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VARIATIONAL PROBLEMS AND PDES IN AFFINE DIFFERENTIAL GEOMETRY

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Abstract. This paper is part of the autumn school on "Variational problems and higher order PDEs for affine hypersurfaces". We discuss variational problems in equiaffine differential geometry, centroaffine differential geometry and relative differential geometry, which have been studied by Blaschke [Bla], Chern [Ch], C. P. Wang [W], Li-Li-Simon [LLS], and Calabi [Ca-II]. We first derive the Euler-Lagrange equations in these settings; these equations are complicated, strongly nonlinear fourth order PDEs. We consider classes of solutions satisfying these equations together with completeness conditions. We also formulate Bernstein problems and give partial solutions.

Contents

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Many geometric problems in analytic formulation lead to important classes of PDEs. The famous Minkowski and Bernstein problems are just two classical examples of such problems which stimulated major developments in the theory of second order nonlinear PDEs. Naturally, since the equations arise in geometric context, geometric methods play a crucial role in these developments.

In affine differential geometry one often encounters fourth order nonlinear PDEs which are far from being well understood. Consequently, new and significant efforts are required for their investigation. Again, the natural approaches are typically based on geometric ideas. The purpose of this talk is to study the fourth order equations associated with the Bernstein problems in equiaffine differential geometry, centroaffine differential geometry and relative differential geometry. Let us recall the following Euclidean Bernstein problem.

THEOREM A (see [SSY]). *Let $x : M \to R^{n+1}$ be an n-dimensional minimal graph given by*

$$x_{n+1} = f(x_1, \ldots, x_n), \qquad (x_1, \ldots, x_n) \in R^n;$$

if $n \leq 7$ then f is a linear function.

For $n \geq 8$ there also exist other solutions. Many similar results were proved in different geometries.

1. A VARIATIONAL PROBLEM IN EQUIAFFINE GEOMETRY

1.1. The first variation of the equiaffine volume. Let $x : M \to A^{n+1}$ $(n \geq 2)$ be a locally strongly convex affine hypersurface and D be a sufficiently small domain of M with compact support and boundary ∂D. With respect to a local frame field $e_1, \ldots, e_n$ and its dual frame field $\omega^1, \ldots, \omega^n$ we can express the Blaschke metric h by $h = \sum h_{ij}\omega^i \otimes \omega^j$. In a local notation we raise and lower indices modulo h. Its affine volume (with respect to the Blaschke metric h) is

$$V(D) = \int_D dV, \tag{1.1.1}$$

where $H := \det(h_{ij})$ and $dV = |H|^{1/(n+2)}\, \omega^1 \wedge \cdots \wedge \omega^n$ (see [Ch] or [LSZ-I]). We consider the first variation $\delta V(D)$ under an infinitesimal displacement of D, with ∂D kept fixed. To express this situation analytically, let I be the interval $-\frac{1}{2} < t < \frac{1}{2}$. Let $f : M \times I \to A^{n+1}$ be a smooth mapping such that its restriction to $M \times t, t \in I$, is an immersion and $f(m, 0) = xt(m), m \in M$. We consider a frame field $e_\alpha(m, t)$ over $M \times I$ such that, for every $t \in I, e_i(m, t)$ are tangent vectors and $e_{n+1}(m, t)$ is in direction of the affine normal to $f(M \times t)$ at (m, t). Pulling the forms $\omega^\alpha, \omega_\alpha^\beta$ in the frame manifold back to $M \times I$, we have, since e_i span the tangent hyperplane at $f(m, t)$,

$$\omega^{n+1} = a dt.$$

Exterior differentiation gives

$$\sum \omega^i \wedge \omega_i^{n+1} + dt \wedge (a\omega_{n+1}^{n+1} + da) = 0.$$

It follows that we can set

$$\omega_i^{n+1} = \sum h_{ij}\omega^j + h_i dt,$$
$$a\omega_{n+1}^{n+1} + da = \sum h_i \omega^i + hdt,$$

where

$$h_{ij} = h_{ji}.$$

The first variation of the volume (see [Bla], [Ch]) is given by

$$V'(0) = \frac{n(n+1)}{n+2} \int_D |H|^{-1/(n+2)} L_1 adV \mid_{t=0}, \tag{1.1.2}$$

where L_1 is the affine mean curvature of x.

If $V'(0)$ is zero for arbitrary functions $a(m,t), m \in D, t \in I$, satisfying $a(m,0) = 0$, $h_i(m,0) = 0, m \in \partial D$, we must have that the affine mean curvature satisfies $L_1 = 0$.

Let $x : M \to A^{n+1}$ be a locally strongly convex hypersurface, where the parameter manifold M may be open, or compact with possibly empty, smooth boundary ∂M.

An allowable interior deformation of x is a differentiable map $f : M \times I \to A^{n+1}$, where I is an open interval $(-\epsilon, \epsilon)$, $\epsilon > 0$, with the following properties:
(i) For each $t \in I$ the map $x_t : M \to A^{n+1}$, defined by $x_t(p) = f(p,t)$, is a locally strongly convex hypersurface such that, for $t = 0$, $x_0 = x$.
(ii) There exists a compact subdomain $\sum' \subset M$ (the closure of a connected, open subset of M) with smooth boundary $\partial \sum'$, where $\partial \sum'$ may contain, meet, or be disjoint from ∂M such that, for each $p \in M \setminus \sum'$ and all $t \in I, f(p,t) = x(p)$, and the tangent hyperplane $dx_t(p)$ coincides with $dx(p)$.

In the sequel, when we study variations of the affinely invariant volume of $x(M)$ under interior deformations, we may replace M, without loss of generality, by the compact subdomain $\sum' \subset M$, otherwise from the beginning we assume that M is compact with smooth boundary.

Definition 1.1.1. Let $x : M \to A^{n+1}$ be a locally strongly convex hypersurface. If $L_1 = 0$ on M, then $x(M)$ is called an *affine maximal hypersurface.*

It follows easily from (1.1.2) that affine maximal hypersurfaces are extremals of the interior variation of the affinely invariant volume. Historically the hypersurface with $L_1 = 0$ were called "affine minimal hypersurfaces". Calabi (see [Ca-II]) suggested to call locally strongly convex hypersurfaces with $L_1 = 0$ "affine maximal hypersurfaces" because of the following result (in fact, Calabi's result is a little more general than this result, see[Ca-II]).

Theorem 1.1.1. *Let $x, x^* : \Omega \to A^{n+1}$ be two graphs defined on a compact domain by locally strongly convex functions f, f^*, namely $x^{n+1} = f(x^1, \dots, x^n)$ and $x^{n+1} = f^*(x^1, \dots, x^n)$, respectively. Suppose that $f = f^*$ and $\frac{\partial f}{\partial x^i} = \frac{\partial f^*}{\partial x^i}$, $i = 1, 2, \dots, n$ on $\partial\Omega$. Denote by L_1, L_1^*, respectively, the affine mean curvatures of x and x^*, and by dV, dV^**

their equi-affinely invariant volume elements, respectively. If $L_1 = 0$ on Ω, then

$$\int_\Omega dV \geq \int_\Omega dV^*,$$

and equality holds if and only if $f = f^*$ on Ω.

1.2. The PDE of affine maximal hypersurfaces. In this section, we derive the partial differential equation of an affine maximal hypersurface. Let $x : M \to A^{n+1}$ be the graph of a strictly convex function

$$x^{n+1} = f(x^1, \dots, x^n), \qquad (x^1, \dots, x^n) \in \Omega \subset A^n.$$

Choose the following unimodular affine frame field:

$$\begin{aligned} e_1 &= \left(1, 0, \dots, 0, \frac{\partial f}{\partial x^1}\right), \\ e_2 &= \left((0, 1, \dots, 0, \frac{\partial f}{\partial x^2}\right), \\ &\vdots \\ e_n &= \left((0, 0, \dots, 0, \frac{\partial f}{\partial x^n}\right), \\ e_{n+1} &= (0, 0, \dots, 0, 1). \end{aligned}$$

Then the Blaschke metric h is given by

$$h = \left[\det\left(\frac{\partial^2 f}{\partial x^j x^i}\right)\right]^{-1/(n+2)} \sum_{i,j} \frac{\partial^2 f}{\partial x^j x^i} dx^i dx^j.$$

The affine conormal vector field U can be identified with

$$\left[\det\left(\frac{\partial^2 f}{\partial x^j x^i}\right)\right]^{-1/(n+2)} \left(-\frac{\partial f}{\partial x^1}, \dots, -\frac{\partial f}{\partial x^n}, 1\right).$$

The formula $\triangle U + nL_1 U = 0$ implies now the following

THEOREM 1.2.1. *Let $x : M \to A^{n+1}$ be a locally strongly convex hypersurface, given as graph of a function f; x is an affine maximal hypersurface (which means $L_1 \equiv 0$ on M) if and only if f satisfies the PDE*

$$\Delta\left\{\left[\det\left(\frac{\partial^2 f}{\partial x^j \partial x^i}\right)\right]^{-1/(n+2)}\right\} = 0, \tag{1.2.1}$$

where the Laplacian, in local coordinates, is defined by

$$\Delta = \frac{1}{\sqrt{\det(h_{kl})}} \sum_{i,j=1}^n \frac{\partial}{\partial x^i}\left(h^{ij}\sqrt{\det(h_{kl})}\frac{\partial}{\partial x^j}\right).$$

Obviously, any parabolic affine hypersphere is an affine maximal hypersurface. In particular, the elliptic paraboloid

$$x^{n+1} = \frac{1}{2}[(x^1)^2 + \cdots + (x^n)^2], \qquad (x^1, \dots, x^n) \in A^n$$

is an affine-complete maximal hypersurface. Here we call $x : M \to A^{n+1}$ an affine-complete hypersurface, if x is complete with respect to the Blaschke metric h.

Denote

$$\rho := \left[\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)\right]^{-1/(n+2)}.$$

Then (1.2.1) is equivalent to

$$\Delta\rho = 0. \tag{1.2.1$'$}$$

Note that in terms of the coordinates $x_1, \ldots, x_n$, (1.2.1) can be written as

$$\sum_{i,j} f^{ij} \frac{\partial^2(\rho^{n+1})}{\partial x_i \partial x_j} = 0. \tag{1.2.1$''$}$$

We can rewrite the PDE (1.2.1) in an equivalent form by using the Legendre function. It follows from the convexity of f that the Hessian $(f_{x_i x_j})$ is positive definite. The Legendre transformation relative to f is defined by (see chapter 2 of [LSZ-I])

$$F : D \to R^n, \quad (x_1, \ldots, x_n) \to (\xi_1, \ldots, \xi_n),$$

where $D \subset R^n$ is the Legendre transform domain, and

$$\xi_i = f_{x_i} = \frac{\partial f}{\partial x_i}, \quad i = 1, \ldots, n.$$

The Legendre function u is defined by

$$u(\xi_1, \ldots, \xi_n) = \sum_i x_i f_{x_i}(x_1, \ldots, x_n) - f(x_1, \ldots, x_n). \tag{1.2.2}$$

We know that $(\frac{\partial^2 u}{\partial \xi_i \partial \xi_j})$ is the inverse matrix of the Hessian matrix $(f_{x_i x_j})$ (see [LSZ-I]). Then the hypersurface can be represented in terms of $\xi_1, \ldots, \xi_n$ as follows

$$x = (x_1, \ldots, x_n, f(x_1, \ldots, x_n)) = \left(\frac{\partial u}{\partial \xi_1}, \ldots, \frac{\partial u}{\partial \xi_n}, -u + \sum_i \xi_i \frac{\partial}{\partial \xi_i}\right).$$

In terms of the coordinates $\xi_1, \ldots, \xi_n$, (1.2.1) can be written as

$$\sum u^{ij} \frac{\partial^2(\rho^{-1})}{\partial \xi_i \partial \xi_j} = 0. \tag{1.2.3}$$

1.3. The second variation of the affine volume. We use the same notation as in section 1.2. Calabi calculated the second variation of the affine volume for an affine maximal hypersurface $x : M \to A^{n+1}$ and got

$$V''(0) = -(n+1)\int_M \left\{(\Delta a)^2 - (n+2)\sum_{i,j} B^{ij} a_{,i} a_{,j} + (n+2)\sum_{i,j} B^{ij} B_{ij} a^2\right\} dV, \tag{1.3.1}$$

where coefficients of the Weingarten form B are denoted by B_{ij}; see p. 51 of [LSZ-I].

