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IMPROPER AFFINE SPHERES AND δ-INVARIANTS

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Abstract. We introduce an inequality for graph hypersurfaces and prove a decomposition theorem in case equality holds.

1. Introduction. A hypersurface $f\colon M \to \mathbf{R}^{n+1}$ is called a graph hypersurface if the affine normal vector field is some constant transversal vector field ξ. Then for any vector fields X, Y tangent to M, one can decompose $D_X f_*(Y)$ into its tangential and transversal components, where D is the canonical flat connection on $\mathbf{R}^{n+1}$. This is written as $D_X f_*(Y) = f_*(\nabla_X Y) + h(X,Y)\xi$, where h is a symmetric tensor of type $(0,2)$. If h is non-degenerate, then h can be considered as semi-Riemannian metric on M, called the Calabi metric [3]. Let $\hat{\nabla}$ denote Levi-Civita connection of (M,h) and K be the difference tensor $\nabla - \hat{\nabla}$ on M. By taking the trace of K, one obtains a so-called Tchebychev form $T(X) := (1/n)\,\mathrm{trace}\{\, Y \to K(X,Y)\,\}$. The Tchebychev vector field $T^\#$ can then be defined by $h(T^\#, X) = T(X)$. The vanishing of T implies that the hypersurface, considered as a hypersurface of the equiaffine space, is an improper affine sphere.

In this article, we assume that h is definite. In case that h is negative definite, we can change the sign of ξ in order to make h positive definite. Hence we will assume that h always defines a Riemannian metric on M.

In section 2 we recall the basic facts about graph hypersurfaces in $\mathbf{R}^{n+1}$. In section 3 we consider the δ-invariant for the curvature of the Riemannian manifold (M,h). We apply a general result from [1] and prove inequalities which involve the δ-invariants and

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the Tchebychev vector field. If equality is achieved in one of the inequalities, then M is an improper affine sphere. In section 4 we show that, if in addition $\dim(\operatorname{Im} K) = n$, then M can be decomposed into lower-dimensional improper affine spheres.

2. Preliminaries on graph hypersurfaces. We recall some basic facts about graph hypersurfaces. For the details, see [4] or [3].

Let M be an n-dimensional C^∞-manifold and let $f : M \to \mathbf{R}^{n+1}$ be a hypersurface. If ξ is a constant vector field which is nowhere tangent to M, then ξ can be regarded as affine normal vector field along f. We call f together with this normalization ξ a graph hypersurface.

The Gauss formula is given by

$$D_X f_*(Y) = f_*(\nabla_X Y) + h(X,Y)\xi, \tag{1}$$

where D denotes the canonical flat connection of $\mathbf{R}^{n+1}$, ∇ is a torsion-free connection on M, called the induced connection, and h is a symmetric $(0,2)$-tensor field. The corresponding equations of Gauss and Codazzi are given by

$$R(X,Y)Z = 0, \tag{2}$$

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z). \tag{3}$$

The totally symmetric (0,3)-tensor field $C(X,Y,Z) = (\nabla_X h)(Y,Z)$ is called the cubic form.

From now on we assume that the graph hypersurface is definite. Following [3], h is called the Calabi metric on M.

Denote by $\hat{\nabla}$ the Levi-Civita connection of h and by $\hat{R}$ and $\hat{\kappa}$ the curvature tensor and the normalized scalar curvature of h, respectively. The difference tensor K is then defined by

$$K_X Y = K(X,Y) = \nabla_X Y - \hat{\nabla}_X Y, \tag{4}$$

which is a symmetric $(1,2)$-tensor field. The difference tensor K and the cubic form C are related by

$$C(X,Y,Z) = -2h(K_X Y, Z). \tag{5}$$

Thus, for each X, K_X is self-adjoint with respect to h.

The Tchebychev form T, the Tchebychev vector field $T^\#$ and the Pick invariant J of the centroaffine hypersurface M are defined by

$$T(X) = \frac{1}{n}\operatorname{trace} K_X, \tag{6}$$

$$h(T^\#, X) = T(X), \tag{7}$$

$$h(C,C) = 4h(K,K) = 4n(n-1)J. \tag{8}$$

If $T = 0$ and if we consider M as a hypersurface of the equiaffine space, then M is a so-called *improper affine sphere*, with Blaschke normal in the direction of ξ and the Calabi metric is homothetic to the Blaschke metric. If the difference tensor K vanishes identically, then M is a paraboloid with axis in the direction of ξ.

