AFFINE MAXIMAL HYPERSURFACES

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Abstract. This paper is part of the autumn school on "Variational problems and higher order PDEs for affine hypersurfaces". We discuss affine Bernstein problems and complete constant mean curvature surfaces in equiaffine differential geometry.

Contents

1. Affine Weierstrass representation. We use the notation for equiaffine hypersurfaces in real affine space A^{n+1} from [L-S-Z]. Let M be a connected, oriented differentiable manifold, and let $x : M \to A^{n+1}$ be a locally strongly convex hypersurface. If the unimodular-affine mean curvature L_1 satisfies $L_1 = 0$ on M then $x(M)$ is called an *affine maximal hypersurface*.

The conormal field U satisfies the PDE $\Delta U + nL_1 U = 0$; this implies the following

THEOREM 1.1. *Let* $x : M \to A^{n+1}$ *be a locally strongly convex hypersurface, given as the graph of a function* f; x *is an affine maximal hypersurface (which means* $L_1 \equiv 0$ *on* M) *if and only if* f *satisfies the PDE*

$$\Delta\left\{\left[\det\left(\frac{\partial^2 f}{\partial x^j \partial x^i}\right)\right]^{-1/(n+2)}\right\} = 0,$$

where the Laplacian, in local coordinates, satisfies:

$$\Delta = \frac{1}{\sqrt{\det(G_{kl})}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left(G^{ij} \sqrt{\det(G_{kl})} \frac{\partial}{\partial x^j} \right).$$

EXAMPLE 1.2. Any parabolic affine hypersphere is an affine maximal hypersurface. In particular, the elliptic paraboloid

$$x^{n+1} = \frac{1}{2}[(x^1)^2 + \cdots + (x^n)^2], \quad (x^1, \ldots, x^n) \in A^n$$

is an affine-complete affine maximal hypersurface.

An affine analogue of the Weierstrass representation. Let V be the $\mathbb{R}$-vector space associated to A^{n+1}. We consider a Euclidean inner product $(\cdot, \cdot) \colon V \times V \to \mathbb{R}$ with associated normed determinant forms on V and its dual space V^*; we use the same notation Det. For simplicity, we study maximal surfaces in $\mathbb{R}^3$. Let $x : M \to \mathbb{R}^3$ be a locally strongly convex surface. Choose isothermal parameters u, v on M with respect to the Blaschke metric G, and let $e_1 = \partial_u x = x_u, e_2 = \partial_v x = x_v$, and denote $\partial U/\partial u =: U_u, \partial U/\partial v =: U_v$. Then $G_{11} = G_{22} =: F > 0, G_{12} = G_{21} = 0$. We have

$$(U, x_u) = 0, \quad (U_u, x_u) = -F, \quad (U_v, x_u) = 0,$$
$$(U, x_v) = 0, \quad (U_u, x_v) = 0, \quad (U_v, x_v) = -F,$$
$$(U, Y) = 1, \quad (U_u, Y) = 0, \quad (U_v, Y) = 0,$$

where Y denotes the affine normal.

We use the cross product construction; then (see [L-S-Z], p. 3)

$$x_u = \lambda[U, U_v], \quad x_v = \mu[U, U_u],$$
$$\mathrm{Det}(x_u, x_v, Y) \, \mathrm{Det}(U_u, U_v, U) = F^2,$$

where λ, μ are differentiable functions. Since

$$\mathrm{Det}(x_u, x_v, Y) = |\det(h_{ij})|^{1/4} = F,$$

we have

$$\mathrm{Det}(U_u, U_v, U) = F.$$

Then

$$-F = (U_u, x_u) = -\lambda \, \mathrm{Det}(U_u, U_v, U) = -\lambda F.$$

It follows that $\lambda = 1$. Similarly, we have $\mu = -1$.

Thus we obtain the following affine analogue of the well known Weierstrass representation for Euclidean minimal surfaces:

(1.1) $$x = \int [U, U_v] du - [U, U_u] dv.$$

If $x(M)$ is an affine maximal surface then

$$\Delta U = 0,$$

where, in the given coordinate system, the Laplacian reads:

$$\Delta = \frac{1}{\mathrm{Det}(U_u, U_v, U)} \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right).$$

It follows that the components $U^1(u, v), U^2(u, v)$ and $U^3(u, v)$ of U are harmonic functions.

Conversely, given a triple of functions

$$U = (U^1(u, v), U^2(u, v), U^3(u, v))$$

defined on $\Omega \subset \mathbb{R}^2$, satisfying the two conditions:

(i) U^1, U^2, U^3 are harmonic with respect to the canonical metric of $\mathbb{R}^2$;
(ii) $\mathrm{Det}(U_u, U_v, U) > 0$ in Ω, where Ω is a simply connected domain and $(u_0, v_0), (u, v) \in \Omega$.

Then we can construct an affine maximal surface $x : \Omega \to A^3$ as follows:

$$(1.2) \qquad x = \int_{(u_0, v_0)}^{(u, v)} [U, U_v] du - [U, U_u] dv.$$

The surface is well defined because the integrability conditions are satisfied:

$$[U, U_v]_v + [U, U_u]_u = [U, U_{uu} + U_{vv}] = 0.$$

From the point of view of local differential geometry the formula (1.2) admits a construction of any affine maximal surface.

In the following we give some further examples of affine maximal surfaces.

EXAMPLE 1.3. Consider $\Omega := \mathbb{R}^2$ and define $U : \mathbb{R}^2 \to \mathbb{R}^3$ by $U := (1, u, v)$; we get

$$x = \left(\frac{1}{2}(u^2 + v^2), -u, -v \right),$$

which is an elliptic paraboloid.

EXAMPLE 1.4. Consider $\Omega := \{(u, v) \in \mathbb{R}^2 | u > 0\}$ and define $U : \Omega \to \mathbb{R}^3$ by $U := (1, u^2 - v^2, v)$, then

$$\mathrm{Det}(U_u, U_v, U) = 2u.$$

The construction above gives

$$x = \left(\frac{1}{3}(u^3 + uv^2), -u, -2uv \right), \quad u > 0.$$

EXAMPLE 1.5. Let $\Omega := \{(u, v) \in \mathbb{R}^2 | u > 0, v < 0\}$ and define $U := (u, v, 2uv)$, then

$$\mathrm{Det}(U_u, U_v, U) = -2uv,$$

$$x = \left(-\frac{2}{3}v^3, -\frac{2}{3}u^3, \frac{1}{2}(u^2 + v^2) \right), \quad u > 0, \ v < 0.$$

Affine completeness and Euclidean completeness. In affine differential geometry there are two notions of completeness:

(1) *affine completeness*, that is, the completeness of the Blaschke metric G;
(2) *Euclidean completeness*, that is, the completeness of the Riemannian metric on M induced from a Euclidean metric on A^{n+1}.

Affine Bernstein problem. About complete affine maximal surfaces there are two conjectures, due to Chern and Calabi, both are called an "affine Bernstein problem" (see [CH], [CA]):

CHERN'S CONJECTURE. *Let $x_3 = f(x_1, x_2)$ be a strictly convex function defined for all $(x_1, x_2) \in A^2$. If $M = \left\{ (x_1, x_2, f(x_1, x_2)) | (x_1, x_2) \in A^2 \right\}$ is an affine maximal surface, then M must be an elliptic paraboloid.*

CALABI'S CONJECTURE. *A locally strongly convex affine complete surface $x : M \to A^3$ with affine mean curvature $L_1 \equiv 0$ is an elliptic paraboloid.*

These conjectures were generalized to higher dimensions (see [L-S-Z], [CA], [SO-2]). Recently Calabi's conjecture was solved, see [L-J-1], [T-W-2]; Chern's conjecture was solved by Trudinger and Wang X.-J. (see [T-W-1]). In [L-J-2] we proved the following result:

THEOREM 1.6. *Let M be a locally strongly convex, Euclidean complete surface in A^3 with constant affine mean curvature L_1.*

(a) *If $L_1 > 0$, then M is an ellipsoid.*
(b) *If $L_1 = 0$, then M is an elliptic paraboloid.*
(c) *If $L_1 < 0$, then M has "finite geometry".*

As a corollary of Theorem 1.6, we present a new proof of Chern's conjecture about affine maximal surfaces.

Note that the two versions of the affine Bernstein problem assume different completeness conditions. Calabi's conjecture assumes affine completeness; Chern's conjecture assumes Euclidean completeness. Generally, the affine completeness and the Euclidean completeness are not equivalent (see [SCH], [NO]).

2. Proof of Calabi's Conjecture

Preliminaries. Let M be a C^∞ manifold of dimension 2 and $x : M \to A^3$ a locally strongly convex surface. We choose a local unimodular affine frame field x, e_1, e_2, e_3 on M such that

$$e_1, e_2 \in T_x M, \quad \det(e_1, e_3, e_3) = 1, \quad e_3 = Y.$$

Denote by A_{ij}^k and B_{ij} the local components of the affine Fubini-Pick tensor and the affine Weingarten tensor with respect to the frame field x, e_1, e_2, e_3. We have the following local formulas (see [L-S-Z], pp. 52–60); the comma indicates covariant differentiation w.r.t. the Blaschke metric:

$$(2.1) \qquad x_{,ij} = \sum A_{ij}^k e_k + G_{ij} Y,$$

$$(2.2) \qquad U_{,ij} = -\sum A_{ij}^k U_{,k} - B_{ij} U,$$

$$(2.3) \qquad \Delta x = 2Y,$$

$$(2.4) \qquad \Delta U = -2L_1 U,$$

$$(2.5) \qquad \sum G^{ij} A_{ijk} = 0,$$

$$(2.6) \qquad R = 2(J + L_1),$$

where R, J denote the scalar curvature and the Pick invariant, resp. For an affine maximal surface we have $R \geq 0$.

