PDES, SUBMANIFOLDS AND
AFFINE DIFFERENTIAL GEOMETRY
BANACH CENTER PUBLICATIONS, VOLUME 69
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2005

INVARIANCE GROUPS OF RELATIVE NORMALS

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Abstract. We investigate a two-parameter family of relative normals that contains Manhart's one-parameter family and the centroaffine normal. The invariance group of each of these normals is classified, and variational problems are studied. The results are Euler-Lagrange equations for the hypersurfaces that are critical with respect to the area functionals of the induced and semi-Riemannian volume forms and a classification of the critical hyperovaloids in the two-parameter family.

1. Introduction. F. Manhart [4] introduced a one-parameter family of relative normals that contains the Euclidean and the Blaschke normal. Obviously, for any two given conormals on a non-degenerate hypersurface $\mathfrak{x}$, there is a one-parameter family connecting them. This family is unique up to affine reparametrizations.

For example, suppose $\mathfrak{x}$ is a non-degenerate centroaffine hypersurface immersion, then the Euclidean support function ρ^E never vanishes. One can add another parameter to Manhart's family which joins the centroaffine normal, i.e.

$$\mathfrak{Y}^{(a,b)} = (\rho^E)^{-b}|H_n^E|^{-a}\mu, \qquad a,b \in \mathbb{R}. \tag{1}$$

Here μ, ρ^E, H_n^E denote the Euclidean conormal, support function, and Gauss-Kronecker curvature, respectively. (The sign is fixed by $\rho^E > 0$.)

For a relative normal $\mathfrak{y}$ for $\mathfrak{x}$ we define the area functionals with respect to the induced and semi-Riemannian volume forms ω and $\hat{\omega}$ by

$$A = A(M,\mathfrak{x},\mathfrak{y}) := \int_M \omega, \quad \text{and} \quad \hat{A} = \hat{A}(M,\mathfrak{x},\mathfrak{y}) := \int_M \hat{\omega}.$$

Let $A^{(a,b)} := A(M,\mathfrak{x},\mathfrak{y}^{(a,b)})$ and $\hat{A}^{(a,b)} := \hat{A}(M,\mathfrak{x},\mathfrak{y}^{(a,b)})$.

2000 *Mathematics Subject Classification*: Primary 53A15; Secondary 53A10.
The paper is in final form and no version of it will be published elsewhere.

In affine differential geometry the Blaschke area $A^e := A^{(\frac{1}{4},0)}$ is one of the best analysed functionals. Work in this direction was started by Blaschke [1] for dimension two. The first and second variation of A^e for arbitrary dimension had been studied by Calabi [2]; further contributions include [5] and [10]. Another approach is to use $\hat{A}$, which was followed by [3], [4]. Wang [11] studies the first and second variation of the centroaffine area $\hat{A}^c := \hat{A}^{(0,1)}$.

The first and second variation of $A^{(a,0)}$ in Manhart's one-parameter family were studied by the second author in [12]. Results for the first variation of $A^{(0,b)}$ can be found in [7]. In this paper we derive Euler-Lagrange equations for the first variation of $A^{(a,b)}$ and $\hat{A}^{(a,b)}$ and prove

THEOREM 1. *Let $\mathfrak{x} : M^n \to A^{n+1}$ be a hyperovaloid which is $A^{(a,b)}$-critical and suppose $(a,b) \neq (1,0)$. Then $\mathfrak{x}(M)$ is a sphere.*

REMARK 2. Any hypersurface is $A^{(1,0)}$-critical.

