THE GAPS IN THE SPECTRUM OF THE SCHRÖDINGER OPERATOR

HAIZHONG LI and LINLIN SU
Department of Mathematical Sciences, Tsinghua University
Beijing, 100084, People’s Republic of China
E-mail: hli@math.tsinghua.edu.cn

Abstract. We obtain inequalities between the eigenvalues of the Schrödinger operator on a compact domain \( \Omega \) of a submanifold \( M \) in \( \mathbb{R}^N \) with boundary \( \partial \Omega \), which generalize many existing inequalities for the Laplacian on a bounded domain of a Euclidean space. We also establish similar inequalities for a closed minimal submanifold in the unit sphere, which generalize and improve Yang-Yau’s result.

1. Introduction. Let \( x : M \rightarrow \mathbb{R}^N \) be an \( m \)-dimensional submanifold in \( N \)-dimensional Euclidean space with the mean curvature vector \( H \). Let \( \Omega \subset M \) be a compact domain on \( M \) with boundary \( \partial \Omega \). We consider the Schrödinger operator

\[
L = \Delta - V,
\]

where \( V \) is a nonnegative smooth function on \( \bar{\Omega} \). We shall consider the following Dirichlet eigenvalue problem of the Schrödinger operator \( L \) on \( \Omega \):

\[
\begin{cases}
Lu = -\lambda u & \text{on } \Omega, \\
u|_{\partial \Omega} = 0.
\end{cases}
\]

As is well-known, the eigenvalues \( \{\lambda_k\} \) of (1.2) are nonnegative, and can be arranged in nondecreasing order as follows:

\[
0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots
\]

Assume we know the first \( n \) eigenfunctions of \( L \)

\[
u_1, u_2, \ldots, u_n
\]

Mathematics Subject Classification: Primary 35P15; Secondary 58G25.
Key words and phrases: eigenvalues of the Schrödinger operator, inequality, bounded domain of submanifold, mean curvature.

This work is partially supported by the grant No. 10131020 of NSFC.
The paper is in final form and no version of it will be published elsewhere.
i.e., $Lu_i = -\lambda_i u_i$, having the properties

$$u_i = 0 \quad \text{on } \partial \Omega,$$

(1.3)

$$\int_{\Omega} u_i u_j = \begin{cases} 1, & \text{when } i = j, \\ 0, & \text{when } i \neq j. \end{cases}$$

(1.4)

To obtain information about the next eigenvalue, we shall need some appropriate "trial functions". In fact, following [5], we can use the following trial functions

$$\phi_1^\alpha, \phi_2^\alpha, \ldots, \phi_n^\alpha.$$

Any one of these trial functions, say $\phi_i^\alpha$, must satisfy two conditions:

$$\phi_i^\alpha = 0 \quad \text{on } \partial \Omega,$$

(1.5)

$$\int_{\Omega} \phi_i^\alpha u_j = 0$$

(1.6)

where $i, j = 1, 2, \ldots, n$. We choose (following [5])

$$\phi_i^\alpha = x_\alpha u_i - \sum_{j=1}^{n} a_{ij}^\alpha u_j, \quad \alpha = 1, \ldots, N,$$

(1.7)

where

$$a_{ij}^\alpha = \int_{\Omega} x_\alpha u_i u_j = a_{ji}^\alpha, \quad x(p) = (x_1(p), \ldots, x_N(p)) \in R^N, \quad \forall p \in \Omega.$$

(1.8)

It is easy to see that the functions $\phi_i^\alpha$ satisfy (1.5) and (1.6), i.e., $\phi_i^\alpha$ are all orthogonal to $u_1, \ldots, u_n$ and vanish at the boundary. We also have, by use of (1.7) and (1.6),

$$\int_{\Omega} (\phi_i^\alpha)^2 = \int_{\Omega} \phi_i^\alpha x_\alpha u_i = \int_{\Omega} x_\alpha^2 u_i^2 - \sum_{j=1}^{n} (a_{ij}^\alpha)^2,$$

(1.9)

$$-\Delta \phi_i^\alpha = -u_i \Delta x_\alpha - 2 \nabla u_i \cdot \nabla x_\alpha + \lambda_i x_\alpha u_i - \sum_{j=1}^{n} a_{ij}^\alpha \lambda_j u_j - V \phi_i^\alpha,$$

(1.10)

$$-\int_{\Omega} \phi_i^\alpha L(\phi_i^\alpha) = -\int_{\Omega} \phi_i^\alpha (-\Delta \phi_i^\alpha) + \int_{\Omega} V(\phi_i^\alpha)^2$$

$$= -\int_{\Omega} \phi_i^\alpha \Delta x_\alpha \cdot u_i - 2 \int_{\Omega} \phi_i^\alpha \nabla u_i \cdot \nabla x_\alpha + \lambda_i \int_{\Omega} (\phi_i^\alpha)^2. \quad \text{(1.11)}$$

Hence by the variational principle for $\lambda_{n+1}$, we have

$$\lambda_{n+1} \int_{\Omega} (\phi_i^\alpha)^2 \leq -\int_{\Omega} \phi_i^\alpha L(\phi_i^\alpha) = -\int_{\Omega} \phi_i^\alpha \Delta x_\alpha \cdot u_i - 2 \int_{\Omega} \phi_i^\alpha \nabla u_i \cdot \nabla x_\alpha + \lambda_i \int_{\Omega} (\phi_i^\alpha)^2,$$

that is,

$$0 \leq (\lambda_{n+1} - \lambda_i) \int_{\Omega} (\phi_i^\alpha)^2 \leq -\int_{\Omega} \phi_i^\alpha \Delta x_\alpha \cdot u_i - 2 \int_{\Omega} \phi_i^\alpha \nabla u_i \cdot \nabla x_\alpha.$$

