NONLOCAL AND ABSTRACT PARABOLIC EQUATIONS
AND THEIR APPLICATIONS
BANACH CENTER PUBLICATIONS, VOLUME 86
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2009

# OPTIMAL CONTROL OF NONLINEAR EVOLUTION EQUATIONS ASSOCIATED WITH TIME-DEPENDENT SUBDIFFERENTIALS AND APPLICATIONS

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**Abstract.** In this paper we consider optimal control problems for abstract nonlinear evolution equations associated with time-dependent subdifferentials in a real Hilbert space. We prove the existence of an optimal control that minimizes the nonlinear cost functional. Also, we study approximating control problems of our equations. Then, we show the relationship between the original optimal control problem and the approximating ones. Moreover, we give some applications of our abstract results.

**1. Introduction.** This paper is concerned with an optimal control problem for a nonlinear evolution equation in a real Hilbert space $H$, of the form:

$$\begin{cases} u'(t) + \partial\varphi^t(u(t)) + g(t, u(t)) \ni f(t) \quad \text{in } H, \quad \text{for } t \in (0, T); \\ u(0) = u_0; \end{cases} \tag{1}$$

where $T > 0$ is a fixed finite time, $u' = \frac{du}{dt}$, $\partial\varphi^t$ is the subdifferential of a time-dependent convex function $\varphi^t$ on $H$, $g(t, \cdot)$ is a perturbation small relative to $\varphi^t$, $f$ is a given forcing term, and $u_0 \in H$ is a given initial condition.

The main object of this paper is to study the following optimal control problem to (1), denoted by (OP).

Problem (OP). Find a function (optimal control) $f^* \in L^2(0, T; H)$ such that

$$J(f^*) = \inf_{f \in L^2(0,T;H)} J(f).$$

2000 *Mathematics Subject Classification*: Primary 49J24; Secondary 35K90.
*Key words and phrases*: optimal control, evolution equations, subdifferentials.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc86-0-20

Here, $J$ is a cost functional defined by:

$$(2) \qquad J(f) := \frac{1}{2}\int_0^T |(u-u_d)(t)|_H^2 dt + \frac{1}{2}\int_0^T |f(t)|_H^2 dt \quad \text{for any } f \in L^2(0,T;H),$$

where $|\cdot|_H$ denotes the norm of $H$, the main parameter $f$ is the control, $u_d$ is a given desired target profile in $L^2(0,T;H)$, and $u$ is a unique solution to (1).

Many mathematicians have already studied the optimal control problems of abstract evolution equations (cf. [1, 3, 6, 7, 9, 13, 14, 15]). In particular, if $g(t,\cdot)$ is a continuous operator in $H$, Hu–Papageorgiou [7] studied the optimal control problem (OP). Also, Cardinali–Papageorgiou [3] studied the min-max problem for subdifferential evolution inclusions. For the related topics of optimal control problems for subdifferential evolution inclusions, we refer to the series of papers by Papageorgiou (cf. [3, 6, 7, 13]).

Now, the aim of the present paper is to consider the approximating problems of (1) and (OP). Then, the main novelties found in this paper are:

(i) to prove the existence of an optimal control for (OP);
(ii) to show the existence of optimal controls for the approximating problems of (OP);
(iii) to show the relationship between the limits of sequences of approximating optimal controls and the optimal controls of the limiting problem (OP).

The plan of this paper is as follows. In the next Section 2, we state the main results (Theorems 2.1–2.3) concerning (OP) and its approximating problems. In Section 3, we prove Theorem 2.1 concerned with the existence of an optimal control to (OP). In Section 4, we study the approximating problems of (OP). Then, by using the relationship between (1) and its approximating equations, we prove Theorems 2.2 and 2.3. In Section 5, we give some applications of our abstract results (Theorems 2.1–2.3).

**1.1.** *Notations.* Throughout this paper, let $H$ be a real Hilbert space with norm $|\cdot|_H$ and inner product $(\cdot,\cdot)$. For a proper (i.e., not identically equal to infinity), l.s.c. (lower semi-continuous) and convex function $\psi : H \to \mathbb{R}\cup\{\infty\}$, the effective domain $D(\psi)$ of $\psi$ is defined by

$$D(\psi) := \{z \in H;\ \psi(z) < \infty\}.$$

We denote by $\partial\psi$ the subdifferential of $\psi$ in the topology of $H$. In general, the subdifferential is a possibly multi-valued operator from $H$ into itself, and for any $z \in H$, the value $\partial\psi(z)$ is defined as:

$$\partial\psi(z) := \{z^* \in H\ ;\ (z^*, y-z) \le \psi(y) - \psi(z) \text{ for all } y \in H\}.$$

Then, the set $D(\partial\psi) := \{z \in H\ ;\ \partial\psi(z) \ne \emptyset\}$ is called the domain of $\partial\psi$. We refer to the monograph by Brézis [2] for detailed properties and related notions of convex functions and their subdifferentials.

## 2. Assumptions and main results.

We begin by defining the solutions of our state problem (1). In the following sections, we denote by (CP;$f,u_0$) the state problem (1) when the data of the control $f$ and the initial condition $u_0$ are specified:

$$(\mathrm{CP};f,u_0)\begin{cases} u'(t) + \partial\varphi^t(u(t)) + g(t,u(t)) \ni f(t) & \text{in } H, \quad \text{for } t\in(0,T); \\ u(0) = u_0. \end{cases}$$

Definition 2.1. Given $f \in L^2(0,T;H)$ and $u_0 \in H$, the function $u : [0,T] \to H$ is called a solution to (CP; $f, u_0$), if $u \in C([0,T];H)$, $u' \in L^2_{loc}((0,T];H)$, $\varphi^{(\cdot)}(u) \in L^1(0,T)$, $u(0) = u_0$, $u(t) \in D(\partial\varphi^t)$ and $f(t) - u'(t) - g(t,u(t)) \in \partial\varphi^t(u(t))$ for a.e. $t \in [0,T]$, namely

$$(f(t) - u'(t) - g(t,u(t)),\ y - u(t)) \le \varphi^t(y) - \varphi^t(u(t)) \quad \text{for any } y \in H, \text{ a.e. } t \in [0,T].$$

Let $\{a_r\} := \{a_r; r \ge 0\}$ and $\{b_r\} := \{b_r; r \ge 0\}$ be two families of real functions in $W^{1,2}(0,T)$ and $W^{1,1}(0,T)$, respectively. We introduce a class $\Phi(\{a_r\},\{b_r\})$ of families of time-dependent proper l.s.c. and convex functions $\varphi^t$ on $H$.