2. Relative geometry of hypersurfaces. For a detailed introduction to the subject see e.g. [6] or [9].

Consider a non-degenerate C^∞-immersion $\mathfrak{x} : M^n \to A^{n+1}$ of an n-dimensional, $n \geq 2$, connected oriented C^∞-manifold into real flat affine space with standard flat connection $\overline{\nabla}$. Suppose that $\mathfrak{y} : M^n \to \mathbb{R}^{n+1}$ is a C^∞ transversal vector field along $\mathfrak{x}$, i.e. $d\mathfrak{x}(T_pM) \oplus \mathbb{R}\mathfrak{y}(p) = \mathbb{R}^{n+1}$ at each $p \in M$. The vector space associated to A^{n+1} is denoted by $\mathbb{R}^{n+1}$. The *structure equations* of $\mathfrak{x}$ with respect to $\mathfrak{y}$ read as follows:

$$\overline{\nabla}_u d\mathfrak{x}(v) = d\mathfrak{x}(\nabla_u v) + h(u,v)\mathfrak{y}, \quad d\mathfrak{y}(u) = -d\mathfrak{x}(Su) + \theta(u)\mathfrak{y}$$

for all vector fields $u, v \in \mathfrak{X}(M)$. If $\mathfrak{y}$ has vanishing connection form θ, then it is called a *relative normal.* From now on we will always assume that $\mathfrak{y}$ is a relative normal. In this case the pair $(\mathfrak{x}, \mathfrak{y})$ is called a *relative hypersurface.*

h is a symmetric bilinear form which is also non-degenerate since $\mathfrak{x}$ is non-degenerate; it is hence called the *relative metric* induced by $\mathfrak{y}$. We denote the Levi-Civita connection of h and the positive valued semi-Riemannian volume form of h by $\hat{\nabla}$ and $\hat{\omega}$, respectively. ∇ is a torsion-free Ricci-symmetric affine connection called the *induced connection.* S is called the *shape operator*. Its trace $nH := \operatorname{trace} S$ is the *relative mean curvature* and its determinant $H_n := \det S$ is the *relative Gauss-Kronecker curvature.* The *induced volume form* ω is defined by

$$\omega(u_1, \ldots, u_n) := \det(d\mathfrak{x}(u_1), \ldots, d\mathfrak{x}(u_n), \mathfrak{y});$$

it is parallel with respect to the induced connection: $\nabla\omega = 0$.

Define the $(1,2)$-*difference tensor* C, the *Tchebychev vector field* T and the *Tchebychev form* $T^\flat$ by

$$C(u,v) = C_u v := \nabla_u v - \hat{\nabla}_u v, \qquad nh(T,u) = nT^\flat(u) := \operatorname{trace}\{v \mapsto C(v,u)\}.$$

Generally, $^\flat$ denotes the operation of lowering an index with respect to h.

Often it will be convenient to consider the *conormal* $\mathfrak{Y}$ to describe the normalization of a hypersurface, which is defined as a section of the cotangent line bundle satisfying

$$\langle \mathfrak{Y}, d\mathfrak{x} \rangle = 0 \qquad \text{and} \qquad \langle \mathfrak{Y}, \mathfrak{y} \rangle = 1,$$

where $\langle \cdot, \cdot \rangle : (\mathbb{R}^{n+1})^* \times \mathbb{R}^{n+1} \to \mathbb{R}$ denotes the standard scalar product. When talking about a *relative normalization*, we mean that either $\mathfrak{Y}$ or $\mathfrak{y}$ is given on $\mathfrak{x}$. This makes sense, since for relative hypersurfaces there is a bijective correspondence between normals and conormals.

The *relative support function* with respect to a point $\mathfrak{x}_0 \in A^{n+1}$ is defined by $\rho_{\mathfrak{x}_0} := \langle \mathfrak{Y}, \mathfrak{x}_0 - \mathfrak{x} \rangle$. Let Δ denote the Laplacian with respect to $\hat{\nabla}$. We define the Laplace-type operators

$$\Box f := \Delta f + nT^\flat(\operatorname{grad}_h f), \quad \Box^* f := \Delta f - nT^\flat(\operatorname{grad}_h f).$$

The induced quantities are *invariant* with respect to the full affine group $GL(n+1, \mathbb{R}) \oplus \mathbb{R}^{n+1}$ acting on $\mathbb{R}^{n+1}$ in the following sense: For any given relative hypersurface $(\mathfrak{x}, \mathfrak{y})$ and $(B, \mathfrak{b}) \in GL(n+1, \mathbb{R}) \oplus \mathbb{R}^{n+1}$, the coefficients of the structure equations of $(\mathfrak{x}, \mathfrak{y})$ and $(\mathfrak{x}^\natural := B\mathfrak{x} + \mathfrak{b}, \mathfrak{y}^\natural := B\mathfrak{y})$ coincide: $\nabla = \nabla^\natural$, $h = h^\natural$, and $S = S^\natural$.

