$.

(a4) $\displaystyle\sum_{i=1}^{N}(a_i(x,\eta,\xi) - a_i(x,\eta,\overline{\xi}))(\xi_i - \overline{\xi_i}) \ge 0$
for a.e. $x \in \Omega,\ \eta \in \mathbb{R},\ \xi = (\xi_1, \xi_2, \cdots, \xi_N),\ \overline{\xi} = (\overline{\xi_1}, \overline{\xi_2}, \cdots, \overline{\xi_N})$.

Our applications are formulated with these functions.

4.1. *Application 1 (Gradient obstacle problem).* Let $X = W_0^{1,p}(\Omega)$ with $1 < p < \infty$. For any u, v, and $w \in X$, let us define the operator $\widetilde{A}$ by

$$\langle \widetilde{A}(v,u), w\rangle = \sum_{i}^{N} \int_\Omega a_i(x, v, \nabla u)\frac{\partial w}{\partial x_i}dx + \int_\Omega a_o(x,v)w dx$$

and write simply $A(u) = \widetilde{A}(u,u)$ for each $u \in X$. We put $K(v) = \{w \in X \mid |\nabla w| \le k_c(v)$ a.e. on $\Omega\}$. Let $k_c(\cdot) : \mathbb{R} \to \mathbb{R}$ be a positive function which is Lipschitz continuous and bounded with upper bound k_c^*, and φ be defined by

$$\varphi(v,u) = \int_\Omega I_{K(v)}(u)dx \qquad (v, u \in X).$$

LEMMA 4.1.

(i) *The operator* $\widetilde{A} : X \to X^*$ *is bounded and semi-monotone.*

(ii) $\varphi(\cdot,\cdot)$ and $K_0 = \{w \in X \mid |\nabla w| \le k_c^*$ *a.e. on* $\Omega\}$ *satisfies the assumptions of Theorem 2.1.*

This lemma is proved in [14].

By Lemma 4.1, all the assumptions of Theorem 2.1 are checked. Applying Theorem 2.1, we can get a solution u of the following quasi-variational inequality for any $f \in L^q(\Omega)$:

$$\begin{cases} u \in X, \\ |\nabla u| \le k_c(u) \quad \text{a.e. on} \quad \Omega, \\ \displaystyle\sum_{i=1}^{N}\int_\Omega a_i(x,u,\nabla u)\Big(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i}\Big)dx + \int_\Omega a_o(x,u)(u-v)dx \le \int_\Omega f(u-v)dx, \\ \qquad\qquad\qquad\qquad \forall v \in X, \quad |\nabla v| \le k_c(u) \text{ a.e. on } \Omega. \end{cases}$$

This gradient obstacle problem is a mathematical model of vibration of a string in $\mathbb{R}$, vibration of a membrane in $\mathbb{R}^2$, and elastic-plastic torsion problem for visco-elastic materials in $\mathbb{R}^3$. In our model we treat the case when the threshold value of $|\nabla u|$ depends upon the displacement u.

4.2. *Application 2 (Non-local constraints in the interior).* Let $X = W^{1,p}(\Omega)$ with $1 < p < \infty$, $f \in L^q(\Omega)$ when q is the conjugate exponent of p, $k_c \in C^1(\mathbb{R})$ with upper bound $k_c^* > 0$, ρ be a C^1-class function on $\mathbb{R}^N \times \mathbb{R}^N$. We define an integral operator Λ as

$$\Lambda v(x) = \int_\Omega \rho(x,y)v(y)dy, \qquad \forall v \in X \ \text{ and } \ \forall x \in \Omega.$$

Let us consider the same operator $\widetilde{A}(\cdot,\cdot) : X \times X \to X^*$ as in Application 1, and A be the pseudo-monotone operator generated by $\widetilde{A}$ which is semi-monotone on X by Lemma 4.1. Moreover, we define a proper l.s.c. convex function φ,

$$\varphi(u,v) = \int_\Omega I_{[0,\infty)}(u - k_c(\Lambda v))dx + \int_\Omega \hat{\beta}(v)dx$$

where $\hat{\beta}$ is a proper l.s.c. convex function on $\mathbb{R}$ such that $k_c^* \in D(\hat{\beta})$ and $D(\hat{\beta})$ has no-empty interior.

Lemma 4.2. *$\varphi(\cdot,\cdot)$ and $G_0 = \{k_c^*\}$ satisfy all the assumptions of Theorem 2.2.*

Proof. We show that the Mosco convergence property is satisfied. For every $w \in D(\varphi(v,\cdot))$, we see

$$w \geq k_c(\Lambda v) \quad \text{a.e. on } \Omega, \qquad \int_\Omega \hat{\beta}(w)dx < \infty.$$

Now, there exists numbers a and b such that $\overline{D(\hat{\beta})} = [a,b]$ with $a < b$. Next choose small constant $\overline{\varepsilon} > 0$ such that $a + \overline{\varepsilon} < b - \overline{\varepsilon}$. Since $k_c(\Lambda v_n) \to k_c(\Lambda v)$ in $C(\overline{\Omega})$,

$$\exists n_\varepsilon \ \text{ s.t. } \ |k_c(\Lambda v_n) - k_c(\Lambda v)| < \varepsilon \quad \text{on } \Omega, \quad n \geq n_\varepsilon. \tag{24}$$