Definition 2.2. We denote by $\Phi(\{a_r\},\{b_r\})$ the class of all families $\{\varphi^t\} := \{\varphi^t; t \in [0,T]\}$ of proper l.s.c. and convex functions $\varphi^t$ on $H$ such that

$$\{z \in H; |z|_H \le k,\ \varphi^t(z) \le k\} \text{ is compact in } H \text{ for every } k \ge 0 \text{ and } t \in [0,T],$$

and the following property $(*)$ is fulfilled:

$(*)$ for each $r \ge 0$, $s$, $t \in [0,T]$ with $s \le t$, and $z \in D(\varphi^s)$ with $|z|_H \le r$, there exists $\tilde{z} \in D(\varphi^t)$ such that

$$|\tilde{z} - z|_H \le |a_r(t) - a_r(s)|(1 + |\varphi^s(z)|^{\frac{1}{2}})$$

and

$$\varphi^t(\tilde{z}) - \varphi^s(z) \le |b_r(t) - b_r(s)|(1 + |\varphi^s(z)|).$$

Remark 2.1 (cf. [8, Chapter 1]). Assume $\{\varphi^t\} \in \Phi(\{a_r\},\{b_r\})$, $f \in L^2(0,T;H)$ and $u_0 \in \overline{D(\varphi^0)}$, where $\overline{D(\varphi^0)}$ denotes the closure of $D(\varphi^0)$ in $H$. Then, Kenmochi [8, Chapter 1] has already proved that the following Cauchy problem has a unique solution $u$ on $[0,T]$:

$$\begin{cases} u'(t) + \partial\varphi^t(u(t)) \ni f(t) & \text{in } H, \qquad \text{for } t \in (0,T); \\ u(0) = u_0. \end{cases}$$

Next, we introduce the class $\mathcal{G}(\{\varphi^t\})$ of time-dependent perturbations $g(t,\cdot)$ associated with $\{\varphi^t\} \in \Phi(\{a_r\},\{b_r\})$.

Definition 2.3. We denote by $\mathcal{G}(\{\varphi^t\})$ the class of all families $\{g(t,\cdot)\} := \{g(t,\cdot);\ t \in [0,T]\}$ of single-valued operators $g(t,\cdot)$ from $D(g(t,\cdot)) \subset H$ into $H$ which fulfill the following conditions (g1)–(g4).

(g1) $D(\varphi^t) \subset D(g(t,\cdot)) \subset H$ for all $t \in [0,T]$, and $g(\cdot, v(\cdot))$ is (strongly) measurable on $J$ for any interval $J \subset [0,T]$ and $v \in L^2_{loc}(J;H)$ with $v(t) \in D(\varphi^t)$ for a.e. $t \in J$.

(g2) There are positive constants $C_0$, $C_1$ and $C_2$ such that

$$|g(t,z)|_H^2 \le C_0\varphi^t(z) + C_1|z|_H^2 + C_2, \quad \forall t \in [0,T],\ \forall z \in D(\varphi^t).$$

(g3) (Demi-closedness) If $\{t_n\} \subset [0,T]$, $z_n \in D(\varphi^{t_n})$, $t_n \to t$, $z_n \to z$ in $H$ (as $n \to \infty$) and $\{\varphi^{t_n}(z_n)\}$ is bounded, then $g(t_n, z_n) \to g(t,z)$ weakly in $H$ as $n \to \infty$.

(g4) For each $\delta > 0$, there exists a positive constant $C_\delta > 0$ such that

$$|(g(t,z_1) - g(t,z_2), z_1 - z_2)| \le \delta(z_1^* - z_2^*, z_1 - z_2) + C_\delta|z_1 - z_2|_H^2,$$
$$\forall t \in [0,T],\ \forall z_i \in D(\varphi^t),\ \forall z_i^* \in \partial\varphi^t(z_i),\ i = 1,2.$$

We recall that existence and uniqueness of solutions for (CP; $f, u_0$) was obtained in [12, Theorem III] and [17, Theorem 2.1].

Proposition 2.1 (cf. [12, Theorem III], [17, Theorem 2.1]). *Assume* $\{\varphi^t\}\in\Phi(\{a_r\},\{b_r\})$, $\{g(t,\cdot)\}\in\mathcal{G}(\{\varphi^t\})$, $f\in L^2(0,T;H)$ *and* $u_0\in\overline{D(\varphi^0)}$. *Then, the Cauchy problem* (CP; $f, u_0$) *has one and only one solution* $u$ *on* $[0,T]$. *In particular, if* $u_0 \in D(\varphi^0)$, *then the solution* $u$ *of* (CP; $f, u_0$) *satisfies that* $u' \in L^2(0,T;H)$.

Now, we state the first main result in this paper, which is concerned with the existence of an optimal control for (OP).

Theorem 2.1. *Assume* $\{\varphi^t\} \in \Phi(\{a_r\},\{b_r\})$, $\{g(t,\cdot)\} \in \mathcal{G}(\{\varphi^t\})$, $u_0 \in \overline{D(\varphi^0)}$ *and* $u_d \in L^2(0,T;H)$. *Then, the problem* (OP) *has at least one optimal control* $f^* \in L^2(0,T;H)$ *so that*

$$J(f^*) = \inf_{f\in L^2(0,T;H)} J(f),$$

*where* $J(\cdot)$ *is the cost functional given in* (2).

The proof of Theorem 2.1 is given in Section 3 by using the well-posedness of the state problem (CP; $f, u_0$).

Next, we study approximating problems for (CP; $f, u_0$) and (OP). In fact, for each $\varepsilon \in (0,1]$, we consider the following nonlinear evolution equation in a Hilbert space $H$, denoted by (CP; $f, u_{0,\varepsilon})_\varepsilon$:

$$\text{(3)} \quad (\mathrm{CP}; f, u_{0,\varepsilon})_\varepsilon \begin{cases} u_\varepsilon'(t) + \partial\varphi_\varepsilon^t(u_\varepsilon(t)) + g_\varepsilon(t,u_\varepsilon(t)) \ni f(t) & \text{in } H, \quad \text{for } t\in(0,T); \\ u(0) = u_{0,\varepsilon}; \end{cases}$$

where $\partial\varphi_\varepsilon^t$ is the subdifferential of the time-dependent proper l.s.c. convex function $\varphi_\varepsilon^t$ on $H$, $g_\varepsilon(t,\cdot)$ is a perturbation small relative to $\varphi_\varepsilon^t$, $f$ is a given forcing term, and $u_{0,\varepsilon}\in H$ is a given initial condition.

Clearly, we observe from Proposition 2.1 that for each $\varepsilon\in(0,1]$, the Cauchy problem (CP; $f, u_{0,\varepsilon})_\varepsilon$ has a unique solution $u_\varepsilon$ on $[0,T]$, if $\{\varphi_\varepsilon^t\} \in \Phi(\{a_r\},\{b_r\})$, $\{g_\varepsilon(t,\cdot)\} \in \mathcal{G}(\{\varphi_\varepsilon^t\})$, $f\in L^2(0,T;H)$ and $u_{0,\varepsilon}\in\overline{D(\varphi_\varepsilon^0)}$.

Now, for each $\varepsilon\in(0,1]$, we consider the following approximating optimal control problem, denoted by $(\mathrm{OP})_{1,\varepsilon}$, of the original problem (OP).