(1.12)

From (1.12), by use of (1.6) we get

$$(\lambda_{n+1} - \lambda_i) \int_{\Omega} (\phi_i^\alpha)^2 \cdot \left[ -\int_{\Omega} \phi_i^\alpha \Delta x_\alpha \cdot u_i - 2 \int_{\Omega} \phi_i^\alpha \nabla u_i \cdot \nabla x_\alpha \right]$$

(1.13)
\[ \leq \left[ -\int_{\Omega} \phi_i^\alpha \Delta x_\alpha \cdot u_i - 2 \int_{\Omega} \phi_i^\alpha \nabla u_i \cdot \nabla x_\alpha \right]^2 = \left[ \int_{\Omega} \phi_i^\alpha (\Delta x_\alpha \cdot u_i + 2\nabla u_i \cdot \nabla x_\alpha) \right]^2 \]

\[ = \left[ \int_{\Omega} \phi_i^\alpha (\Delta x_\alpha \cdot u_i + 2\nabla u_i \cdot \nabla x_\alpha - 2 \sum_{j=1}^{n} b_{ij}^\alpha u_j) \right]^2. \]

Define
\[ (1.14) \quad I_{\alpha i} = \int_{\Omega} |\nabla x_\alpha|^2 u_i^2, \]
\[ (1.15) \quad b_{ij}^\alpha = \int_{\Omega} u_j \nabla x_\alpha \cdot \nabla u_i + \frac{1}{2} \int_{\Omega} \Delta x_\alpha \cdot u_i u_j, \]

By use of the Stokes formula and (1.3), it is easy to check
\[ b_{ij}^\alpha = -b_{ji}^\alpha, \quad \alpha = 1, 2, \ldots, N; \quad i, j = 1, 2, \ldots, n. \]

We have
\[ (1.16) \quad -2 \int_{\Omega} \phi_i^\alpha \nabla x_\alpha \cdot \nabla u_i = -\int_{\Omega} x_\alpha \nabla x_\alpha \cdot \nabla (u_i^2) + 2 \sum_{j=1}^{n} a_{ij}^\alpha \int_{\Omega} u_j \nabla x_\alpha \cdot \nabla u_i \]
\[ = I_{\alpha i} + \int_{\Omega} u_i^2 \Delta x_\alpha + 2 \sum_{j=1}^{n} a_{ij}^\alpha \int_{\Omega} u_j \nabla x_\alpha \cdot \nabla u_i, \]
\[ (1.17) \quad -\int_{\Omega} u_i \phi_i^\alpha \Delta x_\alpha - 2 \int_{\Omega} \phi_i^\alpha \nabla u_i \cdot \nabla x_\alpha = I_{\alpha i} + 2 \sum_{j=1}^{n} a_{ij}^\alpha b_{ij}^\alpha. \]

From the definition of \( a_{ij}^\alpha \),
\[ \lambda_i a_{ij}^\alpha = \int_{\Omega} (-L u_i) x_\alpha u_j = \int_{\Omega} (-\Delta u_i) x_\alpha u_j + \int_{\Omega} V u_i u_j x_\alpha \]
\[ = \int_{\Omega} u_i \Delta (-x_\alpha u_j) + \int_{\Omega} V u_i u_j x_\alpha \]
\[ = -\int_{\Omega} u_i u_j \Delta x_\alpha - 2 \int_{\Omega} u_i \nabla x_\alpha \cdot \nabla u_j + \lambda_j \int_{\Omega} u_i u_j x_\alpha \]
\[ = -2b_{ji}^\alpha + \lambda_j a_{ij}^\alpha = 2b_{ij}^\alpha + \lambda_j a_{ij}^\alpha, \]

that is,
\[ (1.18) \quad 2b_{ij}^\alpha = (\lambda_i - \lambda_j) a_{ij}^\alpha. \]

Putting (1.17) and (1.18) into (1.13), we obtain for any real number \( \epsilon > 0 \)
\[ (1.19) \quad (\lambda_{n+1} - \lambda_i) \int_{\Omega} (\phi_i^\alpha)^2 \cdot [I_{\alpha i} + \sum_{j=1}^{n} (\lambda_i - \lambda_j)(a_{ij}^\alpha)^2] \]
\[ \leq \left[ \int_{\Omega} \phi_i^\alpha (\Delta x_\alpha \cdot u_i + 2\nabla u_i \cdot \nabla x_\alpha - 2 \sum_{j=1}^{n} b_{ij}^\alpha u_j) \right]^2 \]
\[ \leq \int_{\Omega} (\phi_i^\alpha)^2 \cdot \int_{\Omega} \left[ \Delta x_\alpha \cdot u_i + 2\nabla u_i \cdot \nabla x_\alpha - 2 \sum_{j=1}^{n} b_{ij}^\alpha u_j \right]^2. \]
\[
\begin{align*}
= \int_\Omega (\phi_i^\alpha)^2 \cdot \left[ \int_\Omega (\Delta x \alpha u_i + 2\nabla u_i \cdot \nabla x \alpha)^2 - 4 \sum_{j=1}^{n} (b_{ij}^\alpha)^2 \right] \\
\leq \int_\Omega (\phi_i^\alpha)^2 \cdot \left[ (1 + 1/\epsilon) \int_\Omega (\Delta x \alpha)^2 u_i^2 + 4(1 + \epsilon) \int_\Omega |\nabla u_i \cdot \nabla x \alpha|^2 - 4 \sum_{j=1}^{n} (b_{ij}^\alpha)^2 \right],
\end{align*}
\]

where we used the Schwarz inequality in the last step.