We now list some formulas describing the change of relative normalization.

LEMMA 1. *For a hypersurface $\mathfrak{x} : M^n \to A^{n+1}$, any two conormals $\mathfrak{Y}$ and $\mathfrak{Y}^\natural$ with the same orientation are related by $\mathfrak{Y}^\natural = e^\varphi \mathfrak{Y}$, where $\varphi \in C^\infty(M)$. Under this transition, the relative metric changes conformally: $h^\natural = e^\varphi h$. Moreover, we compute (see e.g. [9])*

$$\begin{aligned}
\mathfrak{y}^\natural &= e^{-\varphi}(\mathfrak{y} + dx(\operatorname{grad}\varphi)),\\
\nabla^\natural_u v &= \nabla_u v - h(u,v)\operatorname{grad}\varphi,\\
\Delta^\natural f &= e^{-\varphi}\left(\Delta f + \frac{n-2}{2} d\varphi(\operatorname{grad} f)\right),\\
\hat{\omega}^\natural &= e^{\frac{n}{2}\varphi}\hat{\omega},\\
\omega^\natural &= e^{-\varphi}\omega,\\
S^\natural u &= e^{-\varphi}(Su - \nabla_u \operatorname{grad}\varphi + u(\varphi)\operatorname{grad}\varphi),\\
H^\natural &= e^{-\varphi}\left(H - \frac{1}{n}\Delta\varphi - T^\flat(\operatorname{grad}\varphi) + \frac{1}{n}\|\operatorname{grad}\varphi\|^2\right),\\
T^{\flat\natural} &= T^\flat - \frac{n+2}{2n} d\varphi,\\
T^\natural &= e^{-\varphi}\left(T - \frac{n+2}{2n}\operatorname{grad}\varphi\right).
\end{aligned}$$

Finally, let us mention some special relative normals.

(i) The *Blaschke normal* $\mathfrak{y}^e$ is determined up to sign by $|\omega| = \hat{\omega}$, which is called the *apolarity condition*; it is also characterized up to a non-vanishing constant factor by $T = 0$. The Blaschke normal is invariant with respect to unimodular affine transformations $SL(n+1, \mathbb{R}) \oplus \mathbb{R}^{n+1}$, meaning that for any unimodular affine transformation $(B, \mathfrak{b})$ the Blaschke normal of $\mathfrak{x}^\natural = B\mathfrak{x} + \mathfrak{b}$ is $\mathfrak{y}^{\natural e} = B\mathfrak{y}^e$. Invariants induced by the Blaschke normal will be denoted by ^{e}.

(ii) For an appropriate choice of an origin, any non-degenerate hypersurface locally can be endowed with $\mathfrak{y}^c := -\mathfrak{x}$, which is the *centroaffine normal.* It is characterized by $S = \mathrm{id}$. Therefore, a proper relative sphere is exactly the underlying hypersurface with its centroaffine normal up to a constant factor. The centroaffine normal is invariant with respect to $GL(n+1,\mathbb{R})$. Centroaffine invariants will be marked by ^{c} if ambiguous.

(iii) Locally, we can normalize any hypersurface with a constant transversal field, which is always a relative normal. The hypersurface will be an improper relative sphere with respect to this normal.

(iv) The Euclidean normal is a relative normal which is invariant with respect to the group of Euclidean motions $SO(n+1,\mathbb{R})\oplus\mathbb{R}^{n+1}$. Euclidean invariants will be marked by ^{E} if ambiguous. Moreover, we denote fundamental forms by I, $\mathrm{II} := h^E$ and write $\mu := \mathfrak{Y}^E = \mathfrak{y}^E$.