Problem $(\mathrm{OP})_{1,\varepsilon}$. Find a function (optimal control) $f_{1,\varepsilon}^* \in L^2(0,T;H)$ such that

$$J_{1,\varepsilon}(f_{1,\varepsilon}^*) = \inf_{f\in L^2(0,T;H)} J_{1,\varepsilon}(f).$$

Here, $J_{1,\varepsilon}$ is a cost functional defined by:

$$\text{(4)} \quad J_{1,\varepsilon}(f) := \frac{1}{2}\int_0^T |(u_\varepsilon - u_d)(t)|_H^2 dt + \frac{1}{2}\int_0^T |f(t)|_H^2 dt \quad \text{for any } f\in L^2(0,T;H),$$

where $u_\varepsilon$ is the unique solution of the approximating state problem (CP; $f, u_{0,\varepsilon})_\varepsilon$ on $[0,T]$.

The next object of this paper is to show the relationship between (OP) and $(\mathrm{OP})_{1,\varepsilon}$. To do so, we recall a notion of convergence for convex functions, developed by Mosco [10].

DEFINITION 2.4 (cf. [10]). Let $\psi$, $\psi_n$ ($n \in \mathbb{N}$) be proper, l.s.c. and convex functions on a Hilbert space $H$. Then, we say that $\psi_n$ converges to $\psi$ on $H$ in the sense of Mosco [10] as $n \to \infty$ if the following two conditions are satisfied:

(i) for any subsequence $\{\psi_{n_k}\} \subset \{\psi_n\}$, if $z_k \to z$ weakly in $H$ as $k \to \infty$, then

$$\liminf_{k\to\infty} \psi_{n_k}(z_k) \geq \psi(z);$$

(ii) for any $z \in D(\psi)$, there is a sequence $\{z_n\}$ in $H$ such that

$$z_n \to z \text{ in } H \text{ as } n \to \infty \quad \text{and} \quad \lim_{n\to\infty} \psi_n(z_n) = \psi(z).$$

Now, we state our second main result in this paper, which is concerned with the relationship between problems (OP) and $(\mathrm{OP})_{1,\varepsilon}$ ($\varepsilon \in (0,1]$).

THEOREM 2.2. *Assume $u_d \in L^2(0,T;H)$, $\varepsilon \in (0,1]$, $\{\varphi_\varepsilon^t\} \in \Phi(\{a_r\},\{b_r\})$, $\{g_\varepsilon(t,\cdot)\} \in \mathcal{G}(\{\varphi_\varepsilon^t\})$, and $u_{0,\varepsilon} \in \overline{D(\varphi_\varepsilon^0)}$. Then, for each $\varepsilon \in (0,1]$, the approximating problem $(\mathrm{OP})_{1,\varepsilon}$ has at least one optimal control $f_{1,\varepsilon}^* \in L^2(0,T;H)$ so that*

$$J_{1,\varepsilon}(f_{1,\varepsilon}^*) = \inf_{f\in L^2(0,T;H)} J_{1,\varepsilon}(f).$$

*Furthermore, assume that*

(A1) *$\varphi_\varepsilon^t$ converges to $\varphi^t$ on $H$ in the sense of Mosco [10] for each $t \in [0,T]$ (as $\varepsilon \to 0$), and $\bigcup_{\varepsilon\in[0,1]}\{z \in H; |z|_H \leq k,\ \varphi_\varepsilon^t(z) \leq k\}$ is relatively compact in $H$ for every real $k \geq 0$ and $t \in [0,T]$, where $\{\varphi_0^t\} = \{\varphi^t\} \in \Phi(\{a_r\},\{b_r\})$ when $\varepsilon = 0$;*

(A2) *$g_\varepsilon(t_\varepsilon, z_\varepsilon) \to g(t,z)$ weakly in $H$ (as $\varepsilon \to 0$), if $t_\varepsilon \in [0,T]$, $t_\varepsilon \to t$, $z_\varepsilon \to z$ in $H$ and $\{\varphi_\varepsilon^{t_\varepsilon}(z_\varepsilon)\}$ is bounded, where $\{g(t,\cdot)\} \in \mathcal{G}(\{\varphi^t\})$;*

(A3) *$u_{0,\varepsilon} \to u_0$ in $H$ for some $u_0 \in \overline{D(\varphi^0)}$.*

*Then, there is a subsequence $\{\varepsilon_k\} \subset \{\varepsilon\}$ and a function $f^{**} \in L^2(0,T;H)$ such that $f^{**}$ is an optimal control of* (OP)*, $\varepsilon_k \to 0$, and*

$$f_{1,\varepsilon_k}^* \to f^{**} \quad \text{weakly in } L^2(0,T;H) \text{ as } k \to \infty. \tag{5}$$

In order to show the strong convergence of optimal controls, we consider another type of approximating optimal control problems for each $\varepsilon \in (0,1]$, denoted by $(\mathrm{OP})_{2,\varepsilon}$, as follows.

PROBLEM $(\mathrm{OP})_{2,\varepsilon}$. Find a function (optimal control) $f_{2,\varepsilon}^* \in L^2(0,T;H)$ such that

$$J_{2,\varepsilon}(f_{2,\varepsilon}^*) = \inf_{f\in L^2(0,T;H)} J_{2,\varepsilon}(f).$$

Here, $J_{2,\varepsilon}$ is a cost functional defined by:

$$J_{2,\varepsilon}(f) := \frac{1}{2}\int_0^T |(u_\varepsilon - u_d)(t)|_H^2 dt + \frac{1}{2}\int_0^T |f(t)|_H^2 dt + \frac{1}{2}\int_0^T |(f - f^*)(t)|_H^2 dt, \tag{6}$$
$$\text{for any } f \in L^2(0,T;H)$$

where $f^*$ is the optimal control of (OP) found in Theorem 2.1, and $u_\varepsilon$ is the unique solution of the approximating state problem $(\mathrm{CP}; f, u_{0,\varepsilon})_\varepsilon$ on $[0,T]$.

Now, we state our final main result in this paper, which is concerned with the relationship between problems (OP) and $(\mathrm{OP})_{2,\varepsilon}$ ($\varepsilon \in (0,1]$).

