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THE GAPS IN THE SPECTRUM OF THE SCHRÖDINGER OPERATOR

HAIZHONG LI and LINLIN SU

Department of Mathematical Sciences, Tsinghua University
Beijing, 100084, People's Republic of China
E-mail: hli@math.tsinghua.edu.cn

Abstract. We obtain inequalities between the eigenvalues of the Schrödinger operator on a compact domain Ω of a submanifold M in R^N with boundary $\partial\Omega$, which generalize many existing inequalities for the Laplacian on a bounded domain of a Euclidean space. We also establish similar inequalities for a closed minimal submanifold in the unit sphere, which generalize and improve Yang-Yau's result.

1. Introduction. Let $x : M \to R^N$ be an m-dimensional submanifold in N-dimensional Euclidean space with the mean curvature vector $\mathbf{H}$. Let $\Omega \subset M$ be a compact domain on M with boundary $\partial\Omega$. We consider the Schrödinger operator

$$L = \Delta - V, \tag{1.1}$$

where V is a nonnegative smooth function on $\bar{\Omega}$. We shall consider the following Dirichlet eigenvalue problem of the Schrödinger operator L on Ω:

$$\begin{cases} Lu = -\lambda u & \text{on } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{1.2}$$

As is well-known, the eigenvalues $\{\lambda_k\}$ of (1.2) are nonnegative, and can be arranged in nondecreasing order as follows:

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$$

Assume we know the first n eigenfunctions of L

$$u_1, u_2, \ldots, u_n$$

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i.e., $Lu_i = -\lambda_i u_i$, having the properties

$$u_i = 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

$$\int_\Omega u_i u_j = \begin{cases} 1, & \text{when} \quad i = j, \\ 0, & \text{when} \quad i \neq j. \end{cases} \tag{1.4}$$

To obtain information about the next eigenvalue, we shall need some appropriate "trial functions". In fact, following [5], we can use the following trial functions

$$\phi_1^\alpha, \phi_2^\alpha, \ldots, \phi_n^\alpha.$$

Any one of these trial functions, say ϕ_i^α, must satisfy two conditions:

$$\phi_i^\alpha = 0 \quad \text{on} \quad \partial\Omega, \tag{1.5}$$

$$\int_\Omega \phi_i^\alpha u_j = 0 \tag{1.6}$$

where $i, j = 1, 2, \ldots, n$. We choose (following [5])

$$\phi_i^\alpha = x_\alpha u_i - \sum_{j=1}^n a_{ij}^\alpha u_j, \quad \alpha = 1, \ldots, N, \tag{1.7}$$

where

$$a_{ij}^\alpha = \int_\Omega x_\alpha u_i u_j = a_{ji}^\alpha, \quad x(p) = (x_1(p), \ldots, x_N(p)) \in R^N, \quad \forall\, p \in \Omega. \tag{1.8}$$

It is easy to see that the functions ϕ_i^α satisfy (1.5) and (1.6), i.e., ϕ_i^α are all orthogonal to $u_1, \ldots, u_n$ and vanish at the boundary. We also have, by use of (1.7) and (1.6),

$$\int_\Omega (\phi_i^\alpha)^2 = \int_\Omega \phi_i^\alpha x_\alpha u_i = \int_\Omega x_\alpha^2 u_i^2 - \sum_{j=1}^n (a_{ij}^\alpha)^2, \tag{1.9}$$

$$-\Delta\phi_i^\alpha = -u_i \Delta x_\alpha - 2\nabla u_i \cdot \nabla x_\alpha + \lambda_i x_\alpha u_i - \sum_{j=1}^n a_{ij}^\alpha \lambda_j u_j - V\phi_i^\alpha, \tag{1.10}$$

$$\begin{aligned} -\int_\Omega \phi_i^\alpha L(\phi_i^\alpha) &= \int_\Omega \phi_i^\alpha(-\Delta\phi_i^\alpha) + \int_\Omega V(\phi_i^\alpha)^2 \\ &= -\int_\Omega \phi_i^\alpha \Delta x_\alpha \cdot u_i - 2\int_\Omega \phi_i^\alpha \nabla u_i \cdot \nabla x_\alpha + \lambda_i \int_\Omega (\phi_i^\alpha)^2. \end{aligned} \tag{1.11}$$

Hence by the variational principle for λ_{n+1}, we have

$$\lambda_{n+1} \int_\Omega (\phi_i^\alpha)^2 \leq -\int_\Omega \phi_i^\alpha L(\phi_i^\alpha) = -\int_\Omega \phi_i^\alpha \Delta x_\alpha \cdot u_i - 2\int_\Omega \phi_i^\alpha \nabla u_i \cdot \nabla x_\alpha + \lambda_i \int_\Omega (\phi_i^\alpha)^2,$$

that is,

$$0 \leq (\lambda_{n+1} - \lambda_i) \int_\Omega (\phi_i^\alpha)^2 \leq -\int_\Omega \phi_i^\alpha \Delta x_\alpha \cdot u_i - 2\int_\Omega \phi_i^\alpha \nabla u_i \cdot \nabla x_\alpha. \tag{1.12}$$