By use of (1.3.1), Calabi proved

THEOREM 1.3.1 (Calabi [Ca-II]). *Let $x : M \to A^3$ be an affine maximal surface, then the second variation of the affine volume (with respect to Blaschke metric) under all interior deformations of x is negative definite.*

We end this chapter with the following conjectures:

Chern's affine Bernstein conjecture ([Ch]). *Consider a locally strongly convex graph $x\colon R^n \to A^{n+1}$ with vanishing affine mean curvature $L_1 = 0$. Then x is an elliptic paraboloid.*

Calabi's affine Bernstein conjecture ([Ca-I]). *Consider a locally strongly convex hypersurface $x\colon M^n \to A^{n+1}$ with vanishing affine mean curvature $L_1 = 0$ and complete Blaschke metric. Then x is an elliptic paraboloid.*

2. A VARIATIONAL PROBLEM IN CENTROAFFINE GEOMETRY

2.0. Introduction. From the point of view of PDEs the affine hypersurface theories are attractive topics as the study of curvature properties and variational problems leads to difficult PDEs of order at least four. The serious difficulties as well as the challenges are reflected by the history of famous problems such as the global classification of all locally strongly convex affine spheres or the solution of the affine Bernstein conjectures of Calabi and Chern in Blaschke's unimodular theory.

In centroaffine differential geometry one studies the properties of hypersurfaces in R^{n+1} which are invariant under the centroaffine transformation group $G = GL(n+1, R)$, where G keeps the origin $O \in R^{n+1}$ fixed. In this chapter, we consider centroaffine Bernstein problems. C. P. Wang [W] studied the Euler-Lagrange equation for the area functional of a so called centroaffine hypersurface. As there are no general results about the sign of the second variation of the centroaffine area integral, we use the terminology *centroaffine extremal hypersurface* in case the Euler-Lagrange equation is satisfied. This equation is given by a fourth order PDE, namely, trace$\mathcal{T} = 0$, where $\mathcal{T}$ is the so called Tchebychev operator; in contrast to the Euclidean and also to the above mentioned unimodular Bernstein problems, the operator $\mathcal{T}$ is not related to something similar to "extrinsic curvature". In terms of a local representation of a hypersurface as a graph, the Euler-Lagrange equation is given by (2.3.12) below, where the Laplacian is defined in terms of the centroaffine metric; its expression for graphs is well known. The equation (2.3.12) is strongly nonlinear, and, presently, any attempt of a classification of all its solutions seems to be hopeless.

What about known examples of centroaffine extremal hypersurfaces? All proper affine spheres satisfy the equation trace$\mathcal{T} = 0$; chapter 2 in [LSZ-I] contains many local results and the global classification of all locally strongly convex affine spheres. For proper affine spheres the Blaschke geometry and the centroaffine geometry coincide, and, in particular, the completeness conditions for their metrics are the same. Thus, metrically complete proper affine hyperspheres are centroaffine extremal and complete, with the ellipsoid being the only compact affine hypersphere. Besides affine spheres there are more examples of centroaffine extremal hypersurfaces [W]. In section 2.3 we study classes of such examples and give a generalized Calabi composition to produce again a family of centroaffine extremal hypersurfaces from two given centroaffine extremal hypersurfaces. Moreover, we derive the fourth order equation (2.3.12) for an extremal graph. The study of large classes of examples of complete hypersurfaces in centroaffine geometry shows that the situation here is quite different from that in the Euclidean and the affine geometries, resp., where, at least in low dimensions, there is only one candidate for a solution in any of the Bernstein problems. A detailed study of the examples leads to the formulation of different centroaffine Bernstein problems in section 2.5. For partial subclassifications, additional assumptions on the curvature and the Tchebychev form are quite natural in the centroaffine context, and examples show that they obviously are needed for further subclassifications. In the last sections 2.5-2.7 we formulate and prove our main results which give partial solutions of the centroaffine Bernstein problems.

2.1. Centroaffine hypersurfaces in R^{n+1}. We summarize basic formulas of centroaffine hypersurface theory in terms of Cartan's moving frames (compare [LSZ-I], chapters 1-2; for an approach in the invariant calculus see [SS], chapters 4-6). We restrict to locally strongly convex hypersurfaces as in this case the so called centroaffine metric is a Riemannian metric; see section 4.3.3 in [SS].

Let $x : M \to R^{n+1}$ be a locally strongly convex hypersurface and assume that the position vector x is transversal to the tangent hyperplane $x_*(TM)$ at each point $p \in M$. In particular, this implies that $O \notin x(M)$. In a standard terminology, a hypersurface normalized by its transversal position vector is called a centroaffine hypersurface. According to the type of the hypersurface one uses different orientations for the normalization to get a positive definite centroaffine metric:

1. Hyperbolic type: For any point $x(p) \in R^{n+1}$, the origin of R^{n+1} and the hypersurface are on different sides of the tangent hyperplane $x_*(TM)$; the centroaffine normal vector field is given by $e_{n+1} = x$ (examples are hyperbolic affine hyperspheres in R^{n+1} centered at $O \in R^{n+1}$).

2. Elliptic type: For any point $x(p) \in R^{n+1}$, the origin of R^{n+1} and the hypersurface are on the same side of the tangent hyperplane $x_*(TM)$; the centroaffine normal vector field is given by $e_{n+1} = -x$ (examples are elliptic affine hyperspheres in R^{n+1} centered at $O \in R^{n+1}$).

As already stated in the introduction, in centroaffine differential geometry we study the properties of hypersurfaces in R^{n+1} that are invariant under the centroaffine transformation group G. For the hypersurface, we choose a centroaffine frame field $\{e_1, \ldots, e_n, e_{n+1}\}$ with $e_{n+1} = -\epsilon x$ ($\epsilon = 1$ for elliptic type, $\epsilon = -1$ for hyperbolic type) and $e_1, \ldots, e_n \in T_xM$; we denote by $\{\omega^1, \ldots, \omega^n\}$ the dual frame field of the tangential frame field. The structure equations read

$$dx = \sum_i \omega^i e_i, \quad \omega^{n+1} = 0, \tag{2.1.1}$$

$$de_i = \sum_j \omega_i^j e_j + \omega_i^{n+1} e_{n+1}, \tag{2.1.2}$$

$$de_{n+1} = \sum_i \omega_{n+1}^i e_i, \quad \omega_{n+1}^{n+1} = 0, \quad \omega_{n+1}^i = -\epsilon \omega^i. \tag{2.1.3}$$

Differentiation of (2.1.1) – (2.1.3) gives the integrability conditions (2.1.4)–(2.1.6):

$$d\omega^i = \sum_j \omega^j \wedge \omega_j^i, \quad \sum_i \omega^i \wedge \omega_i^{n+1} = 0, \tag{2.1.4}$$

$$d\omega_i^j = \sum_k \omega_i^k \wedge \omega_k^j - \epsilon\omega_i^{n+1} \wedge \omega^j, \quad d\omega_i^{n+1} = \sum_j \omega_i^j \wedge \omega_j^{n+1}, \tag{2.1.5}$$

$$d\omega_{n+1}^i = \sum_j \omega_{n+1}^j \wedge \omega_j^i. \tag{2.1.6}$$

From the second equation of (2.1.4), we have

$$\omega_i^{n+1} = \sum_{i,j} h_{ij}\omega^j, \quad h_{ij} = h_{ji}. \tag{2.1.7}$$

For locally strongly convex hypersurfaces, the quadratic form

$$h = \sum_{i,j} h_{ij}\omega^i\omega^j \tag{2.1.8}$$

is positive definite by appropriate choice of the orientation; h is called the centroaffine metric of the hypersurface. It is well known that h is independent of the choice of the frame $\{e_1, \ldots, e_n\}$ and that h is invariant under transformations of the group G. The centroaffine metric is the first fundamental invariant of centroaffine hypersurface theory.

We sketch how to derive a second fundamental invariant. We choose a centroaffine tangential frame $\{e_1, \ldots, e_n\}$ on M such that $h_{ij} = \delta_{ij}$, i.e.,

$$\omega_i^{n+1} = \omega^i. \tag{2.1.9}$$

Differentiate (2.1.9) and use (2.1.5); this implies

$$d\omega^i = \sum_j \omega_{ij} \wedge \omega^j. \tag{2.1.10}$$

(2.1.4) and (2.1.10) give

$$d\omega^i = \sum_j \omega^j \wedge \left[\frac{1}{2}(\omega_{ji} - \omega_{ij})\right]. \tag{2.1.11}$$

The expression $\frac{1}{2}(\omega_{ji} - \omega_{ij})$ is skew-symmetric and $\{\omega^1, \ldots, \omega^n\}$ is an orthonormal coframe of h. (2.1.11) and the fundamental theorem of Riemannian geometry imply that the Levi-Civita connection of h satisfies

$$\tilde{\omega}_{ji} = \frac{1}{2}(\omega_{ji} - \omega_{ij}), \quad \tilde{\omega}_{ji} = -\tilde{\omega}_{ij}. \tag{2.1.12}$$

Define

$$\omega_{ij} - \tilde{\omega}_{ij} = \frac{1}{2}(\omega_{ij} + \omega_{ji}) = \sum_k A_{ijk}\omega^k. \tag{2.1.13}$$

This gives the symmetry relation

$$A_{ijk} = A_{jik}. \tag{2.1.14}$$

Combine (2.1.10) with (2.1.11) and use (2.1.13):

$$\sum_{j,k} A_{ijk}\omega_j \wedge \omega_k = 0,$$

this implies the total symmetry of the form

$$A = \sum_{i,j,k} A_{ijk}\omega^i\omega^j\omega^k,$$

namely

$$A_{ijk} = A_{ikj} = A_{jik}. \tag{2.1.15}$$

The form A is called the centroaffine cubic form of the hypersurface. Again it is well known that this form is independent of the choice of the frame and invariant under transformations of the group G. The vanishing of its traceless part characterizes hyperquadrics (see [SS], section 7.1; [LLSSW], Lemma 2.1 and Remark 2.2).

The uniqueness part of the fundamental theorem of centroaffine hypersurface theory states that the forms h and A together build a fundamental system of centroaffine invariants of the hypersurface, that means that they completely describe the geometry of x which is invariant under the transformations of G. Considering integrability conditions, one also can state an existence theorem using the forms h and A.

We need the following two important geometric invariants built from h and A:

$$J = \frac{1}{n(n-1)} \sum_{i,j,k} A_{ijk}^2 \tag{2.1.16}$$

is called the centroaffine Pick invariant. The tangent vector field

$$T = \sum_i T_i e_i, \quad T_i = \frac{1}{n} \sum_{j=1}^{n} A_{jji} \tag{2.1.17}$$

is called the centroaffine Tchebychev vector field of x. For locally strongly convex hypersurfaces the metric is positive definite, thus the vanishing of J implies that of A and T, and therefore that of the traceless part of A; the hypersurface must be a quadric. In the context of relative geometry and in terms of volume forms, the geometric meaning of T was studied in section 4.4.8, 4.4.9 in [SS]. In the centroaffine case, there is an additional well known relation between T, the so called centroaffine Tchebychev function ψ and the support function ρ of the Blaschke geometry. To state this relation, we recall the following definition from section 2 of [LSZ-II].

DEFINITION 2.2.1. The positive function ψ, given by

$$\psi = \frac{\det(h_{ij})}{[e_1, \ldots, e_n, x]^2}, \tag{2.1.18}$$

is independent of the choice of the frame $\{e_1, \ldots, e_n\}$ and is invariant under transformations of G, where $[\cdots]$ is the determinant. We call the function ψ the Tchebychev function of x.

Choosing $i = j$ in (2.1.13) and summing up over i, we get

$$\sum_{i,k} A_{iik}\omega^k = \sum_i \omega_{ii} = d(\log[e_1, \ldots, e_n, x]) = -\frac{1}{2} d \log \psi. \tag{2.1.19}$$

One can compare invariants from different relative geometries of a hypersurface (see section 5 in [SS]); from (2.19) (cf. formula (2) in [LSZ-II]) it follows that the equiaffine support function ρ (section 4.13 in [SS]), the centroaffine Tchebychev function ψ defined above, and the Tchebychev vector field T satisfy the relation

$$T_i = -\frac{1}{2n}(\log \psi)_i = \frac{(n+2)}{2n}(\log \rho)_i, \tag{2.2.20}$$

The relation

$$\rho = const$$

characterizes proper affine spheres (section 7.2 in [SS]); this is equivalent to the centroaffine relation $T = 0$. Our foregoing remarks clarify the geometric meaning of the invariants J and T.

