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CURVATURE CONDITIONS ON HYPERSURFACES WITH TWO DISTINCT PRINCIPAL CURVATURES

MAŁGORZATA GŁOGOWSKA

Department of Mathematics, Agricultural University of Wrocław
Grunwaldzka 53, 50-357 Wrocław, Poland
E-mail: mglog@ozi.ar.wroc.pl

Abstract. We investigate curvature properties of hypersurfaces in semi-Riemannian spaces of constant curvature with the minimal polynomial of the second fundamental tensor of second degree. We present suitable examples of hypersurfaces.

1. Introduction. A semi-Riemannian manifold (M, g), $n = \dim M \geq 3$, is said to be an *Einstein manifold* if

$$S = \frac{\kappa}{n}\, g \tag{1}$$

on M. The Einstein manifolds form a natural subclass of the class of quasi-Einstein manifolds. A semi-Riemannian manifold (M, g), $n \geq 3$, is called a *quasi-Einstein manifold* if at every $x \in M$ its Ricci tensor S has the form

$$S = \alpha g + \epsilon w \otimes w, \qquad \epsilon = \pm 1, \tag{2}$$

where $w \in T_x^*M$ and $\alpha \in \mathbb{R}$. For precise definitions of the symbols used we refer to Sections 2 and 3 of [28] (see also [3] and [15]). Another subclass of quasi-Einstein manifolds form *Ricci-simple manifolds*, i.e. semi-Riemannian manifolds having the Ricci tensor of rank at most one. Quasi-Einstein manifolds arose in the study of exact solutions of the Einstein field equations and in the study of quasi-umbilical hypersurfaces of conformally flat spaces. Quasi-Einstein hypersurfaces were studied among others in [14], [17], and [19], see also references therein. We refer to [3] for a review of results on quasi-Einstein manifolds. We mention that the problem of equivalence of the conditions of semisymmetry

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($R{\cdot}R = 0$, [27]) and Ricci-semisymmetry ($R{\cdot}S = 0$), on hypersurfaces in Euclidean spaces, named the *problem of P. J. Ryan*, leads to consideration of quasi-Einstein hypersurfaces (see e.g. [1], [12] and [18]).

A semi-Riemannian manifold (M, g), $n \geq 3$, is said to be *pseudosymmetric* ([3], [10]) if, at every point of M, the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent. This is equivalent to

$$R \cdot R = L_R Q(g, R) \tag{3}$$

on $U_R = \{x \in M | R - \frac{\kappa}{(n-1)n} G \neq 0 \text{ at } x\}$, where L_R is some function on U_R. The class of pseudosymmetric manifolds is an extension of the class of semisymmetric manifolds (see e.g. [3], sections 3 and 4). In [25] (see also [26]) a special subclass of pseudosymmetric manifolds was introduced. Namely, according to [25], a pseudosymmetric manifold is called a *pseudo-symmetric space of constant type* if the function L_R is constant.

A semi-Riemannian manifold (M, g), $n \geq 3$, is said to be *Ricci-pseudosymmetric* ([3], [10]) if, at every point of M, the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent. This is equivalent to

$$R \cdot S = L_S Q(g, S) \tag{4}$$

on $U_S = \{x \in M | S - \frac{\kappa}{n} g \neq 0 \text{ at } x\}$, where L_S is some function on U_S. The class of Ricci-pseudosymmetric manifolds is an extension of the class of Ricci-semisymmetric manifolds, as well as of the class of pseudosymmetric manifolds (see [3] and [10]). A Ricci-pseudosymmetric manifold is called a *Ricci-pseudosymmetric manifold of constant type* if the function L_S is constant. The Cartan hypersurfaces of dimension ≥ 6 are Ricci-pseudosymmetric manifolds of constant type which are non-pseudosymmetric (see Section 2). Another example of a Ricci-pseudosymmetric manifold of constant type which is non-pseudosymmetric is given in Section 4 of [19].