From (1.12), by use of (1.6) we get

$$(\lambda_{n+1} - \lambda_i) \int_\Omega (\phi_i^\alpha)^2 \cdot \left[-\int_\Omega \phi_i^\alpha \Delta x_\alpha \cdot u_i - 2\int_\Omega \phi_i^\alpha \nabla u_i \cdot \nabla x_\alpha\right] \tag{1.13}$$

$$\leq \left[-\int_\Omega \phi_i^\alpha \Delta x_\alpha \cdot u_i - 2\int_\Omega \phi_i^\alpha \nabla u_i \cdot \nabla x_\alpha\right]^2 = \left[\int_\Omega \phi_i^\alpha (\Delta x_\alpha \cdot u_i + 2\nabla u_i \cdot \nabla x_\alpha)\right]^2$$
$$= \left[\int_\Omega \phi_i^\alpha (\Delta x_\alpha \cdot u_i + 2\nabla u_i \cdot \nabla x_\alpha - 2\sum_{j=1}^n b_{ij}^\alpha u_j)\right]^2.$$

Define

$$I_{\alpha i} = \int_\Omega |\nabla x_\alpha|^2 u_i^2, \tag{1.14}$$

$$b_{ij}^\alpha = \int_\Omega u_j \nabla x_\alpha \cdot \nabla u_i + \frac{1}{2}\int_\Omega \Delta x_\alpha \cdot u_i u_j, \tag{1.15}$$

By use of the Stokes formula and (1.3), it is easy to check

$$b_{ij}^\alpha = -b_{ji}^\alpha, \qquad \alpha = 1, 2, \ldots, N; \quad i, j = 1, 2, \cdots, n.$$

We have

$$-2\int_\Omega \phi_i^\alpha \nabla x_\alpha \cdot \nabla u_i = -\int_\Omega x_\alpha \nabla x_\alpha \cdot \nabla(u_i^2) + 2\sum_{j=1}^n a_{ij}^\alpha \int_\Omega u_j \nabla x_\alpha \cdot \nabla u_i \tag{1.16}$$
$$= I_{\alpha i} + \int_\Omega u_i^2 x_\alpha \Delta x_\alpha + 2\sum_{j=1}^n a_{ij}^\alpha \int_\Omega u_j \nabla x_\alpha \cdot \nabla u_i,$$

$$-\int_\Omega u_i \phi_i^\alpha \Delta x_\alpha - 2\int_\Omega \phi_i^\alpha \nabla u_i \cdot \nabla x_\alpha = I_{\alpha i} + 2\sum_{j=1}^n a_{ij}^\alpha b_{ij}^\alpha. \tag{1.17}$$

From the definition of a_{ij}^α,

$$\begin{aligned}
\lambda_i a_{ij}^\alpha &= \int_\Omega (-Lu_i) x_\alpha u_j = \int_\Omega (-\Delta u_i) x_\alpha u_j + \int_\Omega V u_i u_j x_\alpha \\
&= \int_\Omega u_i \Delta(-x_\alpha u_j) + \int_\Omega V u_i u_j x_\alpha \\
&= -\int_\Omega u_i u_j \Delta x_\alpha - 2\int_\Omega u_i \nabla x_\alpha \cdot \nabla u_j + \lambda_j \int_\Omega u_i u_j x_\alpha \\
&= -2b_{ji}^\alpha + \lambda_j a_{ij}^\alpha = 2b_{ij}^\alpha + \lambda_j a_{ij}^\alpha,
\end{aligned}$$

that is,

$$2b_{ij}^\alpha = (\lambda_i - \lambda_j) a_{ij}^\alpha. \tag{1.18}$$

Putting (1.17) and (1.18) into (1.13), we obtain for any real number $\epsilon > 0$

$$(\lambda_{n+1} - \lambda_i)\int_\Omega (\phi_i^\alpha)^2 \cdot [I_{\alpha i} + \sum_{j=1}^n (\lambda_i - \lambda_j)(a_{ij}^\alpha)^2] \tag{1.19}$$
$$\leq \left[\int_\Omega \phi_i^\alpha \Big(\Delta x_\alpha \cdot u_i + 2\nabla u_i \cdot \nabla x_\alpha - 2\sum_{j=1}^n b_{ij}^\alpha u_j\Big)\right]^2$$
$$\leq \int_\Omega (\phi_i^\alpha)^2 \cdot \int_\Omega \left[\Delta x_\alpha \cdot u_i + 2\nabla u_i \cdot \nabla x_\alpha - 2\sum_{j=1}^n b_{ij}^\alpha u_j\right]^2$$

$$= \int_\Omega (\phi_i^\alpha)^2 \cdot \left[\int_\Omega (\Delta x_\alpha u_i + 2\nabla u_i \cdot \nabla x_\alpha)^2 - 4\sum_{j=1}^n (b_{ij}^\alpha)^2 \right]$$

$$\leq \int_\Omega (\phi_i^\alpha)^2 \cdot \left[(1+1/\epsilon)\int_\Omega (\Delta x_\alpha)^2 u_i^2 + 4(1+\epsilon)\int_\Omega |\nabla u_i \cdot \nabla x_\alpha|^2 - 4\sum_{j=1}^n (b_{ij}^\alpha)^2 \right],$$

where we used the Schwarz inequality in the last step.