It is well-known in (relative) affine geometry for graph hypersurfaces that

$$h(K_X Y, Z) = h(Y, K_X Z), \tag{9}$$

$$\hat{R}(X,Y)Z = K_Y K_X Z - K_X K_Y Z, \tag{10}$$

$$(\hat{\nabla} K)(X,Y,Z) = (\hat{\nabla} K)(Y,Z,X) = (\hat{\nabla} K)(Z,X,Y), \tag{11}$$

$$\hat{\kappa} = J - \frac{n}{n-1} h(T^{\#}, T^{\#}). \tag{12}$$

If we assume that $\xi = (0, \dots, 0, 1)$ then we can assume that locally M is given by $x_{n+1} = F(x_1, \dots, x_n)$. It turns out that then $(x_1, \dots, x_n)$ are ∇-flat coordinates on M and that the Calabi metric is given by

$$h\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \frac{\partial^2 F}{\partial x_i \partial x_j}.$$

Moreover, M is an improper affine sphere if and only if the Hessian determinant $\det[\frac{\partial^2 F}{\partial x_i \partial x_j}]$ is constant.

Now let M_1 and M_2 be improper affine spheres, with equations $x_{p+1} = F_1(x_1, \dots, x_p)$ and $y_{q+1} = F_2(y_1, \dots, y_q)$. Then we define a new improper affine sphere M in $\mathbb{R}^{p+q+1}$ by

$$z = F_1(x_1, \dots, x_p) + F_2(y_1, \dots, y_q),$$

where $(x_1, \dots, x_p, y_1, \dots, y_q, z)$ are the coordinates on $\mathbb{R}^{p+q+1}$. The affine normal of M is given by $(0, \dots, 0, 1)$. Obviously, the Calabi metric is the product metric. Following [2] we call this composition the Calabi composition of M_1 and M_2.

3. δ-invariants. Let (M, g) be any n-dimensional Riemannian manifold and let T be a curvature-like $(0,4)$-tensor field on M. Then we can talk about the sectional curvature $T(\pi)$ associated with a 2-plane section $\pi \subset T_pM$, $p \in M$. Further, let L be a linear subspace of T_pM of dimension $r \geq 2$ and $\{e_1, \dots, e_r\}$ an orthonormal basis of L. We define the scalar curvature $\tau(L)$ of the r-plane section L by

$$\tau(L) = \sum_{\alpha<\beta} \hat{T}(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha, \beta \leq r. \tag{13}$$

For an integer $k \geq 0$ denote by $\mathcal{S}(n,k)$ the finite set consisting of unordered k-tuples $(n_1, \dots, n_k)$ of integers ≥ 2 satisfying $n_1 < n$ and $n_1 + \dots + n_k \leq n$. Denote by $\mathcal{S}(n)$ the set of unordered k-tuples with $k \geq 0$ for a fixed n. For each $(n_1, \dots, n_k) \in \mathcal{S}(n)$ we define the δ-invariant $\delta(n_1, \dots, n_k)$ by

$$\delta(n_1, \dots, n_k)(p) = \tau(p) - \inf\{\tau(L_1) + \dots + \tau(L_k)\}, \tag{14}$$

where $L_1, \dots, L_k$ run over all k mutually orthogonal subspaces of T_pM such that $\dim L_j = n_j$, $j = 1, \dots, k$.

Let (M, g) be any n-dimensional Riemannian manifold and μ be a symmetric $(1,2)$-tensor field on M. If T is a $(0,4)$-tensor field on M such that

$$T(X,Y,Z,W) = g(\mu(Y,Z), \mu(X,W)) - g(\mu(X,Z), \mu(Y,W)) \tag{15}$$

for all tangent vector vector fields X, Y, Z, W, then obviously T is curvature-like. From [1] we have the following theorem.

THEOREM 1. *For each k-tuple $(n_1,\dots,n_k)\in\mathcal{S}(n)$, we have*

$$\delta(n_1,\dots,n_k)\le\frac{n^2(n+k-1-\sum_{j=1}^k n_j)}{2(n+k-\sum_{j=1}^k n_j)}g(\operatorname{trace}\mu,\operatorname{trace}\mu),\tag{16}$$

where $\operatorname{trace}\mu=\sum_{i=1}^n\mu(e_i,e_i)$

Equality holds in (16) *at a point $p\in M$ if and only if there exists an orthonormal basis $\{e_1,\dots,e_n\}$ at p such that with respect this basis every linear map μ_ξ, $\xi\in T_pM$ of the tangent space T_pM, defined by $g(\mu_\xi X,Y)=g(\mu(X,Y),\xi)$ for all $X,Y\in T_pM$ takes the following form:*