Let (M, g), $n \geq 4$, be a semi-Riemannian manifold such that its curvature tensor R satisfies on $U_C \cap U_S \subset M$ the equation

$$R = \phi \overline{S} + \mu g \wedge S + \eta G, \tag{5}$$

where ϕ, μ and η are some functions on this set and $U_C = \{x \in M | C \neq 0 \text{ at } x\}$. According to [11], (5) is called the *Roter type equation.* Consequently, a manifold (M, g), $n \geq 4$, satisfying (5) on $U_C \cap U_S \subset M$, will be called a *Roter type manifold.* Obviously, we consider manifolds (M, g) with nonempty set $U_C \cap U_S \subset M$. We mention that the decomposition of R on $U_C \cap U_S$ in terms $\overline{S}$, $g \wedge S$ and G is unique ([17], Lemma 3.2). If (5) holds on an open set $U \subset U_C \cap U_S$ then we say that the Roter type equation is satisfied on U. Roter type manifolds were recently defined in [11], although investigations on these manifolds were initiated earlier in [16]. Examples of such manifolds are presented in [11] and [20].

It is easy to prove that every Roter type manifold is a non-Einstein and non-conformally flat semi-Riemannian manifold of dimension ≥ 4. The class of Roter type manifolds forms an essential subclass of the class of pseudosymmetric manifolds. Roter type manifolds satisfy also other curvature conditions of pseudosymmetry type (see Section 2). For a survey of results on manifolds satisfying pseudosymmetry type curvature conditions we refer to [3] and [15]. A hypersurface which is a Roter type manifold is called a Roter type hypersurface. We can prove that every Roter type manifold satisfies

$$S \cdot R = L_1 \overline{S} + L_2 g \wedge S + L_3 G, \tag{6}$$

$$R \cdot R - Q(S, R) = L_4 Q(g, C), \tag{7}$$

$$S^2 = L_5 S + L_6 g, \tag{8}$$

where $L_1, \ldots, L_6$ are some functions on $U_C \cap U_S$.

According to [11], a semi-Riemannian manifold (M, g), $n \geq 4$, is said to be an *Akivis-Goldberg type manifold* if (6), (7) and (8) hold on $U_C \cap U_S \subset M$. Every Roter type manifold is an Akivis-Goldberg type manifold. The converse statement is not true ([11]). We refer to [11] for a survey of results on Akivis-Goldberg type manifolds. Again, investigations on these manifolds were initiated in earlier papers: [8] and [23] (see also [2]).

Investigations of curvature properties of Ricci-pseudosymmetric hypersurfaces, and in particular, of the Cartan hypersurfaces of dimension ≥ 6 (see [13] and references therein), as well as the considerations presented above lead to the definition of the class of Cartan type manifolds. A semi-Riemannian manifold (M, g), $n \geq 4$, is said to be a *manifold of Cartan type* if on $U_C \cap U_S \subset M$ we have (7), (8) and

$$S \cdot R = L_0 R + L_1 \overline{S} + L_2 g \wedge S + L_3 G, \tag{9}$$

where $L_0, \ldots, L_6$ are some functions on $U_C \cap U_S$. Every Akivis-Goldberg type manifold is a Cartan type manifold. The converse statement is not true (see Theorem 3.4).

Hypersurfaces which are Akivis-Goldberg type manifolds and hypersurfaces which are Cartan type manifolds will be called Akivis-Goldberg type hypersurfaces and Cartan type hypersurfaces, respectively. In Section 3 we prove that every Cartan hypersurface of dimension ≥ 6 is a Cartan type hypersurface which is not an Akivis-Goldberg type hypersurface.

In [22] (Theorem 1), among other things, it was proved that if the minimal polynomial of the second fundamental tensor H of a hypersurface immersed isometrically in a semi-Riemannian space of constant curvature is of second degree then such a hypersurface is a pseudosymmetric manifold. In this paper we improve that result. Namely, we prove that such a hypersurface is a Roter type hypersurface (see Theorem 3.1). As an immediate consequence of Theorem 3.1 we have

THEOREM 1.1. *Let M be a hypersurface in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$. If at every point of $U_C \cap U_S$ there are exactly two distinct principal curvatures then M is a Roter type hypersurface.*

The converse statement is not true. Namely, we can prove that the hypersurface considered in [17](Example 5.2) is a Roter type hypersurface with three distinct principal curvatures. We mention that quasi-umbilical hypersurfaces of dimension ≥ 4 in Riemannian spaces of constant curvature are non-Roter type pseudosymmetric hypersurfaces (see Remark 3.1). But on the other hand, every non-Einstein and non-conformally flat Clifford torus of dimension ≥ 4 is a Roter type hypersurface (see Corollary 3.2). It is well-known that every Clifford torus is a semisymmetric manifold. Furthermore in Example 3.1 we present an example of a warped product Roter type manifold which can be locally realized as a hypersurface in a semi-Euclidean space.