Dividing by $\int_\Omega (\phi_i^\alpha)^2$, we have for any real number $\epsilon > 0$

$$(1.20)\quad (\lambda_{n+1} - \lambda_i)[I_{\alpha i} + \sum_{j=1}^n (\lambda_i - \lambda_j)(a_{ij}^\alpha)^2] \leq (1+1/\epsilon)\int_\Omega (\Delta x_\alpha)^2 u_i^2 + 4(1+\epsilon)\int_\Omega |\nabla u_i \cdot \nabla x_\alpha|^2 - 4\sum_{j=1}^n (b_{ij}^\alpha)^2.$$

Inequality (1.20) holds even in the case that $\phi_i^\alpha \equiv 0$, since in that case the left-hand side vanishes from (1.17) and (1.18), while the right-hand side is nonnegative (it is not less than the integral of the norm square of $\Delta x_\alpha u_i + 2\nabla u_i \cdot \nabla x_\alpha - 2\sum_{j=1}^n b_{ij}^\alpha u_j$).

Putting (1.18) into (1.20), we have

$$(1.21)\quad (\lambda_{n+1} - \lambda_i) I_{\alpha i} + \sum_{j=1}^n (\lambda_i - \lambda_j)(\lambda_{n+1} - \lambda_j)(a_{ij}^\alpha)^2$$
$$hfill \leq (1+1/\epsilon)\int_\Omega (\Delta x_\alpha)^2 u_i^2 + 4(1+\epsilon)\int_\Omega |\nabla u_i \cdot \nabla x_\alpha|^2.$$

By making summation of the terms of $(\lambda_{n+1} - \lambda_i) \times$ (1.21), we get

$$(1.22)\quad \sum_{i=1}^n (\lambda_{n+1} - \lambda_i)^2 I_{\alpha i}$$
$$hfill \leq (1+1/\epsilon)\sum_{i=1}^n (\lambda_{n+1} - \lambda_i)\int_\Omega (\Delta x_\alpha)^2 u_i^2 + 4(1+\epsilon)\sum_{i=1}^n (\lambda_{n+1} - \lambda_i)\int_\Omega |\nabla u_i \cdot \nabla x_\alpha|^2.$$

By use of the identities

$$\Delta x = -m\mathbf{H}, \quad \sum_{\alpha=1}^N I_{\alpha i} = \sum_{\alpha=1}^N \int_\Omega |\nabla x_\alpha|^2 u_i^2 = m\int_\Omega u_i^2 = m,$$

$$\sum_{\alpha=1}^N \int_\Omega |\nabla u_i \cdot \nabla x_\alpha|^2 = \int_\Omega |\nabla u_i|^2 = -\int_\Omega u_i \Delta u_i = \int_\Omega u_i(\lambda_i u_i - V u_i) = \lambda_i - \int_\Omega V u_i^2,$$

and by making summation of (1.22) over α from 1 to N, we obtain

$$(1.23)\quad m\sum_{i=1}^n (\lambda_{n+1} - \lambda_i)^2$$
$$\leq (1+1/\epsilon)m^2 \sum_{i=1}^n (\lambda_{n+1} - \lambda_i)\int_\Omega |\mathbf{H}|^2 u_i^2 + 4(1+\epsilon)\sum_{i=1}^n (\lambda_{n+1} - \lambda_i)\Big(\lambda_i - \int_\Omega V u_i^2\Big).$$

Writing

$$(1.24)\qquad |\mathbf{H}|_\infty^2 = \sup_{\bar\Omega} |\mathbf{H}|^2, \qquad V_0 = \inf_{\bar\Omega} V,$$

we have from (1.23) and (1.24) for any real number $\epsilon > 0$

$$\sum_{i=1}^{n}(\lambda_{n+1}-\lambda_i)^2 \le (1+1/\epsilon)m^2|\mathbf{H}|_\infty^2\sum_{i=1}^{n}(\lambda_{n+1}-\lambda_i)+4(1+\epsilon)\sum_{i=1}^{n}(\lambda_{n+1}-\lambda_i)(\lambda_i-V_0). \tag{1.24}$$

Thus we have proved the following main result of this section:

THEOREM 1.1. *Let $x : M \to R^N$ be an m-dimensional submanifold in N-dimensional Euclidean space. Let $\Omega \subset M$ be a bounded domain. Then the spectrum of the Schrödinger operator $L = \Delta - V$ satisfies the following inequality, for any real number $\epsilon > 0$, and for any $n = 1, 2, \ldots$:*

$$m\sum_{i=1}^{n}(\lambda_{n+1}-\lambda_i)^2 \le (1+1/\epsilon)m^2|\mathbf{H}|_\infty^2\sum_{i=1}^{n}(\lambda_{n+1}-\lambda_i)+4(1+\epsilon)\sum_{i=1}^{n}(\lambda_{n+1}-\lambda_i)(\lambda_i-V_0). \tag{1.26}$$