Dividing by \( \int_\Omega (\phi_i^\alpha)^2 \), we have for any real number \( \epsilon > 0 \)
\[(1.20) \quad (\lambda_{n+1} - \lambda_i) I_{\alpha i} + \sum_{j=1}^{n} (\lambda_i - \lambda_j)(\lambda_{n+1} - \lambda_j)(a_{ij}^\alpha)^2 \]
\[
\leq (1 + 1/\epsilon) \int_\Omega (\Delta x \alpha)^2 u_i^2 + 4(1 + \epsilon) \int_\Omega |\nabla u_i \cdot \nabla x \alpha|^2.
\]
Inequality (1.20) holds even in the case that \( \phi_i^\alpha \equiv 0 \), since in that case the left-hand side vanishes from (1.17) and (1.18), while the right-hand side is nonnegative (it is not less than the integral of the norm square of \( \Delta x \alpha u_i + 2\nabla u_i \cdot \nabla x \alpha - 2 \sum_{j=1}^{n} b_{ij}^\alpha u_j \)).

Putting (1.18) into (1.20), we have
\[(1.21) \quad (\lambda_{n+1} - \lambda_i) I_{\alpha i} + \sum_{j=1}^{n} (\lambda_i - \lambda_j)(\lambda_{n+1} - \lambda_j)(a_{ij}^\alpha)^2 \]
\[
hf ill \leq (1 + 1/\epsilon) \int_\Omega (\Delta x \alpha)^2 u_i^2 + 4(1 + \epsilon) \int_\Omega |\nabla u_i \cdot \nabla x \alpha|^2.
\]

By making summation of the terms of \( (\lambda_{n+1} - \lambda_i) \times (1.21) \), we get
\[(1.22) \quad \sum_{i=1}^{n} (\lambda_{n+1} - \lambda_i)^2 I_{\alpha i} \]
\[
hf ill \leq (1 + 1/\epsilon) \sum_{i=1}^{n} (\lambda_{n+1} - \lambda_i) \int_\Omega (\Delta x \alpha)^2 u_i^2 + 4(1 + \epsilon) \sum_{i=1}^{n} (\lambda_{n+1} - \lambda_i) \int_\Omega |\nabla u_i \cdot \nabla x \alpha|^2.
\]

By use of the identities
\[
\Delta x = -mH, \quad \sum_{\alpha=1}^{N} I_{\alpha i} = \sum_{\alpha=1}^{N} \int_\Omega |\nabla x \alpha|^2 u_i^2 = m \int_\Omega u_i^2 = m,
\]
\[
\sum_{\alpha=1}^{N} \int_\Omega |\nabla u_i \cdot \nabla x \alpha|^2 = \int_\Omega |\nabla u_i|^2 = - \int_\Omega u_i \Delta u_i = \int_\Omega u_i (\lambda_i u_i - V u_i) = \lambda_i - \int_\Omega V u_i^2,
\]
and by making summation of (1.22) over \( \alpha \) from 1 to \( N \), we obtain
\[(1.23) \quad m \sum_{i=1}^{n} (\lambda_{n+1} - \lambda_i)^2 \]
\[
\leq (1 + 1/\epsilon) m^2 \sum_{i=1}^{n} (\lambda_{n+1} - \lambda_i) \int_\Omega |H|^2 u_i^2 + 4(1 + \epsilon) \sum_{i=1}^{n} (\lambda_{n+1} - \lambda_i) (\lambda_i - \int_\Omega V u_i^2).
\]

Writing
\[(1.24) \quad |H|_\infty^2 = \sup_\Omega |H|^2, \quad V_0 = \inf_\Omega V,
\]
we have from (1.23) and (1.24) for any real number $\epsilon > 0$

\[
(1.24) \quad \sum_{i=1}^{n} (\lambda_{n+1} - \lambda_i)^2 \leq (1 + 1/\epsilon)m^2|H|_\infty^2 \sum_{i=1}^{n} (\lambda_{n+1} - \lambda_i) + 4(1 + \epsilon) \sum_{i=1}^{n} (\lambda_{n+1} - \lambda_i)(\lambda_i - V_0).
\]

Thus we have proved the following main result of this section:

**Theorem 1.1.** Let $x : M \to \mathbb{R}^N$ be an $m$-dimensional submanifold in $N$-dimensional Euclidean space. Let $\Omega \subset M$ be a bounded domain. Then the spectrum of the Schrödinger operator $L = \Delta - V$ satisfies the following inequality, for any real number $\epsilon > 0$, and for any $n = 1, 2, \ldots$

\[
(1.26) \quad m \sum_{i=1}^{n} (\lambda_{n+1} - \lambda_i)^2 \leq (1 + 1/\epsilon)m^2|H|_\infty^2 \sum_{i=1}^{n} (\lambda_{n+1} - \lambda_i) + 4(1 + \epsilon) \sum_{i=1}^{n} (\lambda_{n+1} - \lambda_i)(\lambda_i - V_0).
\]