3. Invariance groups of constructions of relative normals. The *construction* of a relative normal is a mapping which assigns a relative normal $\mathfrak{y}$ to a given non-degenerate hypersurface $\mathfrak{x}$. The *invariance group* of such a construction is the maximal subgroup $I \subseteq GL(n+1,\mathbb{R}) \oplus \mathbb{R}^{n+1}$ such that the order of construction and transformation does not matter, i.e. for any $g \in I$ with linear part B we have $c \circ g = B \circ c$ on the set of all non-degenerate hypersurfaces.

Examples of constructions are E, e and c. Of course, we are only interested in a small subset of all constructions, namely those with big invariance groups. In the generic case, invariance groups will be $\{(id,0)\}$. The invariance groups of relative normals in the two-parameter family will be denoted by $I^{(a,b)}$.

LEMMA 2. *Let $\mathfrak{x}\colon M^n \to A^{n+1}$ be a non-degenerate hypersurface. For a given conormal $\mathfrak{Y}$ and $q \in C^\infty(M)$, let $\mathfrak{Y}^{(a)} = q^a\mathfrak{Y}$ be a one-parameter family of relative conormals. Assume $\rho = \rho^{(0)} \neq 0$. Let G be a subgroup of the full affine group such that $G \subseteq I^{(a_0)}$ and $G \subseteq I^{(a_1)}$ for two values $a_0 \neq a_1$. Then $G \subseteq I^{(a)}$ for all $a \in \mathbb{R}$.*

Proof. Without loss of generality we can assume $a_0 = 0$ and $a_1 = 1$, for otherwise the one-parameter family $\mathfrak{Y}^{(a)} = \tilde{q}^{\tilde{a}}\tilde{\mathfrak{Y}}$ where $\tilde{q} = q^{a_1-a_0}$, $\tilde{a} = \frac{a-a_0}{a_1-a_0}$ and $\tilde{\mathfrak{Y}} = q^{a_0}\mathfrak{Y}$ satisfies this condition. The proof is trivial for a pure translation, so assume the affine map from G fixes the origin and has matrix B. Let $\mathfrak{x}^\natural = B\mathfrak{x}$, $\mathfrak{y}^\natural = B\mathfrak{y}^{(a)}$ be the transformed hypersurface. It suffices to prove $\mathfrak{y}^{\natural(a)} = \mathfrak{y}^\natural$. We know that $\mathfrak{y}^{\natural(0)} = B\mathfrak{y}^{(0)}$ and $\mathfrak{y}^{\natural(1)} = B\mathfrak{y}^{(1)}$, thus $\mathfrak{Y}^{\natural(0)} = B^{*-1}\mathfrak{Y}^{(0)}$ and $\mathfrak{Y}^{\natural(1)} = B^{*-1}\mathfrak{Y}^{(1)}$. From the assumption we can express $q = \frac{\rho^{(1)}}{\rho^{(0)}}$. We get $\rho^{\natural(0)} = \rho^{(0)}$ and $\rho^{\natural(1)} = \rho^{(1)}$, hence $q^\natural = q$. Finally we get

$$\langle \mathfrak{Y}^{\natural(a)}, \mathfrak{y}^\natural \rangle = \langle q^{\natural a}B^{*-1}\mathfrak{Y}, Bq^{-a}(\mathfrak{y} + d\mathfrak{x}\,\mathrm{grad}\log q^a)\rangle = 1. \;\blacksquare$$

COROLLARY 3. *(i) In the particular case that $I^{(a_0)} \subseteq I^{(a_1)}$ it follows that $I^{(a)} \cap I^{(a_1)} = I^{(a_0)}$ for each $a \in \mathbb{R}\backslash\{a_1\}$.*

(ii) If $I^{(a_0)} \subseteq I^{(a_1)} = GL(n+1,\mathbb{R}) \oplus \mathbb{R}^{n+1}$, then $I^{(a)} = I^{(a_0)}$ for each $a \in \mathbb{R}\backslash\{a_1\}$.

The following theorem is an extension of [12], Theorem 5.7.