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2. Preliminaries. It is easy to check that (2) implies

$$S^2 = (\kappa - (n-2)\alpha)S + \alpha((n-1)\alpha - \kappa)g.$$

Thus we have

Proposition 2.1. *Let (M,g), $n \geq 4$, be a semi-Riemannian manifold satisfying (2). Then (8), with $L_5 = \kappa - (n-2)\alpha$ and $L_6 = \alpha((n-1)\alpha - \kappa)$, is satisfied on M. Moreover, if (6) and (7) hold on $U_C \cap U_S \subset M$ then (M,g) is an Akivis-Goldberg type manifold.*

Further, we have

Proposition 2.2. *Let (M,g), $n \geq 4$, be a semi-Riemannian manifold. The following conditions: (2) and*

$$\overline{S} = \alpha g \wedge S - \alpha^2 G = \alpha^2 G + \alpha\epsilon g \wedge (w \otimes w) \tag{10}$$

are equivalent on $U_S \subset M$.

Proof. Our assertion is an immediate consequence of Lemma 3.1 of [24] and (2).

We can also prove the following

Proposition 2.3. *Let (M,g), $n \geq 4$, be a semi-Riemannian manifold satisfying (9). Let V be a set of all points of $U_C \cap U_S \subset M$ at which (2) or (5) is fulfilled. Then the decomposition of the tensor $S \cdot R$ in terms R, $\overline{S}$, $g \wedge S$ and G is unique on $(U_C \cap U_S) \setminus V$.*

Proof. Let $x \in (U_C \cap U_S) \setminus V$ and let

$$S \cdot R = L_0' R + L_1' \overline{S} + L_2' g \wedge S + L_3' G \tag{11}$$

at x, where $L_0', \ldots, L_3' \in \mathbb{R}$. We suppose that at x we have $L_0 - L_0' \neq 0$. Now (9) and (11) imply (5), a contradiction. Thus at x we have $L_0 = L_0'$. Further, we suppose that at x we have $L_1 - L_1' \neq 0$. Now (9) and (11) imply (10). This, in view of Lemma 3.1 [24], turns into (2), a contradiction. Thus at x we have $L_1 = L_1'$. Similarly, we prove that $L_2 = L_2'$ and $L_3 = L_3'$ at x. Our proposition is thus proved.

We can check that (5) implies (3) with $L_R = (n-2)(\frac{\mu}{\phi}(\mu - \frac{1}{n-2}) - \eta)$ ([16], Theorem 4.2) and

$$R \cdot R - Q(S,R) = \left(L_R + \frac{\mu}{\phi}\right) Q(g,C), \tag{12}$$

$$S^2 = \alpha S + \beta g, \tag{13}$$

$$S \cdot R = -4(\alpha\phi + \mu)\overline{S} - 2(\alpha\mu + \eta + \beta\phi) g \wedge S - 4\beta\mu G, \tag{14}$$

$$\alpha = \kappa + \frac{(n-2)\mu - 1}{\phi}, \qquad \beta = \frac{\mu\kappa + (n-1)\eta}{\phi}.$$

Furthermore, we have $C \cdot C = L_C Q(g,C)$ and $C \cdot R = L_C Q(g,R)$, where $L_C = L_R + \frac{1}{n-2}(\frac{\kappa}{n-1} - \alpha)$. We have

Theorem 2.1 ([11], Theorem 3.1). *Every Roter type semi-Riemannian manifold (M,g), $n \geq 4$, is an Akivis-Goldberg type manifold.*

PROPOSITION 2.4. *On every semi-Riemannian Einstein manifold* (M, g), $n \geq 4$, *(8) and (9) are satisfied on* M. *Moreover, if* (M, g) *is pseudosymmetric then (7) holds on* M.