Writing

\[
\tilde{\lambda}_{n+1} = \lambda_{n+1} - V_0, \quad \tilde{\lambda}_i = \lambda_i - V_0, \quad i = 1, 2, \ldots, n,
\]

we have

\[
\tilde{\lambda}_n \geq \tilde{\lambda}_{n-1} \geq \cdots \geq \tilde{\lambda}_2 \geq \tilde{\lambda}_1 = \lambda_1 - V_0 = \frac{\int |\nabla u_1|^2}{\int u_1^2} + \frac{\int V u_1^2}{\int u_1^2} - V_0 \geq \frac{\int |\nabla u_1|^2}{\int u_1^2} > 0.
\]

Then (1.26) can be rewritten as

\[
(1.26)' \quad m \sum_{i=1}^{n} (\tilde{\lambda}_{n+1} - \tilde{\lambda}_i)^2 \leq (1 + 1/\epsilon)m^2|H|_\infty^2 \sum_{i=1}^{n} (\tilde{\lambda}_{n+1} - \tilde{\lambda}_i) + 4(1 + \epsilon) \sum_{i=1}^{n} (\tilde{\lambda}_{n+1} - \tilde{\lambda}_i)\tilde{\lambda}_i.
\]

It is easy to see that (1.26)' is equivalent to

\[
(1.27) \quad \tilde{\lambda}_{n+1} \leq \frac{1}{2}m(1 + 1/\epsilon)|H|_\infty^2 + \frac{1}{n}(1 + \frac{2}{m}(1 + \epsilon)) \sum_{i=1}^{n} \tilde{\lambda}_i + \frac{1}{2n} \left\{ \left[ (1 + 1/\epsilon)mn|H|_\infty^2 + \left( 2 + \frac{4}{m}(1 + \epsilon) \right) \sum_{i=1}^{n} \tilde{\lambda}_i \right] \right\} \frac{1}{2} - 4n \left( 1 + \frac{4}{m}(1 + \epsilon) \right) \sum_{i=1}^{n} \tilde{\lambda}_i^2 - 4mn(1 + 1/\epsilon)|H|_\infty^2 \sum_{i=1}^{n} \lambda_i.
\]

Noting $(\sum_{i=1}^{n} \tilde{\lambda}_i)^2 \leq n \sum_{i=1}^{n} \tilde{\lambda}_i^2$, we have from (1.27)

\[
(1.28) \quad \tilde{\lambda}_{n+1} \leq m(1 + 1/\epsilon)|H|_\infty^2 + \frac{1}{n}(1 + \frac{4}{m}(1 + \epsilon)) \sum_{i=1}^{n} \tilde{\lambda}_i.
\]

Thus we have proved the following weak, but simpler version of Theorem 1.1:

**Theorem 1.2.** Let $x : M \to \mathbb{R}^N$ be an $m$-dimensional submanifold in $N$-dimensional Euclidean space. Let $\Omega \subset M$ be a bounded domain. Then the spectrum of the Schrödinger
operator $L = \Delta - V$ satisfies the following inequality, for any real number $\epsilon > 0$:

$$(1.28)' \quad \lambda_{n+1} - V_0 \leq m(1 + 1/\epsilon)|H|_{\infty}^2 + \frac{1}{n}(1 + \frac{4}{m}(1 + \epsilon)) \sum_{i=1}^{n} (\lambda_i - V_0), \quad n = 1, 2, \ldots$$

Now we prove that Theorem 1.2 implies the following conclusion:

**Theorem 1.3.** Let $x : M \to \mathbb{R}^N$ be an $m$-dimensional submanifold in $N$-dimensional Euclidean space. Let $\Omega \subset M$ be a bounded domain. Then the spectrum of the Schrödinger operator $L = \Delta - V$ satisfies the following inequality, for any real number $\epsilon > 0$, and for any $n \in \mathbb{N}$ satisfying $\lambda_{n+1} > \lambda_n$:

$$(1.29) \quad \frac{4(1 + \epsilon)(\lambda_i - V_0) + (1 + 1/\epsilon)m^2|H|_{\infty}^2}{\lambda_{n+1} - \lambda_i} \geq mn, \quad n = 1, 2, \ldots$$

and equivalently

$$(1.29)' \quad \sum_{i=1}^{n} \frac{4(1 + \epsilon)\tilde{\lambda}_i + (1 + 1/\epsilon)m^2|H|_{\infty}^2}{\tilde{\lambda}_{n+1} - \tilde{\lambda}_i} \geq mn, \quad n = 1, 2, \ldots$$

**Proof.** (1.28) or (1.28)' can be rewritten as

$$(1.30) \quad \frac{1}{mn} \left[ 4(1 + \epsilon) \sum_{i=1}^{n} \tilde{\lambda}_i + m^2n(1 + 1/\epsilon)|H|_{\infty}^2 \right] \geq \tilde{\lambda}_{n+1} - \frac{1}{n} \sum_{i=1}^{n} \tilde{\lambda}_i,$$

that is,

$$(1.30)' \quad mn \leq \frac{4(1 + \epsilon) \sum_{i=1}^{n} \tilde{\lambda}_i + m^2n(1 + 1/\epsilon)|H|_{\infty}^2}{\tilde{\lambda}_{n+1} - \frac{1}{n} \sum_{i=1}^{n} \tilde{\lambda}_i}.$$  