Theorem 2.3. *Assume* $u_d \in L^2(0,T;H)$, $\varepsilon \in (0,1]$, $\{\varphi_\varepsilon^t\} \in \Phi(\{a_r\},\{b_r\})$, $\{g_\varepsilon(t,\cdot)\} \in \mathcal{G}(\{\varphi_\varepsilon^t\})$, *and* $u_{0,\varepsilon} \in \overline{D(\varphi_\varepsilon^0)}$. *Let* $\{\varphi^t\} \in \Phi(\{a_r\},\{b_r\})$, $\{g(t,\cdot)\} \in \mathcal{G}(\{\varphi^t\})$, $u_0 \in \overline{D(\varphi^0)}$, *and* $f^*$ *be the optimal control obtained in Theorem 2.1. Then, for each* $\varepsilon \in (0,1]$, *the approximating problem* $(\mathrm{OP})_{2,\varepsilon}$ *has at least one optimal control* $f_{2,\varepsilon}^* \in L^2(0,T;H)$ *so that*

$$J_{2,\varepsilon}(f_{2,\varepsilon}^*) = \inf_{f\in L^2(0,T;H)} J_{2,\varepsilon}(f).$$

*Furthermore, suppose the convergence assumptions* (A1), (A2) *and* (A3) *in Theorem 2.2 hold. Let* $u_\varepsilon^*$ *and* $u^*$ *be the unique solutions of* $(\mathrm{CP};f_{2,\varepsilon}^*,u_{0,\varepsilon})_\varepsilon$ *and* $(\mathrm{CP};f^*,u_0)$ *on* $[0,T]$, *respectively. Then, there is a subsequence* $\{\varepsilon_k\} \subset \{\varepsilon\}$ *such that* $\varepsilon_k \to 0$,

$$f_{2,\varepsilon_k}^* \to f^* \quad \text{strongly in } L^2(0,T;H) \tag{7}$$

*and*

$$u_{\varepsilon_k}^* \to u^* \quad \text{strongly in } C([0,T];H) \tag{8}$$

*as* $k \to \infty$.

The proof of Theorems 2.2 and 2.3 is given in Section 4. Roughly, the convergences (5) and (7) are proved by using the continuous dependence between solutions of $(\mathrm{CP};f,u_0)$ and the approximating solutions of $(\mathrm{CP};f,u_{0,\varepsilon})_\varepsilon$.

**3. Optimal control problem (OP).** In this section, we prove Theorem 2.1 concerned with the existence of an optimal control for (OP). Throughout this section, we assume all the conditions of Theorem 2.1.

First, we recall the result of continuous dependence of solutions for $(\mathrm{CP};f,u_0)$, stated as follows.

Proposition 3.1 (cf. [17, Lemma 4.1]). *Assume* $\{\varphi^t\} \in \Phi(\{a_r\},\{b_r\})$ *and* $\{g(t,\cdot)\} \in \mathcal{G}(\{\varphi^t\})$. *Let* $f \in L^2(0,T;H)$, $u_0 \in \overline{D(\varphi^0)}$, *and* $u$ *be the unique solution to* $(\mathrm{CP};f,u_0)$ *on* $[0,T]$. *Also, let* $\{f_n\} \subset L^2(0,T;H)$, $\{u_{0,n}\} \subset \overline{D(\varphi^0)}$, *and* $u_n$ *be the unique solution to* $(\mathrm{CP};f_n,u_{0,n})$ *on* $[0,T]$. *Assume that*

$$u_{0,n} \to u_0 \text{ in } H \quad \text{and} \quad f_n \to f \text{ weakly in } L^2(0,T;H) \quad \text{as } n \to \infty.$$

*Then, the following convergence holds:*

$$u_n \to u \quad \text{strongly in } C([0,T];H) \text{ as } n \to \infty.$$

Proposition 3.1 has already been proved in [17, Lemma 4.1].

Now, let us prove our main Theorem 2.1, which is concerned with the existence of an optimal control for (OP).

*Proof of Theorem 2.1.* By a quite standard method, we can prove Theorem 2.1. In fact, let $\{f_n\} \subset L^2(0,T;H)$ be a minimizing sequence so that

$$\lim_{n\to\infty} J(f_n) = \inf_{f\in L^2(0,T;H)} J(f).$$

Then, by the definition (2) of $J(\cdot)$, we see that $\{f_n\}$ is bounded in $L^2(0,T;H)$. Hence, there is a subsequence $\{n_k\} \subset \{n\}$ and a function $f^* \in L^2(0,T;H)$ such that $n_k \to \infty$ and

$$f_{n_k} \to f^* \text{ weakly in } L^2(0,T;H) \quad \text{as } k \to \infty. \tag{9}$$

For any $k \in \mathbb{N}$, let $u_{n_k}$ be the unique solution to (CP;$f_{n_k}, u_0$) on $[0,T]$. Then, from (9) and Proposition 3.1, we observe that

$$u_{n_k} \to u^* \quad \text{strongly in } C([0,T];H) \ \text{ as } \ k \to \infty, \tag{10}$$

where $u^*$ is the unique solution to (CP;$f^*, u_0$) on $[0,T]$.

Hence, it follows from (9)–(10) and the weak lower semicontinuity of $L^2$-norm that

$$J(f^*) \le \lim_{k\to\infty} J(f_{n_k}) = \inf_{f\in L^2(0,T;H)} J(f).$$

The above inequality implies that $f^* \in L^2(0,T;H)$ is an optimal control for (OP). Thus, the proof of Theorem 2.1 has been completed. ■

**4. Approximating problems (OP)$_{1,\varepsilon}$ and (OP)$_{2,\varepsilon}$.** In this section, we consider the approximating optimal control problems (OP)$_{1,\varepsilon}$ and (OP)$_{2,\varepsilon}$ ($\varepsilon \in (0,1]$). Then, by using the convergence result for solutions to (CP; $f, u_0$) and (CP;$f, u_{0,\varepsilon}$)$_\varepsilon$, we prove Theorem 2.2 (resp. Theorem 2.3) concerned with the relationship between (OP) and its approximating problems (OP)$_{1,\varepsilon}$ (resp. (OP)$_{2,\varepsilon}$).

Here, we give the key proposition to showing Theorems 2.2 and 2.3.

PROPOSITION 4.1 (cf. [17, Lemma 4.2]). *Assume* $\varepsilon\in(0,1]$, $\{\varphi^t_\varepsilon\}\in\Phi(\{a_r\},\{b_r\})$, $\{g_\varepsilon(t,\cdot)\} \in \mathcal{G}(\{\varphi^t_\varepsilon\})$, $u_{0,\varepsilon} \in \overline{D(\varphi^0_\varepsilon)}$, $\{\varphi^t\} \in \Phi(\{a_r\},\{b_r\})$, $\{g(t,\cdot)\} \in \mathcal{G}(\{\varphi^t\})$ *and* $u_0 \in \overline{D(\varphi^0)}$. *Also, suppose the convergence assumptions* (A1), (A2) *and* (A3) *in Theorem 2.2 hold. Furthermore, assume* $\{f_\varepsilon\} \subset L^2(0,T;H)$, $f \in L^2(0,T;H)$ *and*

$$f_\varepsilon \to f \ \textit{weakly in } L^2(0,T;H) \quad \textit{as } \varepsilon \to 0.$$

*Then, the solution* $u_\varepsilon$ *of* (CP;$f_\varepsilon, u_{0,\varepsilon}$)$_\varepsilon$ *converges to the solution* $u$ *of* (CP;$f, u_0$) *on* $[0,T]$ *in the following sense:*

$$u_\varepsilon \to u \quad \textit{strongly in } C([0,T];H) \ \textit{as} \ \varepsilon \to 0.$$

By a slight modification of the proof of [17, Lemma 4.2], we can show Proposition 4.1, so we omit its proof.