$$\mu_\xi=\begin{pmatrix}A_1^\xi & & & 0\\ & \ddots & & \\ & & A_k^\xi & \\ 0 & & & \mu_\xi I\end{pmatrix},\tag{17}$$

where I is an identity submatrix and $\{A_j^\xi\}_{j=1}^k$ are symmetric $n_j\times n_j$ submatrices satisfying

$$\operatorname{trace}(A_1^\xi)=\cdots=\operatorname{trace}(A_k^\xi)=\mu_\xi\tag{18}$$

for some μ_ξ.

For any k-tuple $(n_1,\dots,n_k)\in\mathcal{S}(n)$, we put

$$\Delta_1=\{1,\dots,n_1\},\ \dots,\ \Delta_k=\{n_1+\cdots+n_{k-1}+1,\dots,n_1+\cdots+n_k\},\tag{19}$$

Throughout this paper, we assume the following convention of indices:

$$i_1,j_1\in\Delta_1,\ \dots,\ i_k,j_k\in\Delta_k.\tag{20}$$

If equality holds in (16) for some k-tuple $(n_1,\dots,n_k)\in\mathcal{S}(n)$, then there exists an orthonormal basis $\{e_1,\dots,e_n\}$ of T_pM such that, with respect to this basis, each linear map μ_ξ takes the form of (17)-(18). With respect to the orthonormal basis $\{e_1,\dots,e_n\}$ so chosen, we put

$$L_j=\operatorname{Span}\{e_\alpha\,:\,\alpha\in\Delta_j\},\tag{21}$$

where $\Delta_1,\dots,\Delta_k$ are defined by (19). Also, let L_{k+1} be the linear subspace of T_pM spanned by $e_{\sigma_k+1},\dots,e_n$.

From [1] we also obtain that if equality holds in (16) for some $(n_1,\dots,n_k)\in\mathcal{S}(n)$ at a point $p\in M$ and in addition that $g(\mu(X,Y),Z)$ is totally symmetric, then $\operatorname{trace}\mu=0$ at p.

If equality is achieved in (16) identically for some k-tuple $(n_1,\dots,n_k)\in\mathcal{S}(n)$, then we may locally define $\mathcal{D}_j, j=1,\dots,k$, to be the distributions given by $L_j=\operatorname{Span}\{e_\alpha : \alpha\in\Delta_j\}$, where $\Delta_1,\dots,\Delta_k$ are given by (19). Finally we also need the following result from [1]:

THEOREM 2. *Let (M,g) be a Riemannian manifold, T a curvature-like $(0,4)$-tensor field and μ a symmetric $(1,2)$-tensor field satisfying the algebraic Gauss equation* (15) *and the* *total symmetry conditions*

$$g(\mu(X,Y),Z) = g(Y,\mu(X,Z)) \tag{22}$$

$$(\nabla_X\mu)(Y,Z) = (\nabla_Y\mu)(X,Z). \tag{23}$$

If $\dim(\operatorname{Im}\mu) = n$, *then the leaves of the integrable distributions* $\mathcal{D}_i$ *are totally geodesic submanifolds of* M. *Moreover,* M *is locally isometric to the Riemannian product* $M_1 \times \cdots \times M_k$, *where* M_i *denotes the leaf of the integrable distribution* $\mathcal{D}_i$ *for each* i.

4. A general inequality for graph hypersurfaces. The purpose of this section is to specify and extend the results mentioned in section 3 for graph hypersurfaces, considered as Riemannian manifolds, equipped with their Calabi metric h. If we take $\mu = K$ and $T = -\hat{R}$, then (10) shows that the algebraic Gauss equation (15) is satisfied. Moreover, (9) and (11) show that the two symmetry conditions are satisfied.

However, since we prefer to formulate the inequality using the curvature tensor $\hat{R}$, we propose the following definition for the affine δ-invariant:

$$\delta^{\#}(n_1,\dots,n_k)(p) = \hat{\tau}(p) - \sup\{\hat{\tau}(L_1) + \cdots + \hat{\tau}(L_k)\}, \tag{24}$$

where $L_1,\dots,L_k$ run over all k mutually h-orthogonal subspaces of T_pM such that $\dim L_j = n_j$, $j = 1,\dots,k$.

Then we immediately have the following theorem.