For later applications we list the integrability conditions in terms of the metric and the cubic form. In a standard local notation, by a comma we indicate covariant differentiation in terms of the Levi-Civita connection. The sign of the Riemannian curvature tensor $\Omega = \sum R_{ijkl}\omega^i \otimes \omega^j \otimes \omega^k \otimes \omega^l$ of h is fixed by

$$d\tilde{\omega}_{ij} - \sum_k \tilde{\omega}_{ik} \wedge \tilde{\omega}_{kj} = -\frac{1}{2}\sum_{k,l} R_{ijkl}\omega^k \wedge \omega^l. \tag{2.1.21}$$

In terms of the frame considered ($h_{ij} = \delta_{ij}$), the Gauss equations read

$$R_{ijkl} = \sum(A_{jkm}A_{mil} - A_{ikm}A_{mjl}) + \epsilon(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}), \tag{2.1.22}$$

while the cubic form satisfies Codazzi equations, that means the covariant derivative is totally symmetric:

$$A_{ijk,l} = A_{ijl,k}. \tag{2.1.23}$$

Here, as mentioned above, $A_{ijk,l}$ are the components of the covariant derivative of A with respect to the Levi-Civita connection of h. Contraction of (2.1.22) gives the important relations

$$R_{ik} = \sum A_{iml}A_{mlk} - n\sum_m T_m A_{mik} + \epsilon(n-1)\delta_{ik}, \tag{2.1.24}$$

where R_{ik} denote the components of the Ricci tensor, and the "centroaffine theorema egregium"

$$n(n-1)\kappa = R = n(n-1)(J+\epsilon) - n^2|T|^2, \quad |T|^2 = \sum(T_i)^2, \tag{2.1.25}$$

where κ denotes the normalized scalar curvature.

Later we will need the Ricci identities

$$A_{ijk,lm} - A_{ijk,ml} = \sum A_{rjk}R_{rilm} + \sum A_{irk}R_{rjlm} + \sum A_{ijr}R_{rklm}. \tag{2.1.26}$$

The Codazzi equations for A (or the relations between T and the Tchebychev function) imply

$$T_{i,j} = T_{j,i}. \tag{2.1.27}$$

If $T_{i,j} = 0$, we say that the Tchebychev vector field T is parallel.

As stated above, for a centroaffine hypersurface the position vector is used for a normalization; from this a Weingarten type equation is trivial, and there is no shape operator describing "exterior curvature" in the standard way. But studies of Wang [W] and other authors ([LW1], [LLSSW]) show that there is another important operator in centroaffine geometry. Wang called this operator originally shape operator, but for the reasons just stated, later the notion was changed to Tchebychev operator. This operator $\mathcal{T} : TM \to TM$ of x is defined by

$$\mathcal{T}(v) := \nabla_v T, \quad v \in TM. \tag{2.1.28}$$

The foregoing relation $T_{i,j} = T_{j,i}$ implies that $\mathcal{T}$ is a self-adjoint operator with respect to the centroaffine metric h. Moreover, $\mathcal{T} \equiv 0$ if and only if T is parallel.

2.2. The first and second variation of the centroaffine volume. Let $x : M \to R^{n+1}$ be a compact locally strongly convex centroaffine hypersurface with boundary ∂M. We consider the variation $f : M \times R \to R^{n+1}$ such that (i) $f_0 = x$ on M; (ii) for each (small) t, $f_t := f(\cdot, t) : M \to R^{n+1}$ is a locally strongly convex centroaffine hypersurface; (iii) $f_t = x$ and $df_t(TM) = dx(TM)$ on ∂M for each (small) t. Such f will be called an admitted variation of x.

Let f be an admitted variation of x. Let $\{E_i\}$ be a local orthonormal basis for the centroaffine metric h_t induced by f_t and $\{\omega_i\}$ the dual basis for $\{E_i\}$. We can identify $T(M \times R)$ with $TM \oplus TR$. Then $\{E_1, \dots, E_n, \partial/\partial t\}$ is a local basis for $T(M \times R)$ with the dual basis $\{\omega_1, \dots, \omega_n, dt\}$. We denote $e_i = E_i(f_t)$, then $\{e_1, \dots, e_n, f\}$ is a moving frame in R^{n+1} along $M \times R$. Thus we can find 1-forms $\{\theta_\alpha, \theta_{ij}, \theta_i^*\}$ on $M \times R$ such that

$$df = \sum_{i=1}^{n} \theta_i e_i + \theta_0 f, \tag{2.2.1}$$

$$de_i = \sum_{i=1}^{n} \theta_{ij} e_j + \theta_i^* f. \tag{2.2.2}$$

We denote the variational vector field in R^{n+1} by

$$\frac{\partial f}{\partial t} = \phi f + \sum_i \psi_i e_i \tag{2.2.3}$$

for some smooth functions ϕ and ψ_i with $\phi = \psi_i = 0$ on ∂M. By (2.2.3) and the fact $df_t(TM) = dx(TM)$ on ∂M we have $\phi_i = E_i(\phi) = 0$ on ∂M. From (2.1) we get

$$\frac{\partial f}{\partial t} = df\left(\frac{\partial}{\partial t}\right) = \sum_i \theta_i\left(\frac{\partial}{\partial t}\right) e_i + \theta_0\left(\frac{\partial}{\partial t}\right) f.$$

Thus (2.2.3) implies that $\theta_i(\partial/\partial t) = \psi_i$ and $\theta_0(\partial/\partial t) = \phi$. Furthermore, from (2.2.1) we have

$$e_j = E_j(f) = df(E_j) = \sum_i \theta_i(E_j) e_i + \theta_0(E_j) f,$$

so we get $\theta_i(E_j) = \delta_{ij}$ and $\theta_0(E_j) = 0$. Therefore

$$\theta_i = \omega_i + \psi_i dt, \qquad \theta_0 = \phi dt. \tag{2.2.4}$$

In similar way, we can write

$$\theta_i^* = -\epsilon\omega_i + a_i dt \tag{2.2.5}$$

for some 1-forms $\omega_i \in T^*M$ and some smooth functions a_i. Since $T^*(M \times R) = T^*M \oplus T^*R$, we can write

$$\theta_{ij} = \omega_{ij} + B_{ij} dt \tag{2.2.6}$$

for some 1-forms $\omega_{ij} \in T^*M$ and some smooth functions B_{ij}. By taking exterior differentiation of (2.2.1) and (2.2.2), we obtain

$$d\theta_0 = \sum_i \theta_i \wedge \theta_i^*, \tag{2.2.7}$$

$$d\theta_i = \sum_j \theta_j \wedge \theta_{ji} + \theta_0 \wedge \theta_i, \tag{2.2.8}$$

$$d\theta_i^* = \sum_j \theta_{ij} \wedge \theta_j^* + \theta_i^* \wedge \theta_0, \tag{2.2.9}$$

$$d\theta_{ij} = \sum_k \theta_{ik} \wedge \theta_{kj} + \theta_i^* \wedge \theta_j. \tag{2.2.10}$$

Since the exterior differential operator on $T^*(M \times R) = T * M \oplus T^*R$ is given by

$$d = d_M + dt \wedge \frac{\partial}{\partial t}, \tag{2.2.11}$$

where d_M is the exterior differential operator on T^*M, from (2.2.7), (2.2.4) and (2.2.5) we get

$$\left(d_M + dt \wedge \frac{\partial}{\partial t}\right)(\phi dt) = \sum_i (\omega_i + \psi_i dt) \wedge (-\epsilon\omega_i + a_i dt).$$

Thus $d_M\phi = \sum_i (a_i + \epsilon\psi)\omega_i$. If we write $d_M\phi = \sum_i \phi_i\omega_i$, then we have

$$a_i = \phi_i - \epsilon\psi_i. \tag{2.2.12}$$

Similarly, using (2.2.8), (2.2.4) and (2.2.6), we get

$$\frac{\partial\omega_i}{\partial t} = d_M\psi_i + \sum_j \psi_j\omega_{ji} + \phi\omega_i - \sum_j \omega_j B_{ji}, \tag{2.2.13}$$

and using (2.2.9), (2.2.5), (2.2.4) and (2.2.6), we get

$$d_M a_i + \epsilon\frac{\partial\omega_i}{\partial t} = \sum_j a_j\omega_{ij} + \epsilon\sum_j \omega_j B_{ij} - \epsilon\phi\omega_i. \tag{2.2.14}$$

We denote by $\tilde{\omega}_{ij}$ the Levi-Civita connection for the centroaffine metric h_t of f_t. Like in (2.1.13) we can define the cubic form $A = \sum_{i,j,k} A_{ijk}\omega_i\omega_j\omega_k$ for f_t by $\sum_k A_{ijk}\omega_k = \omega_{ij} - \tilde{\omega}_{ij}$. For any 1-form $\alpha = \sum_i \alpha_i\omega_i$ on M we denote by $\nabla\alpha = \sum_{i,j} \alpha_{i,j}\omega_i \otimes \omega_j$ the covariant derivative of α with respect to $\tilde{\omega}_{ij}$, where $\alpha_{i,j}$ are defined by

$$d_M\alpha_i + \sum_j \alpha_j\tilde{\omega}_{ji} = \sum_j \alpha_{i,j}\omega_j.$$

Thus we can rewrite (2.2.13) and (2.2.14) as

$$\frac{\partial\omega_i}{\partial t} = \sum_j (\psi_{i,j} + \sum_k \psi_k A_{ijk} - B_{ji} + \phi\delta_{ij})\omega_j, \tag{2.2.15}$$

$$\frac{\partial\omega_i}{\partial t} = \sum_j \Big(- \epsilon a_{i,j} + \epsilon\sum_k a_k A_{ijk} + B_{ij} - \phi\delta_{ij}\Big)\omega_j. \tag{2.2.16}$$

By adding these two equations and using (2.2.12) we obtain

$$\frac{\partial\omega_i}{\partial t} = \frac{1}{2}\sum_j (\psi_{i,j} - \epsilon a_{i,j} + \epsilon\sum_k \phi_k A_{ijk} + B_{ij} - B_{ji})\omega_j. \tag{2.2.17}$$

Thus we have

$$\frac{\partial}{\partial t}(\omega_1 \wedge \cdots \wedge \omega_n) = \frac{1}{2}\Big\{ \sum_i (\psi_{i,i} - \epsilon a_{i,i}) + n\epsilon\sum_k \phi_k T_k\Big\} dM, \tag{2.2.18}$$

where $dM := \omega_1 \wedge \cdots \wedge \omega_n$. We note that all terms in (2.2.18) are globally defined on M. Since $\partial f/\partial t = 0$ on ∂M, by Green's formula we get

$$\begin{aligned} V'(t) &= \frac{\partial}{\partial t}\Big(\int_M \omega_1 \wedge \cdots \wedge \omega_n\Big) = \int_M \frac{\partial}{\partial t}(\omega_1 \wedge \cdots \wedge \omega_n)) \\ &= \frac{n\epsilon}{2}\int_M \Big(\sum_k T_k\phi_k\Big) dM = -\frac{n\epsilon}{2}\int_M \Big(\sum_k T_{k,k}\Big)\phi dM. \end{aligned} \tag{2.2.19}$$

We proved

THEOREM 2.2.1 (Wang [W]). *The relation* trace$\mathcal{T} = 0$ *is the Euler-Lagrange equation for the centroaffine area functional.*

As there is no general statement about the sign of the second variation, we call the critical points of the area functional "extremal centroaffine hypersurfaces" (other authors call them minimal centroaffine hypersurfaces).

By (1.2.20), we obtain

THEOREM 2.2.2 (Wang [W]). *Let $x : M \to R^{n+1}$ $(n \geq 2)$ be a centroaffine hypersurface with Tchebychev function ψ. Then x is an extremal centroaffine hypersurface if and only if*

$$\Delta(\log\psi) = 0, \tag{2.2.20}$$

where Δ is the Laplacian of the centroaffine metric h of x.