Now, by using Proposition 4.1, we prove Theorem 2.2, which is concerned with the relationship between (OP) and (OP)$_{1,\varepsilon}$ ($\varepsilon \in (0,1]$).

*Proof of Theorem 2.2.* First, note from Proposition 3.1 that we obtain the convergence of solutions for (CP; $f, u_{0,\varepsilon}$)$_\varepsilon$. In fact, we have only to replace $\varphi^t$ (resp. $g(t,\cdot)$) with $\varphi^t_\varepsilon$ (resp. $g_\varepsilon(t,\cdot)$) in Proposition 3.1. Thus, for each $\varepsilon \in (0,1]$, by the same proof of Theorem 2.1, we can show the existence of an optimal control $f^*_{1,\varepsilon}$ of (OP)$_{1,\varepsilon}$ such that

$$J_{1,\varepsilon}(f^*_{1,\varepsilon}) = \inf_{f\in L^2(0,T;H)} J_{1,\varepsilon}(f),$$

where $J_{1,\varepsilon}(\cdot)$ is the cost functional defined in (4).

Now, we show (5). Let $f$ be any function in $L^2(0,T;H)$. Also, let $u_\varepsilon$ be the unique solution for (CP;$f, u_{0,\varepsilon}$)$_\varepsilon$ on $[0,T]$, and let $u$ be the unique solution for (CP;$f, u_0$) on $[0,T]$. Then, we observe from the assumptions (A1)–(A3) and Proposition 4.1 that

$$u_\varepsilon \to u \quad \text{strongly in } C([0,T];H) \ \text{ as } \varepsilon \to 0. \tag{11}$$

Since $f^*_{1,\varepsilon}$ is the optimal control of $(\mathrm{OP})_{1,\varepsilon}$, we see that

$$J_{1,\varepsilon}(f^*_{1,\varepsilon}) \le J_{1,\varepsilon}(f) = \frac{1}{2}\int_0^T |(u_\varepsilon - u_d)(t)|_H^2 dt + \frac{1}{2}\int_0^T |f(t)|_H^2 dt. \tag{12}$$

Clearly, it follows from (4), (11)–(12) that $\{f^*_{1,\varepsilon}\}$ is bounded in $L^2(0,T;H)$ with respect to $\varepsilon \in (0,1]$. Thus, there is a subsequence $\{\varepsilon_k\} \subset \{\varepsilon\}$ and a function $f^{**} \in L^2(0,T;H)$ such that $\varepsilon_k \to 0$ and

$$f^*_{1,\varepsilon_k} \to f^{**} \quad \text{weakly in } L^2(0,T;H) \quad \text{as } k \to \infty. \tag{13}$$

For any $k \in \mathbb{N}$, let $u^*_{\varepsilon_k}$ be the unique solution of $(\mathrm{CP}; f^*_{1,\varepsilon_k}, u_{0,\varepsilon_k})_{\varepsilon_k}$ on $[0,T]$. Then, by (13), the assumptions (A1)–(A3) and Proposition 4.1, we see that $u^*_{\varepsilon_k}$ converges to the unique solution $u^{**}$ of $(\mathrm{CP}; f^{**}, u_0)$ on $[0,T]$ in the sense that

$$u^*_{\varepsilon_k} \to u^{**} \quad \text{strongly in } C([0,T];H) \quad \text{as } k \to \infty. \tag{14}$$

Now, by using (11)–(14) and the weak lower semicontinuity of $L^2$-norm, we see that

$$J(f^{**}) \le \liminf_{k\to\infty} J_{1,\varepsilon_k}(f^*_{1,\varepsilon_k}) \le J(f).$$

Since $f$ is any function in $L^2(0,T;H)$, we infer from the above inequality that $f^{**}$ is the optimal control of (OP) satisfying the convergence (13) (i.e. (5)). Thus, the proof of Theorem 2.2 has been completed. ■

Next, by using Proposition 4.1, we show Theorem 2.3, which is concerned with the relationship between the optimal control problems (OP) and $(\mathrm{OP})_{2,\varepsilon}$ ($\varepsilon \in (0,1]$).

*Proof of Theorem 2.3.* First, note that by the same argument in Theorem 2.1, namely, by using the continuous dependence of solutions for $(\mathrm{CP}; f, u_{0,\varepsilon})_\varepsilon$ (cf. Proposition 3.1), we can get the existence of an optimal control $f^*_{2,\varepsilon}$ of $(\mathrm{OP})_{2,\varepsilon}$ for each $\varepsilon \in (0,1]$, such that

$$J_{2,\varepsilon}(f^*_{2,\varepsilon}) = \inf_{f \in L^2(0,T;H)} J_{2,\varepsilon}(f),$$

where $J_{2,\varepsilon}(\cdot)$ is the cost functional defined in (6).

Now, we show (7)–(8). Let $f^*$ be the optimal control of (OP) obtained in Theorem 2.1. Also, let $u_\varepsilon$ be the unique solution to $(\mathrm{CP}; f^*, u_{0,\varepsilon})_\varepsilon$ on $[0,T]$, and let $u^*$ be the unique solution to $(\mathrm{CP}; f^*, u_0)$ on $[0,T]$. Then, we observe from the assumptions (A1)–(A3) and Proposition 4.1 that

$$u_\varepsilon \to u^* \quad \text{strongly in } C([0,T];H) \ \text{ as } \varepsilon \to 0. \tag{15}$$

On the other hand, since $f^*_{2,\varepsilon}$ is the optimal control of $(\mathrm{OP})_{2,\varepsilon}$, we see that

$$J_{2,\varepsilon}(f^*_{2,\varepsilon}) \le J_{2,\varepsilon}(f^*) = \frac{1}{2}\int_0^T |(u_\varepsilon - u_d)(t)|_H^2 dt + \frac{1}{2}\int_0^T |f^*(t)|_H^2 dt. \tag{16}$$

Clearly, it follows from (6) and (15)–(16) that $\{f^*_{2,\varepsilon}\}$ is bounded in $L^2(0,T;H)$ with respect to $\varepsilon \in (0,1]$. Thus, there is a subsequence $\{\varepsilon_k\} \subset \{\varepsilon\}$ and a function $f^\circ \in L^2(0,T;H)$ such that $\varepsilon_k \to 0$ and

$$f^*_{2,\varepsilon_k} \to f^\circ \quad \text{weakly in } L^2(0,T;H) \quad \text{as } k \to \infty. \tag{17}$$