Writing

$$\tilde\lambda_{n+1} = \lambda_{n+1} - V_0, \quad \tilde\lambda_i = \lambda_i - V_0, \quad i = 1, 2, \ldots, n,$$

we have

$$\tilde\lambda_n \ge \tilde\lambda_{n-1} \ge \cdots \ge \tilde\lambda_2 \ge \tilde\lambda_1 = \lambda_1 - V_0 = \frac{\int|\nabla u_1|^2}{\int u_1^2} + \frac{\int V u_1^2}{\int u_1^2} - V_0 \ge \frac{\int|\nabla u_1|^2}{\int u_1^2} > 0.$$

Then (1.26) can be rewritten as

$$m\sum_{i=1}^{n}(\tilde\lambda_{n+1}-\tilde\lambda_i)^2 \le (1+1/\epsilon)m^2|\mathbf{H}|_\infty^2\sum_{i=1}^{n}(\tilde\lambda_{n+1}-\tilde\lambda_i)+4(1+\epsilon)\sum_{i=1}^{n}(\tilde\lambda_{n+1}-\tilde\lambda_i)\tilde\lambda_i. \tag{1.26$'$}$$

It is easy to see that $(1.26)'$ is equivalent to

$$\begin{aligned}\tilde\lambda_{n+1} \le{}& \frac{1}{2}m(1+1/\epsilon)|\mathbf{H}|_\infty^2 + \frac{1}{n}\Big(1+\frac{2}{m}(1+\epsilon)\Big)\sum_{i=1}^{n}\tilde\lambda_i \\ &+ \frac{1}{2n}\Bigg\{\Bigg[(1+1/\epsilon)mn|\mathbf{H}|_\infty^2 + \Big(2+\frac{4}{m}(1+\epsilon)\Big)\sum_{i=1}^{n}\tilde\lambda_i\Bigg]^2 \\ &- 4n\Big(1+\frac{4}{m}(1+\epsilon)\Big)\sum_{i=1}^{n}\tilde\lambda_i^2 - 4mn(1+1/\epsilon)|\mathbf{H}|_\infty^2\sum_{i=1}^{n}\tilde\lambda_i\Bigg\}^{\frac{1}{2}}.\end{aligned} \tag{1.27}$$

Noting $(\sum_{i=1}^{n}\tilde\lambda_i)^2 \le n\sum_{i=1}^{n}\tilde\lambda_i^2$, we have from (1.27)

$$\tilde\lambda_{n+1} \le m(1+1/\epsilon)|\mathbf{H}|_\infty^2 + \frac{1}{n}\Big(1+\frac{4}{m}(1+\epsilon)\Big)\sum_{i=1}^{n}\tilde\lambda_i. \tag{1.28}$$

Thus we have proved the following weak, but simpler version of Theorem 1.1:

THEOREM 1.2. *Let $x : M \to R^N$ be an m-dimensional submanifold in N-dimensional Euclidean space. Let $\Omega \subset M$ be a bounded domain. Then the spectrum of the Schrödinger* *operator $L = \Delta - V$ satisfies the following inequality, for any real number $\epsilon > 0$:*

$$(1.28)' \quad \lambda_{n+1} - V_0 \leq m(1 + 1/\epsilon)|\mathbf{H}|_\infty^2 + \frac{1}{n}(1 + \frac{4}{m}(1+\epsilon))\sum_{i=1}^{n}(\lambda_i - V_0), \qquad n = 1, 2, \ldots$$

Now we prove that Theorem 1.2 implies the following conclusion:

THEOREM 1.3. *Let $x : M \to R^N$ be an m-dimensional submanifold in N-dimensional Euclidean space. Let $\Omega \subset M$ be a bounded domain. Then the spectrum of the Schrödinger operator $L = \Delta - V$ satisfies the following inequality, for any real number $\epsilon > 0$, and for any $n \in N$ satisfying $\lambda_{n+1} > \lambda_n$:*

$$(1.29) \qquad \sum_{i=1}^{n} \frac{4(1+\epsilon)(\lambda_i - V_0) + (1 + 1/\epsilon)m^2|\mathbf{H}|_\infty^2}{\lambda_{n+1} - \lambda_i} \geq mn, \qquad n = 1, 2, \ldots$$

and equivalently

$$(1.29)' \qquad \sum_{i=1}^{n} \frac{4(1+\epsilon)\tilde{\lambda}_i + (1 + 1/\epsilon)m^2|\mathbf{H}|_\infty^2}{\tilde{\lambda}_{n+1} - \tilde{\lambda}_i} \geq mn, \qquad n = 1, 2, \ldots$$

Proof. (1.28) or (1.28)′ can be rewritten as

$$(1.30) \qquad \frac{1}{mn}\Big[4(1+\epsilon)\sum_{i=1}^{n}\tilde{\lambda}_i + m^2 n(1 + 1/\epsilon)|\mathbf{H}|_\infty^2\Big] \geq \tilde{\lambda}_{n+1} - \frac{1}{n}\sum_{i=1}^{n}\tilde{\lambda}_i,$$

that is,

$$(1.30)' \qquad mn \leq \frac{4(1+\epsilon)\sum_{i=1}^{n}\tilde{\lambda}_i + m^2 n(1+1/\epsilon)|\mathbf{H}|_\infty^2}{\tilde{\lambda}_{n+1} - \frac{1}{n}\sum_{i=1}^{n}\tilde{\lambda}_i}.$$