THEOREM 3. *Let* M *be graph hypersurface in* $\mathbf{R}^{n+1}$ *with positive definite Calabi metric. Then, for each* k*-tuple* $(n_1,\dots,n_k) \in \mathcal{S}(n)$, *we have*

$$\delta^{\#}(n_1,\dots,n_k) \geq -\frac{n^2\left(n+k-1-\sum_{j=1}^k n_j\right)}{2\left(n+k-\sum_{j=1}^k n_j\right)} h(T^{\#},T^{\#}). \tag{25}$$

Equality holds in (25) *at a point* $p \in M$ *if and only if* $T^{\#} = 0$, *implying that* M *is an improper affine sphere, and there exists an orthonormal basis* $\{e_1,\dots,e_n\}$ *at* p *such that with respect this basis every linear map* K_X, $X \in T_pM$, *takes the following form:*

$$K_X = \begin{pmatrix} A_1^X & & & 0 \\ & \ddots & & \\ & & A_k^X & \\ 0 & & & 0 \end{pmatrix}, \tag{26}$$

where $\{A_j^X\}_{j=1}^k$ *are symmetric* $n_j \times n_j$ *submatrices satisfying*

$$\operatorname{trace}(A_1^X) = \cdots = \operatorname{trace}(A_k^X) = 0. \tag{27}$$

From Theorem 3 we immediately obtain the following corollary.

COROLLARY 4. *If* (M,h) *is a Riemannian manifold and for some* k*-tuple* $(n_1,\dots,n_k) \in \mathcal{S}(n)$ *the* δ*-invariant satisfies* $\delta^{\#}(n_1,\dots,n_k) < 0$ *at some point, then* (M,h) *cannot be realized as improper affine sphere in some affine space.*

If we assume additionally that $\dim(\operatorname{Im} K) = n$, we obtain the following theorem.

THEOREM 5. *Suppose that M is a graph hypersurface in $\mathbf{R}^{n+1}$, $n \geq 3$, such that the Calabi metric is positive definite and satisfies the equality case of one of the inequalities in* (25) *for some k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$. If $\dim(\operatorname{Im} K) = n$, then M is locally the Calabi composition of k improper affine spheres of dimension $n_1, \ldots, n_k$.*

Proof. We use the notations of section 3. From Theorem 2 we know that M is locally the Riemannian product $M_1 \times \cdots \times M_k$, where each M_j is a leaf of $\mathcal{D}_j$. Since $\mathcal{D}_j \oplus \xi$ is parallel along M_j, each M_j is contained in an $n_j + 1$-dimensional affine subspace $\mathbb{R}^{n_j+1}$ spanned by $\mathcal{D}_j$ and ξ. Hence we can consider M_j as graph hypersurface with ξ as affine normal. Since M_j is totally geodesic in (M, h), it is easy to see that the induced connection, Calabi metric and difference tensor of M_j are the restrictions of the induced connection, Calabi metric and difference tensor of M to M_j. Hence each M_j is an improper affine sphere with affine normal in the direction of ξ.

If we choose coordinates on $(x_1, \ldots, x_n, x_{n+1})$ on $\mathbb{R}^{n+1}$ such that $\xi = (0, \ldots, 0, 1)$ and each $\mathbb{R}^{n_j+1}$ is given by the equations $x_i = 0$, $i \notin \Delta_j, 1 \leq i \leq n$. Then M_j is given locally as the graph of a function F_j.

If $X_i \in \mathcal{D}_i$ and $X_j \in \mathcal{D}_j$ for $i \neq j$, then from (18) it follows easily that $K(X_i, X_j) = 0$, see [1]. Moreover from (1) and (3) it then follows that

$$D_{X_i} f_*(X_j) = 0,$$

which implies that after translation M is given by

$$(x_1, \ldots, x_n, F_1(x_1, \ldots, x_{n_1}) + \cdots + F_k(x_{n-n_k+1}, \ldots, x_n)). \blacksquare$$

References

[1] B.-Y. Chen, F. Dillen, L. Verstraelen, *Affine δ-invariants and their applications to centroaffine geometry*, Differential Geom. Appl. 22 (2005), 341–354.

[2] F. Dillen and L. Vrancken, *Calabi-type composition of affine spheres*, Differential Geom. Appl. 4 (1994), 303–328.

[3] H. Z. Li, *Variational problems and PDEs in affine differential geometry*, this volume, 9–41.

[4] K. Nomizu and T. Sasaki, *Affine Differential Geometry. Geometry of Affine Immersions*, Cambridge Tracts in Mathematics 111, Cambridge University Press, Cambridge, 1994.