We define $w_\varepsilon(x) = \min\{\max\{a+\varepsilon, w(x)\}, b-\varepsilon\}$ for a.e. $x \in \Omega$. Then $w_\varepsilon \in [a+\varepsilon, b-\varepsilon]$ on Ω, $w_\varepsilon \to w$ in X as $\varepsilon \searrow 0$, and hence $w_\varepsilon \in D(\varphi(v,\cdot))$ for any $\varepsilon \in (0,\overline{\varepsilon})$. Moreover, we see

$$\int_\Omega \hat{\beta}(w_\varepsilon)dx \to \int_\Omega \hat{\beta}(w)dx \quad \text{as } \varepsilon \searrow 0, \tag{25}$$

because $|\hat{\beta}(w_\varepsilon)| \leq |\hat{\beta}(w)| + \beta_0$ on Ω, where β_0 is a positive constant independent of ε. Next, we define $w_{\varepsilon,n} = w_\varepsilon - k_c(\Lambda v) + k_c(\Lambda v_n)$ for each $\varepsilon \in (0,\overline{\varepsilon})$ and $n = 1,2,\cdots$. On account of (24) we have for each $\varepsilon \in (0,\overline{\varepsilon})$,

$$\begin{cases} w_{\varepsilon,n} \to w_\varepsilon \ \text{ in } \ X \quad (n\to\infty), \\ \displaystyle\int_\Omega \hat{\beta}(w_{\varepsilon,n})dx \to \int_\Omega \hat{\beta}(w_\varepsilon)dx \quad (n\to\infty), \\ w_{\varepsilon,n} \in D(\varphi(v_n,\cdot)), \ \forall n \geq n_\varepsilon. \end{cases} \tag{26}$$

Making use of (24) to (26), by the diagonal argument we easily construct a sequence $\{\widetilde{w}_n\}$ such that

$$\widetilde{w}_n \to w \ \text{ in } \ X, \quad \int_\Omega \hat{\beta}(\widetilde{w}_n)dx \to \int_\Omega \hat{\beta}(w)dx.$$

Hence (MC1) holds for $\varphi(v_n,\cdot)$. Also, (MC2) is shown by

$$\begin{aligned} \liminf_{n\to\infty} \varphi(v_n,w_n) &= \liminf_{n\to\infty} \left(\int_\Omega I_{K(v_n)}(w_n)dx + \int_\Omega \hat{\beta}(v_n)dx \right) \\ &\geq 0 + \int_\Omega \liminf_{n\to\infty} \hat{\beta}(v_n)dx = \varphi(v,w). \ \blacksquare \end{aligned}$$

By Lemmas 4.1, 4.2, and Theorem 2.2, we are able to find a solution u of the following quasi-variational inequality for every $f \in L^q(\Omega)$:

$$\begin{cases} u \in X, \quad u \geq k_c(\Lambda u) \;\text{ a.e. on }\; \Omega, \\ \displaystyle\sum_{i=1}^{N} \int_\Omega a_i(x,u,\nabla u)(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i})dx + \int_\Omega a_0(x,u)(u-v)dx + \int_\Omega \hat{\beta}(u)dx \\ \qquad\qquad\qquad \leq \displaystyle\int_\Omega f(u-v)dx + \int_\Omega \hat{\beta}(v)dx, \\ \qquad\qquad\qquad \forall v \in X, \quad v \geq k_c(\Lambda u) \;\text{ a.e. on }\; \Omega. \end{cases}$$

4.3. *Application 3 (Non-local constraints on the boundary).* Let $X = W^{1,p}(\Omega)$ with $1 < p < \infty$, $\Gamma = \partial\Omega$, $k_c(\cdot)$ be the same function as in Application 2, $\rho(\cdot,\cdot) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, be of class C^1, $G_0 = \{k_c^*\}$, and $K(v) = \{w \in X \mid w \geq k_c(\Lambda v) \;\text{ a.e. on }\; \Gamma\}$, where

$$\begin{cases} \Lambda v(x) = \displaystyle\int_\Gamma \rho(x,y)v(y)d\Gamma_y, \; \forall x \in \Gamma, \\ \varphi(v,u) = \displaystyle\int_\Gamma I_{[0,\infty)}(u - k_c(\Lambda v))d\Gamma. \end{cases}$$

It is easy to see that all the assumptions of Theorem 2.2 are satisfied in the same way as in Section 4.1 and 4.2. Now applying Theorem 2.2, we see that the following quasi-variational inequality has at least one solution u:

$$\begin{cases} u \in X, \quad u \geq k_c(\Lambda u) \;\text{ a.e. on }\; \Gamma, \\ \displaystyle\sum_{i=1}^{N} \int_\Omega a_i(x,u,\nabla u)\Big(\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i}\Big)dx + \int_\Omega a_0(x,u)(u-v)dx \leq \int_\Omega f(u-v)dx, \\ \qquad\qquad\qquad \forall v \in X, \quad v \geq k_c(\Lambda u) \;\text{ a.e. on }\; \Gamma. \end{cases}$$

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