THEOREM 4. *Let* $\mathfrak{x}\colon M^n \to A^{n+1}$ *be a non-degenerate hypersurface and* $a, b \in \mathbb{R}$. *The invariance group of the relative normal* $\mathfrak{y}^{(a,b)}$ *is*

(i) $SL(n+1,\mathbb{R}) \oplus \mathbb{R}^{n+1}$ *if* $(a,b) = (\frac{1}{n+2}, 0)$,
(ii) $\mathbb{R}^+SO(n+1,\mathbb{R}) \oplus \mathbb{R}^{n+1}$ *if* $(a,b) = (-\frac{1}{n}, 0)$ *(in this case* $\mathfrak{y}^{(a,b)}$ *is called the* conformal relative normal*)*,
(iii) $SO(n+1,\mathbb{R}) \oplus \mathbb{R}^{n+1}$ *if* $a \notin \{\frac{1}{n+2}, -\frac{1}{n}\}, b = 0$,
(iv) $GL(n+1,\mathbb{R})$ *if* $(a,b) = (0,1)$,
(v) $SL(n+1,\mathbb{R})$ *if* $a(n+2)+b = 1$ *and* $y^{(a,b)}$ *is neither the Blaschke nor the centroaffine normal,*
(vi) $\mathbb{R}^+SO(n+1,\mathbb{R})$ *if* $b = an+1$ *and* $y^{(a,b)}$ *is neither the conformal nor the centroaffine normal,*
(vii) $SO(n+1,\mathbb{R})$ *otherwise.*

Proof. As ρ^E is not invariant with respect to translations (origin is fixed), we know that translations are not part of the invariance group for $b \neq 0$. We exclude the line $a(n+2)+b = 1$. This part of the proof follows easily from Lemma 2, where one has to use invariance groups of the centroaffine and the Blaschke normalizations.

Invariance. Suppose we have a linear transformation $\mathfrak{x}^\natural = B\mathfrak{x}$ of $\mathfrak{x}$ such that $B = cD$ for some $D \in O(n+1,\mathbb{R})$ and $0 \neq c \in \mathbb{R}$. Define $\mathfrak{y}^\natural = B\mathfrak{y}^{(a,b)}$. It is our aim to find conditions under which $\mathfrak{y}^\natural = \mathfrak{y}^{(a,b)\natural}$, where $\mathfrak{y}^{(a,b)\natural}$ is the relative normal of $\mathfrak{x}^\natural$ belonging to the parameter (a,b). We have $d\mathfrak{x}^\natural = cDd\mathfrak{x}$ and thus $\mu^\natural = D\mu$. The sign of $\mu^\natural$ is chosen such that

$$\rho^{E\natural} = -\langle \mu^\natural, \mathfrak{x}^\natural\rangle = -\langle D\mu, cD\mathfrak{x}\rangle = -c\langle \mu, \mathfrak{x}\rangle = c\rho^E.$$

Moreover,

$$-d\mathfrak{x}^\natural(S^{E\natural}u) = d\mu^\natural(u) = Dd\mu(u) = -\epsilon Dd\mathfrak{x}(S^E u) = -c^{-1}d\mathfrak{x}^\natural(S^E u).$$

We get $S^{\natural E} = c^{-1}S^E$, hence $H_n{}^{\natural E} = c^{-n}H_n^E$. Finally,

$$\mathfrak{Y}^{(a,b)\natural} = \rho^{E\natural - b}|H_n^{E\natural}|^{-a}\mu^\natural = c^{an-b}D\mathfrak{Y}^{(a,b)}$$

and

$$\langle \mathfrak{Y}^{(a,b)\natural}, \mathfrak{y}^\natural\rangle = c^{an-b}\langle D\mathfrak{Y}^{(a,b)}, B\mathfrak{y}^{(a,b)}\rangle = c^{an-b+1}\langle D\mathfrak{Y}^{(a,b)}, D\mathfrak{y}^{(a,b)}\rangle = c^{an-b+1} = 1.$$

This works only for $c = 1$ or $b = an+1$.