Again we use isothermal parameters u,v. Suppose that M is an affine maximal surface. As above U is harmonic with respect to u,v. Let $z := u + iv$. Define α and β by

$$\alpha := \frac{1}{2}(A_{111} + iA_{222}), \quad \beta := \frac{1}{2}\left(\frac{B_{11} - B_{22}}{2} - iB_{12}\right).$$

Let $\| \, . \, \|_G$ denote the norm with respect to the Blaschke metric. We have (see [L-S-Z], p. 155)

$$(2.11) \qquad \|\alpha\|^2 := \left(\frac{F}{2}\right)^{-3} \cdot \alpha\bar{\alpha} = \frac{1}{2}\|A\|_G^2, \quad \|\beta\|^2 := \left(\frac{F}{2}\right)^{-2} \cdot \beta\bar{\beta} = \frac{1}{2}\|B\|_G^2,$$

$$(2.12) \qquad \alpha = i\,\text{Det}(U, U_z, U_{z^2}),$$

$$(2.13) \qquad \beta = iF^{-1}\,\text{Det}(U_z, U_{\bar{z}}, U_{z^2}),$$

$$(2.14) \qquad \frac{F}{2} = -i\,\text{Det}(U, U_z, U_{\bar{z}}).$$

Calabi calculated the Laplacian of $1/2(J + \|B\|^2) = 1/2J + \|\beta\|$, he got the following differential inequality (compare [L-S-Z], p. 231):

$$(2.15) \qquad \Delta\left(\frac{1}{2}J + \|\beta\|\right) \geq 3J^2.$$

Proof of the conjecture. Using (2.15) we can easily prove the following result of Martinez and Milan (see [M-M] or [L-S-Z], p. 231):

THEOREM 2.1. *Let* $x : M \to A^3$ *be a locally strongly convex affine maximal surface. If M is complete with respect to the Blaschke metric and if there is a constant $N > 0$ such that $\|B\|_G^2 \leq N$ everywhere, then M must be an elliptic paraboloid.*

Proof. From (2.15) and the assumptions we have

$$\Delta\left(\frac{1}{2}J + \|\beta\|\right) \geq 3J^2 \geq 6\left(\frac{1}{2}J + \|\beta\|\right)^2 - 12\|\beta\|^2 \geq 6\left(\frac{1}{2}J + \|\beta\|\right)^2 - 6N.$$

By a well-known result from geometric analysis it follows that $\frac{1}{2}J + \|\beta\|$ is bounded from above. Being a bounded subharmonic function on a complete 2-dim Riemannian manifold with $R \geq 0$, $\frac{1}{2}J + \|\beta\|$ must be a constant. It follows that $J = 0$ everywhere. Thus $x(M)$ is an elliptic paraboloid. ∎

From the assumption, $x : M \to A^3$ is a locally strongly convex affine maximal surface, which is complete with respect to the Blaschke metric. We want to show that there is a constant $N > 0$ such that $\|B\|_G^2 \leq N$ everywhere. We need the following lemma (see [H], p. 635, Lemma 26).

LEMMA 1 (Hofer). *Let (X, d) be a complete metric space with metric d, and $B_a(p) = \{x | d(p, x) \leq a\}$ be a ball with center p and radius a. Let Φ be a positive continuous function defined on $B_{2a}(p)$. Then there is a point $q \in B_a(p)$ and a positive number $\epsilon \leq a/2$ such that*

$$\Phi(x) \leq 2\Phi(q) \quad \text{for all } x \in B_\epsilon(q) \quad \text{and} \quad \epsilon\Phi(q) \geq \frac{a}{2}\Phi(p). \quad \blacksquare$$

Now we assume that $\|B\|_G^2$ is not bounded above. Then there is a sequence of points $p_\ell \in M$ such that $\|B\|_G^2(p_\ell) \to \infty$. We may assume that M is simply connected, otherwise

we consider its universal covering space. As M is noncompact, complete with $R \geq 0$, it is conformally equivalent to $\mathbb{C}$. Then we may choose global isothermal parameters u,v on M such that the Blaschke metric is given by $G = F(du^2 + dv^2)$. Let $B_1(p_\ell)$ be the geodesic ball with center p_ℓ and radius 1. Consider a family $\Phi(\ell) : B_2(p_\ell) \to \mathbb{R}$ of functions, $\ell \in \mathbb{N}$, defined by

$$\Phi(\ell) = \|\nabla \log F\|_G + \|A\|_G + \|B\|_G^{1/2}.$$

In terms of u, v we have

$$\|\nabla \log F\|_G^2 = \frac{1}{F}\left(\left(\frac{\partial \log F}{\partial u}\right)^2 + \left(\frac{\partial \log F}{\partial v}\right)^2\right),$$

$$\|A\|_G^2 = \frac{1}{F}\sum (A_{ij}^k)^2, \quad \|B\|_G^2 = \frac{1}{F^2}\sum (B_{ij})^2.$$

Using Hofer's Lemma we find a sequence of points q_ℓ and positive numbers ϵ_ℓ such that

$$(2.16) \qquad\qquad \Phi(x) \leq 2\Phi(q_\ell) \quad \forall \ x \in B_{\epsilon_\ell}(q_\ell)$$

$$(2.17) \qquad\qquad \epsilon_\ell \Phi(q_\ell) \geq \frac{1}{2}\Phi(p_\ell) \to \infty.$$

The restriction of the surface x to the balls $B_{\epsilon_\ell}(q_\ell)$ defines a family $M(\ell)$ of maximal surfaces. For every ℓ, we normalize $M(\ell)$ as follows:

STEP 1. Denote by $u(\ell)$, $v(\ell)$ the restriction of the isothermal parameters of M to $M(\ell)$. First we take a parameter transformation on $M(\ell)$:

$$(2.18) \qquad\qquad \hat{u}(\ell) = c(\ell)u(\ell), \quad \hat{v}(\ell) = c(\ell)v(\ell), \quad c(\ell) > 0,$$

where $c(\ell)$ is a constant. Choosing $c(\ell)$ appropriately and using an obvious notation $\hat{F}$, we may assume that, for every ℓ, we have $\hat{F}(q_\ell) = 1$. Note that, under the parameter transformation (2.18), Φ is invariant.

STEP 2. We use the Weierstrass representation for affine maximal surfaces (see Section 1) to define, for every ℓ, a new surface $\tilde{M}(\ell)$ from $M(\ell)$ via its conormal by

$$\tilde{U}(\ell) = \lambda(\ell)U(\ell), \quad \lambda(\ell) > 0;$$

we introduce new parameters $\tilde{u}(\ell)$, $\tilde{v}(\ell)$ by

$$\tilde{u}(\ell) = b(\ell)\hat{u}(\ell), \quad \tilde{v}(\ell) = b(\ell)\hat{v}(\ell), \quad b(\ell) > 0,$$

where $\lambda(\ell)$ and $b(\ell)$ are appropriate constants. From the foregoing conormal equation one easily verifies that each $\tilde{M}(\ell)$ again is a locally strongly convex maximal surface. We now choose $\lambda(\ell) = (b(\ell))^{2/3}$, $b(\ell) = \Phi(q_\ell)$. Using again an obvious notation $\tilde{F}$, $\tilde{\Phi}$, from (2.11)-(2.14) one can see that

$$\tilde{F} = \hat{F}, \qquad \tilde{\Phi}(\ell) = \frac{1}{b(\ell)}\Phi(\ell).$$

The first equation is trivial. We calculate the second one. We can easily get

$$\|\nabla \log \tilde{F}\|_{\tilde{G}} = \frac{1}{b}\|\nabla \log \hat{F}\|_G.$$

From (2.11)–(2.14) and our choice $\lambda^3 = b^2$ we have

$$\|\tilde{B}\|_{\tilde{G}}^2 = 2\|\tilde{\beta}\|^2 = 2\frac{1}{b^4}\|\beta\|^2 = \frac{1}{b^4}\|B\|_G^2,$$

$$\|\tilde{A}\|_{\tilde{G}}^2 = 2\|\tilde{\alpha}\|^2 = 2\frac{\lambda^6}{b^6}\|\alpha\|^2 = 2\frac{1}{b^2}\|\alpha\|^2 = \frac{1}{b^2}\|A\|_G^2.$$

Then the second equality follows.

We denote $\tilde{B}_a(q_\ell) = \{x \in \tilde{M}(\ell) \mid \tilde{d}(\ell)(x, q_\ell) \le a\}$, where $\tilde{d}(\ell)$ is the geodesic distance function with respect to the Blaschke metric on $\tilde{M}(\ell)$. Then $\tilde{\Phi}(\ell)$ is defined on the geodesic ball $\tilde{B}_{r(\ell)}(q_\ell)$ with $r(\ell) = \epsilon_\ell \Phi(q_\ell) \ge \frac{1}{2}\Phi(p_\ell) \to \infty$. From (2.16) we have

(2.19) $$\tilde{\Phi}(q_\ell) = 1,$$

(2.20) $$\tilde{\Phi}(x) \le 2, \quad \forall x \in \tilde{B}_{r(\ell)}(q_\ell).$$

STEP 3. For any ℓ we introduce new parameters $\xi_1(\ell), \xi_2(\ell)$ as follows:

$$\xi_1(\ell) = \tilde{u}(\ell) - \tilde{u}(\ell)(q_\ell), \quad \xi_2(\ell) = \tilde{v}(\ell) - \tilde{v}(\ell)(q_\ell).$$

Then at q_ℓ, $(\xi_1, \xi_2) = (0, 0)$ for any ℓ, and we can identify the parametrization (ξ_1, ξ_2) for any index ℓ. Let $\tilde{x}(\ell)$ denote the position vector of $\tilde{M}(\ell)$. An appropriate unimodular affine transformation gives