For any $k \in \mathbb{N}$, let $u^*_{\varepsilon_k}$ be the unique solution of (CP;$f^*_{2,\varepsilon_k}, u_{0,\varepsilon_k})_{\varepsilon_k}$ on $[0,T]$. Then, by the assumptions (A1)–(A3), (17) and Proposition 4.1, we see that $u^*_{\varepsilon_k}$ converges to the unique solution $u^\circ$ of (CP;$f^\circ, u_0$) on $[0,T]$ in the sense that

$$u^*_{\varepsilon_k} \to u^\circ \quad \text{strongly in } C([0,T];H) \quad \text{as } k \to \infty. \tag{18}$$

Now, by using (15)–(18) and the weak lower semicontinuity of $L^2$-norm, we see that

$$\begin{aligned}
&\frac{1}{2}\limsup_{k\to\infty}\int_0^T |(f^*_{2,\varepsilon_k} - f^*)(t)|_H^2 dt \\
&\le \limsup_{k\to\infty}\left( J_{2,\varepsilon_k}(f^*) - \frac{1}{2}\int_0^T |(u^*_{\varepsilon_k} - u_d)(t)|_H^2 dt - \frac{1}{2}\int_0^T |f^*_{2,\varepsilon_k}(t)|_H^2 dt \right) \\
&\le \frac{1}{2}\int_0^T |(u^* - u_d)(t)|_H^2 dt + \frac{1}{2}\int_0^T |f^*(t)|_H^2 dt \\
&\qquad -\frac{1}{2}\liminf_{k\to\infty}\int_0^T |(u^*_{\varepsilon_k} - u_d)(t)|_H^2 dt - \frac{1}{2}\liminf_{k\to\infty}\int_0^T |f^*_{2,\varepsilon_k}(t)|_H^2 dt \\
&\le J(f^*) - \frac{1}{2}\int_0^T |(u^\circ - u_d)(t)|_H^2 dt - \frac{1}{2}\int_0^T |f^\circ(t)|_H^2 dt \\
&= J(f^*) - J(f^\circ).
\end{aligned}$$

Thus, we have

$$J(f^\circ) + \frac{1}{2}\limsup_{k\to\infty}\int_0^T |(f^*_{2,\varepsilon_k} - f^*)(t)|_H^2 dt \le J(f^*).$$

Since $f^*$ is the optimal control to (OP), we see that

$$\frac{1}{2}\limsup_{k\to\infty}\int_0^T |(f^*_{2,\varepsilon_k} - f^*)(t)|_H^2 dt = 0. \tag{19}$$

Therefore, we observe from (17) and (19) that $f^\circ = f^*$ and the convergence (7) holds, i.e.,

$$f^*_{2,\varepsilon_k} \to f^* \quad \text{strongly in } L^2(0,T;H) \quad \text{as } k \to \infty.$$

Also, we infer from (18) and the uniqueness of solutions to (P;$f^*, u_0$) that $u^\circ = u^*$ and the convergence (8) holds, i.e.,

$$u^*_{\varepsilon_k} \to u^* \quad \text{strongly in } C([0,T];H) \quad \text{as } k \to \infty.$$

Thus, the proof of Theorem 2.3 has been completed. ∎

**5. Applications.** In this section, we give some applications of our abstract results (Theorems 2.1–2.3).

**5.1.** *Mixed boundary condition.* Let us consider the following initial-boundary value problem with a Signorini–Dirichlet–Neumann type mixed boundary condition, denoted by (P).

PROBLEM (P). Find a function $u \in C([0,T];L^2(\Omega)) \cap W^{1,2}_{loc}((0,T];L^2(\Omega))$ which fulfills the following system:

$$\begin{aligned}
u' - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + g(u) = f(t,x) \qquad & \text{in } (0,T)\times\Omega;\\
u \le h(t), \qquad \nu\cdot(|\nabla u|^{p-2}\nabla u) \le 0 \qquad & \\
\text{and} \qquad (u-h(t))\nu\cdot(|\nabla u|^{p-2}\nabla u) = 0 \qquad & \text{on } (0,T)\times\Gamma_S;\\
u = h(t) \qquad & \text{on } (0,T)\times\Gamma_D;\\
\nu\cdot(|\nabla u|^{p-2}\nabla u) = 0 \qquad & \text{on } (0,T)\times\Gamma_N;\\
u(0,\cdot) = u_0 \qquad & \text{in } \Omega.
\end{aligned}$$

Here, $p$ is a fixed number with $2 \le p < \infty$, $\Omega$ is a bounded domain in $\mathbb{R}^m$ $(m \ge 1)$, and the boundary $\Gamma$ of $\Omega$ is smooth and admits a mutually disjoint decomposition such as

$$\Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_S.$$

Also, $\nu$ is the outward normal vector on the boundary $\Gamma$, $g : \mathbb{R} \to \mathbb{R}$ is a given Lipschitz function, $f$ and $u_0$ are given data, and $h(t,x)$ is a given function satisfying

$$h \in W^{1,2}(0,T;W^{1,p}(\Omega)).$$

The main object of this subsection is to consider the optimal control problem of (P) by applying the abstract result (Theorem 2.1). To do so, for each $t \in [0,T]$, define a convex set $K(t)$ by

$$K(t) := \{z \in W^{1,p}(\Omega)\ ;\ z \le h(t) \text{ on } \Gamma_S \text{ and } z = h(t) \text{ on } \Gamma_D\}.$$

Also, we choose $L^2(\Omega)$ as a real Hilbert space $H$, and we define a family $\{\varphi^t\}$ of proper l.s.c. convex functions $\varphi^t : L^2(\Omega) \to \mathbb{R}\cup\{\infty\}$ by

$$\varphi^t(z) := \begin{cases} \dfrac{1}{p}\displaystyle\int_\Omega |\nabla z(x)|^p dx & \text{if } z \in K(t),\\ \infty & \text{if } z \in L^2(\Omega)\setminus K(t). \end{cases} \tag{20}$$

Then, by similar calculations to [8, Proposition 3.2.2], we can get the following lemma.

LEMMA 5.1 (cf. [8, Proposition 3.2.2]). *Put for any $r \ge 0$ and $t \in [0,T]$*

$$a_r(t) = b_r(t) := M\int_0^t |h'(\tau)|_{W^{1,p}(\Omega)} d\tau,$$

*where $M$ is a (sufficiently large) positive constant. Then, $\{\varphi^t\} \in \Phi(\{a_r\},\{b_r\})$ and $\{g(\cdot)\} \in \mathcal{G}(\{\varphi^t\})$.*

Clearly, the initial-boundary value problem (P) can be transformed into the following evolution equation (CP; $f, u_0$):

$$(\mathrm{CP}; f, u_0) \begin{cases} u'(t) + \partial\varphi^t(u(t)) + g(u(t)) \ni f(t) & \text{in } L^2(\Omega), \qquad \text{for } t \in (0,T);\\ u(0) = u_0. \end{cases}$$

Therefore, by applying Proposition 2.1, we see that the initial-boundary value problem (P) has one and only one solution $u$ on $[0,T]$ for each $f \in L^2(0,T;L^2(\Omega))$ and $u_0 \in \overline{K(0)}^{L^2(\Omega)} = \overline{D(\varphi^0)}^{L^2(\Omega)}$, where $\overline{K(0)}^{L^2(\Omega)}$ denotes the closure of $K(0)$ in $L^2(\Omega)$.