From (1.29) and (1.30)', we only need to prove

$$(1.31) \quad \frac{4(1 + \epsilon) \sum_{i=1}^{n} \tilde{\lambda}_i + m^2n(1 + 1/\epsilon)|H|_{\infty}^2}{\tilde{\lambda}_{n+1} - \frac{1}{n} \sum_{i=1}^{n} \tilde{\lambda}_i} \leq \sum_{i=1}^{n} \frac{4(1 + \epsilon)\tilde{\lambda}_i + (1 + 1/\epsilon)m^2|H|_{\infty}^2}{\tilde{\lambda}_{n+1} - \tilde{\lambda}_i}.$$  

We consider the function

$$f(x) = \frac{4(1 + \epsilon)x + (1 + 1/\epsilon)m^2|H|_{\infty}^2}{\tilde{\lambda}_{n+1} - x} = -4(1 + \epsilon) + \frac{(1 + 1/\epsilon)m^2|H|_{\infty}^2 + 4(1 + \epsilon)\tilde{\lambda}_{n+1}}{\tilde{\lambda}_{n+1} - x}.$$  

It is convex when $x < \tilde{\lambda}_{n+1}$. Thus

$$(1.32) \quad f\left( \frac{\tilde{\lambda}_1 + \tilde{\lambda}_2 + \cdots + \tilde{\lambda}_n}{n} \right) \leq \frac{1}{n} [f(\tilde{\lambda}_1) + f(\tilde{\lambda}_2) + \cdots + f(\tilde{\lambda}_n)].$$  

It is easy to check that (1.32) is equivalent to (1.31). Thus we have completed the proof of Theorem 1.3.

Theorem 1.3 has the following corollary:

**Theorem 1.4.** Let $x : M \to \mathbb{R}^N$ be an $m$-dimensional submanifold in $N$-dimensional Euclidean space. Let $\Omega \subset M$ be a bounded domain. Then the spectrum of the Schrödinger
operator $L = \Delta - V$ satisfies the following inequality, for any real number $\epsilon > 0$:

\[
\lambda_{n+1} - \lambda_n \leq \frac{4(1 + \epsilon)}{mn} \sum_{i=1}^{n} (\lambda_i - V_0) + (1 + 1/\epsilon)m|H|_{\infty}^2, \quad n = 1, 2, \ldots
\]

and equivalently

\[
\tilde{\lambda}_{n+1} - \tilde{\lambda}_n \leq \frac{4(1 + \epsilon)}{mn} \sum_{i=1}^{n} \tilde{\lambda}_i + (1 + 1/\epsilon)m|H|_{\infty}^2, \quad n = 1, 2, \ldots
\]

Proof. Replacing the $\lambda_i$ in the denominator of the left-hand side of (1.29) by $\lambda_n$, we obtain (1.33).

Remark 1.1. If $x : M \to R^N$ is an $m$-dimensional minimal submanifold and $V \equiv 0$, letting $\epsilon \to 0$ in Theorem 1.4, we recover S. Y. Cheng’s result [2].

When $M = R^m$ and $V \equiv 0$, in this case $x : M \to R^N$ is an $m$-dimensional totally geodesic submanifold, we have $H \equiv 0$. We have, from Theorems 1.1 to 1.4 by letting $\epsilon \to 0$:

Corollary 1.1 (Yang [8]). Let $\Omega \subset R^m$ be a bounded domain. Then the spectrum of the Laplacian $\Delta$ satisfies the inequality

\[
m \sum_{i=1}^{n} (\lambda_{n+1} - \lambda_i)^2 \leq 4 \sum_{i=1}^{n} (\lambda_{n+1} - \lambda_i) \lambda_i, \quad n = 1, 2, \ldots
\]

Corollary 1.2 (Ashbaugh [1], Yang [8]). Let $\Omega \subset R^m$ be a bounded domain. Then the spectrum of the Laplacian $\Delta$ satisfies the inequality

\[
\lambda_{n+1} \leq \frac{1}{n} \left(1 + \frac{4}{m}\right) \sum_{i=1}^{n} \lambda_i, \quad n = 1, 2, \ldots
\]

Corollary 1.3 (Hile-Protter [3]). Let $\Omega \subset R^m$ be a bounded domain. Then the spectrum of the Laplacian $\Delta$ satisfies the inequality

\[
4 \sum_{i=1}^{n} \frac{\lambda_i}{\lambda_{n+1} - \lambda_i} \geq mn, \quad n = 1, 2, \ldots
\]

Corollary 1.4 (Payne–Pólya–Weinberger [5], Thompson [7]). Let $\Omega \subset R^m$ be a bounded domain. Then the spectrum of the Laplacian $\Delta$ satisfies the inequality

\[
\lambda_{n+1} - \lambda_n \leq \frac{4}{mn} \sum_{i=1}^{n} \lambda_i, \quad n = 1, 2, \ldots
\]

2. Minimal submanifold in the unit sphere. In this section, let $x : M \to S^{N-1}(1)$ be an $m$-dimensional closed (i.e., compact without boundary) minimal submanifold in the Euclidean unit sphere. As in the previous section, we consider the Schrödinger operator

\[
L = \Delta - V,
\]

where $V$ is a nonnegative smooth function on $M$. We shall consider the following Dirichlet eigenvalue problem for the Schrödinger operator $L$ on $M$:

\[
Lu = -\lambda u \quad \text{on} \quad M.
\]
As is well-known, the eigenvalues \( \{ \lambda_k \} \) of (2.2) are nonnegative, and can be arranged in nondecreasing order as follows:

\[
0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots
\]