Let $x : M \to R^{n+1}$ be a centroaffine extremal hypersurface. Let $f : M \times R \to R^{n+1}$ be an admitted variation with compact support which fixes the boundary ∂M. By Theorem 2.2.1, we may assume that $\partial f/\partial t = \phi f$. Wang calculated the second variation of the centroaffine area functional and proved

THEOREM 2.2.3 (Wang [W]). *Let $x : M \to R^{n+1}$ be a centroaffine extremal hypersurface, then*

$$\begin{aligned} V''(0) = &-\frac{1}{4}\int_M \Big\{\Delta\phi(\Delta\phi + 2(n+1)\epsilon\phi) - n^2\Big(\sum_i T_i\phi_i\Big)^2 \\ &+ 2n\sum_{i,j,k} A_{ijk}T_i\phi_j\phi_k\Big\} dM. \end{aligned} \tag{2.2.21}$$

COROLLARY 2.2.1 (Wang [W]). *The hyperbolic equiaffine hypersurfaces in R^{n+1} centered at $0 \in R^{n+1}$ are stable centroaffine extremal hypersurfaces.*

Proof. For hyperbolic equiaffine hypersurfaces we have $T_i = 0$ and $\epsilon = -1$. Thus

$$\begin{aligned} V''(0) &= -\frac{1}{4}\int_M \{\Delta\phi(\Delta\phi - 2(n+1)\phi) dM \\ &= -\frac{1}{4}\int_M \{\Delta\phi(\Delta\phi + 2(n+1)\|\nabla\phi\|^2) dM \leq 0. \end{aligned}$$

Moreover, $V''(0) \equiv 0$ if and only if $\phi = 0$.

COROLLARY 2.2.2 (Wang [W]). *The ellipsoid in R^{n+1} centered at $0 \in R^{n+1}$ is unstable.*

Proof. For the ellipsoid we have $T_i = 0$ and $\epsilon = 1$. By (2.2.21) we get

$$V''(0) = -\frac{1}{4}\int_M \Delta\phi(\Delta\phi + 2(n+1)\phi)dM.$$

Let ψ_k be the k-th eigenfunction of Δ. Since (M, h) is isometric to the standard sphere S^n, we have $\Delta\phi_k = -k(k+n-1)\psi_k$. Thus

$$V_k''(0) = -\frac{1}{4}k(k+n-1)\{k(k+n-1) - 2(n+1)\}\int_M (\psi_k)^2 dM.$$

So $V_1''(0) > 0; V_2''(0) = 0$ and $V_k''(0) < 0, k = 3, 4, \ldots$.

2.3. Examples of extremal centroaffine hypersurfaces. In this section, we recall examples of locally strongly convex, extremal centroaffine hypersurfaces; some already were listed in [W]. The convexity condition implies that the centroaffine metric is positive definite for an appropriate orientation of the normalization. It is well known that the hyperellipsoids are the only closed (compact without boundary), centroaffine extremal hypersurfaces; this result is due to C. P. Wang.

PROPOSITION 2.3.1 (Theorem 1 of [W]). *Let $x : M \to R^{n+1}$ $(n \geq 2)$ be a compact centroaffine hypersurface with constant trace of the Tchebychev operator. Then $x(M)$ is centroaffinely equivalent to a hyperellipsoid centered at $0 \in R^{n+1}$.*

In this section we consider non-compact examples which satisfy at least one of the following completeness conditions:

(i) the centroaffine metric is complete;

(ii) the hypersurface can be represented as graph over a hyperplane.

We will come back to the completeness conditions in section 4 below.

EXAMPLE 2.3.1 (*Proper affine spheres*). According to C. P. Wang [W], any locally strongly convex, proper affine hypersphere is centroaffine extremal. This is a trivial consequence of the fact that the vanishing of the Tchebychev field characterizes proper affine spheres in centroaffine geometry. In the Blaschke geometry, it is well known that hyperbolic affine hyperspheres can be described in terms of solutions of some Monge-Ampère equations; therefore there are many proper affine hyperspheres, and thus this gives a very large class of centroaffine extremal hypersurfaces. For proper affine hyperspheres the unimodular (equiaffine) theory (sometimes called Blaschke theory) and the centroaffine theory coincide modulo a nonzero constant factor. In particular this implies that the notions of completeness with respect to the metrics coincide in both theories. The classification of the locally strongly convex affine hyperspheres, which are complete with respect to the affine metric, was finished about a decade ago; see e.g. [LSZ-I], chapter 2. Considering proper affine hyperspheres, there are two subclasses, namely the elliptic ones and the hyperbolic ones. While there is only one type of complete elliptic affine hyperspheres, namely the hyperellipsoid, the class of complete hyperbolic affine hyperspheres is described by what Calabi originally stated as a conjecture (see [LSZ-I], section 2.7); all examples in this latter class are non compact, but they satisfy both completeness conditions (i) and (ii) (in fact, in this case the two completeness conditions are equivalent).

From this, any hyperbolic affine hypersphere is an example of a noncompact, centroaffine extremal hypersurface satisfying the two different completeness conditions (i) and (ii). Moreover, their Ricci tensor is bounded below: $\mathrm{Ric} \geq -(n-1)h$.

A particular example in this class is one sheet of a two-sheeted hyperboloid $H(c,n)$:

$$(x_{n+1})^2 = c^2 + (x_1)^2 + \cdots + (x_n)^2, \quad (x_1, \ldots, x_n) \in R^n, \quad c > 0. \tag{2.3.1}$$

We have (see [LSZ-I])

$$A_{ijk} = 0, \quad 1 \leq i, j, k \leq n.$$

Thus it is a centroaffine extremal hypersurface satisfying two different completeness conditions; for a hyperboloid the Pick invariant vanishes: $J \equiv 0$. The Riemannian curvature tensor of the centroaffine metric and its Ricci curvature tensor satisfy

$$R_{ijkl} = -c^{-\frac{2n+2}{n+2}}(h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.3.2}$$

$$R_{ik} = -(n-1)c^{-\frac{2n+2}{n+2}}h_{ik}. \tag{2.3.3}$$

Obviously the sectional curvature, the Ricci curvature and the scalar curvature of the metric of $H(c,n)$ are negative constants.

EXAMPLE 2.3.2 (*Centroaffine graphs with constant trace of the Tchebychev operator*). Let $x : M \to R^{n+1}$ be a locally strongly convex hypersurface with transversal position vector x at each point M. Then we have a local representation of x as graph:

$$x_{n+1} = f(x_1, x_2, \ldots, x_n). \tag{2.3.4}$$

We have the centroaffine frame

$$e_i = (0, \ldots, 1, \ldots, 0, f_{x_i}), \quad 1 \leq i \leq n, \quad e_{n+1} = (x_1, x_2, \ldots, x_n, f), \tag{2.3.5}$$

where $f_{x_i} = \frac{\partial f}{\partial x_i}$. The structure equations read

$$dx = \sum_i \omega^i e_i, \tag{2.3.6}$$

$$de_i = \sum_j \omega_i^j e_i + \sum_j h_{ij}\omega^j e_{n+1}, \tag{2.3.7}$$

thus we have

$$[e_1, \ldots, e_n, x] = \begin{vmatrix} 1 & 0 & \cdots & 0 & f_{x_1} \\ 1 & 1 & \cdots & 0 & f_{x_2} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & f_{x_n} \\ x_1 & x_2 & \cdots & x_n & f \end{vmatrix} = f - \sum_i x_i f_{x_i},$$

$$(h_{ij}) = \left(\frac{f_{x_i x_j}}{f - x_1 f_{x_1} - \cdots x_n f_{x_n}} \right) \tag{2.3.8}$$

and

$$\det(h_{ij}) = \frac{1}{(f - \sum_i x_i f_{x_i})^n} \cdot \det(f_{x_i x_j}). \tag{2.3.9}$$

The Tchebychev function ψ is given by

$$\psi = \frac{\det(h_{ij})}{[e_1, \dots, e_n, x]^2} = \frac{1}{(f - \sum_i x_i f_{x_i})^{n+2}} \cdot \det(f_{x_i x_j}). \tag{2.3.10}$$

Therefore x is a centroaffine local graph with constant value a for the trace of the Tchebychev operator if and only if the graph function f satisfies the following nonlinear PDE of fourth order:

$$\Delta\left\{ \log\left(\frac{\det(h_{ij})}{[e_1, \dots, e_n, x]^2} \right) \right\} = \Delta\left\{ \log\left(\frac{\det(f_{x_i x_j})}{(f - \sum_i x_i f_{x_i})^{n+2}} \right) \right\} = a. \tag{2.3.11}$$

As above, Δ is the Laplacian of the centroaffine metric h of x. In particular, we get a nonlinear PDE of fourth order for centroaffine extremal hypersurfaces. This allows us to consider a centroaffine Bernstein problem using this PDE.

PROPOSITION 2.3.2. *Let x be a locally strongly convex graph given by the function f in* (2.3.4)*. Then x is centroaffine extremal if and only if f satisfies the PDE*

$$\Delta\left\{ \log\left(\frac{\det(f_{x_i x_j})}{(f - \sum_i x_i f_{x_i})^{n+2}} \right) \right\} = 0. \tag{2.3.12}$$

REMARK 2.3.1. (i) We can rewrite the PDE (2.3.12) in a simpler form using the Legendre function. It follows from the convexity of f that the Hessian $(f_{x_i x_j})$ is positive definite. The Legendre transformation relative to f is defined by (see chapter 2 of [LSZ-I])

$$F : D \to R^n, \quad (x_1, \dots, x_n) \to (\xi_1, \dots, \xi_n),$$

where $D \subset R^n$ is the Legendre transform domain, and

$$\xi_i = f_{x_i} = \frac{\partial f}{\partial x_i}, \quad i = 1, \dots, n.$$

The Legendre function u is defined by

$$u(\xi_1, \dots, \xi_n) = \sum_i x_i f_{x_i}(x_1, \dots, x_n) - f(x_1, \dots, x_n). \tag{2.3.13}$$

We know that $(\frac{\partial^2 u}{\partial \xi_i \partial \xi_j})$ is the inverse matrix of the Hessian $(f_{x_i x_j})$ (see [LSZ-I]). Thus the PDE (2.3.12) of the centroaffine extremal graph can be rewritten as

$$\Delta\left\{ \log\left((-u)^{n+2} \cdot \det\left(\frac{\partial^2 u}{\partial \xi_i \partial \xi_j} \right) \right) \right\} = 0. \tag{2.3.14}$$

Equations (2.3.12) and (2.3.14) show the following: in terms of a graph function, the Euler-Lagrange equation for centroaffine extremal hypersurfaces is a highly complicated nonlinear fourth order PDE. From the global classification of locally strongly convex hyperbolic affine spheres we know about earlier difficulties to solve the much simpler equation (2.3.15).

(ii) We recall that the PDE of a hyperbolic hypersphere with constant affine mean curvature L_1, in terms of the Legendre function, is (see [LSZ-I], p. 132)

$$(-u)^{n+2} \cdot \det\left(\frac{\partial^2 u}{\partial \xi_i \partial \xi_j} \right) = (-L_1)^{-n-2}. \tag{2.3.15}$$

EXAMPLE 2.3.3 (*Wang's class of centroaffine extremal hypersurfaces*). Li-Wang [LW] and Wang [W] also listed the following type of hypersurfaces, and Wang proved that they are centroaffine extremal:

$$(x_1)^{\beta_1}(x_2)^{\beta_2}\cdots(x_{n+1})^{\beta_{n+1}} = c, \quad c>0, \quad \beta_i>0, \quad 1\le i\le n+1.$$

It is easy to see that the above hypersurfaces also can be represented by

$$Q(c;\alpha_1,\dots,\alpha_n;n):\ x_{n+1} = cx_1^{-\alpha_1}x_2^{-\alpha_2}\cdots x_n^{-\alpha_n}, \quad c>0, \quad 1\le i\le n, \tag{2.3.16}$$

where $\alpha_i = \beta_i/\beta_{n+1} > 0$.