(2.21) $$\tilde{x}(\ell)(o) = (0, 0, 0),$$

(2.22) $$\tilde{x}_{\xi_1}(\ell)(o) = e_1 = (1, 0, 0),$$

(2.23) $$\tilde{x}_{\xi_2}(\ell)(o) = e_2 = (0, 1, 0),$$

(2.24) $$\tilde{Y}(\ell)(o) = (0, 0, 1).$$

Consider the open geodesic balls

$$\Omega(\ell) := \{(\xi_1, \xi_2) \in \mathbb{R}^2 \mid \tilde{d}(\ell)(o, \xi) < r(\ell)\}$$

and the sequence $\tilde{M}(\ell)$ of maximal surfaces $\tilde{x}(\ell) : \Omega(\ell) \to A^3$. They satisfy (2.21)-(2.24) and the conditions

(2.25) $$\tilde{F}(\ell)(o) = 1,$$

(2.26) $$\tilde{\Phi}(\ell)(o) = 1, \quad \tilde{\Phi}(\ell)(\xi) \le 2 \quad \forall \xi \in \Omega(\ell),$$

(2.27) $$r(\ell) \to \infty.$$

It follows from (2.7)-(2.10) and (2.22)-(2.25) that, for any ℓ, $(\tilde{U}_{\xi_1}, \tilde{U}_{\xi_2}, \tilde{U})(0) = I$, where I is the unit matrix. We need the following lemma

LEMMA 2. *Let M be an affine maximal surface defined in a neighborhood of $0 \in \mathbb{R}^2$. Suppose that, with the notations from above,*

(i) $$F(o) = 1, \quad (U_{\xi_1}, U_{\xi_2}, U)(o) = I,$$

(ii) $$\left(\frac{1}{F}\sum\left(\frac{\partial(\log F)}{\partial \xi_i}\right)^2\right)^{1/2} + \left(\frac{1}{F}\sum(A_{ij}^k)^2\right)^{1/2} + \left(\frac{1}{F}\left(\sum(B_{ij})^2\right)^{1/2}\right)^{1/2} \le 2.$$

Denote $D := \{(\xi_1, \xi_2) \mid \xi_1^2 + \xi_2^2 \le \frac{1}{2}\}$. Then there is a constant $C_1 > 0$ such that, for $(\xi_1, \xi_2) \in D$, the following estimates hold:

(1)

$$\frac{4}{9} \le F \le 4;$$

(2)
$$|U| + |U_{\xi_1}| + |U_{\xi_2}| \leq C_1$$

where $|\cdot|$ denotes the canonical norm in $\mathbb{R}^3$;

(3) denote $r_o = \frac{1}{3}$; then $\overline{\Omega}_{r_o} \subset \{\xi_1^2 + \xi_2^2 < \frac{1}{4}\} \subset D$, where Ω_{r_o} is the geodesic ball with center o and radius r_o with respect to the Blaschke metric.

Proof of Lemma 2. **(1)** Consider an arbitrary curve $\Gamma = \{\xi_1 = a_1 s, \ \xi_2 = a_2 s, \ a_1^2 + a_2^2 = 1, \ s \geq 0\}$. By assumption we have

$$\frac{1}{F}\left(\frac{\partial \log F}{\partial s}\right)^2 \leq 2, \qquad F(0) = 1.$$

Solving this differential inequality with $F(0) = 1$, we get

$$\left(\frac{1}{1 + \frac{\sqrt{2}}{2}s}\right)^2 \leq F(s) \leq \left(\frac{1}{1 - \frac{\sqrt{2}}{2}s}\right)^2.$$

From the assumption we have $s \leq \frac{1}{\sqrt{2}}$, then **(1)** follows.

(2) Note that the Christoffel symbols are given by $\partial \log F/\partial \xi_i$. Along the curve Γ the structure equation $U_{,ij} = -\sum A_{ij}^k U_{,k} - B_{ij} U$ gives an ODE which can be written in matrix form:

$$(2.28) \qquad \frac{dX}{ds} = XH,$$

where $X = (U_{\xi_1}, U_{\xi_2}, U)$, and H is a matrix, whose elements depend on B_{ij}, A_{ij}^k and $\partial \log F/\partial \xi_i$. It follows from (2.27) that

$$(2.29) \qquad \frac{dX^t}{ds} = H^t X^t,$$

where we use an obvious notation for the transpose of a matrix. Then

$$(2.30) \qquad \frac{d(X^t X)}{ds} = H^t X^t X + X^t X H.$$

Denote $f = Tr(X^t X)$. Taking the trace of (2.30) we get

$$(2.31) \qquad \frac{df}{ds} = Tr(H^t X^t X) + Tr(X^t X H) \leq Cf,$$

where C is a constant. Deriving the last inequality we used **(1)** and the condition (ii). Solving (2.30) with the condition (i) we get **(2)**. From **(1)** we immediately get **(3)**. ∎

We continue with the proof of the conjecture. Since $r(\ell) \to \infty$, we have $D \subset \Omega(\ell)$ for ℓ big enough. In fact, by **(1)**, the geodesic distance from 0 to the boundary of D with respect to the Blaschke metric on $\tilde{M}(\ell)$ is less than $\sqrt{2}$. Using **(2)** and a standard elliptic estimate we get a C^k-estimate, independent of ℓ, for any k. It follows that there is a ball $\{\xi_1^2 + \xi_2^2 \leq C_2\}$ and a subsequence (still indexed by ℓ) such that $\tilde{U}(\ell)$ converges to $\tilde{U}$ on the ball, and correspondingly all derivatives, where $C_2 < \frac{1}{2}$ is very close to $\frac{1}{2}$. Thus, as limit, we get a maximal surface $\tilde{M}$, defined on the ball, which contains a geodesic ball $\overline{\Omega}_{r_o}$. We now extend the surface $\tilde{M}$ as follows: For every boundary point $p = (\xi_{1o}, \xi_{2o})$ of the geodesic ball $\overline{\Omega}_{r_o}$ we first make the parameter transformation: $\tilde{\xi}_i = b(\xi_i - \xi_{io})$ such

that at p, $(\tilde{\xi}_1, \tilde{\xi}_2) = (0,0)$, and for the limit surface $\tilde{M}$ we have $\tilde{F}(p) = 1$. We choose frame e_1, e_2, e_3 at p such that $e_1 = \tilde{x}_{\tilde{\xi}_1}$, $e_2 = \tilde{x}_{\tilde{\xi}_2}$, $e_3 = \tilde{Y}$. We have

(i′) $\qquad \tilde{F}(\ell)(p) \to \tilde{F}(p) = 1, \quad (\tilde{U}_{\tilde{\xi}_1}(\ell), \tilde{U}_{\tilde{\xi}_2}(\ell), \tilde{U}(\ell))(p) \to I \quad \text{as} \ \ell \to \infty.$

It is easy to see that under the conditions (i′) and (ii) in Lemma 2, the estimates **(1)**, **(2)** and **(3)** in Lemma 2 remain true. By the same argument as above we conclude that there is a ball around p and a subsequence ℓ_k, such that $\tilde{U}(\ell_k)$ converges to $\tilde{U}'$ on the ball, and correspondingly all derivatives. As limit, we get a maximal surface $\tilde{M}'$, which contains a geodesic ball of radius r_o around p. Then we return to the original parameters ξ_1, ξ_2 and the original frame e_1, e_2, e_3 at 0. Note that the geodesic distance is independent of the choice of the parameters and the frames. It is obvious that $\tilde{M}$ and $\tilde{M}'$ agree on the common part. We repeat this procedure to extend $\tilde{M}$ to be defined on $\overline{\Omega}_{2r_o}$, etc. In this way we may extend $\tilde{M}$ to be an affine complete maximal surface defined in a domain $\Omega \subset \mathbb{R}^2$; using (24) and (25) we get

$$\|\tilde{B}\|_{\tilde{G}} \le 2, \quad \tilde{\Phi}(0) = 1.$$

By Theorem 1, $\tilde{M}$ must be an elliptic paraboloid, given by

$$x_3 = \frac{1}{2}(x_1^2 + x_2^2),$$

where x_1, x_2, x_3 are the coordinates in A^3 with respect to the frame e_1, e_2, e_3. For a paraboloid we have $\|\tilde{A}\|_{\tilde{G}} = 0$, $\|\tilde{B}\|_{\tilde{G}}^2 = 0$, $\tilde{R} = 0$ identically, and $\tilde{G} = dx_1^2 + dx_2^2$. Thus

(2.32) $\qquad\qquad\qquad\qquad \|\nabla \log \tilde{F}\|_{\tilde{G}}(0) = 1.$

We consider $\log \tilde{F}$ as a function of x_1, x_2. Since the scalar curvature vanishes identically, $\tilde{R} = 0$, from the formula

$$\Delta \log \tilde{F} = -\tilde{R}$$

we conclude that $\log \tilde{F}$ is a harmonic function. As $\|\nabla \log \tilde{F}\|_{\tilde{G}} \le 2$, $\log \tilde{F}$ must be a linear function. In view of (2.32), without loss of generality, we may assume that $\log \tilde{F} = x_1$. We introduce complex coordinates and write $w = \xi_1 + i\xi_2$, $z = x_1 + ix_2$, then $w(z)$ is a holomorphic or anti-holomorphic function. We consider the case that w is holomorphic. For the case that w is anti-holomorphic, the discussion is similar. Since $\tilde{G} = |dz|^2 = \tilde{F}|dw|^2$ we have $|w'|^2 = \tilde{F}^{-1} = e^{-x_1}$. Let $Q = e^{\frac{z}{2}}$. Then $|w'Q| = 1$. From the maximum principle we get $w'Q = C$ where C is a constant with $|C| = 1$. So $w' = Ce^{-\frac{z}{2}}$. It follows that $w = -2Ce^{-\frac{z}{2}} + E$, where E is a constant. Since $e^{-\frac{z}{2}}$ has period 2π for x_2, we have a covering map $A^2 \to \Omega$; this is impossible. We get a contradiction. So $\|B\|_G$ must be bounded above on M. By Theorem 1 M is an elliptic paraboloid. We have proved the following