Let \( x = (x_1, \ldots, x_N) : M \rightarrow S^{N-1}(1) \subset R^N \) be the minimal immersion. Let \( \{(u_i, \lambda_i)\}_{i=1, \ldots, n} \) be the normalized first \( n \) eigenfunctions and their corresponding eigenvalues. Then

\[
\Delta x_\alpha = -mx_\alpha, \quad \sum_{\alpha=1}^{N} x_\alpha^2 = 1,
\]

(2.3)

\[
Lu_i = -\lambda_i u_i, \quad i = 1, \ldots, n.
\]

(2.4)

Noting that all formulas from preceding section still hold, we have from (1.13) and (1.17) by use of (2.3)

\[
(\lambda_{n+1} - \lambda_i) \int_M (\phi_i^{\alpha})^2 \cdot \left[ I_{\alpha i} + \sum_{j=1}^{n} (\lambda_i - \lambda_j)(a_{ij}^{\alpha})^2 \right]
\]

\[
\leq \int_M (\phi_i^{\alpha})^2 \cdot \left[ \int_M (\Delta x_\alpha u_i + 2\nabla u_i \cdot \nabla x_\alpha)^2 - 4 \sum_{j=1}^{n} (b_{ij}^{\alpha})^2 \right]
\]

\[
= \int_M (\phi_i^{\alpha})^2 \cdot \left[ \int_M (\Delta x_\alpha)^2 u_i^2 + 4 \int_M |\nabla u_i \cdot \nabla x_\alpha|^2 - 2m \int_M u_i \nabla u_i \cdot \nabla x_\alpha^2 - 4 \sum_{j=1}^{n} (b_{ij}^{\alpha})^2 \right]
\]

\[
= \int_M (\phi_i^{\alpha})^2 \cdot \left[ \int_M m^2 x_\alpha^2 u_i^2 + 4 \int_M |\nabla u_i \cdot \nabla x_\alpha|^2 - 2m \int_M u_i \nabla u_i \cdot \nabla x_\alpha^2 - 4 \sum_{j=1}^{n} (b_{ij}^{\alpha})^2 \right],
\]

thus we get

\[
(\lambda_{n+1} - \lambda_i) \left[ I_{\alpha i} + \sum_{j=1}^{n} (\lambda_i - \lambda_j)(a_{ij}^{\alpha})^2 \right]
\]

\[
\leq \int_M m^2 x_\alpha^2 u_i^2 + 4 \int_M |\nabla u_i \cdot \nabla x_\alpha|^2 - 2m \int_M u_i \nabla u_i \cdot \nabla x_\alpha^2 - 4 \sum_{j=1}^{n} (b_{ij}^{\alpha})^2.
\]

(2.6)

Putting (1.18) into (2.6), we have

\[
(\lambda_{n+1} - \lambda_i) I_{\alpha i} + \sum_{j=1}^{n} (\lambda_i - \lambda_j)(\lambda_{n+1} - \lambda_j)(a_{ij}^{\alpha})^2
\]

\[
\leq \int_M m^2 x_\alpha^2 u_i^2 + 4 \int_M |\nabla u_i \cdot \nabla x_\alpha|^2 - 2m \int_M u_i \nabla u_i \cdot \nabla x_\alpha^2.
\]

(2.7)

By making summation of the terms of \( (\lambda_{n+1} - \lambda_i) \times (2.7) \), we get

\[
\sum_{i=1}^{n} (\lambda_{n+1} - \lambda_i)^2 I_{\alpha i}
\]

\[
\leq \sum_{i=1}^{n} (\lambda_{n+1} - \lambda_i) \int_M m^2 x_\alpha^2 u_i^2 + 4 \sum_{i=1}^{n} (\lambda_{n+1} - \lambda_i) \int_M |\nabla u_i \cdot \nabla x_\alpha|^2
\]

\[
- 2m \sum_{i=1}^{n} (\lambda_{n+1} - \lambda_i) \int_M u_i \nabla u_i \cdot \nabla x_\alpha^2.
\]

(2.8)
By use of the identities
\[ V_0 = \inf_{\Omega} V, \quad \sum_{\alpha=1}^{N} I_{\alpha} = \sum_{\alpha=1}^{N} \int_M |\nabla x_\alpha|^2 u_i^2 = m \int_M u_i^2 = m, \]
\[ \sum_{\alpha=1}^{N} \int_M |\nabla u_i \cdot \nabla x_\alpha|^2 = \int_M |\nabla u_i|^2 = -\int_M u_i \Delta u_i \]
\[ = \int_M u_i (\lambda_i u_i - V u_i) = \lambda_i - \int_M V u_i^2, \]
and by making summation of (2.8) over \( \alpha \) from 1 to \( N \) and using (2.3), we get
\[ m \sum_{i=1}^{n} (\lambda_{n+1} - \lambda_i)^2 \leq m^2 \sum_{i=1}^{n} (\lambda_{n+1} - \lambda_i) + 4 \sum_{i=1}^{n} (\lambda_{n+1} - \lambda_i) (\lambda_i - V_0). \]