Proof. Our proposition is an immediate consequence of (11) of [28] and (1), (8) and (9).

From Propositions 2.3 and 2.4 it follows that consideration of Einstein manifolds, as well as of conformally flat manifolds, satisfying (6) (or (9)), (7) and (8) is rather uninteresting.

Using Propositions 2.1 and 2.3 and Theorems 3.3 and 3.4 of [9] we can prove the following

PROPOSITION 2.5. *If (7) is satisfied on a conformally flat semi-Riemannian manifold* (M, g), $n \geq 4$, *then (8) and (9) hold on* M.

3. Ricci-pseudosymmetric hypersurfaces. Let M be a hypersurface immersed isometrically in a semi-Riemannian manifold (N, g^N). If (5), resp. (6), (7) and (8) or (7), (8) and (9), hold on $U_C \cap U_S \subset M$ then M is said to be a *Roter type hypersurface*, resp. an *Akivis-Goldberg type hypersurface* ([11]) or a *Cartan type hypersurface*.

Let $N_s^{n+1}(c)$, $n \geq 4$, denote a semi-Riemannian space of constant curvature, with signature $(s, n+1-s)$, where $c = \frac{\widetilde{\kappa}}{n(n+1)}$ and $\widetilde{\kappa}$ being its scalar curvature. Further, let M be a hypersurface immersed isometrically in $N_s^{n+1}(c)$. The Gauss equation of M in $N_s^{n+1}(c)$ reads (see e.g. [15], [28])

$$R_{hijk} = \varepsilon(H_{hk}H_{ij} - H_{hj}H_{ik}) + \frac{\widetilde{\kappa}}{n(n+1)}G_{hijk}, \tag{15}$$

where R_{hijk}, G_{hijk} and H_{ij} denote the local components of the curvature tensor R, the tensor G and the second fundamental tensor H of M. Contracting (15) with g^{ij} and g^{kh}, respectively, we obtain

$$S_{hk} = \varepsilon(tr(H)H_{hk} - H^2_{hk}) + \frac{(n-1)\widetilde{\kappa}}{n(n+1)}g_{hk}, \tag{16}$$
$$\kappa = \varepsilon((tr(H))^2 - tr(H^2)) + \frac{(n-1)\widetilde{\kappa}}{n+1},$$

respectively, where κ is the scalar curvature of M, $tr(H) = g^{hk}H_{hk}$, $tr(H^2) = g^{hk}H^2_{hk}$ and S_{hk} are the local components of the Ricci tensor S of M. For the definition of the tensor H^2 see e.g. [28] (Section 1). Further, we denote by U_H the set of all points of M at which the tensor H^2 is not a linear combination of the metric tensor g and the second fundamental tensor H of M. Using (16) and Theorem 4.1 of [21] we can deduce that $U_H \subset U_C \cap U_S \subset M$. Evidently, on $(U_C \cap U_S) \setminus U_H$ we have

$$H^2 = \alpha H + \beta g, \tag{17}$$

where α and β are some functions on $(U_C \cap U_S) \setminus U_H$.

We recall that a Cartan hypersurface in the sphere $S^{n+1}(c)$ is a compact minimal hypersurface with constant principal curvatures $-(3c)^{\frac{1}{2}}$, 0, $(3c)^{\frac{1}{2}}$ of the same multiplicity $\frac{n}{3}$. It is known that the Cartan hypersurfaces are tubes of constant radius over the standard Veronese embeddings $i : \mathbb{E}P^2 \to S^{3d+1}(c) \to \mathbb{E}^{3d+2}$, $d = 1, 2, 4, 8$, of the projective plane $\mathbb{E}P^2$ into the sphere $S^{3d+1}(c)$ in a Euclidean space $\mathbb{E}^{3d+2}$, where $\mathbb{E} = \mathbb{R}$ (real numbers),