Consider the connected component

$$x_{n+1} = \frac{c}{(x_1)^{\alpha_1}(x_2)^{\alpha_2}\cdots(x_n)^{\alpha_n}}, \qquad \text{for } x_1>0,\dots,x_n>0.$$

This representation of the hypersurface in terms of a graph function

$$f(x_1,\dots,x_n) = cx_1^{-\alpha_1}\cdots x_n^{-\alpha_n}$$

admits us to apply the calculations from Example 2.3.2:

$$\begin{aligned}
h_{ii} &= \frac{\alpha_i(1+\alpha_i)}{1+\alpha_1+\cdots+\alpha_n}\cdot x_i^{-2}, \quad 1\le i\le n,\\
h_{ij} &= \frac{\alpha_i\alpha_j}{1+\alpha_1+\cdots+\alpha_n}\cdot x_i^{-1}x_j^{-1}, \quad 1\le i\ne j\le n,\\
\det(h_{ij}) &= \frac{\alpha_1\cdots\alpha_n}{(1+\alpha_1+\cdots+\alpha_n)^{n-1}}x_1^{-2}\cdots x_n^{-2},\\
[e_1,e_2,\dots,e_n,x] &= c(1+\alpha_1+\cdots+\alpha_n)x_1^{-\alpha_1}\cdots x_n^{-\alpha_n}.
\end{aligned}$$

We calculate the Tchebychev function:

$$\begin{aligned}
\psi = \frac{\det(h_{ij})}{[e_1,\dots,e_n,x]^2} &= \frac{1}{(f-\sum_i x_if_{x_i})^{n+2}}\cdot\det(f_{x_ix_j})\\
&= \frac{1}{c^2}\cdot\frac{\alpha_1\cdots\alpha_n}{(1+\alpha_1+\cdots+\alpha_n)^{n+1}}x_1^{-2+2\alpha_1}\cdots x_n^{-2+2\alpha_n}.
\end{aligned}\tag{2.3.17}$$

We easily see that the Tchebychev field has constant norm for any hypersurface of this class and that it satisfies $|T|=0$ if and only if

$$\alpha_1=\alpha_2=\cdots=\alpha_n=1.$$

Thus there is exactly one affine hypersphere in the class $Q(c;\alpha_1,\dots,\alpha_n;n)$. As mentioned, it is well known that proper affine spheres, in terms of centroaffine invariants, can be characterized by the vanishing of the Tchebychev field. Thus Wang's large class of centroaffine extremal hypersurfaces contains exactly one proper affine sphere, and within the example 2.3.3 the nonvanishing of the Tchebychev field characterizes the hypersurfaces not belonging to the class 3.1. Again, all hypersurfaces of the class 3.3 satisfy both completeness conditions (i) and (ii), stated at the beginning of this section.

To calculate the curvature tensor easily, we introduce new parameters $u_1,u_2,\dots,u_n$:

$$x_i = e^{u_i}, \quad 1\le i\le n.$$

Then $Q(c;\alpha_1,\dots,\alpha_n;n)$ can be represented as graph in terms of $u_1,\dots,u_n$ by

$$(x_1,\dots,x_n,x_{n+1}) = (e^{u_1},e^{u_2},\dots,e^{u_n},ce^{-\alpha_1u_1-\alpha_2u_2-\cdots-\alpha_nu_n}).$$

The coefficients of the centroaffine metric

$$h = \sum_{i,j} h_{ij} dx_i dx_j = \sum_{i,j} \tilde{h}_{ij} du_i du_j$$

satisfy

$$(\tilde{h}_{ij}) = \frac{1}{1 + \alpha_1 + \cdots + \alpha_n} \begin{pmatrix} \alpha_1(1+\alpha_1) & \alpha_1\alpha_2 & \cdots & \alpha_1\alpha_n \\ \alpha_2\alpha_1 & \alpha_2(1+\alpha_2) & \cdots & \alpha_2\alpha_n \\ \ldots & \vdots & \ddots & \vdots \\ \alpha_n\alpha_1 & \alpha_n\alpha_2 & \cdots & \alpha_n(1+\alpha_n) \end{pmatrix}.$$

Since $(\tilde{h}_{ij})$ is a constant matrix, we immediately get that the metric is flat. From [LW] we also know

$$A_{ijk,l} = 0, \quad \text{but} \quad J = \text{constant} \neq 0.$$

The properties just stated characterize the class $Q(c; \alpha_1, \ldots, \alpha_n; n)$. A.-M. Li and C. P. Wang proved

Proposition 2.3.3 (see Theorem 1.3 in [LW]). *Let $x : M \to R^{n+1}$ be an n-dimensional ($n \geq 2$) centroaffine hypersurface. If its centroaffine metric is flat and its centroaffine Pick form is parallel with respect to its centroaffine metric, then $x(M)$ is centroaffinely equivalent to one of the following hypersurfaces in R^{n+1}:*

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n+1}^{\alpha_{n+1}} = 1, \quad \alpha_1 > 0, \ldots, \alpha_{n+1} > 0.$$

In particular, any hypersurface of type $Q(c; \alpha_1, \ldots, \alpha_n; n)$ is an extremal centroaffine hypersurface with flat centroaffine metric and parallel centroaffine cubic form; contraction gives that the Tchebychev operator vanishes and thus the square of the norm of T is constant (and non-zero for all such hypersurfaces which are not affine spheres). Moreover, the two completeness conditions (i) and (ii) are satisfied.

Example 2.3.4 (*Generalized Calabi composition* ([LLS])). We extend the well-known Calabi composition for hyperbolic affine hypersurfaces to centroaffine extremal hypersurfaces.

Proposition 2.3.4 ([LLS]). *Given two centroaffine hyperbolic extremal hypersurfaces $x : M_1 \to R^{p+1}$ and $y : M_2 \to R^{q+1}$, the generalized Calabi composition $z : R \times M_1 \times M_2 \to R^{p+q+2}$:*

$$z = (C_1 e^u x, C_2 e^{-\lambda u} y), \quad u \in R, \tag{2.3.18}$$

defines a centroaffine extremal hypersurface, where λ, C_1, C_2 are arbitrary positive real numbers.

When x and y are two hyperbolic affine spheres, choosing $\lambda = \frac{p+1}{q+1}$ in Proposition 2.3.4, we recover the Calabi composition of two hyperbolic affine spheres:

Corollary 2.3.1 (see [LSZ-I]). *Given two hyperbolic affine spheres $x : M_1 \to R^{p+1}$ and $y : M_2 \to R^{q+1}$, the Calabi composition $x : R \times M_1 \times M_2 \to R^{p+q+2}$*

$$z = (C_1 e^u x, C_2 e^{-\frac{p+1}{q+1} u} y), \quad u \in R, \tag{2.3.19}$$

defines a hyperbolic affine sphere, where C_1, C_2 are any positive real numbers.

Proof of Proposition 2.3.4. Consider the given centroaffine extremal hypersurfaces x and y in Proposition 3.4. We construct the generalized Calabi composition z defined by (2.3.18). Let $\{u_1,\dots,u_p\}$ and $\{u_{p+1},\dots,u_{p+q}\}$ be local coordinates for M_1 and M_2, respectively. We denote $u_0=u$ and use the following range of indices:

$$1\le i,j,k\le p;\quad p+1\le\alpha,\beta,\gamma\le p+q;\quad 0\le A,B,C\le p+q.$$

We mark quantities of the hypersurface z by a tilde. Then $e_i=\frac{\partial x}{\partial u_i}$ form a basis for $x_*(TM_1)$, $e_\alpha=\frac{\partial y}{\partial u_\alpha}$ form a basis for $y_*(TM_2)$. Let $\tilde e_A=\frac{\partial z}{\partial u_A}$, i.e.,

$$\tilde e_0=(C_1e^ux,-C_2\lambda e^{-\lambda u}y),\quad \tilde e_i=(C_1e^ue_i,0),\quad \tilde e_\alpha=(0,C_2e^{-\lambda u}e_\alpha). \tag{2.3.20}$$

Then $\{\tilde e_A\}$ form a basis for $z_*(TR\oplus TM_1\oplus TM_2)$. We have

$$\begin{aligned}&[\tilde e_0,\tilde e_1,\dots,\tilde e_{p+q},z]\\ &\quad=(-1)^pC_1^{p+1}C_2^{q+1}(\lambda+1)e^{[(p+1)-(q+1)\lambda]u}[e_1,\dots,e_p,x]\cdot[e_{p+1},\dots,e_{p+q},y]\neq0.\end{aligned}$$

x and y are centroaffine hypersurfaces, thus z is also a centroaffine hypersurface.

We denote by h_x, h_y, h_z the centroaffine metrics and ∇_x, ∇_y, ∇_z the Levi-Civita connections for x,y,z, respectively. Then, by a direct calculation, we have

$$\begin{aligned}&\frac{\partial^2z}{\partial^2u_0}=(1-\lambda)\tilde e_0+\lambda z;\quad \frac{\partial^2z}{\partial u_0\partial u_i}=\tilde e_i;\quad \frac{\partial^2z}{\partial u_0\partial u_\alpha}=-\lambda\tilde e_\alpha,\\ &\frac{\partial^2z}{\partial u_i\partial u_j}=\frac{1}{\lambda+1}(h_x)_{ij}\tilde e_0+\sum_{k=1}^p(\nabla_x)_{ij}^k\tilde e_k+\frac{\lambda}{\lambda+1}(h_x)_{ij}\cdot z,\\ &\frac{\partial^2z}{\partial u_i\partial u_\alpha}=\frac{\partial^2z}{\partial u_\alpha\partial u_i}=0,\\ &\frac{\partial^2z}{\partial u_\alpha\partial u_\beta}=-\frac{1}{\lambda+1}(h_y)_{\alpha\beta}\tilde e_0+\sum_{\gamma=p+1}^{p+q}(\nabla_y)_{\alpha\beta}^\gamma\tilde e_\gamma+\frac{1}{\lambda+1}(h_y)_{\alpha\beta}\cdot z.\end{aligned} \tag{2.3.21}$$

By definition, the centroaffine metric of z is

$$h_z=\lambda(du_0)^2+\frac{\lambda}{\lambda+1}h_x+\frac{1}{\lambda+1}h_y=:\sum_{A,B=0}^{p+q}\tilde h_{AB}du_Adu_B. \tag{2.3.22}$$

If h_x, h_y are complete metrics, h_z is a complete metric. The Tchebychev function $\tilde\psi$ of z is

$$\begin{aligned}\tilde\psi&=\frac{\det(\tilde h_{AB})}{[\tilde e_0,\tilde e_1,\dots,\tilde e_{p+q},z]^2}\\ &=\frac{\lambda(\frac{\lambda}{\lambda+1})^p(\frac{1}{\lambda+1})^q\det(h^x_{ij})\cdot\det(h^y_{\alpha\beta})}{C_1^{2(p+1)}C_2^{2(q+1)}(\lambda+1)^2e^{2[(p+1)-(q+1)\lambda]u}[e_1,\dots,e_p,x]^2\cdot[e_{p+1},\dots,e_{p+q},y]^2}\\ &=C_1^{-2(p+1)}C_2^{-2(q+1)}\lambda^{p+1}(\lambda+1)^{-2-p-q}\cdot e^{-2[(p+1)-(q+1)\lambda]u}\psi_x\cdot\psi_y,\end{aligned} \tag{2.3.23}$$

where ψ_x and ψ_y are the Tchebychev functions of x and y, respectively. Thus

$$\begin{aligned}\log\tilde\psi&=[(p+1)\log\lambda-(2+p+q)\log(\lambda+1)-2(p+1)\log C_1-(q+1)\log C_2]\\ &\quad-2[(p+1)-(q+1)\lambda]u+\log\psi_x+\log\psi_y.\end{aligned} \tag{2.3.24}$$

The Laplacian $\tilde{\Delta}$ of h_z is given by

$$\tilde{\Delta} = \frac{1}{\lambda}\frac{\partial^2}{\partial^2 u} \oplus \frac{\lambda+1}{\lambda}\Delta_x \oplus (1+\lambda)\Delta_y, \tag{2.3.25}$$

thus we have

$$\tilde{\Delta}(\log\tilde{\psi}) = 0, \tag{2.3.26}$$

where Δ_x (resp. Δ_y) is the Laplacian of h_x (rep. h_y). From Theorem 2.2.2 and (2.3.18), $z : R\times M_1\times M_2 \to R^{p+q+2}$ is a $(p+q+1)$-dimensional centroaffine extremal hypersurface. In particular, if the Tchebychev operators of x and y vanish, then $\tilde{\mathcal{T}} \equiv 0$.

Proof of Corollary 2.3.1. If $x : M_1 \to R^{p+1}$ and $y : M_2 \to R^{q+1}$ are two hyperbolic affine spheres, then

$$(\log\psi)_x = \text{constant}, \quad (\log\psi)_y = \text{constant}. \tag{2.3.27}$$

Choosing

$$\lambda = \frac{p+1}{q+1}, \tag{2.3.28}$$

from (2.3.24) we have

$$\log\tilde{\psi} = \text{constant}.$$

Thus $x : R \times M_1 \times M_2 \to R^{p+q+2}$:

$$z = (C_1 e^u x, C_2 e^{-\frac{p+1}{q+1}u} y), \quad u \in R,$$

is a hyperbolic affine sphere.