THEOREM 2.2. *Let $x : M \to A^3$ be a locally strongly convex affine maximal surface. If M is complete with respect to the Blaschke metric, then M must be an elliptic paraboloid.*

3. Euclidean complete affine surfaces with constant affine mean curvature

Introduction. The classification of locally strongly convex, affine-complete affine hyperspheres had attracted many geometers during the last decades. For the history of this problem and the contributions of different authors we refer to the monograph [L-S-Z],

pp. 84-85. Obviously, every affine hypersphere has constant affine mean curvature L_1. Thus an interesting and important problem is the classification of locally strongly convex, complete affine hypersurfaces with constant affine mean curvature L_1. We shall show that the study of locally strongly convex, Euclidean complete hypersurfaces with constant affine mean curvature $L_1 = L$ is equivalent to the study of the convex solutions of the fourth order P.D.E.

$$(*) \qquad \Delta\left[\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)\right]^{-1/(n+2)} = -nL\left[\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)\right]^{-1/(n+2)}.$$

In this paper, we consider the convex solutions of the equation $(*)$ for $n = 2$. We shall prove that a) if $L > 0$, then there is no convex solution of $(*)$ which is defined for all $(x_1, x_2) \in \mathbb{R}^2$; b) if $L = 0$ and $f(x_1, x_2)$ is a convex solution of $(*)$, which is defined for all $(x_1, x_2) \in \mathbb{R}^2$, then $f(x_1, x_2)$ must be a quadratic polynomial. In the language of affine differential geometry, our main theorems can be stated as follows:

THEOREM 3.1. *Every locally strongly convex, Euclidean complete surface with constant affine mean curvature is affine complete.*

To state our Theorem 3.2 we introduce a terminology. A locally strongly convex hypersurface is said to have "*finite geometry*" if $\|B\|_k$ and $\|A\|_k$ are bounded, where

$$\|B\|_k := \|B\| + \|\nabla B\| + \|\nabla^2 B\| + \cdots + \|\nabla^k B\|,$$
$$\|A\|_k := \|A\| + \|\nabla A\| + \|\nabla^2 A\| + \cdots + \|\nabla^k A\|.$$

Using Theorem 3.1 we immediately get the following result:

THEOREM 3.2. *Let M be a locally strongly convex, Euclidean complete surface in A^3 with constant affine mean curvature L_1.*

(a) *If $L_1 > 0$, then M is an ellipsoid.*
(b) *If $L_1 = 0$, then M is an elliptic paraboloid.*
(c) *If $L_1 < 0$, then M has "finite geometry".*

As a corollary of Theorem 3.2, we present a new proof of Chern's conjecture about affine maximal surfaces.

Fundamental formulas. Consider a locally strongly convex hypersurface $x \colon M \to A^{n+1}$, which is given as a graph by a strictly convex function

$$x_{n+1} = f(x_1, \ldots, x_n).$$

Then, the Blaschke metric G is given as in Section 1 and the affine conormal vector field U can be identified with

$$U = \left[\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)\right]^{-1/(n+2)}\left(-\frac{\partial f}{\partial x_1}, \ldots, -\frac{\partial f}{\partial x_n}, 1\right).$$

The formula $\Delta U = -nL_1 U$ implies that $x(M)$ is a locally strongly convex hypersurface with constant affine mean curvature $L_1 \equiv L$ if and only if f satisfies the following P.D.E.:

$$(3.1) \qquad \Delta\left[\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)\right]^{-1/(n+2)} = -nL\left[\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)\right]^{-1/(n+2)}.$$

Denote

$$\rho := \left[\det \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right) \right]^{-1/(n+2)}.$$

Then (3.1) gives

(3.2) $$\Delta \rho = -n L_1 \rho.$$

Note that in terms of $x_1, \ldots, x_n$ we have $(\det(G_{kl}))^{1/2} = \frac{1}{\rho}$. By a direct calculation we get

(3.3) $$\Delta = \frac{1}{\rho} \sum f^{ij} \frac{\partial^2}{\partial x_i \partial x_j} - \frac{2}{\rho^2} \sum f^{ij} \frac{\partial \rho}{\partial x_j} \frac{\partial}{\partial x_i} + \frac{1}{\rho} \sum \frac{\partial f^{ij}}{\partial x_i} \frac{\partial}{\partial x_j}$$

where (f^{ij}) denotes the inverse matrix of (f_{ij}) and $f_{ij} = \partial^2 f / \partial x_i \partial x_j$. Differentiate the equality $\sum f^{ik} f_{kj} = \delta^i_j$; one finds

$$\sum \frac{\partial f^{ik}}{\partial x_i} f_{kj} = -\sum f^{ik} \frac{\partial f_{kj}}{\partial x_i} = \frac{(n+2)}{\rho} \frac{\partial \rho}{\partial x_j}.$$

It follows that

(3.4) $$\sum \frac{\partial f^{ik}}{\partial x_i} = \frac{(n+2)}{\rho} \sum f^{jk} \frac{\partial \rho}{\partial x_j}.$$

Inserting (3.4) into (3.3) we obtain

(3.5) $$\Delta = \frac{1}{\rho} \sum f^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \frac{n}{\rho^2} \sum f^{ij} \frac{\partial \rho}{\partial x_j} \frac{\partial}{\partial x_i}.$$

To find the affine normal Y and calculate the affine Weingarten tensor B_{ij}, we let (see [CH])

$$e_i^* = e_i, \quad 1 \le i \le n, \quad e_{n+1}^* = e_{n+1} + \sum a_{n+1}^i e_i,$$

where e_{n+1}^* is in the affine normal direction. Then the coefficients a_{n+1}^i are determined by

$$\sum a_{n+1}^j f_{ji} = \frac{\partial}{\partial x_i} \log \rho.$$

It follows that

$$a_{n+1}^i = \sum f^{ji} \frac{\partial}{\partial x_j} \log \rho,$$

and hence

$$e_{n+1}^* = e_{n+1} + \sum f^{ji} \frac{\partial}{\partial x_j} \log \rho \cdot e_i.$$

Therefore

$$Y = H^{1/(n+2)} e_{n+1}^* = H^{1/(n+2)} \sum f^{ji} \frac{\partial}{\partial x_j} \log \rho \cdot e_i + H^{1/(n+2)} e_{n+1},$$

where $H = \det(\partial^2 f / \partial x_i \partial x_j)$.

Let x be the position vector of the hypersurface M. We can write

$$dx = \sum w^\alpha e_\alpha^*, \quad de_\alpha^* = \sum w_\alpha^\beta e_\beta^*.$$

w^α, w_α^β denote the Maurer-Cartan forms of the unimodular affine group. We compute

$$w_{n+1}^i = da_{n+1}^i - a_{n+1}^i d\log \rho = \sum \left(\frac{\partial}{\partial x_j} \left(f^{ki} \frac{\partial}{\partial x_k} \log \rho \right) - f^{ki} \frac{\partial}{\partial x_k} \log \rho \cdot \frac{\partial}{\partial x_j} \log \rho \right) w^j.$$

Therefore the affine Weingarten tensor is

$$(3.6) \quad B_{ij} = \sum \left(-\frac{\partial}{\partial x_i} \left(f^{lk} \frac{\partial}{\partial x_l} \log \rho \right) + f^{lk} \frac{\partial}{\partial x_l} \log \rho \frac{\partial}{\partial x_i} \log \rho \right) f_{kj}$$

$$= -\frac{1}{\rho} \frac{\partial^2 \rho}{\partial x_i \partial x_j} + \frac{2}{\rho^2} \frac{\partial \rho}{\partial x_i} \frac{\partial \rho}{\partial x_j} + \sum \frac{f^{kl}}{\rho} \frac{\partial \rho}{\partial x_l} \frac{\partial f_{ij}}{\partial x_k}.$$

Consider the Legendre transformation relative to f:

$$\xi_i = \frac{\partial f}{\partial x_i}(x_1, \ldots, x_n),$$

$$u(\xi_1, \ldots, \xi_n) = \sum x_i \frac{\partial f}{\partial x_i}(x_1, \ldots, x_n) - f(x_1, \ldots, x_n),$$

and denote by Ω^* the Legendre transformation domain of f, i.e. $u : \Omega^* \to \mathbb{R}$ and

$$\Omega^* = \{(\xi_1(x), \ldots, \xi_n(x)) | x \in \Omega\}.$$

In terms of the coordinates $(\xi_1, \ldots, \xi_n)$ the Blaschke metric is given by

$$G_{ij} = \rho \frac{\partial^2 u}{\partial \xi_i \partial \xi_j},$$

and $(\partial^2 u / \partial \xi_i \partial \xi_j)$ is the inverse matrix of $(\partial^2 f / \partial x_i \partial x_j)$. We have

$$(3.7) \quad \rho = \left[\det \left(\frac{\partial^2 u}{\partial \xi_i \partial \xi_j} \right) \right]^{1/(n+2)},$$

$$\sqrt{\det(G_{kl})} = \rho^{n+1},$$

$$\Delta = \frac{1}{\sqrt{\det(G_{kl})}} \sum \frac{\partial}{\partial \xi_i} \left(G^{ij} \sqrt{\det(G_{kl})} \frac{\partial}{\partial \xi_j} \right)$$

(see [L-S-Z], p. 91). By a similar calculation as above we get

$$(3.8) \quad \Delta = \frac{1}{\rho} \sum u^{ij} \frac{\partial^2}{\partial \xi_i \partial \xi_j} - \frac{2}{\rho^2} \sum u^{ij} \frac{\partial \rho}{\partial \xi_j} \frac{\partial}{\partial \xi_i}.$$