Also, by applying Theorem 2.1 we see that for each $u_d \in L^2(0,T;L^2(\Omega))$ and $u_0 \in \overline{K(0)}^{L^2(\Omega)}$, the following optimal control problem to (P) has at least one solution $f^* \in L^2(0,T;L^2(\Omega))$:

$$J(f^*) = \inf_{f \in L^2(0,T;L^2(\Omega))} J(f), \tag{21}$$

where $J(\cdot)$ is the cost functional given in (2) with $H = L^2(\Omega)$.

Next, we study the approximating problem of (P) from the view-point of numerical analysis. In fact, for each $\varepsilon \in (0,1]$ let us consider the following approximating problem of (P), denoted by $(\mathrm{P})_\varepsilon$.

PROBLEM $(\mathrm{P})_\varepsilon$. Find a function $u_\varepsilon \in C([0,T];L^2(\Omega)) \cap W^{1,2}_{loc}((0,T];L^2(\Omega))$ which fulfills the following system:

$$\begin{aligned} u'_\varepsilon - \operatorname{div}(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon) + g(u_\varepsilon) &= f(t,x) && \text{in } (0,T)\times\Omega; \\ \nu\cdot(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon) &= -\frac{u_\varepsilon - h(t)}{\varepsilon}\cdot\chi_D - \frac{[u_\varepsilon - h(t)]^+}{\varepsilon}\cdot\chi_S && \text{on } (0,T)\times\Gamma; \\ u_\varepsilon(0,\cdot) &= u_{0,\varepsilon} && \text{in } \Omega; \end{aligned}$$

where $u_{0,\varepsilon} \in L^2(\Omega)$, $\chi_j$ is the characteristic function of $(0,T)\times\Gamma_j$ $(j = D, S)$, and $[z]^+$ is the positive part of $z$.

The next object is to consider the optimal control problem of $(\mathrm{P})_\varepsilon$ by applying the abstract results (Theorems 2.2 and 2.3). To do so, for each $\varepsilon \in (0,1]$ we define a family $\{\varphi^t_\varepsilon\}$ of proper l.s.c. convex functions $\varphi^t_\varepsilon : L^2(\Omega) \to \mathbb{R}\cup\{\infty\}$ by

$$\varphi^t_\varepsilon(z) := \begin{cases} \dfrac{1}{p}\displaystyle\int_\Omega |\nabla z(x)|^p dx + \dfrac{1}{2\varepsilon}\int_{\Gamma_D} |z - h(t)|^2 d\Gamma & \\ \qquad\qquad + \dfrac{1}{2\varepsilon}\displaystyle\int_{\Gamma_S} ([z-h(t)]^+)^2 d\Gamma & \text{if } z \in H^1(\Omega), \\ \infty & \text{if } z \in L^2(\Omega)\setminus H^1(\Omega). \end{cases}$$

By using the same function $a_r(t) = b_r(t)$ in Lemma 5.1, we observe that $\{\varphi^t_\varepsilon\} \in \Phi(\{a_r\},\{b_r\})$ and $\{g(\cdot)\} \in \mathcal{G}(\{\varphi^t_\varepsilon\})$. Also, we get the following property.

LEMMA 5.2. *$\varphi^t_\varepsilon$ converges to $\varphi^t$ in the sense of Mosco [10] for each $t \in [0,T]$ as $\varepsilon \to 0$, where $\varphi^t$ is the proper, l.s.c. convex function defined in* (20).

It is very easy to prove Lemma 5.2, since we employ the standard approximation of the boundary by the penalty method in the problem $(\mathrm{P})_\varepsilon$. So, we omit the detailed proof of Lemma 5.2.

Also, we easily see that the approximating initial-boundary value problem $(\mathrm{P})_\varepsilon$ can be reformulated as the following evolution equation $(\mathrm{CP}; f, u_{0,\varepsilon})_\varepsilon$:

$$(\mathrm{CP}; f, u_{0,\varepsilon})_\varepsilon \begin{cases} u'_\varepsilon(t) + \partial\varphi^t_\varepsilon(u_\varepsilon(t)) + g(u_\varepsilon(t)) \ni f(t) \quad \text{in } L^2(\Omega), \quad \text{for } t \in (0,T); \\ u_\varepsilon(0) = u_{0,\varepsilon}. \end{cases}$$

Therefore, by taking account of Lemma 5.2, and applying Proposition 2.1 and Theorems 2.2–2.3, we get the existence of a unique solution of $(\mathrm{P})_\varepsilon$, and the relationship between the problem (21) and the optimal control problems of $(\mathrm{P})_\varepsilon$.

**5.2.** *Allen-Cahn type equation with constraint.* In this subsection, let us consider the optimal control problem of the following Allen-Cahn type equation with constraint, denoted by (P).

PROBLEM (P). Find a function $u : [0,T] \to L^2(0,1)$ which fulfills the following singular diffusion equation with constraint:

$$u' - \kappa \left( \frac{u_x}{|u_x|} \right)_x + \partial I_{[-1,1]}(u) \ni u + f(t,x) \quad \text{in } Q := (0,T) \times (0,1);$$
$$u_x(t,0) = u_x(t,1) = 0, \quad t \in (0,T);$$
$$u(0,x) = u_0(x), \quad x \in (0,1).$$

Here, $\kappa > 0$ is a given (small) constant, $f$ and $u_0$ are given data, and $\partial I_{[-1,1]}(\cdot)$ is the subdifferential of the indicator function $I_{[-1,1]}(\cdot)$ on the closed interval $[-1,1]$, that is defined as:

$$I_{[-1,1]}(\tau) := \begin{cases} 0 & \text{if } \tau \in [-1,1], \\ \infty & \text{otherwise.} \end{cases}$$

In order to transform the problem (P) into an abstract nonlinear evolution equation, let us recall the definition of the total variation and bounded variation functions.

DEFINITION 5.1. (I) Let $z \in L^1(0,1)$. Then, $z$ is called a bounded variation function, or simply a BV-function, in $(0,1)$, if and only if:

$$V_0(z) := \sup \left\{ \int_0^1 z\eta_x dx \ ; \ \begin{array}{l} \eta \in C^1[0,1] \text{ with a compact support in } (0,1), \\ |\eta| \le 1 \text{ on } [0,1] \end{array} \right\} < \infty.$$

Here, we call $V_0(z)$ the total variation of $z$.
(II) We denote by $BV(0,1)$ the space of all BV-functions in $(0,1)$.