From (1.29) and (1.30)′, we only need to prove

$$(1.31) \qquad \frac{4(1+\epsilon)\sum_{i=1}^{n}\tilde{\lambda}_i + m^2 n(1+1/\epsilon)|\mathbf{H}|_\infty^2}{\tilde{\lambda}_{n+1} - \frac{1}{n}\sum_{i=1}^{n}\tilde{\lambda}_i} \leq \sum_{i=1}^{n}\frac{4(1+\epsilon)\tilde{\lambda}_i + (1+1/\epsilon)m^2|\mathbf{H}|_\infty^2}{\tilde{\lambda}_{n+1} - \tilde{\lambda}_i}.$$

We consider the function

$$f(x) = \frac{4(1+\epsilon)x + (1+1/\epsilon)m^2|\mathbf{H}|_\infty^2}{\tilde{\lambda}_{n+1} - x} = -4(1+\epsilon) + \frac{(1+1/\epsilon)m^2|\mathbf{H}|_\infty^2 + 4(1+\epsilon)\tilde{\lambda}_{n+1}}{\tilde{\lambda}_{n+1} - x}.$$

It is convex when $x < \tilde{\lambda}_{n+1}$. Thus

$$(1.32) \qquad f\left(\frac{\tilde{\lambda}_1 + \tilde{\lambda}_2 + \cdots + \tilde{\lambda}_n}{n}\right) \leq \frac{1}{n}[f(\tilde{\lambda}_1) + f(\tilde{\lambda}_2) + \cdots + f(\tilde{\lambda}_n)].$$

It is easy to check that (1.32) is equivalent to (1.31). Thus we have completed the proof of Theorem 1.3.

Theorem 1.3 has the following corollary:

THEOREM 1.4. *Let $x : M \to R^N$ be an m-dimensional submanifold in N-dimensional Euclidean space. Let $\Omega \subset M$ be a bounded domain. Then the spectrum of the Schrödinger* *operator $L = \Delta - V$ satisfies the following inequality, for any real number $\epsilon > 0$:*

$$\lambda_{n+1} - \lambda_n \le \frac{4(1+\epsilon)}{mn} \sum_{i=1}^{n} (\lambda_i - V_0) + (1 + 1/\epsilon) m |\mathbf{H}|_\infty^2, \quad n = 1, 2, \dots \tag{1.33}$$

and equivalently

$$\tilde{\lambda}_{n+1} - \tilde{\lambda}_n \le \frac{4(1+\epsilon)}{mn} \sum_{i=1}^{n} \tilde{\lambda}_i + (1 + 1/\epsilon) m |\mathbf{H}|_\infty^2, \quad n = 1, 2, \dots \tag{1.33$'$}$$

Proof. Replacing the λ_i in the denominator of the left-hand side of (1.29) by λ_n, we obtain (1.33).

REMARK 1.1. If $x : M \to R^N$ is an m-dimensional minimal submanifold and $V \equiv 0$, letting $\epsilon \to 0$ in Theorem 1.4, we recover S. Y. Cheng's result [2].

When $M = R^m$ and $V \equiv 0$, in this case $x : M \to R^N$ is an m-dimensional totally geodesic submanifold, we have $H \equiv 0$. We have, from Theorems 1.1 to 1.4 by letting $\epsilon \to 0$:

COROLLARY 1.1 (Yang [8]). *Let $\Omega \subset R^m$ be a bounded domain. Then the spectrum of the Laplacian Δ satisfies the inequality*

$$m \sum_{i=1}^{n} (\lambda_{n+1} - \lambda_i)^2 \le 4 \sum_{i=1}^{n} (\lambda_{n+1} - \lambda_i) \lambda_i, \quad n = 1, 2, \dots$$

COROLLARY 1.2 (Ashbaugh [1], Yang [8]). *Let $\Omega \subset R^m$ be a bounded domain. Then the spectrum of the Laplacian Δ satisfies the inequality*

$$\lambda_{n+1} \le \frac{1}{n}(1 + \frac{4}{m}) \sum_{i=1}^{n} \lambda_i, \quad n = 1, 2, \dots$$

COROLLARY 1.3 (Hile-Protter [3]). *Let $\Omega \subset R^m$ be a bounded domain. Then the spectrum of the Laplacian Δ satisfies the inequality*

$$4 \sum_{i=1}^{n} \frac{\lambda_i}{\lambda_{n+1} - \lambda_i} \ge mn, \quad n = 1, 2, \dots$$

COROLLARY 1.4 (Payne–Pólya–Weinberger [5], Thompson [7]). *Let $\Omega \subset R^m$ be a bounded domain. Then the spectrum of the Laplacian Δ satisfies the inequality*

$$\lambda_{n+1} - \lambda_n \le \frac{4}{mn} \sum_{i=1}^{n} \lambda_i, \quad n = 1, 2, \dots$$

2. Minimal submanifold in the unit sphere. In this section, let $x : M \to S^{N-1}(1)$ be an m-dimensional closed (i.e., compact without boundary) minimal submanifold in the Euclidean unit sphere. As in the previous section, we consider the Schrödinger operator

$$L = \Delta - V, \tag{2.1}$$

where V is a nonnegative smooth function on M. We shall consider the following Dirichlet eigenvalue problem for the Schrödinger operator L on M:

$$Lu = -\lambda u \quad \text{on} \quad M. \tag{2.2}$$

As is well-known, the eigenvalues $\{\lambda_k\}$ of (2.2) are nonnegative, and can be arranged in nondecreasing order as follows:

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$$

Let $x = (x_1, \ldots, x_N) : M \to S^{N-1}(1) \subset R^N$ be the minimal immersion. Let $\{(u_i, \lambda_i)\}_{i=1,\ldots,n}$ be the normalized first n eigenfunctions and their corresponding eigenvalues. Then

$$\Delta x_\alpha = -m x_\alpha, \qquad \sum_{\alpha=1}^{N} x_\alpha^2 = 1, \tag{2.3}$$

$$L u_i = -\lambda_i u_i, \quad i = 1, \ldots, n. \tag{2.4}$$

Noting that all formulas from preceding section still hold, we have from (1.13) and (1.17) by use of (2.3)

$$\begin{aligned} (2.5) \quad & (\lambda_{n+1} - \lambda_i) \int_M (\phi_i^\alpha)^2 \cdot \Big[I_{\alpha i} + \sum_{j=1}^{n} (\lambda_i - \lambda_j)(a_{ij}^\alpha)^2 \Big] \\ & \leq \int_M (\phi_i^\alpha)^2 \cdot \Big[\int_M (\Delta x_\alpha u_i + 2 \nabla u_i \cdot \nabla x_\alpha)^2 - 4 \sum_{j=1}^{n} (b_{ij}^\alpha)^2 \Big] \\ & = \int_M (\phi_i^\alpha)^2 \cdot \Big[\int_M (\Delta x_\alpha)^2 u_i^2 + 4 \int_M |\nabla u_i \cdot \nabla x_\alpha|^2 - 2m \int_M u_i \nabla u_i \cdot \nabla x_\alpha^2 - 4 \sum_{j=1}^{n} (b_{ij}^\alpha)^2 \Big] \\ & = \int_M (\phi_i^\alpha)^2 \cdot \Big[\int_M m^2 x_\alpha^2 u_i^2 + 4 \int_M |\nabla u_i \cdot \nabla x_\alpha|^2 - 2m \int_M u_i \nabla u_i \cdot \nabla x_\alpha^2 - 4 \sum_{j=1}^{n} (b_{ij}^\alpha)^2 \Big], \end{aligned}$$

thus we get

$$\begin{aligned} (2.6) \quad & (\lambda_{n+1} - \lambda_i) \Big[I_{\alpha i} + \sum_{j=1}^{n} (\lambda_i - \lambda_j)(a_{ij}^\alpha)^2 \Big] \\ & \leq \int_M m^2 x_\alpha^2 u_i^2 + 4 \int_M |\nabla u_i \cdot \nabla x_\alpha|^2 - 2m \int_M u_i \nabla u_i \cdot \nabla x_\alpha^2 - 4 \sum_{j=1}^{n} (b_{ij}^\alpha)^2. \end{aligned}$$

Putting (1.18) into (2.6), we have

$$\begin{aligned} (2.7) \quad & (\lambda_{n+1} - \lambda_i) I_{\alpha i} + \sum_{j=1}^{n} (\lambda_i - \lambda_j)(\lambda_{n+1} - \lambda_j)(a_{ij}^\alpha)^2 \\ & \leq \int_M m^2 x_\alpha^2 u_i^2 + 4 \int_M |\nabla u_i \cdot \nabla x_\alpha|^2 - 2m \int_M u_i \nabla u_i \cdot \nabla x_\alpha^2. \end{aligned}$$

By making summation of the terms of $(\lambda_{n+1} - \lambda_i) \times (2.7)$, we get

$$\begin{aligned} (2.8) \quad & \sum_{i=1}^{n} (\lambda_{n+1} - \lambda_i)^2 I_{\alpha i} \\ & \leq \sum_{i=1}^{n} (\lambda_{n+1} - \lambda_i) \int_M m^2 x_\alpha^2 u_i^2 + 4 \sum_{i=1}^{n} (\lambda_{n+1} - \lambda_i) \int_M |\nabla u_i \cdot \nabla x_\alpha|^2 \\ & \quad - 2m \sum_{i=1}^{n} (\lambda_{n+1} - \lambda_i) \int_M u_i \nabla u_i \cdot \nabla x_\alpha^2. \end{aligned}$$

By use of the identities

$$V_0 = \inf_{\bar\Omega} V, \quad \sum_{\alpha=1}^N I_{\alpha i} = \sum_{\alpha=1}^N \int_M |\nabla x_\alpha|^2 u_i^2 = m\int_M u_i^2 = m,$$

$$\sum_{\alpha=1}^N \int_M |\nabla u_i \cdot \nabla x_\alpha|^2 = \int_M |\nabla u_i|^2 = -\int_M u_i \Delta u_i$$

$$= \int_M u_i(\lambda_i u_i - V u_i) = \lambda_i - \int_M V u_i^2,$$

and by making summation of (2.8) over α from 1 to N and using (2.3), we get

$$m\sum_{i=1}^n (\lambda_{n+1} - \lambda_i)^2 \le m^2 \sum_{i=1}^n (\lambda_{n+1} - \lambda_i) + 4\sum_{i=1}^n (\lambda_{n+1} - \lambda_i)(\lambda_i - V_0).$$