$\mathbb{C}$ (complex numbers), $\mathbb{Q}$ (quaternions) or $\mathbb{O}$ (Cayley numbers), respectively ([4]). Every Cartan hypersurface satisfies (4). Precisely, we have (see e.g. [13], Theorem 4.3)

$$R \cdot S = \frac{\widetilde{\kappa}}{n(n+1)} Q(g, S). \tag{18}$$

Thus such a hypersurface is a Ricci-pseudosymmetric manifold of constant type. In addition, the Cartan hypersurface in $S^4(c)$ is a pseudosymmetric manifold satisfying ([22], Example 2)

$$R \cdot R = \frac{\widetilde{\kappa}}{12} Q(g, R).$$

If on the set U_H of a hypersurface M in $N_s^{n+1}(c)$, $n \geq 3$, we have $H^3 = tr(H)H^2 + \lambda H$, for some function λ, then (18) holds on U_H ([7], Proposition 3.2). It is also known that if rank$H = 2$ at every point of a hypersurface M in $N_s^{n+1}(c)$, $n \geq 3$, then it is a pseudosymmetric space of constant type ([7], Theorem 4.2). Precisely, on such a hypersurface we have

$$R \cdot R = \frac{\widetilde{\kappa}}{n(n+1)} Q(g, R).$$

On every hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, we have ([21])

$$R \cdot R - Q(S, R) = -\frac{(n-2)\widetilde{\kappa}}{n(n+1)} Q(g, C). \tag{19}$$

Thus on $U_C \cap U_S \subset M$ we have (8) with $L_4 = -\frac{(n-2)\widetilde{\kappa}}{n(n+1)}$. Clearly, if (6) and (8), resp. (8) and (9), hold on $U_C \cap U_S \subset M$ then M is an Akivis-Goldberg type hypersurface, resp. a Cartan type hypersurface.

Theorem 4.3 of [13] states that every Cartan hypersurface of dimension $n \geq 6$ is a Cartan type hypersurfce. They are non-pseudosymmetric Ricci-pseudosymmetric hypersurfaces. Further, (9), with $L_1 = 0$, is satisfied on $U_H \subset U_C \cap U_S \subset M$ for every Ricci-pseudosymmetric hypersurface M in $N_s^{n+1}(c)$, $n \geq 4$, ([13], Theorem 3.2). In other words, on U_H we have (8) and

$$S \cdot R = L_0 R + L_2 g \wedge S + L_3 G.$$

In this section we prove that such hypersurfaces are Cartan type hypersurfaces. We also mention that examples of quasi-Einstein Ricci-pseudosymmetric hypersurfaces were found recently in [17] and [19].

Since (19) is a relation of the form (7), we have

Proposition 3.1. *Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$.*

(i) The relation (7) is satisfied on M.
(ii) If (6) and (8) hold on $U_C \cap U_S \subset M$ then M is an Akivis-Goldberg type hypersurface.

Propositions 2.1 and 3.1 imply

Proposition 3.2. *If M is a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, satisfying (2) and (6) on $U_C \cap U_S \subset M$ then M is an Akivis-Goldberg type hypersurface.*

PROPOSITION 3.3. *Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$. Then on $(U_C \cap U_S) \setminus U_H \subset M$ we have*

$$R = \varepsilon(tr(H) - \alpha)^{-2}\Big(\overline{S} - \Big(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \varepsilon\beta\Big) g \wedge S\Big) + \Big(\varepsilon(tr(H)-\alpha)^{-2}\Big(\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \varepsilon\beta\Big)^2 + \frac{\widetilde{\kappa}}{n(n+1)}\Big)G, \tag{20}$$

where α and β are defined by (17).

Proof. First of all, we note that (16) and (17) yield

$$S = \varepsilon(tr(H) - \alpha)H + (\frac{(n-1)\widetilde{\kappa}}{n(n+1)} - \varepsilon\beta)g. \tag{21}$$

Thus we see that $tr(H) - \alpha \neq 0$ at every point of $(U_C \cap U_S) \setminus U_H$. Further, using (15) and (21) we get (20), completing the proof.