Now, let us choose $L^2(0,1)$ as a real Hilbert space $H$, and let us define a functional $\varphi : L^2(0,1) \to \mathbb{R} \cup \{\infty\}$ by

$$\varphi(z) = \kappa V_0(z) + \int_0^1 I_{[-1,1]}(z)dx \quad \text{for any } z \in L^2(0,1). \tag{22}$$

Then, it follows from [4, Chapter 5] that $\varphi$ is proper, l.s.c. and convex on $L^2(0,1)$, and its effective domain is

$$D(\varphi) = \{ z \in BV(0,1) \ ; \ |z| \le 1, \text{ a.e. in } (0,1) \}.$$

Clearly, the initial-boundary value problem (P) can be reformulated as the following evolution equation $(\mathrm{CP}; f, u_0)$:

$$(\mathrm{CP}; f, u_0) \begin{cases} u'(t) + \partial\varphi(u(t)) - u(t) \ni f(t) & \text{in } L^2(0,1), \quad \text{for } t \in (0,T); \\ u(0) = u_0. \end{cases}$$

Here, we take the function $a_r(t) = b_r(t) \equiv 0$ for any $r \ge 0$ and $t \in [0,T]$, and define the operator $g(u) := -u$ in $L^2(0,1)$. Then, we easily see that $\{\varphi\} \in \Phi(\{a_r\}, \{b_r\})$ and

$\{g(\cdot)\} \in \mathcal{G}(\{\varphi\})$. Therefore, by applying Proposition 2.1, we see that the problem (P) has one and only one solution $u \in C([0,T]; L^2(0,1)) \cap W^{1,2}_{loc}((0,T]; L^2(0,1))$ for each $f \in L^2(0,T; L^2(0,1))$ and $u_0 \in \overline{D(\varphi)}^{L^2(0,1)}$, where $\overline{D(\varphi)}^{L^2(0,1)}$ denotes the closure of $D(\varphi)$ in $L^2(0,1)$.

Also, by applying Theorem 2.1, we observe that for each $u_d \in L^2(0,T; L^2(0,1))$ and $u_0 \in \overline{D(\varphi)}^{L^2(0,1)}$, the following optimal control problem to (P) has at least one solution $f^* \in L^2(0,T; L^2(0,1))$:

$$J(f^*) = \inf_{f \in L^2(0,T;L^2(0,1))} J(f), \tag{23}$$

where $J(\cdot)$ is the cost functional given in (2) with $H = L^2(0,1)$.

Next, for each $\varepsilon \in (0,1]$ let us consider the following approximating problem of (P), denoted by $(\mathrm{P})_\varepsilon$.

Problem $(\mathrm{P})_\varepsilon$. Find a function $u_\varepsilon : [0,T] \to L^2(0,1)$ which fulfills the following equations:

$$u_\varepsilon' - \kappa \left( \frac{(u_\varepsilon)_x}{\sqrt{|(u_\varepsilon)_x|^2 + \varepsilon^2}} + \varepsilon (u_\varepsilon)_x \right)_x + F_\varepsilon(u_\varepsilon) = u_\varepsilon + f(t,x) \quad \text{in } Q;$$
$$(u_\varepsilon)_x(t,0) = (u_\varepsilon)_x(t,1) = 0, \quad t \in (0,T);$$
$$u_\varepsilon(0,x) = u_{0,\varepsilon}(x), \quad x \in (0,1).$$

Here, $F_\varepsilon$ is a nondecreasing function on $\mathbb{R}$ defined by

$$F_\varepsilon(r) := \mathrm{sign}(r) \int_0^{|r|} \min\left\{ \frac{1}{\varepsilon}, \frac{[s-1]^+}{\varepsilon^2} \right\} ds \quad \text{for } r \in \mathbb{R},$$

where $[s]^+$ denotes the positive part of $s$. Clearly, $F_\varepsilon$ is a $C^1$-function with derivative $F_\varepsilon' \in W^{1,\infty}(\mathbb{R})$. Note that for each $\varepsilon \in (0,1]$, the singular diffusion term $(u_x/|u_x|)_x$ and the constraint $\partial I_{[-1,1]}(u)$ in the problem (P) are approximated by

$$\left( \frac{(u_\varepsilon)_x}{\sqrt{|(u_\varepsilon)_x|^2 + \varepsilon^2}} + \varepsilon (u_\varepsilon)_x \right)_x \quad \text{and} \quad F_\varepsilon(u_\varepsilon),$$

respectively.

Now, we fix a primitive $\widehat{F}_\varepsilon$ of $F_\varepsilon$ such that

$$\widehat{F}_\varepsilon(0) = 0 \quad \text{and} \quad \widehat{F}_\varepsilon(r) \geq 0 \quad \text{for all } r \in \mathbb{R}.$$

Then, for any $\varepsilon \in (0,1]$, let us set:

$$\varphi_\varepsilon(z) := \begin{cases} \kappa \displaystyle\int_0^1 \sqrt{|z_x|^2 + \varepsilon^2}\,dx + \frac{\varepsilon\kappa}{2} \int_0^1 |z_x|^2 dx + \int_0^1 \widehat{F}_\varepsilon(z) dx & \text{if } z \in H^1(0,1), \\ \infty & \text{otherwise.} \end{cases}$$

Clearly, each functional $\varphi_\varepsilon$ ($\varepsilon \in (0,1]$) forms a proper, l.s.c. and convex functional on $L^2(0,1)$ such that $\{\varphi_\varepsilon\} \in \Phi(\{a_r\},\{b_r\})$ with $a_r(t) = b_r(t) \equiv 0$ for any $r \geq 0$ and $t \in [0,T]$. Also, we get the following property.

Lemma 5.3 (cf. [16, Lemma 3.1]). *$\varphi_\varepsilon$ converges to $\varphi$ in the sense of Mosco [10] as $\varepsilon \to 0$, where $\varphi$ is the proper, l.s.c. convex function defined in* (22).

By a slight modification of [16, Lemma 3.1], we can prove Lemma 5.3, so we omit its proof. For the detailed arguments, we refer to [16, Lemma 3.1].

We easily see that the approximating initial-boundary value problem $(\mathrm{P})_\varepsilon$ can be reformulated as the following evolution equation $(\mathrm{CP}; f, u_{0,\varepsilon})_\varepsilon$:

$$(\mathrm{CP}; f, u_{0,\varepsilon})_\varepsilon \begin{cases} u_\varepsilon'(t) + \partial\varphi_\varepsilon(u_\varepsilon(t)) - u_\varepsilon(t) \ni f(t) \quad \text{in } L^2(0,1), \qquad \text{for } t \in (0,T); \\ u_\varepsilon(0) = u_{0,\varepsilon}. \end{cases}$$

Here, we define the operator $g_\varepsilon(z) := -z$ in $L^2(0,1)$. Then, we easily see that $\{g_\varepsilon(\cdot)\} \in \mathcal{G}(\{\varphi_\varepsilon\})$. Therefore, by taking account of Lemma 5.3, and applying Proposition 2.1 and Theorems 2.2–2.3, we get the existence of a unique solution of $(\mathrm{P})_\varepsilon$, and the relationship between the problem (23) and the optimal control problems of $(\mathrm{P})_\varepsilon$.

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