Example 2.3.4-A ([LLS]). Taking $x(M_1) = H(1,p)$, $y(M_2) = H(1,q)$ and $C_1 = C_2 = 1$ in Proposition 2.3.4, we obtain a family of centroaffine extremal hypersurfaces $z : R \times M_1 \times M_2 \to R^{p+q+2}$

$$[z_{p+1}^2 - (z_1^2 + \cdots + z_p^2)] \cdot [z_{p+q+2}^2 - (z_{p+2}^2 + \cdots + z_{p+q+1}^2)]^{\frac{1}{\lambda}} = 1, \quad \lambda > 0.$$

We note that z is a hyperbolic affine sphere if and only if $\lambda = \frac{p+1}{q+1}$.

Example 2.3.4-B ([LLS]). Taking $x(M_1) = H(1,p)$, $y(M_2) = Q(1;\alpha_1,\ldots,\alpha_q;q)$ and $C_1 = C_2 = 1$ in Proposition 2.3.4, we obtain a family of centroaffine extremal hypersurfaces $z : R \times M_1 \times M_2 \to R^{p+q+2}$

$$[z_{p+1}^2 - (z_1^2 + \cdots + z_p^2)]^{\frac{(1+\alpha_1+\cdots+\alpha_q)\lambda}{2}} z_{p+2}^{\alpha_1} \cdots z_{p+q+1}^{\alpha_q} \cdot z_{p+q+2} = 1,$$

where $\alpha_1 > 0, \ldots, \alpha_q > 0$. We note that z is a hyperbolic affine sphere if and only if $\lambda = \frac{p+1}{q+1}$ and $\alpha_1 = \cdots = \alpha_q = 1$.

2.4. Centroaffine Bernstein problems. We first give the definition of completeness.

Definition 2.4.1. (i) Euclidean completeness is the completeness of the Riemannian metric on M induced from a Euclidean metric on A^{n+1}; this notion is independent of the specific choice of the Euclidean metric on the affine space and thus it is a notion of affine geometry; see $[LSZ-I]$, p. 110;

(ii) centroaffine completeness is the completeness of the centroaffine metric h.

In section 2.3 we studied large classes of centroaffine extremal hypersurfaces. All the explicit examples have vanishing Tchebychev operator. Comparing the class of hyperbolic affine spheres and the class of examples given in 2.3.3, there is only one type of hypersurfaces in the intersection of both classes, namely the hypersurfaces represented by

$$x_1 x_2 \cdots x_{n+1} = c, \quad c > 0.$$

Concerning completeness conditions, the compact case is solved by Wang's theorem. Thus only complete, non-compact centroaffine extremal hypersurfaces are still of interest. The classes in example 2.3.1 and 2.3.3 can be represented as graphs over R^n, that means they are Euclidean complete. The hypersurfaces in examples 2.3.1 and 2.3.3 are also centroaffine complete.

In the following we list several related versions of centroaffine Bernstein problems for locally strongly convex hypersurfaces; some of the problems are stated in form of conjectures.

Centroaffine Bernstein Problem I. *Let $x : M \to R^{n+1}$ $(n \geq 2)$ be a centroaffine extremal hypersurface satisfying one of the completeness conditions from Definition 2.4.1. Is $\mathcal{T} \equiv 0$?*

Centroaffine Bernstein Conjecture. *Let $x : M \to R^{n+1}$ $(n \geq 2)$ be a centroaffine extremal hyperbolic hypersurface satisfying one of the completeness conditions from Definition 2.4.1. If the Ricci curvature of the centroaffine metric is non-negative, then x is centroaffinely equivalent to one of the hypersurfaces*

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n+1}^{\alpha_{n+1}} = 1, \quad \alpha_1 > 0, \ldots, \alpha_{n+1} > 0.$$

Centroaffine Bernstein Problem II. *Does the class of centroaffine extremal hyperbolic graphs over R^n contain other examples as the ones given in examples 2.3.1 and 2.3.3?*

Centroaffine Bernstein Problem III. *Do there exist extremal centroaffine hypersurfaces with complete centroaffine metric which can not be represented as graphs over R^n?*

Centroaffine Bernstein Problem IV. *Do there exist extremal centroaffine hypersurfaces satisfying one of the completeness conditions such that the Tchebychev field does not have constant norm?*

Centroaffine Bernstein Problem V. *Do there exist extremal elliptic centroaffine hypersurfaces satisfying one of the completeness conditions which are not hyperellipsoids?*

2.5. Statement of the results

Theorem 2.5.1 ([LLS]). *Let $x : M \to R^3$ be a noncompact, hyperbolic extremal centroaffine surface with complete centroaffine metric. If the Gaussian curvature K of the centroaffine metric and the length $|T|$ of the Tchebychev vector field satisfy*

(1) $K \geq 0$,
(2) $|T| < \infty$,

then x is centroaffinely equivalent to one of the surfaces

$$x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}=1,\quad \alpha_1>0,\quad \alpha_2>0,\quad \alpha_3>0. \tag{2.5.1}$$

Corollary 2.5.1 (see [LSZ-I]). *Let $x: M\to R^3$ be an affine complete hyperbolic affine sphere. If the Gaussian curvature K of the centroaffine metric is nonnegative, then x is affinely equivalent to the surface*

$$x_1x_2x_3=1. \tag{2.5.2}$$

Theorem 2.5.2 ([LLS]). *Let $x: M\to R^{n+1}$ $(n\geq 2)$ be a non-compact hyperbolic extremal centroaffine hypersurface with complete centroaffine metric. If the Ricci curvature of the centroaffine metric and the length $|T|$ of the Tchebychev vector field satisfy*

(1) $\mathrm{Ric}\geq 0$,
(2) $|T|=\mathrm{constant}$,

then x is centroaffinely equivalent to one of the hypersurfaces

$$x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_{n+1}^{\alpha_{n+1}}=1,\quad \alpha_1>0,\dots\alpha_{n+1}>0. \tag{2.5.3}$$

Corollary 2.5.2 (see [LSZ-I]). *Let $x: M\to R^{n+1}$ $(n\geq 2)$ be a complete hyperbolic affine hypersphere. If the Ricci curvature of the centroaffine metric is non-negative, then x is affinely equivalent to the hypersurface*

$$x_1x_2\cdots x_{n+1}=1. \tag{2.5.4}$$

Theorem 2.5.3 ([LLS]). *Let $x: M\to R^{n+1}$ $(n\geq 2)$ be a metrically complete, non-compact extremal centroaffine hypersurface. If the Ricci curvature of the centroaffine metric and the length $|T|$ of the Tchebychev vector field satisfy*

(1) $\mathrm{Ric}\geq 0$,
(2) $|T|\in L^p(M)$, *for some* $p>1$,

then x is centroaffinely equivalent to the hypersurface

$$x_1x_2\cdots x_{n+1}=1.$$

Theorem 2.5.4 ([LLS]). *Let $x: M\to R^{n+1}(n\geq 2)$ be a metrically complete, non-compact extremal centroaffine hypersurface. If the Ricci curvature of the centroaffine metric is non-negative and $\log\psi$ is bounded, then x is centroaffinely equivalent to the hypersurface*

$$x_1x_2\cdots x_{n+1}=1.$$

Remark 2.5.1. A hyperboloid $H(c,n)$ satisfies (see Example 2.3.1)

1. the centroaffine metric is complete and centroaffine extremal,
2. the Tchebychev function is a constant function and the Tchebychev vector field vanishes.

On the other hand its Ricci curvature is a negative constant (see (2.3.3)). Thus the assumption in Theorems 2.5.1-2.5.4 that the "Ricci curvature is nonnegative" is necessary.

Remark 2.5.2. For the centroaffine hypersurfaces

$$x_1^{\alpha_1}\cdots x_{n+1}^{\alpha_{n+1}}=c,\ c>0,\ (\alpha_1,\dots,\alpha_{n+1})\neq(1,\dots,1),\ \alpha_i>0,\ 1\leq i\leq n+1,$$

using (3.17), it is easy to check that $\log\psi$ is not bounded. Thus the assumption in Theorem 2.5.4 that "$\log\psi$ is bounded" is essential.

2.6. Lemmas and proofs of Theorem 2.5.1 and Theorem 2.5.2. We will apply the following well known Bochner-Lichnerowicz formula as a tool:

$$\frac{1}{2}\Delta(|T|^2) = \frac{1}{2}\Delta\Big(\sum_i (T_i)^2\Big) = \sum_{i,j}(T_{i,j})^2 + \sum_{i,j} R_{ij}T_iT_j + \sum_i T_i\Big(\sum_k T_{k,k}\Big)_i. \tag{2.6.1}$$

If we assume that the trace of the Tchebychev operator is constant, i.e., $\sum_k T_{k,k} =$ constant, then (2.6.1) becomes

$$\frac{1}{2}\Delta(|T|^2) = \frac{1}{2}\Delta\Big(\sum_i (T_i)^2\Big) = \sum_{i,j}(T_{i,j})^2 + \sum_{i,j} R_{ij}T_iT_j. \tag{2.6.2}$$

Lemma 2.6.1. *Let $x : M \to R^3$ be a metrically complete, noncompact centroaffine surface with constant trace of the Tchebychev operator. If the Gaussian curvature K of the centroaffine metric and the length $|T|$ of the Tchebychev vector field satisfy*

(1) $K \geq 0$,
(2) $|T| < \infty$,

then the Tchebychev vector field is parallel, i.e., $T_{i,j} = 0$.

Proof. As we assume $K \geq 0$, from the Riemann mapping theorem we conclude that either M is conformally equivalent to the Riemannian sphere S^2, or M is conformally equivalent to the Euclidean space R^2. From the assumption the surface is complete, but non-compact, we know that M is conformally equivalent to the Euclidean space R^2.

We apply (2.6.2), $Ric = Kh$ and the assumption $K \geq 0$:

$$\frac{1}{2}\Delta(|T|^2) \geq \sum_{i,j}(T_{i,j})^2 \geq 0,$$

that is, $|T|^2$ is a subharmonic function on M. The assumption $|T|^2 < \infty$ gives $|T|^2 =$ constant (see Leon Karp [Ka]), and (2.6.2) implies $T_{i,j} = 0$, i.e., $\mathcal{T} = 0$.

Lemma 2.6.2. *Let $x : M \to R^{n+1}$ be a complete noncompact centroaffine hypersurface with* trace$\mathcal{T}$ = constant. *If the Ricci curvature of the centroaffine metric and the length $|T|$ of the Tchebychev vector field satisfy*

(1) Ric ≥ 0,
(2) $|T| =$ constant,

then $\mathcal{T} \equiv 0$.

The proof follows again from (2.6.2).

We need the following generalized maximum principle:

Lemma 2.6.3 (Omori-Yau [Om], [Y1]). *Let M be a complete Riemannian manifold with Ricci curvature bounded from below. Let f be a C^2-function which is bounded from below on M. Then there is a sequence of points $\{p_k\}$ in M such that*

$$\lim_{k\to\infty} f(p_k) = \inf(f), \quad \lim_{k\to\infty} |\mathrm{grad}(f)|(p_k) = 0, \quad \lim_{k\to\infty} \Delta f(p_k) \geq 0.$$

Proposition 2.6.1 ([LLS]). *Let $x: M \to R^{n+1}$ be a complete, noncompact hyperbolic centroaffine hypersurface with $\mathcal{T} \equiv 0$. If the Ricci curvature of the centroaffine metric is non-negative, then x is centroaffinely equivalent to one of the hypersurfaces*

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n+1}^{\alpha_{n+1}} = 1, \quad \alpha_1 > 0, \ldots, \alpha_{n+1} > 0.$$