Proposition 3.3 implies

THEOREM 3.1. *Let M be a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, and let $(U_C \cap U_S) \setminus U_H \subset M$ be nonempty. Then the Roter type equation holds on this set. Moreover, if $U_H \subset M$ is empty then M is a Roter type hypersurface.*

It is easy to see that the last theorem implies Theorem 1.1.

REMARK 3.1. Let M be a hypersurface in a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$. Using the well-known Cartan-Schouten result (see e.g. [5], Section 13.1.1.) we can easily deduce that if at a point of $U_C \subset M$ there are exactly two distinct principal curvatures then the multiplicity of each principal curvature is ≥ 2, i.e. M cannot be quasi-umbilical at this point. More generally, if M is a hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, and if (17) is satisfied at a point of $U_C \subset M$ then M is not quasi-umbilical at this point ([21], Theorem 4.1).

THEOREM 3.2 ([13], Theorem 3.2). *If M is a Ricci-pseudosymmetric hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, then (8) and (9) hold on $U_H \subset M$.*

Theorems 3.1 and 3.2 yield

THEOREM 3.3. *Every Ricci-pseudosymmetric hypersurface in $N_s^{n+1}(c)$, $n \geq 4$, is a Cartan type hypersurface.*

THEOREM 3.4. *The Cartan hypersurface M in $S^{n+1}(c)$, $n = 6, 12$ or 24 is a Cartan type hypersurface which is not an Akivis-Goldberg type hypersurface.*

Proof. First of all we note that M is a non-pseudosymmetric Ricci-pseudosymmetric hypersurface (e.g. see [13], section 2). From the definition of the Cartan hypersurface it follows that $U_C \cap U_S = M$. We suppose now that M is an Akivis-Goldberg type hypersurface. Thus on M we have $S \cdot R = L'_1 \overline{S} + L'_2 g \wedge S + L'_3 G$, where L'_1, L'_2 and L'_3 are some functions on M. The last relation, together with the equation (53) of [13], yields (5). But (5) implies (3) (see section 2), a contradiction. Thus our theorem is proved.

EXAMPLE 3.1. Let M_1, resp. M_2, be a nonempty open connected subset of $\mathbb{R}^p$, resp. $\mathbb{R}^{n-p}$, $2 \leq p \leq n-2$, equipped with the standard metric g_1, $g_{1ab} = \varepsilon_a \delta_{ab}$, $\varepsilon_a = \pm 1$, resp.

g_2, $g_{2\alpha\beta} = \varepsilon_\alpha \delta_{\alpha\beta}$, $\varepsilon_\alpha = \pm 1$, where $a, b, c, d, e, f \in \{1, \ldots, p\}$, $\alpha, \beta, \gamma, \delta \in \{p+1, \ldots, n\}$ and $h, i, j, k \in \{1, \ldots, n\}$. We set $F = F(x^1, \ldots, x^p) = k \exp(\xi_a x^a)$, where $k, \xi_1, \ldots, \xi_p \in \mathbb{R}$, $\xi_1^2 + \ldots + \xi_p^2 > 0$, $k > 0$ and $g_1{}^{ef} \xi_e \xi_f \neq 0$. We consider the warped product $M_1 \times_F M_2$. Let τ be a function on $\overline{M} \times_F \widetilde{N}$ defined by (cf. [8], Section 4)

$$\tau^2 = -\frac{\varepsilon}{4} g_1{}^{ef} \xi_e \xi_f, \qquad \varepsilon = \pm 1. \tag{22}$$

It is clear that there exist constants ε, ε_a, and ξ_a such that the right-hand side of (22) is positive at every point x of $\overline{M} \times_F \widetilde{N}$. Using now (44) of [8] and (22) we find

$$\begin{aligned} S_{ab} &= -\frac{n-p}{4} \xi_a \xi_b, \qquad S_{\alpha\beta} = (n-p)\varepsilon\tau^2 g_{\alpha\beta}, \\ S^2 &= \frac{\kappa}{n-p+1} S, \qquad \kappa = (n-p)(n-p+1)\varepsilon\tau^2. \end{aligned} \tag{23}$$