Thus we have proved the following

THEOREM 2.1 *Let $x : M \to S^{N-1}(1)$ be an m-dimensional closed minimal submanifold in an $(N-1)$-dimensional Euclidean sphere. Let V be a nonnegative function on M. Then the spectrum of the Schrödinger operator $L = \Delta - V$ on M satisfies the inequality*

$$m\sum_{i=1}^n (\lambda_{n+1} - \lambda_i)^2 \le m^2 \sum_{i=1}^n (\lambda_{n+1} - \lambda_i) + 4\sum_{i=1}^n (\lambda_{n+1} - \lambda_i)(\lambda_i - V_0), \quad n = 1, 2, \ldots \tag{2.9}$$

Writing

$$\tilde\lambda_{n+1} = \lambda_{n+1} - V_0, \quad \tilde\lambda_i = \lambda_i - V_0, \quad i = 1, 2, \ldots, n,$$

we have

$$\tilde\lambda_{n+1} \ge \tilde\lambda_n \ge \cdots \ge \tilde\lambda_1 \ge 0.$$

Then (2.9) can be rewritten as

$$\sum_{i=1}^n (\tilde\lambda_{n+1} - \tilde\lambda_i)^2 \le m^2 \sum_{i=1}^n (\tilde\lambda_{n+1} - \tilde\lambda_i) + 4\sum_{i=1}^n (\tilde\lambda_{n+1} - \tilde\lambda_i)\tilde\lambda_i. \tag{2.9)$'$}$$

It is easy to see that (2.9)$'$ is equivalent to

$$\tilde\lambda_{n+1} \le \frac{1}{2}m + \frac{1}{n}(1 + 2/m)\sum_{i=1}^n \tilde\lambda_i$$
$$+ \frac{1}{2n}\Big\{\Big[mn + (2 + 4/m)\sum_{i=1}^n \tilde\lambda_i\Big]^2 - 4n(1 + 4/m)\sum_{i=1}^n \tilde\lambda_i^2 - 4mn\sum_{i=1}^n \tilde\lambda_i\Big\}^{\frac{1}{2}}. \tag{2.10}$$

Noting $(\sum_{i=1}^n \tilde\lambda_i)^2 \le n\sum_{i=1}^n \tilde\lambda_i^2$, we have from (2.10)

$$\tilde\lambda_{n+1} \le m + \frac{1}{n}(1 + 4/m)\sum_{i=1}^n \tilde\lambda_i. \tag{2.11}$$

Thus we proved the following weak, but simpler version of Theorem 2.1:

THEOREM 2.2. *Let $x : M \to S^{N-1}(1)$ be an m-dimensional closed minimal submanifold in an $(N-1)$-dimensional Euclidean sphere. Let V be a nonnegative function on M.*

Then the spectrum of the Schrödinger operator $L = \Delta - V$ on M satisfies the inequality

$$(2.11)' \qquad \lambda_{n+1} - V_0 \le m + \frac{1}{n}(1 + 4/m)\sum_{i=1}^{n}(\lambda_i - V_0), \quad n = 1, 2, \dots$$

Now we prove that Theorem 2.2 implies the following conclusion:

THEOREM 2.3. *Let $x : M \to S^{N-1}(1)$ be an m-dimensional closed minimal submanifold in an $(N-1)$-dimensional Euclidean sphere. Let V be a nonnegative function on M. Then the spectrum of the Schrödinger operator $L = \Delta - V$ on M satisfies the following inequality, for any $n \in N$ satisfying $\lambda_{n+1} > \lambda_n$:*

$$(2.12) \qquad \sum_{i=1}^{n} \frac{m^2 + 4(\lambda_i - V_0)}{\lambda_{n+1} - \lambda_i} \ge mn, \qquad n = 1, 2, \dots$$

and equivalently

$$(2.12)' \qquad \sum_{i=1}^{n} \frac{m^2 + 4\tilde{\lambda}_i}{\tilde{\lambda}_{n+1} - \tilde{\lambda}_i} \ge mn, \quad n = 1, 2, \dots$$

Proof. (2.11) or (2.11)′ can be rewritten as

$$\frac{1}{mn} \ge \frac{\tilde{\lambda}_{n+1} - \frac{1}{n}\sum_{i=1}^{n}\tilde{\lambda}_i}{m^2 n + 4\sum_{i=1}^{n}\tilde{\lambda}_i}$$

that is,

$$(2.13) \qquad mn \le \frac{m^2 n + 4\sum_{i=1}^{n}\tilde{\lambda}_i}{\tilde{\lambda}_{n+1} - \frac{1}{n}\sum_{i=1}^{n}\tilde{\lambda}_i}.$$

From (2.12)′ and (2.13), we only need to prove

$$(2.14) \qquad \frac{m^2 n + 4\sum_{i=1}^{n}\tilde{\lambda}_i}{\tilde{\lambda}_{n+1} - \frac{1}{n}\sum_{i=1}^{n}\tilde{\lambda}_i} \le \sum_{i=1}^{n} \frac{m^2 + 4\tilde{\lambda}_i}{\tilde{\lambda}_{n+1} - \tilde{\lambda}_i}.$$