For the proof we need the following lemma

Lemma 2.6.4. *Let $x: M \to R^{n+1}$ be a centroaffine hypersurface with* $\mathrm{Ric} \geq 0$ *and $\mathcal{T} \equiv 0$. Then the normalized scalar curvature satisfies*

$$\Delta\kappa \geq 4\kappa(\kappa - \epsilon). \tag{2.6.3}$$

Proof. By use of (2.1.15), (2.1.23) and (2.1.26), we have the following calculation (cf. [LSZ-I])

$$\begin{aligned}
\Delta A_{ijk} &= \sum_l A_{ijk,ll} = \sum_l A_{ijl,kl} \\
&= \sum_l A_{ijl,lk} + \sum_{r,l} A_{ijr} R_{rlkl} + \sum_{r,l} A_{ril} R_{rjkl} + \sum_{r,l} A_{rjl} R_{rikl} \\
&= nT_{i,jk} + \sum_r A_{ijr} R_{rk} + + \sum_{r,l} A_{ril} R_{rjkl} + \sum_{r,l} A_{rjl} R_{rikl} \\
&= \sum_r A_{ijr} R_{rk} + \sum_{r,l} A_{ril} R_{rjkl} + \sum_{r,l} A_{rjl} R_{rikl},
\end{aligned} \tag{2.6.4}$$

where we used $\mathcal{T} \equiv 0$. (2.6.4) and (2.1.22) give

$$\begin{aligned}
\frac{1}{2} n(n-1)\Delta J &= \Delta\Big(\sum_{i,j,k} (A_{ijk})^2\Big) = \sum_{i,j,k,l} (A_{ijk,l})^2 + \sum_{i,j,k,l} A_{ijk} A_{ijk,ll} \\
&= \sum_{i,j,k,l} (A_{ijk,l})^2 + \sum A_{ijk} A_{ijr} R_{rk} + \sum A_{ijk} A_{ril} R_{rjkl} + \sum A_{ijk} A_{rjl} R_{rikl} \\
&= \sum_{i,j,k,l} (A_{ijk,l})^2 + \sum A_{ijk} A_{ijr} R_{rk} + \sum (A_{ijk} A_{ril} - A_{ijl} A_{irk}) R_{rjkl} \\
&= \sum_{i,j,k,l} (A_{ijk,l})^2 + \sum (R_{rjkl})^2 + A_{ijk} A_{ijr} R_{rk} - 2\epsilon R \\
&\geq \sum (R_{rjkl})^2 - 2\epsilon R \geq \frac{2}{n(n-1)} R^2 - 2\epsilon R,
\end{aligned} \tag{2.6.5}$$

where we used $\mathrm{Ric} \geq 0$ and the well known estimate

$$\sum (R_{rjkl})^2 \geq \frac{2}{n-1} \sum (R_{rk})^2 \geq \frac{2}{n(n-1)} R^2. \tag{2.6.6}$$

From (2.1.25), we have

$$n(n-1)J = n(n-1)(\kappa - \epsilon) + n^2 |T|^2. \tag{2.6.7}$$

The assumption $\mathcal{T} \equiv 0$ implies that $|T|^2$ is constant; we insert (2.6.7) into (2.6.5)

$$\frac{1}{2} n(n-1)\Delta\kappa = \Delta\Big(\sum_{i,j,k} (A_{ijk})^2\Big) \geq \frac{2}{n(n-1)} R^2 - 2\epsilon R = 2n(n-1)\kappa(\kappa - \epsilon). \tag{2.6.8}$$

Proof of Proposition 2.6.1. For any given positive constant δ, define the positive smooth function u on M by

$$u := \frac{1}{\sqrt{\kappa + \delta}}. \tag{2.6.9}$$

Through a direct calculation, by use of (2.6.3) and $\epsilon = -1$, the Laplacian Δu of u satisfies

$$u\Delta u = 3|\text{grad}(u)|^2 - \frac{1}{2(\kappa+\delta)^2}\Delta\kappa \leq 3|\text{grad}(u)|^2 - \frac{2}{(\kappa+\delta)^2}\kappa(\kappa+1). \tag{2.6.10}$$

We have $u \geq 0$; as we assumed that the Ricci curvature is non-negative, we can apply the generalized maximum principle (Lemma 2.6.3) of Omori and Yau to the function u on M. Then there is a sequence of points $\{p_k\}$ on M such that

$$\lim_{k\to\infty} u(p_k) = \inf(u), \quad \lim_{k\to\infty} |\text{grad}(u)|(p_k) = 0, \quad \lim_{k\to\infty} \Delta u(p_k) \geq 0.$$

We claim that $\inf(u) \neq 0$. Otherwise, from the definition of u, the assumption $\inf(u) = 0$ gives $\sup(\kappa) = \infty$. Considering the limit for both sides of the inequality (2.6.10), we get

$$0 = \inf(u) \cdot \lim_{k\to\infty} \Delta u(p_k) \leq -2,$$

which gives a contradiction. Thus $\inf(u) \neq 0$ and then $0 \leq \lim_{k\to\infty} \kappa(p_k) = \sup(\kappa) < \infty$. Considering again the limit for both sides of the inequality (2.6.10), we get

$$\begin{aligned} 0 &\leq \inf(u) \cdot \lim_{k\to\infty} \Delta u(p_k) \\ &\leq 3 \cdot \lim_{k\to\infty} |\text{grad}(u)|^2(p_k) - \frac{2\sup(\kappa)}{(\sup(\kappa)+\delta)^2}(\sup(\kappa)+1) \\ &= -\frac{2\sup(\kappa)}{(\sup(\kappa)+\delta)^2}(\sup(\kappa)+1). \end{aligned} \tag{2.6.11}$$

(2.6.11) implies

$$\sup(\kappa) \leq 0,$$

that is

$$\kappa \leq 0.$$

Thus we conclude that $\kappa \equiv 0$ (because we assumed $Ric \geq 0$). From (2.6.7) and $\epsilon = -1$, we get $J = 1 + \frac{n}{n-1}|T|^2 =$ constant and then (2.6.5) gives

$$R_{ijkl} \equiv 0, \quad A_{ijk,l} = 0, \quad 1 \leq i,j,k,l \leq n. \tag{2.6.12}$$

Thus $x(M)$ has a flat centroaffine metric and its centroaffine Pick form is parallel with respect to its centroaffine metric. The assertion of Proposition 2.6.1 now follows from Proposition 2.3.3.

Proofs of Theorem 2.5.1 and Theorem 2.5.2. Theorem 2.5.1 comes from Lemma 2.6.1 and Proposition 2.6.1. Theorem 2.5.2 comes from Lemma 2.6.2 and Proposition 2.6.1.

REMARK 2.6.1. We also can get the following local uniqueness results, which generalize the result of Li-Wang (see Proposition 2.3.3).

PROPOSITION 2.6.2 ([LLS]). *Let $x : M \to R^{n+1}$ be a centroaffine hypersurface with $\mathcal{T} \equiv 0$. If the Ricci curvature of the centroaffine metric is non-negative (or non-positive),* *then x is locally centroaffinely equivalent to a proper affine sphere or one of the following hypersurfaces*

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n+1}^{\alpha_{n+1}} = 1, \quad \alpha_1 > 0, \ldots, \alpha_{n+1} > 0.$$

Proof. Because we assume $\mathcal{T} \equiv 0$, we have from (2.6.2)

$$\sum_{i,j} R_{ij} T_i T_j \equiv 0. \tag{2.6.13}$$

From the assumption $R_{ij} \geq 0$ (resp. $R_{ij} \leq 0$) we have either $|T| \equiv 0$, or $R_{ij} \equiv 0$. If $|T| \equiv 0$ then $x : M \to R^{n+1}$ is a proper affine sphere. If $R_{ij} \equiv 0$, we get from (2.6.5)

$$R_{ijkl} \equiv 0, \quad A_{ijk,l} = 0, \quad 1 \leq i, j, k, l \leq n.$$

Thus $x(M)$ has a flat centroaffine metric and its centroaffine Pick form is parallel with respect to its centroaffine metric. Proposition 2.6.2 now follows from Proposition 2.3.3.

Corollary 2.6.1. *Let $x : M \to R^{n+1}(n \geq 2)$ be an n-dimensional complete elliptic centroaffine hypersurface with $\mathcal{T} \equiv 0$. If the Ricci curvature of the centroaffine metric is non-negative (resp. non-positive), then x is centroaffinely equivalent to a hyperellipsoid (resp. there does not exist such a hypersurface).*

Proof. From Proposition 2.6.2, it follows that x is an elliptic affine sphere, thus x is centroaffinely equivalent to a hyperellipsoid (resp. there does not exist such a hypersurface).

Proposition 2.6.3 ([LLS]). *Let $x : M \to R^{n+1}$ $(n \geq 2)$ be an n-dimensional hyperbolic centroaffine extremal hypersurface. If the Ricci curvature of the centroaffine metric is nonnegative, the scalar curvature is constant, and the length of the Tchebychev vector field is constant, then x is centroaffinely equivalent to one of the hypersurfaces*

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n+1}^{\alpha_{n+1}} = 1, \quad \alpha_1 > 0, \ldots, \alpha_{n+1} > 0.$$

Proof. As we assume that $x : M \to R^{n+1}$ is a centroaffine extremal hypersurface with $|T| =$ constant, we have from (2.6.2) that

$$T_{i,j} = 0.$$

Our assumptions imply $J =$ constant and $\epsilon = -1$. From (2.6.5) we get

$$R_{ijkl} \equiv 0, \quad A_{ijk,l} = 0, \quad 1 \leq i, j, k, l \leq n.$$

Thus $x(M)$ has a flat centroaffine metric and its centroaffine Pick form is parallel with respect to its centroaffine metric. Proposition 2.6.3 now follows from Proposition 2.3.3.

2.7. Proofs of Theorem 2.5.3 and Theorem 2.5.4. We need the following lemmas:

Lemma 2.7.1 ([Y1]). *Let (M, g) be a complete Riemannian manifold with non-negative Ricci curvature, then any bounded (from below or from above) harmonic function on M must be a constant.*

Lemma 2.7.2 ([Y2]). *Let (M, g) be a complete noncompact Riemannian manifold with non-negative Ricci curvature. If for some $p > 1$*

$$\Delta u \geq 0, \quad u \geq 0, \quad u \in L^p(M),$$

then u is constant.

Proof of Theorem 2.5.3. Under the assumptions of Theorem 2.5.3, we have from (2.6.2)

$$\frac{1}{2}\Delta(|T|^2) = \sum_{i,j}(T_{i,j})^2 + \sum_{ij} R_{ij}T_iT_j \geq \sum_{i,j}(T_{i,j})^2. \tag{2.7.1}$$

Noting

$$\frac{1}{2}\Delta(|T|^2) = |T|\Delta|T| + \sum_i(|T|_i)^2, \tag{2.7.2}$$

we have from (2.7.1) and (2.7.2),

$$|T|\Delta|T| \geq \sum_{i,j}(T_{i,j})^2 - \sum_i(|T|_i)^2. \tag{2.7.3}$$

From (2.7.3) and

$$\begin{aligned} |T|^2\sum_i(|T|_i)^2 &= \sum_i(|T||T|_i)^2 = \frac{1}{2}\sum_i((|T|^2)_i)^2 \\ &= \sum_i(\sum_j T_iT_{i,j})^2 \leq |T|^2 \cdot \sum_{i,j}(T_{i,j})^2, \end{aligned} \tag{2.7.4}$$

we conclude that $\Delta|T| \geq 0$, i.e. $|T|$ is a non-negative subharmonic function. From Lemma 2.7.2, our assumption $|T| \in L^p(M)$ $(p > 1)$ implies that $|T|$ is constant. Thus we get $T_{i,j} = 0$ from (2.7.1). In this case, as the volume of M is infinite (see [SY1] or [SY2]) and as we assume $|T| \in L^p(M)$, we necessarily have $|T| = 0$. Since a complete elliptic affine hypersphere is a hyperellipsoid (compact), Theorem 2.5.3 then directly follows from Proposition 2.6.1 and the remarks in Example 2.3.3.

Proof of Theorem 2.5.4. Let $x : M \to R^{n+1}$ be an n-dimensional centroaffine extremal hypersurface; then we have

$$\Delta(\log\psi) = 0,$$

where ψ is the Tchebychev function of x. From Lemma 2.7.1 it follows that $\log\psi$ is constant and that the Tchebychev vector field vanishes. Since a complete elliptic affine hypersphere is a hyperellipsoid (compact), Theorem 2.5.4 follows from Proposition 2.6.1 and the remarks in Example 2.3.3.