Estimates for the function ρ. In this section we give some estimates which play an important role in our proof. Suppose that $f(x_1, x_2)$ is a strictly convex function defined in a bounded convex domain Ω such that the graph

$$M = \{(x_1, x_2, f(x_1, x_2)) | (x_1, x_2) \in \Omega\}$$

is a locally strongly convex surface with constant affine mean curvature L_1. As above consider the Legendre transformation relative to f with $u : \Omega^* \to \mathbb{R}$. Then u is a convex function defined in Ω^* and M can be represented in terms of ξ_1, ξ_2 as follows:

$$x = (x_1, x_2, f(x_1, x_2)) = \left(\frac{\partial u}{\partial \xi_1}, \frac{\partial u}{\partial \xi_2}, -u + \sum \xi_i \frac{\partial u}{\partial \xi_i} \right).$$

First we estimate $\rho = [\det(\partial^2 u / \partial \xi_i \partial \xi_j)]^{1/4}$ from above. For any $\xi_0 = (\xi_1^0, \xi_2^0) \in \Omega^*$, we choose $r > 0$ such that $0 < r < \text{dist}(\xi_0, \partial \Omega^*)$. We introduce the notations

$$\overline{B_r(\xi_0)} = \left\{ (\xi_1, \xi_2) | \sum (\xi_i - \xi_i^\circ)^2 \le r^2 \right\}, \quad h = \sum (\xi_i - \xi_i^\circ)^2,$$

$$g = \sum \left(\frac{\partial u}{\partial \xi_i} \right)^2, \quad A = \max_{B_r(\xi_0)} g.$$

Consider the function

$$(3.9) \qquad F = \exp\left\{-\frac{m}{r^2 - h} + \frac{1}{\alpha A}g\right\}\rho$$

defined on $\overline{B_r(\xi_0)}$, where m and α are appropriate positive constants to be determined later. Clearly, F attains its supremum at some interior point ξ^* of $\overline{B_r(\xi_0)}$. We choose a local orthonormal frame field e_1, e_2 of the Blaschke metric on M near

$$p^* = \left(\frac{\partial u}{\partial \xi_1}(\xi^*), \frac{\partial u}{\partial \xi_2}(\xi^*), -u(\xi^*) + \sum \xi_i^* \frac{\partial u}{\partial \xi_i}(\xi^*)\right).$$

Then, at p^*,

$$(3.10) \qquad F_{,i} = 0,$$

$$(3.11) \qquad \sum F_{,ii} \leq 0.$$

We calculate both expressions explicitly

$$(3.12) \qquad -\frac{m}{(r^2 - h)^2}h_{,i} + \frac{1}{\alpha A}g_{,i} + \frac{\rho_{,i}}{\rho} = 0,$$

$$(3.13) \qquad -\frac{2m}{(r^2 - h)^3}\sum h_{,i}^2 - \frac{m}{(r^2 - h)^2}\Delta h + \frac{1}{\alpha A}\Delta g - \frac{\sum \rho_{,i}^2}{\rho^2} + \frac{\Delta \rho}{\rho} \leq 0.$$

Using the formulas (3.5) (3.8) and (3.2), we get

$$(3.14) \qquad -\frac{2m}{(r^2 - h)^3}\sum h_{,i}^2 - \frac{2m}{(r^2 - h)^2}\frac{1}{\rho}\sum u^{ii} + \frac{2m}{(r^2 - h)^2}\sum \frac{\rho_{,i}}{\rho}h_{,i}$$
$$+\frac{2}{\alpha A}\frac{1}{\rho}\sum f^{ii} + \frac{2}{\alpha A}\sum \frac{\rho_{,i}}{\rho}g_{,i} - \sum \frac{\rho_{,i}^2}{\rho^2} - 2|L_1| \leq 0.$$

Inserting (3.12) into (3.14) and applying the Cauchy-Schwarz inequality, we obtain

$$(3.15) \qquad -\frac{2m}{(r^2 - h)^3}\sum h_{,i}^2 - \frac{2m}{(r^2 - h)^2}\frac{1}{\rho}\sum u^{ii} + \frac{1}{2}\frac{m^2}{(r^2 - h)^4}\sum h_{,i}^2$$
$$+\frac{2}{\alpha A}\frac{1}{\rho}\sum f^{ii} - \frac{5}{(\alpha A)^2}\sum g_{,i}^2 - 2|L_1| \leq 0.$$

Note that

$$(3.16) \qquad \sum g_{,i}^2 = \sum \frac{u^{ij}}{\rho}\frac{\partial g}{\partial \xi_i}\frac{\partial g}{\partial \xi_j} = \frac{4}{\rho}\sum u^{ij}u_k u_{ki}u_l u_{lj}$$
$$= \frac{4}{\rho}\sum u_{kl}u_k u_l \leq \frac{4}{\rho}\sum u_{ii} \cdot \sum u_i^2.$$

Inserting (3.16) into (3.15), we get

$$(3.17) \qquad -\frac{2m}{(r^2 - h)^3}\sum h_{,i}^2 - \frac{2m}{(r^2 - h)^2}\frac{1}{\rho}\sum u^{ii} + \frac{1}{2}\frac{m^2}{(r^2 - h)^4}\sum h_{,i}^2$$
$$+\frac{2}{\alpha A}\frac{1}{\rho}\sum f^{ii} - \frac{20}{\alpha^2 A}\frac{1}{\rho}\sum u_{ii} - 2|L_1| \leq 0.$$

We choose $m = 4r^2$ and $\alpha = 20$. Then

$$(3.18) \qquad \frac{1}{20A}\sum u_{ii} \leq \frac{2m}{(r^2 - h)^2}\sum u^{ii} + 2|L_1|\rho.$$

Denote by λ_1, λ_2 the eigenvalues of $(\partial^2 u/\partial\xi_i\partial\xi_j) = (u_{ij})$. Recall that $\rho = [\det(\partial^2 u/\partial\xi_i\partial\xi_j)]^{\frac{1}{4}} = (\lambda_1\lambda_2)^{\frac{1}{4}}$. It follows from (3.18) that

$$(3.19) \qquad \frac{1}{20A}(\lambda_1 + \lambda_2) \leq \frac{2m}{(r^2-h)^2}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) + 2|L_1|(\lambda_1\lambda_2)^{\frac{1}{4}}$$

$$= \frac{2m}{(r^2-h)^2}\frac{\lambda_1+\lambda_2}{\lambda_1\lambda_2} + 2|L_1|(\lambda_1\lambda_2)^{\frac{1}{4}},$$

and hence

$$\frac{1}{20A}\lambda_1\lambda_2 \leq \frac{2m}{(r^2-h)^2} + 2|L_1|\frac{(\lambda_1\lambda_2)^{\frac{1}{2}}(\lambda_1\lambda_2)^{\frac{3}{4}}}{\lambda_1+\lambda_2} \leq \frac{2m}{(r^2-h)^2} + |L_1|(\lambda_1\lambda_2)^{\frac{3}{4}},$$

i.e.

$$\rho^4 - 20A|L_1|\rho^3 - 40A\frac{m}{(r^2-h)^2} \leq 0.$$

Consequently, from the Cauchy-Schwarz inequality, we have

$$\rho^4 - 400A^2|L_1|^2\rho^2 - 80A\frac{m}{(r^2-h)^2} \leq 0.$$

The left hand side is a quadratic expression in ρ^2. If one consider its zeroes, it follows that

$$\rho^2 \leq 400A^2|L_1|^2 + \left(80A\frac{m}{(r^2-h)^2}\right)^{\frac{1}{2}}.$$

Therefore

$$\rho \leq (800)^{1/2}A|L_1| + (320)^{\frac{1}{4}}A^{1/4}\frac{m^{1/4}}{(r^2-h)^{1/2}}.$$

With our special choice of m and α, from (3.9) we thus get

$$F \leq \exp\left\{\frac{1}{20}\right\}\left[(800)^{1/2}A|L_1| + (20)^{1/4}\exp\left\{-1/2\right\}\frac{A^{1/4}}{r^{1/2}}\right],$$

which holds at ξ^*, where F attains its supremum. Hence, at any point of $B_{\frac{1}{2}r}(\xi_0)$, we have

$$(3.20) \qquad \rho \leq \exp\left\{\frac{161}{20}\right\}\left[(800)^{1/2}A|L_1| + (20)^{1/4}\exp\left\{-1/2\right\}\frac{A^{1/4}}{r^{1/2}}\right].$$

Using (3.20), we obtain the following lemma:

LEMMA 3.3. *Let Ω be a bounded convex domain with $(0,0) \in \Omega$ and Ω'^* be an arbitrary subdomain of Ω^* with $\mathrm{dist}(\Omega'^*, \partial\Omega^*) > 0$. Then the following estimate holds:*

$$\rho \leq C_1 \quad \text{for } \xi \in \Omega'^*,$$

where C_1 is a constant depending only on $\mathrm{dist}(\Omega'^*, \partial\Omega^*)$, $\mathrm{diam}(\Omega)$ and $|L_1|$.