Consider the function

$$f(x) = \frac{m^2 + 4x}{\tilde{\lambda}_{n+1} - x} = -4 + \frac{m^2 + 4\tilde{\lambda}_{n+1}}{\tilde{\lambda}_{n+1} - x}.$$

It is convex when $x < \tilde{\lambda}_{n+1}$. Thus

$$(2.15) \qquad f\left(\frac{\tilde{\lambda}_1 + \tilde{\lambda}_2 + \cdots + \tilde{\lambda}_n}{n}\right) \le \frac{1}{n}[f(\tilde{\lambda}_1) + f(\tilde{\lambda}_2) + \cdots + f(\tilde{\lambda}_n)].$$

It is easy to check that (2.15) is equivalent to (2.14). Therefore the proof of Theorem 2.3 is complete.

Theorem 2.3 has the following corollary:

THEOREM 2.4. *Let $x : M \to S^{N-1}(1)$ be an m-dimensional closed minimal submanifold in an $(N-1)$-dimensional Euclidean sphere. Let V be a nonnegative function on M. Then the spectrum of the Schrödinger operator $L = \Delta - V$ on M satisfies the inequality*

$$(2.16) \qquad \lambda_{n+1} - \lambda_n \le m + \frac{4}{mn}\sum_{i=1}^{n}(\lambda_i - V_0), \qquad n = 1, 2, \dots$$

and equivalently

$$\tilde{\lambda}_{n+1} - \tilde{\lambda}_n \le m + \frac{4}{mn}\sum_{i=1}^{n}\tilde{\lambda}_i, \qquad n = 1, 2, \ldots \tag{2.16$'$}$$

Proof. Replacing the λ_i by λ_n in the denominator of the left-hand side of (2.12), we obtain (2.16).

In the case $V \equiv 0$ the first eigenvalue is zero, and it is traditional to reindex so that λ_1 is the first nonzero eigenvalue. We indicate this by denoting the eigenvalues by λ'_j, that is,

$$\lambda'_0 = \lambda_1 = 0, \lambda'_1 = \lambda_2, \ldots, \lambda'_j = \lambda_{j+1}, \ldots, \lambda'_n = \lambda_{n+1}.$$

In this case, from Theorems 2.1 to 2.4, we have

COROLLARY 2.1. *Let $x : M \to S^{N-1}(1)$ be an m-dimensional closed minimal submanifold in an $(N-1)$-dimensional Euclidean sphere. Then the spectrum of the Laplacian Δ on M satisfies the inequality*

$$m\sum_{j=0}^{n-1}(\lambda'_n - \lambda'_j)^2 \le m^2\sum_{j=0}^{n-1}(\lambda'_n - \lambda'_j) + 4\sum_{j=0}^{n-1}(\lambda'_n - \lambda'_j)\lambda'_j, \qquad n = 2, 3, \ldots$$

COROLLARY 2.2. *Let $x : M \to S^{N-1}(1)$ be an m-dimensional closed minimal submanifold in an $(N-1)$-dimensional Euclidean sphere. Then the spectrum of the Laplacian Δ on M satisfies the inequality*

$$\lambda'_n \le m + \frac{1}{n}(1 + 4/m)\sum_{j=0}^{n-1}\lambda'_j, \qquad n = 2, 3, \ldots$$

COROLLARY 2.3. *Let $x : M \to S^{N-1}(1)$ be an m-dimensional closed minimal submanifold in an $(N-1)$-dimensional Euclidean sphere. Then the spectrum of the Laplacian Δ on M satisfies the following inequality, for any $n \in N$ satisfying $\lambda'_n > \lambda'_{n-1}$:*

$$\sum_{j=0}^{n-1}\frac{m^2 + 4\lambda'_j}{\lambda'_n - \lambda'_j} \ge mn, \qquad n = 2, 3, \ldots$$

COROLLARY 2.4. *Let $x : M \to S^{N-1}(1)$ be an m-dimensional closed minimal submanifold in an $(N-1)$-dimensional Euclidean sphere. Then the spectrum of the Laplacian Δ on M satisfies the inequality*

$$\lambda'_n - \lambda'_{n-1} \le m + \frac{4}{mn}\sum_{j=0}^{n-1}\lambda'_j, \qquad n = 2, 3, \ldots$$

Noting $\lambda'_0 = 0$, we immediately get the following Yang-Yau's result from Corollary 2.4.

COROLLARY 2.5 (Leung [4], Yang-Yau [9]). *Let $x : M \to S^{N-1}(1)$ be an m-dimensional closed minimal submanifold in an $(N-1)$-dimensional Euclidean sphere. Then the spectrum of the Laplacian Δ on M satisfies the inequality*

$$\lambda'_n - \lambda'_{n-1} \le m + \frac{2}{mn}\left(\sum_{j=1}^{n-1}\lambda'_j + \sqrt{\left(\sum_{j=1}^{n-1}\lambda'_j\right)^2 + m^2 n\sum_{j=1}^{n-1}\lambda'_j}\right).$$

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