3. VARIATIONAL PROBLEMS IN RELATIVE GEOMETRY

3.1. Introduction. In this chapter we study a graph defined by a convex function $x_{n+1} = f(x_1, \ldots, x_n)$. There is a natural metric $G = \sum_{i,j}\frac{\partial^2 f}{\partial x_i \partial x_j}dx_idx_j$, which is called *Calabi metric*. We calculate the first variation of the volume of this metric. The Euler-Lagrange equation for the area functional is

$$\Delta\log(\det(f_{ij})) = 0,$$

where $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$, and Δ is the Laplacian with respect to the metric G. This is a 4-th order PDE. Solutions of the PDE are called affine extremal graphs of this variational problem. It is easy to see that all parabolic equiaffine spheres are affine extremal graphs. We would like to raise the following conjectures:

Conjecture 3.1.1. *Let $f(x_1,\dots,x_n)$ be a convex function defined on a convex domain $\Omega\subset A^n$ and M be the graph determined by f. If M is an affine extremal graph and if M is complete with respect to the metric G, then M must be an elliptic paraboloid.*

Conjecture 3.1.2. *Let $f(x_1,\dots,x_n)$ be a convex function defined on all of A^n. Let M be the graph determined by f. If M is an affine extremal graph, then M must be an elliptic paraboloid.*

In this chapter we give a partial answer to the first conjecture. Precisely, we prove the following result:

Theorem 3.1.1. *Let $f(x_1,\dots,x_n)$ be a convex function defined on a convex domain $\Omega\subset A^n$ and M be the graph determined by f. Suppose that M is an affine extremal graph and M is complete with respect to the metric G. If the Ricci curvature is nonnegative, and if there is a constant $N>0$ such that the so called Tchebychev function (see Definition 2.1 below) satisfies the inequality $\psi\le N$ everywhere on M, then M must be an elliptic paraboloid.*

3.2. Preliminaries. We summarize basic formulas of affine graphs with relative normalization $e_{n+1}=(0,\dots,0,1)$ in terms of Cartan's moving frames. The setup is similar to the centroaffine case. Let $f(x_1,\dots,x_n)$ be a convex function defined on a convex domain $\Omega\subset A^n$. We choose the relative normalization $e_{n+1}=(0,\dots,0,1)$ and study the relative geometry of the graph $M=\{x_1,\dots,x_n,f(x_1,\dots,x_n)\}$. We choose an affine frame field $\{e_1,\dots,e_n,e_{n+1}\}$ with $e_{n+1}=(0,\dots,0,1)$ and $e_1,\dots,e_n\in T_xM$; we denote by $\{\omega^1,\dots,\omega^n\}$ the dual frame field of the tangential frame field. We can write

$$dx=\sum_i\omega^ie_i,\quad \omega^{n+1}=0, \tag{3.2.1}$$

$$de_i=\sum_j\omega_i^je_j+\omega_i^{n+1}e_{n+1},\quad de_{n+1}=0. \tag{3.2.2}$$

Differentiation of (2.1) – (2.2) gives the integrability conditions:

$$d\omega^i=\sum_j\omega^j\wedge\omega_j^i,\quad \sum_i\omega^i\wedge\omega_i^{n+1}=0, \tag{3.2.3}$$

$$d\omega_i^j=\sum_k\omega_i^k\wedge\omega_k^j-\epsilon\omega_i^{n+1}\wedge\omega^j,\quad d\omega_i^{n+1}=\sum_j\omega_i^j\wedge\omega_j^{n+1}. \tag{3.2.4}$$

From the second equation in (2.3), we have

$$\omega_i^{n+1}=\sum_{i,j}h_{ij}\omega^j,\quad h_{ij}=h_{ji}. \tag{3.2.5}$$

As f is a convex function, the quadratic form

$$G=\sum_{i,j}h_{ij}\omega^i\omega^j \tag{3.2.6}$$

is positive definite; h is called the relative affine metric of the graph. It is well known that h is independent of the choice of the frame $\{e_1,\dots,e_n\}$ and that h is invariant under transformations of the group $GL(R,n+1)$. We choose an affine tangential frame

$\{e_1, \ldots, e_n\}$ on M such that $h_{ij} = \delta_{ij}$, i.e.,

$$\omega_i^{n+1} = \omega^i. \tag{3.2.7}$$

Differentiate (3.2.7) and use (3.2.4); this implies

$$d\omega^i = \sum_j \omega_{ij} \wedge \omega^j. \tag{3.2.8}$$

(3.2.3) and (3.2.8) give

$$d\omega^i = \sum_j \omega^j \wedge \left[\frac{1}{2}(\omega_{ji} - \omega_{ij})\right]. \tag{3.2.9}$$

The expression $\frac{1}{2}(\omega_{ji} - \omega_{ij})$ is skew-symmetric and $\{\omega^1, \ldots, \omega^n\}$ is an orthonormal coframe of the relative metric G. (3.2.9) and the fundamental theorem of Riemannian geometry imply that the Levi-Civita connection of G satisfies

$$\tilde{\omega}_{ji} = \frac{1}{2}(\omega_{ji} - \omega_{ij}), \quad \tilde{\omega}_{ji} = -\tilde{\omega}_{ij}. \tag{3.2.10}$$

Define

$$\omega_{ij} - \tilde{\omega}_{ij} = \frac{1}{2}(\omega_{ij} + \omega_{ji}) = \sum_k A_{ijk}\omega^k. \tag{3.2.11}$$

This gives the symmetry relation

$$A_{ijk} = A_{jik}. \tag{3.2.12}$$

Combine (3.2.8) with (3.2.9) and use (3.2.11):

$$\sum_{j,k} A_{ijk}\omega_j \wedge \omega_k = 0,$$

this implies the total symmetry of the form

$$A = \sum_{i,j,k} A_{ijk}\omega^i\omega^j\omega^k,$$

namely

$$A_{ijk} = A_{ikj} = A_{jik}. \tag{3.2.13}$$

The form A is called the relative cubic form of the graph M with the given relative normalization. Again it is well known that this form is independent of the choice of the frame and invariant under transformations of the group $GL(R, n+1)$. We need the following two important geometric invariants built from G and A:

$$J = \frac{1}{n(n-1)} \sum_{i,j,k} A_{ijk}^2 \tag{3.2.14}$$

is called the Pick invariant. The tangent vector field

$$T = \sum_i T_i e_i, \quad T_i = \frac{1}{n}\sum_j A_{jji} \tag{3.2.15}$$

is called the Tchebychev vector field of M. There is an additional well known relation between T and the so called affine Tchebychev function ψ of x. To state this relation, we recall the following definition.

Definition 3.2.1. The positive function ψ, given by

$$\psi = \frac{\det(h_{ij})}{[e_1, \dots, e_n, x]^2}, \tag{3.2.16}$$

is independent of the choice of the frame $\{e_1, \dots, e_n\}$ and is invariant under transformations of $GL(R, n+1)$, where $[\cdots]$ denotes the determinant.

Choosing $i = j$ in (3.2.11) and summing up over i, we get

$$\sum_{i,k} A_{iik}\omega^k = \sum_i \omega_{ii} = d(\log[e_1, \dots, e_n, x]) = -\frac{1}{2} d \log \psi. \tag{3.2.17}$$

The Tchebychev vector field T satisfies the relation

$$T_i = -\frac{1}{2n}(\log \psi)_i = \frac{(n+2)}{2n}(\log \rho)_i. \tag{3.2.18}$$

For later applications we list the integrability conditions in terms of the metric and the cubic form. In a standard local notation, by a comma we indicate covariant differentiation in terms of the Levi-Civita connection. As in (2.1.21) the sign of the Riemannian curvature tensor $\Omega = \sum R_{ijkl}\omega^i \otimes \omega^j \otimes \omega^k \otimes \omega^l$ of h is fixed by

$$d\tilde{\omega}_{ij} - \sum_k \tilde{\omega}_{ik} \wedge \tilde{\omega}_{kj} = -\frac{1}{2} \sum_{k,l} R_{ijkl}\omega^k \wedge \omega^l. \tag{3.2.19}$$

In terms of the frame considered ($h_{ij} = \delta_{ij}$), the Gauss equations read

$$R_{ijkl} = \sum_m (A_{jkm}A_{mil} - A_{ikm}A_{mjl}); \tag{3.2.20}$$

this formula differs from the centroaffine formula (2.1.22). The cubic form satisfies Codazzi equations, that means the covariant derivative is totally symmetric:

$$A_{ijk,l} = A_{ijl,k}. \tag{3.2.21}$$

Here, as mentioned above, $A_{ijk,l}$ are the components of the covariant derivative of A with respect to the Levi-Civita connection of h. Contraction of (2.20) gives the important relation

$$R_{ik} = \sum_{m,l} A_{iml}A_{mlk} - n \sum_m T_m A_{mik}, \tag{3.2.22}$$

where R_{ik} denote the components of the Ricci tensor, and the "relative theorema egregium" for a graph

$$n(n-1)\kappa = R = n(n-1)J - n^2|T|^2, \quad |T|^2 = \sum_i (T_i)^2, \tag{3.2.23}$$

where κ denotes the normalized scalar curvature.

Later we will need the Ricci identities

$$A_{ijk,lm} - A_{ijk,ml} = \sum_r A_{rjk}R_{rilm} + \sum_r A_{irk}R_{rjlm} + \sum_r A_{ijr}R_{rklm}. \tag{3.2.24}$$

The Codazzi equations for A (or the relations between T and the Tchebychev function) imply

$$T_{i,j} = T_{j,i}. \tag{3.2.25}$$

If $T_{i,j} = 0$, we say that the Tchebychev vector field T is parallel.

In the following we choose the following affine frame field:

$$e_1 = \left(1, 0, \dots, 0, \frac{\partial f}{\partial x_1}\right)$$
$$\dots\dots\dots$$
$$e_n = \left(0, 0, \dots, 1, \frac{\partial f}{\partial x_n}\right)$$
$$e_{n+1} = (0, 0, \dots, 0, 1).$$

Then the relative metric is given by

$$G = \sum_{i,j} f_{ij} dx_i dx_j;$$

here and later we write $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$. It is easy to check that, with respect to this frame, we have

$$\omega_i^j = 0, \quad A_{ijk} = f_{ijk}, \quad \psi = \det(f_{ij}),$$
$$J = \frac{1}{n(n-1)} \sum_{i,j,k,m,n} f^{il} f^{jm} f^{kn} f_{ijk} f_{lmn}, \quad T_k = -\frac{1}{2n} (\log \psi)_k.$$

3.3. The first variational formula. As in section 2.2 we study admitted variations of the hypersurface. Consider the one parameter variation: $f(t,x)$, $x = (x_1, \dots, x_n) \in \Omega$. The volume element is given by

$$dV = \sqrt{\det(f_{ij})} dx_1 \wedge \dots \wedge dx_n. \tag{3.3.1}$$

We have

$$\frac{\partial}{\partial t} \sqrt{\det(f_{ij})} = \frac{1}{2} \sqrt{\det(f_{ij})} \sum_{i,j} f^{ij} H_{ij}, \tag{3.3.2}$$

where we denote

$$H(t,x) = \frac{\partial f}{\partial t}.$$

$H(0,x)$ is the variational vector field. Stokes' formula gives

$$\frac{\partial V}{\partial t}|_{t=0} = \frac{1}{2} \int \sum_{i,j} (\sqrt{\det(f_{kl})} f^{ij})_{ij} H dx_1 \wedge \dots \wedge dx_n. \tag{3.3.3}$$

Now we express $\sum_{i,j} (\sqrt{\det(f_{kl})} f^{ij})_{ij}$ in terms of the Tchebychev function. First we have

$$\sum_{i,j} (\sqrt{\det(f_{kl})} f^{ij})_{ij} = \sum_{i,j} \left(\frac{1}{2} \sqrt{\det(f_{kl})} (\log \psi)_i f^{ij} + \sqrt{\det(f_{kl})} f_i^{ij} \right)_j. \tag{3.3.4}$$

Differentiating the equality

$$\sum_k f^{ik} f_{kj} = \delta_j^i$$

we get

$$\sum_i f_i^{ij} = -\sum f^{ij} (\log \psi)_i. \tag{3.3.5}$$

Inserting (3.3.5) into (3.3.4) we have

$$\sum_{i,j}(\sqrt{\det(f_{kl})}f^{ij})_{ij} = -\frac{1}{2}\sum_{i,j}(\sqrt{\det(f_{kl})}f^{ji}(\log\psi)_i)_j = -\frac{1}{2}\sqrt{\det(f_{kl})}\Delta(\log\psi). \quad (3.3.6)$$

Putting (3.3.6) into (3.3.3) we get the first variational formula:

$$\frac{\partial V}{\partial t}|_{t=0} = -\frac{1}{4}\int \Delta(\log\psi)HdV. \quad (3.3.7)$$

A hypersurface M defined by a convex function $x_{n+1} = f(x_1,\ldots,x_n)$ is called an *affine extremal graph* if $\frac{\partial V}{\partial t}|_{t=0} = 0$.

We immediately obtain

THEOREM 3.3.1. *Let $f(x_1,\ldots,x_n)$ be a convex function defined on a convex domain $\Omega \subset A^n$. Let M be the graph determined by f. Then M is an affine extremal graph if and only if*

$$\Delta(\log\psi) = 0. \quad (3.3.8)$$

Proof. By assumption (M,G) is a complete Riemannian manifold with Ric ≥ 0, and $\log\psi$ is a bounded harmonic function. Therefore $\psi = const.$, this means M is a parabolic affine hypersphere. By Pogorelov's theorem (see [LSZ-I]), M must be an elliptic paraboloid.

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