In Section 4 of [8] it was shown that the conditions: $R \cdot R = Q(S, R)$ and $S \cdot R = 0$ are satisfied on $\overline{M} \times_F \widetilde{N}$. Further, we have (see [8], section 4) $R = \varepsilon \overline{H}$, where H is a symmetric $(0, 2)$-tensor with the local components

$$H_{ab} = -\frac{1}{4\tau} \xi_a \xi_b, \qquad H_{a\alpha} = 0, \qquad H_{\alpha\beta} = \varepsilon\tau g_{\alpha\beta}. \tag{24}$$

The tensor H is a Codazzi tensor. Therefore $M_1 \times_F M_2$ locally can be realized as a hypersurface in a semi-Euclidean space $\mathbb{E}_s^{n+1}$. From (23) and (24) it follows that

$$\mathrm{tr} H = -\frac{1}{4\tau} g^{1ef} \xi_e \xi_f + (n-p)\varepsilon\tau = (n-p+1)\varepsilon\tau, \qquad H^2 = \varepsilon\tau H.$$

Applying this and (23) into (16) and (15), on $M_1 \times_F M_2$ we get

$$S = (n-p)\tau H, \qquad R = \frac{n-p+1}{(n-p)\kappa} \overline{S},$$

respectively. Thus we see that $M_1 \times_F M_2$ is a Roter type manifold which locally can be realized as a hypersurface in a semi-Euclidean space $\mathbb{E}_s^{n+1}$.

Remark 3.2. (i) An example of a hypersurface M in $N_s^{n+1}(c)$, $c \neq 0$, $n \geq 4$, satisfying the Roter type equation on the set $(U_C \cap U_S) \setminus U_H \subset M$, is given in [20]. We also mention that warped products satisfying (5) were investigated in [20].
(ii) The warped product defined in Example 4.1 of [14] is a Ricci-simple Akivis-Goldberg type manifold. Clearly, such manifold is not a Roter type manifold. That warped product locally can be realized as a hypersurface in a semi-Euclidean space $\mathbb{E}_s^{n+1}$ ([14], Example 5.1).
(iii) Examples of Ricci-pseudosymmetric quasi-Einstein hypersurfaces in spaces of constant curvature are given in [17] and [19].

We consider the Cartesian product $N_{s_1}^p(c_1) \times N_{s_2}^{n-p}(c_2)$ of two semi-Riemannian spaces of constant curvature $N_{s_1}^p(c_1)$ and $N_{s_2}^{n-p}(c_2)$, $2 \leq p \leq n-2$, where $c_1 = \frac{\kappa_1}{(p-1)p}$, $c_2 = \frac{\kappa_2}{(n-p-1)(n-p)}$ and κ_1 and κ_2 are the scalar curvatures of $N_{s_1}^p(c_1)$ and $N_{s_2}^{n-p}(c_2)$, respectively. This product is a semisymmetric manifold ([27], Theorem 4.5). Further, it is known that the local components R_{hijk} of the curvature tensor and the local components S_{hk} of the Ricci tensor S of $N_{s_1}^p(c_1) \times N_{s_2}^{n-p}(c_2)$ which might not vanish identically, are the following:

$$\begin{aligned} R_{abcd} &= c_1(g_{ad}g_{bc} - g_{ac}g_{bd}),\\ R_{\alpha\beta\gamma\delta} &= c_2(g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\delta}),\\ S_{ad} &= (p-1)c_1 g_{ad},\\ S_{\alpha\delta} &= (n-p-1)c_2 g_{\alpha\delta}, \end{aligned} \tag{25}$$

where g_{hk} are the local components of the metric tensor of $N^p_{s_1}(c_1) \times N^{n-p}_{s_2}(c_2)$, $a, b, c, d \in \{1, 2, \dots, p\}$ and $\alpha, \beta, \gamma, \delta \in \{p+1, p+2, \dots, n\}$. The scalar curvature κ of $N^p_{s_1}(c_1) \times N^{n-p}_{s_2}(c_2)$ is given by

$$\kappa = p(p-1)c_1 + (n-p)(n-p-1)c_2. \tag{26}$$

In addition, we assume that the considered product satisfies

$$(a)\ \ c_1 + c_2 \neq 0 \quad \text{and} \quad (b)\ \ (p-1)c_1 \neq (n-p-1)c_2. \tag{27}$$