Next we estimate $1/\rho = [\det(\partial^2 f/\partial x_i\partial x_j)]^{1/4}$ from above. Let Ω' be an arbitrary subdomain of Ω with $\mathrm{dist}(\Omega', \partial\Omega) = 2r > 0$. It is easy to see that

$$(3.21) \qquad \left(\sum\left(\frac{\partial f}{\partial x_i}\right)^2(x)\right)^{1/2} \leq \frac{\sup_{y,z\in\Omega}|f(y)-f(z)|}{\mathrm{dist}(x,\partial\Omega)} \quad \text{for } x \in \Omega.$$

On the other hand, by a similar calculation as in the proof of Lemma 3.3, one can obtain
(3.22)
$$\frac{1}{\rho} \leq \exp\left\{\frac{33}{8}\right\}\left[2^{1/4}A^{1/4}(2^{1/4}A^{1/4}|L_1| + 2^{3/2}) + 2\exp\{-1/2\}\frac{A^{1/4}}{r^{1/2}}\right] \quad \text{for } x \in B_{\frac{1}{2}r}(x_0),$$

where $\overline{B}_r(x_0) = \left\{(x_1, x_2)| \sum(x_i - x_i^{\circ})^2 \leq r^2\right\} \subset \Omega$ and $A = \max_{\overline{B}_r(x_0)} \sum(\partial f/\partial x_i)^2$.
Combination of (3.21) and (3.22) gives us

LEMMA 3.4. *Let Ω be a bounded domain and Ω' be an arbitrary subdomain of Ω with* $\text{dist}(\Omega', \partial\Omega) > 0$. *Then the following estimate holds:*

$$\frac{1}{\rho} \leq C_2 \quad \text{for } x \in \Omega',$$

where C_2 is a constant depending only on $\text{dist}(\Omega', \partial\Omega), \sup_{y,z \in \Omega}|f(y) - f(z)|,$ *and* $|L_1|$.

Proof of Theorem 3.1. Suppose that M is a locally strongly convex, Euclidean complete affine surface. Obviously, M is affine complete if M is compact. Therefore, it is enough to consider the case when M is a non-compact, Euclidean complete, locally strongly convex surface with constant affine mean curvature L_1. From Hadamard's Theorem(see [WU]) M is the graph of a strictly convex function $x_3 = f(x_1, x_2)$ defined in a convex domain $V \subset A^2$. To prove Theorem 3.1, we need the following result (see [L-S-Z], p. 117):

THEOREM 3.5. *Let M be a locally strongly convex, Euclidean complete hypersurface in* A^{n+1}. *If there is a constant $N > 0$ such that*

$$\|B\|_G^2 \leq N,$$

then M is also affine complete.

We now use blow-up analysis to show that there is a constant $N > 0$ such that $\|B\|_G^2 \leq N$ everywhere. Then, by Theorem 3.5, Theorem 3.1 follows. To this end, suppose that this is false. Then there would exist a sequence of points $\{p_k\} \subset M$ such that

(3.23)
$$\|B\|_G^2(p_k) \to \infty,$$

as $k \to \infty$. For each $p_k \in M$ we may assume that the plane $x_3 = 0$ is parallel to the tangent plane of M at p_k and p_k has the coordinates $(0, 0)$. With respect to this coordinate system we have $f \geq 0$, and for any number $C > 0$ the set

$$M_C = \{(x_1, x_2) \in V | x_3 = f(x_1, x_2) < C\}$$

is a bounded convex domain in A^2. It is well-known that there exists a unique ellipsoid E, which attains the minimum volume among all the ellipsoids that contain M_C and that are centered at the center of mass of M_C, and a positive constant α such that

(3.24)
$$\alpha E \subset M_C \subset E,$$

where αE means the α-dilation of E with respect to its center. By an orthogonal linear transformation, we may suppose that the equation of the minimum ellipsoid E is

$$\frac{(x_1 - x_1^{\circ})^2}{a_1^2} + \frac{(x_2 - x_2^{\circ})^2}{a_2^2} = 1.$$

By the unimodular affine transformation

$$(3.25) \qquad \widehat{x}_1 = \sqrt{C\frac{a_2}{a_1}}x_1, \quad \widehat{x}_2 = \sqrt{C\frac{a_1}{a_2}}x_2, \quad \widehat{x}_3 = \frac{1}{C}x_3,$$

M is given as a graph of a strictly convex function $\widehat{f}(\widehat{x}_1, \widehat{x}_2)$ defined in a convex domain $\widehat{\Omega} \subset A^2$. Denote by L_C the linear transformation

$$\widehat{x}_1 = \sqrt{C\frac{a_2}{a_1}} \cdot x_1, \quad \widehat{x}_2 = \sqrt{C\frac{a_1}{a_2}} \cdot x_2.$$

Then $L_C(E)$ is the ball B_r with center $(\sqrt{C\frac{a_2}{a_1}}x_1^\circ, \sqrt{C\frac{a_1}{a_2}}x_2^\circ)$ and radius $r = \sqrt{Ca_1a_2}$. Setting

$$\Omega_C = L_C(M_C),$$

(3.23) becomes

$$(3.26) \qquad\qquad\qquad \alpha B_r \subset \Omega_C \subset B_r.$$

Obviously, we have

$$\Omega_C = \{(\widehat{x}_1, \widehat{x}_2) \in \widehat{\Omega} | \widehat{f}(\widehat{x}_1, \widehat{x}_2) < 1\}.$$

It is easy to see that the function

$$H : (0, \infty) \to \mathbb{R}, H(C) = \pi \cdot C \cdot a_1(C) a_2(C)$$

is continuous. Note that $H((0, \infty)) = (0, \infty)$. It follows that there exists a number $C^{(k)} > 0$ such that $C^{(k)} a_1(C^{(k)}) a_2(C^{(k)}) = 1$. This implies that, by a unimodular affine transformation (3.25) with $C = C^{(k)}$, M is given by a strictly convex function $f^{(k)}$ defined in a convex domain in A^2 such that

$$B(x^{(k)}, \alpha) \subset \Omega_k \subset B(x^{(k)}, 1), \quad \text{where} \quad \Omega_k = \{(x_1, x_2) | f^{(k)} < 1\}.$$

Thus, we would obtain a sequence of convex functions $\{f^{(k)}\}$ and a sequence of points $\{x^{(k)}\}$ such that $f^{(k)} \geq 0$, and such that $B(x^{(k)}, \alpha) \subset \Omega_k \subset B(x^{(k)}, 1)$. Therefore, we may suppose, by taking subsequences, that $\{\Omega_k\}$ converges to a convex domain Ω and $\{f^{(k)}\}$ converges to a convex function f_∞, locally uniformly in Ω. For $x \in \partial\Omega$, we define $f_\infty(x) = \underline{\lim}_{y \to x, y \in \Omega} f_\infty(y)$.

In the following we shall give the uniform estimates of $\det(\partial^2 u^{(k)}/\partial\xi_i\partial\xi_j)$ from below and above (where $u^{(k)}$ denotes the Legendre transformation relative to $f^{(k)}$), and use the Caffarelli-Gutierrez theory to obtain a Hölder estimate for $\det(\partial^2 u^{(k)}/\partial\xi_i\partial\xi_j)($ see [C-G] or [T-W], Theorem 4.1). Then we use the Caffarelli-Schauder estimate for the Monge-Ampère equation [CAF] to get a $C^{2,\alpha}$ estimate to show that the limit surface is a smooth surface.

Estimate for $\det(\partial^2 u^{(k)}/\partial\xi_i\partial\xi_j)$ *from below.* Let us denote

$$D = \left\{ x \in \overline{\Omega} | f_\infty(x) = 0 \right\},$$

where $\overline{\Omega}$ denotes the closure of Ω. It is easy to see that D is a closed subset of $\overline{\Omega}$ and $(0, 0) \in D$. To estimate $\det(\partial^2 u^{(k)}/\partial\xi_i\partial\xi_j)$, we shall consider different cases according to the location of D:

CASE 1: $D \subset \Omega$.

CASE 2: $D \cap \partial\Omega \neq \emptyset$.

Later we shall prove that Case 2 cannot take place. Now we consider Case 1. In this case, there exists a number $b, 0 < b < 1$, such that the set

$$\overline{\Omega}_{2b} \subset \Omega,$$

where $\Omega_{2b} := \{(x_1, x_2) \in \Omega | f_\infty(x_1, x_2) < 2b\}$. Put $\Omega_{k,b} := \{(x_1, x_2) \in \Omega_k \mid f^{(k)}(x_1, x_2) < b\}$. Since $\{f^{(k)}\}$ converges to f_∞ locally uniformly,

$$\overline{\Omega}_{k,b} \subset \overline{\Omega_{2b}} \subset \Omega$$

for k large enough; it follows that

$$\text{dist}(\Omega_{k,b}, \partial\Omega_k) > d$$

for k large enough, where $d > 0$ is a constant independent of k. Now we use Lemma 3.2 to conclude that

$$(3.27) \qquad \det\left(\frac{\partial^2 f^{(k)}}{\partial x_i \partial x_j}\right) \leq d_1 \quad \text{for } x \in \Omega_{k,b},$$

where $d_1 > 0$ is a constant depending only on $|L_1|$ and d. Consider the Legendre transformation relative to $f^{(k)}$:

$$\xi_i^{(k)} = \frac{\partial f^{(k)}}{\partial x_i}, \quad u^{(k)} = \sum x_i \frac{\partial f^{(k)}}{\partial x_i} - f^{(k)}.$$

Set

$$\Omega_k^* = \{(\xi_1^{(k)}(x), \xi_2^{(k)}(x))|(x_1, x_2) \in \Omega_k\}, \quad \Omega_{k,b}^* = \{(\xi_1^{(k)}(x), \xi_2^{(k)}(x))|(x_1, x_2) \in \Omega_{k,b}\}.$$

Then, by (3.27), we have

$$(3.28) \qquad \det\left(\frac{\partial^2 u^{(k)}}{\partial \xi_i \partial \xi_j}\right) \geq \frac{1}{d_1} \quad \text{for } \xi \in \Omega_{k,b}^*.$$