Thus we have proved the following

**Theorem 2.1** Let \( x : M \to S^{N-1}(1) \) be an \( m \)-dimensional closed minimal submanifold in an \((N - 1)\)-dimensional Euclidean sphere. Let \( V \) be a nonnegative function on \( M \). Then the spectrum of the Schrödinger operator \( L = \Delta - V \) on \( M \) satisfies the inequality
\[ m \sum_{i=1}^{n} (\lambda_{n+1} - \lambda_i)^2 \leq m^2 \sum_{i=1}^{n} (\lambda_{n+1} - \lambda_i) + 4 \sum_{i=1}^{n} (\lambda_{n+1} - \lambda_i) (\lambda_i - V_0), \quad n = 1, 2, \ldots \]

Writing
\[ \tilde{\lambda}_{n+1} = \lambda_{n+1} - V_0, \quad \tilde{\lambda}_i = \lambda_i - V_0, \quad i = 1, 2, \ldots, n, \]
we have
\[ \tilde{\lambda}_{n+1} \geq \tilde{\lambda}_n \geq \cdots \geq \tilde{\lambda}_1 \geq 0. \]

Then (2.9) can be rewritten as
\[ \sum_{i=1}^{n} (\tilde{\lambda}_{n+1} - \tilde{\lambda}_i)^2 \leq m^2 \sum_{i=1}^{n} (\tilde{\lambda}_{n+1} - \tilde{\lambda}_i) + 4 \sum_{i=1}^{n} (\tilde{\lambda}_{n+1} - \tilde{\lambda}_i) \tilde{\lambda}_i. \]

It is easy to see that (2.9)' is equivalent to
\[ \tilde{\lambda}_{n+1} \leq \frac{1}{2} m + \frac{1}{n} (1 + 2/m) \sum_{i=1}^{n} \tilde{\lambda}_i \]
\[ + \frac{1}{2n} \left\{ \left[ mn + (2 + 4/m) \sum_{i=1}^{n} \tilde{\lambda}_i \right]^2 - 4n(1 + 4/m) \sum_{i=1}^{n} \tilde{\lambda}_i^2 - 4mn \sum_{i=1}^{n} \tilde{\lambda}_i \right\}^{\frac{1}{2}}. \]

Noting \( (\sum_{i=1}^{n} \tilde{\lambda}_i)^2 \leq n \sum_{i=1}^{n} \tilde{\lambda}_i^2 \), we have from (2.10)
\[ \tilde{\lambda}_{n+1} \leq m + \frac{1}{n} (1 + 4/m) \sum_{i=1}^{n} \tilde{\lambda}_i. \]

Thus we proved the following weak, but simpler version of Theorem 2.1:

**Theorem 2.2.** Let \( x : M \to S^{N-1}(1) \) be an \( m \)-dimensional closed minimal submanifold in an \((N - 1)\)-dimensional Euclidean sphere. Let \( V \) be a nonnegative function on \( M \).
Then the spectrum of the Schrödinger operator \( L = \Delta - V \) on \( M \) satisfies the inequality
\[
\lambda_{n+1} - V_0 \leq m + \frac{1}{n} (1 + 4/m) \sum_{i=1}^{n} (\lambda_i - V_0), \quad n = 1, 2, \ldots
\]

Now we prove that Theorem 2.2 implies the following conclusion:

**Theorem 2.3.** Let \( x : M \to S^{N-1}(1) \) be an \( m \)-dimensional closed minimal submanifold in an \((N - 1)\)-dimensional Euclidean sphere. Let \( V \) be a nonnegative function on \( M \). Then the spectrum of the Schrödinger operator \( L = \Delta - V \) on \( M \) satisfies the following inequality, for any \( n \in \mathbb{N} \) satisfying \( \lambda_{n+1} > \lambda_n \):
\[
\sum_{i=1}^{n} \frac{m^2 + 4(\lambda_i - V_0)}{\lambda_{n+1} - \lambda_i} \geq mn, \quad n = 1, 2, \ldots
\]
and equivalently
\[
\sum_{i=1}^{n} \frac{\tilde{m}^2 + 4\tilde{\lambda}_i}{\tilde{\lambda}_{n+1} - \tilde{\lambda}_i} \geq mn, \quad n = 1, 2, \ldots
\]

**Proof.** (2.11) or (2.11)' can be rewritten as
\[
\frac{1}{mn} \geq \frac{\tilde{\lambda}_{n+1} - \frac{1}{n} \sum_{i=1}^{n} \tilde{\lambda}_i}{m^2 n + 4 \sum_{i=1}^{n} \tilde{\lambda}_i}
\]
that is,
\[
mn \leq \frac{m^2 n + 4 \sum_{i=1}^{n} \tilde{\lambda}_i}{\tilde{\lambda}_{n+1} - \frac{1}{n} \sum_{i=1}^{n} \tilde{\lambda}_i}.
\]
From (2.12)' and (2.13), we only need to prove
\[
\frac{m^2 n + 4 \sum_{i=1}^{n} \tilde{\lambda}_i}{\tilde{\lambda}_{n+1} - \frac{1}{n} \sum_{i=1}^{n} \tilde{\lambda}_i} \leq \sum_{i=1}^{n} \frac{m^2 + 4\tilde{\lambda}_i}{\tilde{\lambda}_{n+1} - \tilde{\lambda}_i}.
\]
Consider the function
\[
f(x) = \frac{m^2 + 4x}{\tilde{\lambda}_{n+1} - x} = -4 + \frac{m^2 + 4\tilde{\lambda}_{n+1}}{\tilde{\lambda}_{n+1} - x}.
\]
It is convex when \( x < \tilde{\lambda}_{n+1} \). Thus
\[
\left( \frac{\tilde{\lambda}_1 + \tilde{\lambda}_2 + \cdots + \tilde{\lambda}_n}{n} \right) \leq \frac{1}{n} [f(\tilde{\lambda}_1) + f(\tilde{\lambda}_2) + \cdots + f(\tilde{\lambda}_n)].
\]
It is easy to check that (2.15) is equivalent to (2.14). Therefore the proof of Theorem 2.3 is complete.