Using (25) and (27) we can conclude that $N^p_{s_1}(c_1) \times N^{n-p}_{s_2}(c_2)$, $2 \leq p \leq n-2$, is a non-conformally flat and non-Einstein manifold. Further, from (25) it follows that also (5) is satisfied with

$$\begin{aligned} \phi &= \tau(c_1 + c_2),\\ \mu &= -(n-2)\tau c_1 c_2,\\ \eta &= \tau c_1 c_2((p-1)^2 c_1 + (n-p-1)^2 c_2),\\ \tau &= ((p-1)c_1 - (n-p-1)c_2)^{-2}. \end{aligned}$$

Moreover, (26) and (27) yield

$$\frac{\phi\kappa}{n-1} + \mu - \frac{1}{n-2} = \frac{(p-1)(n-p-1)}{(n-2)(n-1)}\tau(c_1+c_2)^2. \tag{28}$$

The above considerations yield

Proposition 3.4. *The Cartesian product $N^p_{s_1}(c_1) \times N^{n-p}_{s_2}(c_2)$, $2 \leq p \leq n-2$, is a Roter type manifold, provided that (27) is satisfied.*

Example 3.2. Let $S^p(r)$ be p-dimensional standard sphere of radius r in $\mathbb{E}^{p+1}$. We note that $r^{-2} = \frac{\kappa}{(p-1)p}$, where κ is the scalar curvature of the given sphere. From the above considerations it follows that the Cartesian product $S^p(r_1) \times S^{n-p}(r_2)$, $2 \leq p \leq n-2$, of two spheres $S^p(r_1)$ and $S^{n-p}(r_2)$ is a Roter type manifold, provided that (27)(b), or equivalently, $(n-p-1)r_1^2 \neq (p-1)r_2^2$ is satisfied. From Theorem 5.1 of [30] (cf. [6]) it follows that $M = S^p(r_1) \times S^{n-p}(r_2)$, $r_1 = \sqrt{\frac{p}{n}}$, $r_2 = \sqrt{\frac{n-p}{n}}$, $2 \leq p \leq n-2$, can be realized as a minimal hypersurface immersed isometrically in the sphere $S^{n+1}(1)$ having at every point exactly two distinct principal curvatures ρ_1 and ρ_2 of multiplicity p and $n-p$, respectively. It is known that $\rho_1\rho_2 + 1 = 0$ and $\rho_i^2 = r_i^{-2} - 1$, $i = 1, 2$. The hypersurface M is called the *Clifford torus.*

Corollary 3.1. *The Clifford torus $S^p(\sqrt{\frac{p}{n}}) \times S^{n-p}(\sqrt{\frac{n-p}{n}})$, $2 \leq p \leq n-2$, $n \neq 2p$, is a Roter type hypersurface.*

We finish this section with the following

REMARK 3.3. Let (M,g), $n \geq 4$, be a semi-Riemannian manifold satisfying

$$C = \frac{L}{2}\left(S - \frac{\kappa}{n-1}g\right) \wedge \left(S - \frac{\kappa}{n-1}g\right) \tag{29}$$

on $U_C \cap U_S \subset M$, where L is some function on $U_C \cap U_S$. Manifolds satisfying (29) were investigated in [24]. Evidently, (29) is equivalent to (5) with $\phi = L$, $\mu = \frac{1}{n-2} - \frac{L\kappa}{n-1}$ and $\eta = \frac{\kappa}{n-1}(\frac{L\kappa}{n-1} - \frac{1}{n-2})$. It is easy to see that on $U_C \cap U_S$ we have $\frac{\phi\kappa}{n-1} + \mu - \frac{1}{n-2} = 0$. Comparing this with (28) we can conclude that a semi-Riemannian manifold (M,g), $n \geq 4$, satisfying (29) on $U_C \cap U_S \subset M$, and $N^p_{s_1}(c_1) \times N^{n-p}_{s_2}(c_2)$, $2 \leq p \leq n-2$, satisfying (27), are not isometric.

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