Estimate for $\det(\partial^2 u^{(k)}/\partial\xi_i\partial\xi_j)$ *from above.* To get the estimate, we need some important results of classical convex body theory (see [BA], [BU]). Let F be a convex hypersurface in A^{n+1} and e be a subset of F. We denote by $\psi_F(e)$ the spherical image of e. If the set e is a Borel set, the spherical image of the set e is also a Borel set and therefore is measurable. Denote by $\sigma_F(e)$ the area (measure) of the spherical image $\psi_F(e)$ of the Borel set e of F and call it the *integral Gaussian curvature* of e. Denote by $A(e)$ the measure (or area) of the Borel set e on F. The ratio $\sigma_F(e)/A(e)$ is called the *specific curvature* of e. The following theorems hold (see [BU], [BA]):

THEOREM 3.6 (A. V. Pogorelov). *A convex surface whose specific curvature is bounded away from zero is strictly convex.*

THEOREM 3.7. *Let a sequence of closed convex hypersurfaces F_k converge to a closed convex hypersurface F and a sequence of closed subset M_k of F_k converge to a closed subset M of F; then*

$$\sigma_F(M) \geq \overline{\lim_{k\to\infty}} \, \sigma_{F_k}(M_k).$$

First of all, we claim that there exists a ball $B(0, r)$ such that

$$B(0, r) \subset \Omega_{k,b}^* \quad \text{for } k = 1, 2, \ldots$$

In fact, since Ω is bounded, there is a ball $B(0, r_1)$ with the center $(0,0)$ and the radius r_1 such that
$$\overline{\Omega}_b := \{(x_1, x_2) \in \Omega | f_\infty(x_1, x_2) \leq b\} \subset B(0, r_1).$$
Since $\{f^{(k)}\}$ converges locally uniformly to f_∞ in Ω, we see that
$$\overline{\Omega}_{k,b/2} := \{(x_1, x_2) \in \Omega_k | f^{(k)}(x_1, x_2) \leq b/2\} \subset B(0, r_1)$$
for k large enough. Consider the convex cone K with vertex $(0,0)$ and the base
$$\{(y_1, y_2, b/2)|(y_1, y_2) \in \partial\Omega_{k,b/2}\}.$$
Then we have (see [BA], p. 115)
$$\chi_{f^{(k)}}(\Omega_{k,b/2}) \supset \chi_K(\Omega_{k,b/2}).$$
On the other hand, since $\Omega_{k,b/2} \subset B(0, r_1)$, we see that (see [BA], p.126) $\chi_K(\Omega_{k,b/2}) \supset B(0, b/2r_1)$ and the claim follows.

Next, we want to prove that u_∞ is strictly convex at $(0,0)$. In fact, since $u^{(k)}$ are convex and bounded, we may assume by taking subsequences, that $\{u^{(k)}\}$ converges locally uniformly to a convex function u_∞ in $\Omega^*_{b/2r_1} = \{(\xi_1, \xi_2) | \sum \xi_i^2 < b^2/4r_1^2\}$. Let e be a closed subset of $\Omega^*_{b/2r_1}$ with $e^0 \neq \emptyset$, where e^0 denotes the interior of e. We denote by F and $F^{(k)}$ the graphs of the functions $u_\infty : \Omega^*_{b/2r_1} \to \mathbb{R}$ and $u^{(k)} : \Omega^*_{b/2r_1} \to \mathbb{R}$, respectively.

Set
$$F_e = \{(\xi_1, \xi_2, u_\infty(\xi_1, \xi_2))|(\xi_1, \xi_2) \in e\}, \quad F_e^{(k)} = \{(\xi_1, \xi_2, u^{(k)}(\xi_1, \xi_2))|(\xi_1, \xi_2) \in e\}.$$
Then, by Theorem 3.7, we get
$$
\begin{aligned}
\sigma_F(F_e) &\geq \varlimsup_{k\to\infty} \sigma_{F^{(k)}}(F_e^{(k)}) = \varlimsup_{k\to\infty} \int_{F_e^{(k)}} \frac{\det(\frac{\partial^2 u^{(k)}}{\partial\xi_i\partial\xi_j})}{(1 + \sum(\frac{\partial u^{(k)}}{\partial\xi_i})^2)^2} dp \\
&\geq b_2 \varlimsup_{k\to\infty} A(F_e^{(k)}) = b_2 A(F_e),
\end{aligned}
$$
where b_2 is a constant depending only d_1 and $\text{diam}(\Omega)$, i.e.,
$$(3.29) \qquad\qquad \frac{\sigma_F(F_e)}{A(F_e)} \geq b_2 > 0.$$
By Theorem 3.6, we conclude that u_∞ is strictly convex at $(0,0)$.

We now estimate $\det(\partial^2 u^{(k)}/\partial\xi_i\partial x\xi_j)$ from above. Since u_∞ is strictly convex at $(0,0)$, there exists a positive constant $0 < h_1 < 1$, such that
$$\Omega^*_{h_1} = \{(\xi_1, \xi_2) \in \Omega^*_{b/2r_1} | u_\infty(\xi_1, \xi_2) < h_1\}$$
is a bounded convex domain. Then, we choose $0 < h_2 < h_1$ such that
$$\overline{\Omega}^*_{2h_2} \subset \Omega^*_{h_1},$$
where $\Omega^*_{2h_2} = \{(\xi_1, \xi_2) \in \Omega^*_{h_1} | u_\infty(\xi_1, \xi_2) < 2h_2\}$. Put
$$\Omega^*_{k,h_2} = \{(\xi_1, \xi_2) \in \Omega^*_k | u^{(k)}(\xi_1, \xi_2) < h_2\}.$$
Since $u^{(k)}$ converges locally uniformly to u_∞, we have
$$\overline{\Omega}^*_{k,h_2} \subset \Omega^*_{h_1}$$

and there exists a constant $d_2 > 0$, such that

$$\text{dist}(\Omega^*_{k,h_2}, \partial\Omega^*_{h_1}) > d_2$$

for k large enough. Clearly, there is a uniform estimate

$$\sum \left(\frac{\partial u^{(k)}}{\partial \xi_i}\right)^2 \le \text{diam}(\Omega) \quad \text{for } \xi \in \Omega^*_{b/2r_1}.$$

Now we use Lemma 3.1 to conclude that

$$(3.30) \qquad \det\left(\frac{\partial^2 u^{(k)}}{\partial\xi_i\partial\xi_j}\right) < d_3 \quad \text{for } \xi \in \Omega^*_{k,h_2},$$

for k large enough, where $d_3 > 0$ is a constant depending only on $\text{diam}(\Omega), d_2$ and $|L_1|$.

We are now ready to prove the following lemma:

LEMMA 3.7. *There exists a neighborhood U of $(0,0)$ such that*

$$(3.31) \qquad D^2 u^{(k)} \ge C_3 I, \quad |D^l u^{(k)}| \le C_4 \quad \text{for } k, l = 1, 2, \dots,$$

where C_3 and C_4 are constants. C_3 depends only on $d_2, \dim(\Omega), |L_1|$ and d and C_4 depends additionally on l.

Proof. We set

$$v = \frac{1}{\rho}, \quad \rho_i = \frac{\partial\rho}{\partial\xi_i}, \quad v_i = \frac{\partial v}{\partial\xi_i} \quad v_{ij} = \frac{\partial^2 v}{\partial\xi_i\partial\xi_j}.$$

Then

$$\Delta\left(\frac{1}{v}\right) = \frac{1}{\rho}\sum u^{ij}\left(-\frac{v_{ij}}{v^2} + 2\frac{v_i v_j}{v^3}\right) - \frac{2}{\rho^2}\sum u^{ij}\frac{v_i v_j}{v^4} = -\rho\sum u^{ij}v_{ij}$$

On the other hand we have $\Delta(1/v) = \Delta\rho = -nL_1\rho$, and it follows that

$$\sum u^{ij}v_{ij} - nL_1 = 0.$$

Therefore, by setting

$$\Psi = v - L_1 u + 2|L_1|,$$

we obtain

$$(3.32) \qquad \sum U^{ij}\Psi_{ij} = 0.$$

where (U^{ij}) is the matrix of cofactors of (u_{ij}). By (4.6), (4.8) and (4.10), we use the Caffarelli-Gutierrez theory to obtain a Hölder estimate for $\det(\partial^2 u^{(k)}/\partial\xi_i\partial\xi_j)$ (see [C-G] or [T-W], Theorem 4.1).Then we use the Caffarelli-Schauder estimate for the Monge-Ampère equation ([CAF]) to get a $C^{2,\alpha}$ estimate. Finally, by bootstrapping, Lemma 3.7 follows.

Consequently, from Lemma 3.7 it follows that u_∞ is a smooth strictly convex function in a neighborhood of $(0,0)$, and hence f_∞ is a smooth strictly convex function in a neighborhood of $(0,0)$.

Now it is our purpose to show that Case 2 cannot take place. Let

$$V_k = \{(x_1, x_2, x_3) \in A^3 | f^{(k)}(x_1, x_2) \le x_3 \le 1, (x_1, x_2) \in \overline{\Omega}_k\},$$
$$V_\infty = \{(x_1, x_2, x_3) \in A^3 | f_\infty(x_1, x_2) \le x_3 \le 1, (x_1, x_2) \in \overline{\Omega}\}.$$

Then the sequence of convex bodies $\{V_k\}$ converges to the convex body V_∞. First we claim that D is a line segment or a single point. To prove this claim, we first show that there exists a ball $B(0, r_0)$ with center (0,0) and the radius r_0 such that

$$B(0, r_0) \subset \Omega_k^* \quad \text{for } k = 1, 2, \ldots.$$

As before, we choose a ball $B(0, l)$ with the center (0,0) and the radius l such that $\Omega \subset B(0, l)$. Since Ω_k converges to Ω, we see that $\Omega_k \subset B(0, l)$, for k large enough. Then it is easy to see that $\Omega_k^* \supset B(0, 1/l)$ for $k = 1, 2, \ldots$.