Theorem 2.3 has the following corollary:

**Theorem 2.4.** Let \( x : M \to S^{N-1}(1) \) be an \( m \)-dimensional closed minimal submanifold in an \((N - 1)\)-dimensional Euclidean sphere. Let \( V \) be a nonnegative function on \( M \). Then the spectrum of the Schrödinger operator \( L = \Delta - V \) on \( M \) satisfies the inequality
\[
\lambda_{n+1} - \lambda_n \leq m + \frac{4}{mn} \sum_{i=1}^{n} (\lambda_i - V_0), \quad n = 1, 2, \ldots
\]
and equivalently
\[(2.16)' \quad \tilde{\lambda}_{n+1} - \tilde{\lambda}_n \leq m + \frac{4}{mn} \sum_{i=1}^{n} \tilde{\lambda}_i, \quad n = 1, 2, \ldots \]

Proof. Replacing the \(\lambda_i\) by \(\lambda_n\) in the denominator of the left-hand side of (2.12), we obtain (2.16).

In the case \(V \equiv 0\) the first eigenvalue is zero, and it is traditional to reindex so that \(\lambda_1\) is the first nonzero eigenvalue. We indicate this by denoting the eigenvalues by \(\lambda'_j\), that is,
\[
\lambda'_0 = \lambda_1 = 0, \lambda'_1 = \lambda_2, \ldots, \lambda'_j = \lambda_{j+1}, \ldots, \lambda'_n = \lambda_{n+1}.
\]

In this case, from Theorems 2.1 to 2.4, we have

Corollary 2.1. Let \(x : M \to S^{N-1}(1)\) be an \(m\)-dimensional closed minimal submanifold in an \((N-1)\)-dimensional Euclidean sphere. Then the spectrum of the Laplacian \(\Delta\) on \(M\) satisfies the inequality
\[
m \sum_{j=0}^{n-1} (\lambda'_n - \lambda'_j)^2 \leq m^2 \sum_{j=0}^{n-1} (\lambda'_n - \lambda'_j) + 4 \sum_{j=0}^{n-1} (\lambda'_n - \lambda'_j) \lambda'_j, \quad n = 2, 3, \ldots
\]

Corollary 2.2. Let \(x : M \to S^{N-1}(1)\) be an \(m\)-dimensional closed minimal submanifold in an \((N-1)\)-dimensional Euclidean sphere. Then the spectrum of the Laplacian \(\Delta\) on \(M\) satisfies the inequality
\[
\lambda'_n \leq m + \frac{1}{n} (1 + 4/m) \sum_{j=0}^{n-1} \lambda'_j, \quad n = 2, 3, \ldots
\]

Corollary 2.3. Let \(x : M \to S^{N-1}(1)\) be an \(m\)-dimensional closed minimal submanifold in an \((N-1)\)-dimensional Euclidean sphere. Then the spectrum of the Laplacian \(\Delta\) on \(M\) satisfies the following inequality, for any \(n \in \mathbb{N}\) satisfying \(\lambda'_n > \lambda'_{n-1}\):
\[
\sum_{j=0}^{n-1} \frac{m^2 + 4\lambda'_j}{\lambda'_n - \lambda'_j} \geq mn, \quad n = 2, 3, \ldots
\]

Corollary 2.4. Let \(x : M \to S^{N-1}(1)\) be an \(m\)-dimensional closed minimal submanifold in an \((N-1)\)-dimensional Euclidean sphere. Then the spectrum of the Laplacian \(\Delta\) on \(M\) satisfies the inequality
\[
\lambda'_n - \lambda'_{n-1} \leq m + \frac{4}{mn} \sum_{j=0}^{n-1} \lambda'_j, \quad n = 2, 3, \ldots
\]

Noting \(\lambda'_0 = 0\), we immediately get the following Yang-Yau’s result from Corollary 2.4.

Corollary 2.5 (Leung [4], Yang-Yau [9]). Let \(x : M \to S^{N-1}(1)\) be an \(m\)-dimensional closed minimal submanifold in an \((N-1)\)-dimensional Euclidean sphere. Then the spectrum of the Laplacian \(\Delta\) on \(M\) satisfies the inequality
\[
\lambda'_n - \lambda'_{n-1} \leq m + \frac{2}{mn} \left( \sum_{j=1}^{n-1} \lambda'_j + \sqrt{ \left( \sum_{j=1}^{n-1} \lambda'_j \right)^2 + m^2 n \sum_{j=1}^{n-1} \lambda'_j} \right).
\]
Acknowledgements. The first author began this research during his stay at the Institute of Mathematics of TU Berlin as an AVH fellow in 2002. He would like to express his thanks to Prof. Dr. Udo Simon and Dr. Martin Wiehe for their help. We would also like to thank the referee for some helpful comments.

References