Now we prove our claim. By contradiction let us assume that there exists a ball

$$B(x_0, \epsilon) = \left\{ (x_1, x_2) \mid \sum (x_i - x_i^\circ)^2 < \epsilon^2 \right\}$$

such that $B(x_0, \epsilon) \subset D$. Since $\{f^{(k)}\}$ converges locally uniformly to f_∞, there is a positive number k_0 such that

$$0 \le f^{(k)}(x) < \frac{r_0^2 \epsilon}{8} \quad \text{for } x \in B(x_0, \epsilon/2) \quad (k > k_0).$$

Clearly, there exists a uniform estimate

$$\sum \left(\frac{\partial f^{(k)}}{\partial x_i} \right)^2 < \frac{r_0^2}{4} \quad \text{for } x \in B(x_0, \epsilon/2) \ (k > k_0).$$

Put

$$\Omega_{k, \epsilon/2}^* = \{ (\xi_1^{(k)}(x), \xi_2^{(k)}(x_1)) | x \in B(x_0, \epsilon/2) \}.$$

Then we have

$$\Omega_{k, \epsilon/2}^* \subset B(0, r_0/2) \subset \Omega_k^* \quad \text{for } k > k_0.$$

Note that $B(0, r_0) \subset \Omega_k^*$ for $k = 1, 2, \ldots$. Hence we use Lemma 3.3 to conclude that there exists a constant $d_4 > 0$, depending only on $r_0, \operatorname{diam}(\Omega)$ and $|L_1|$, such that

$$\det \left(\frac{\partial^2 u^{(k)}}{\partial \xi_i \partial \xi_j} \right) < d_4 \quad \text{for } \xi \in B(0, r_0/2).$$

This implies that

(3.33) $$\det \left(\frac{\partial^2 f^{(k)}}{\partial x_i \partial x_j} \right) > \frac{1}{d_4} \quad \text{for } x \in B(x_0, \epsilon/2) \ (k > k_0).$$

Therefore we can apply the argument of Case 1 to $\{f^{(k)}\}$ and conclude that the function f_∞ is strictly convex at x_0. This contradiction shows that D must be a line segment or a single point.

Now we prove that Case 2 cannot take place. We shall consider the following two cases:

CASE 2.1: $D \cap \partial\Omega$ contains at most two points.

CASE 2.2: D is a line segment with $D \subset \partial\Omega$.

CASE 2.1. Let $p \in D \cap \partial\Omega$ and let l be a supporting line of Ω at p. The line l and the unit normal ν of the (x_1, x_2)-plane determine a plane. We denote the plane by P. Then the plane P and the (x_1, x_2)-plane divide the space $\mathbb{R}^3$ into four closed subsets such that V_∞ lies completely in one of them. Let α be a supporting plane of V_∞ containing the line l such that it intersects P and forms an angle $\angle(\alpha, P) = \theta$ with P, where

$\theta > 0$ is sufficiently small. Since $p \in \partial V_\infty$ and α is a supporting plane of V_∞, there is a neighborhood $U \subset \partial V_\infty$ which projects orthogonally and one-to-one onto a convex domain $\Omega^{(1)} \subset \alpha$. This implies that, near the point $p, \partial V_\infty$ can be represented as a graph of a convex function g defined in $\Omega^{(1)}$. Obviously, g is strictly convex at p, but is not smooth at p. We choose a number $b, 0 < b < 1$, such that

$$\Omega^{(2)} = \{(y_1, y_2) \in \Omega^{(1)} | g(y_1, y_2) < b\}$$

is a bounded convex domain in A^2. Then we choose a new coordinate system $\{y_1, y_2, y_3\}$ such that

1) p has coordinate $(0, 0, 1)$;
2) the equation of α is $y_3 = 1$.

Since the sequence of convex bodies $\{V_k\}$ converges to the convex body V_∞, we see that the boundary ∂V_k of V_k can also be represented as a graph of a convex function $g^{(k)}$ for sufficiently large k. Obviously, $g^{(k)} \to g + 1$ in a bounded convex domain $\Omega^{(3)}$. Note that the graph of $g^{(k)}$ is a locally strongly convex surface with constant affine mean curvature L_1. Therefore we can apply the argument of Case 1 to $\{g^{(k)}\}$ and conclude that the function g is a smooth function near the point p. The contradiction shows that Case 2.1 cannot take place.

CASE 2.2. In this case, we have $p = (0, 0) \in \partial\Omega$. Let l be the line containing D. We choose a new coordinate system $\{y_1, y_2, y_3\}$ as in Case 2.1. Then the boundary ∂V_∞ of V_∞ can be represented as the graph of a convex function g defined in a convex domain $\Omega^{(3)}$. With respect to this coordinate system we have $g \geq 1$. The boundary ∂V_k of V_k can also be represented as the graph of a convex function $g^{(k)}$ for sufficiently large k. Obviously, $g^{(k)} \to g$ in $\Omega^{(3)}$. Note that the graph of $g^{(k)}$ is a locally strongly convex surface with constant affine mean curvature L_1. Again, we shall consider different cases according to the location of

$$D^\# = \{(y_1, y_2) \in \overline{\Omega}^{(3)} | g(y_1, y_2) = 1\}.$$

CASE 2.2.1: $D^\# \subset \Omega^{(3)}$.
CASE 2.2.2. $D^\# \cap \partial\Omega^{(3)} \neq \emptyset$.

CASE 2.2.1. Note that $D^\#$ is a line segment. Since $D^\# \subset \Omega^{(3)}$, we can apply the argument of Case 1 to Case 2.2.1 and conclude that Case 2.2.1 cannot take place.

CASE 2.2.2. In this case, $D^\# \cap \partial\Omega^{(3)}$ contains at most two points. Therefore we can apply the argument of Case 2.1 to Case 2.2.2 and conclude that Case 2.2.2 cannot take place.

We are now in a position to prove that $\|B\|_G^2$ is bounded. By Lemma 3.7, we have

$$D^2 u^{(k)} \geq C_3 I, \quad |D^l u^{(k)}| \leq C_4, \quad l = 0, 1, 2, \ldots,$$

in a neighborhood U of $(0,0)$, where C_3 and C_4 are constants. C_3 depends only on $d, d_2, \text{diam}(\Omega)$ and $|L_1|$, and C_4 depends additionally on l. Note that $\|B\|_G^2$ is equiaffinely invariant. By (3.6) we have

$$\|B\|_G^2(p_k) = \sum G_{(k)}^{is} G_{(k)}^{jt} B_{ij}^{(k)} B_{st}^{(k)} \big|_{(0,0)},$$

where

$$B_{ij}^{(k)} = -\frac{1}{\rho^{(k)}}\frac{\partial^2 \rho^{(k)}}{\partial x_i \partial x_j} + \frac{2}{(\rho^{(k)})^2}\frac{\partial \rho^{(k)}}{\partial x_i}\frac{\partial \rho^{(k)}}{\partial x_j} + \sum \frac{f^{(k)st}}{\rho^{(k)}}\frac{\partial \rho^{(k)}}{\partial x_s}\frac{\partial f_{ij}^{(k)}}{\partial x_t}.$$

$$\rho^{(k)} = \left[\det\left(\frac{\partial^2 f^{(k)}}{\partial x_i \partial x_j}\right)\right]^{-\frac{1}{4}}, \quad G_{ij}^{(k)} = \rho^{(k)}\frac{\partial^2 f^{(k)}}{\partial x_i \partial x_j}.$$

Note that

$$\frac{\partial^2 \rho^{(k)}}{\partial x_i \partial x_j} = \sum \frac{\partial^2 \rho^{(k)}}{\partial \xi_l \partial \xi_s} u^{(k)li} u^{(k)sj} + \sum \frac{\partial \rho^{(k)}}{\partial \xi_l}\frac{\partial u^{(k)li}}{\partial \xi_s} u^{(k)sj},$$

$$\frac{\partial \rho^{(k)}}{\partial x_i} = \sum \frac{\partial \rho^{(k)}}{\partial \xi_l} u^{(k)li}, \quad \frac{\partial f_{ij}^{(k)}}{\partial x_l} = \sum \frac{\partial u^{(k)ij}}{\partial \xi_s} u^{(k)sl}.$$

Consequently, from Lemma 3.7, it follows that there exists a number $N > 0$ such that

$$\|B\|_G^2(p_k) \le N, \quad k = 1, 2, \ldots.$$

On the other hand, by (3.23) we have

$$\|B\|_G^2(p_k) \to \infty \quad \text{as } k \to \infty.$$

The contradiction shows that there must exist a number $N > 0$ such that $\|B\|_G^2 \le N$ on M. Then, by Theorem 3.5, Theorem 3.1 follows.

Proof of Theorem 3.2. Recall that (see [L-S-Z], p. 121) there is no locally strongly convex, compact hypersurface without boundary and with non-positive affine mean curvature. This implies that M is non-compact if M is a locally strongly convex surface with constant affine curvature $L_1 \le 0$.

(a) Denote by R the scalar curvature; we have $R = 2(J + L_1) \ge 2L_1 > 0$ (see [L-S-Z], p. 76). Moreover, by Theorem 3.1, M is affine-complete. This implies that (M, G) is a complete Riemannian manifold with Ricci curvature bounded from below by a positive constant $2L_1 > 0$. By Myer's and Bonnet's Theorem (see [CH-E], p. 27) M is compact. It follows that M is an ellipsoid see ([L-S-Z], p. 121).

(b) Since M is an Euclidean complete and affine complete affine maximal surface, by Calabi's Theorem (see, [L-S-Z], p. 219) M must be an elliptic paraboloid.

(c) Since the tensor norm is equiaffinely invariant, we replace $\|B\|_G^2$ by it, then a similar argument shows that the tensor norms of the Fubini-Pick tensor and the affine Weingarten tensor and tensor norms of the k-th covariant differentiation of them are all bounded. This completes the proof of Theorem 3.2.

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