Maximality. Suppose that $\mathfrak{x}, \mathfrak{x}^\natural\colon M^n \to A^{n+1}$ are non-degenerate hypersurfaces such that $\mathfrak{x}^\natural = B\mathfrak{x}$ and $\mathfrak{y}^{(a,b)\natural} = B\mathfrak{y}^{(a,b)}$, where $B \in GL(n+1,\mathbb{R})$. Then all objects induced on M by $(\mathfrak{x}, \mathfrak{y}^{(a,b)})$ and $(\mathfrak{x}^\natural, \mathfrak{y}^{(a,b)\natural})$ coincide. Let us show that we can write $B = cD$ for some $D \in SO(n+1,\mathbb{R})$ and $c \in \mathbb{R}\backslash\{0\}$, where $c = 1$ follows from the invariance part. This is done if we prove $\nabla(\mathrm{I}^\natural) = \nabla(\mathrm{I})$ and $\mathrm{II}^\natural = c\,\mathrm{II}$. The definition can be rewritten as

$$\mathfrak{Y}^{(a,b)} = \rho^{E-a(n+2)-b}\rho^{ea(n+2)}\mu = \rho^{E-a(n+2)-b+1}\rho^{ea(n+2)-1}\mathfrak{Y}^e.$$

We have

$$T^{\flat(a,b)} = \frac{n+2}{2n}d\log(\rho^{E\,a(n+2)+b-1}\rho^{e\,1-a(n+2)})$$

With $T^{\flat(a,b)} = T^{\flat(a,b)\natural}$ we get

$$\left(\frac{\rho^{e\natural}}{\rho^e}\right)^{1-a(n+2)}\left(\frac{\rho^{E\natural}}{\rho^E}\right)^{a(n+2)+b-1} = \text{const} \neq 0.$$

Under a $GL(n+1,\mathbb{R})$ transformation of the hypersurface the Blaschke support function is changed by a constant factor which equals the determinant of the transformation matrix. Thus, for $a(n+2)+b \neq 1$, we get that $\rho^{\natural E}/\rho^E = \text{const}$. We obtain $\mathrm{II}^\natural = c\,\mathrm{II}$ from

$$\rho^{E-a(n+2)-b}\rho^{ea(n+2)}\,\mathrm{II} = h^{(a,b)} = h^{\natural(a,b)} = \rho^{E\natural-a(n+2)-b}\rho^{e\natural a(n+2)}\mathrm{II}^\natural.$$

The proof is finished by recalling $\nabla^{\natural(a,b)} = \nabla^{(a,b)}$ in

$$\begin{aligned}\nabla^{(a,b)\natural}_u v &= \nabla(\mathrm{I}^\natural)_u v - \mathrm{II}^\natural(u,v)\,\mathrm{grad}(\mathrm{II}^\natural)\log(\rho^{\natural E-a(n+2)-b}\rho^{e\natural a(n+2)})\\ &= \nabla(\mathrm{I}^\natural)_u v - \mathrm{II}(u,v)\,\mathrm{grad}(\mathrm{II})\log(\rho^{E-a(n+2)-b}\rho^{ea(n+2)})\\ &= \nabla(\mathrm{I}^\natural)_u v + \nabla^{(a,b)}_u v - \nabla(\mathrm{I})_u v.\end{aligned}$$

The classification follows from the unification of the two parts. ∎

We conclude this section by mentioning another possibility of a one-parameter family.

THEOREM 5. *Suppose* $\det S^e \neq 0$. *Then* $\mathfrak{Y}^{(a)} = |H_n{}^e|^{-a}\mathfrak{Y}^e$ *is a one-parameter family with invariance group*

(i) $GL(n+1,\mathbb{R}) \oplus \mathbb{R}^{n+1}$ *if* $a = -\frac{2(n+1)}{n^2}$, *and*
(ii) $SL(n+1,\mathbb{R}) \oplus \mathbb{R}^{n+1}$ *otherwise.*

Proof. First observe that translations are included in the invariance groups since the construction is translation independent. For $B = cD \in GL(n+1,\mathbb{R})$ where $c > 0$ and $\det D = \pm 1$ suppose $\mathfrak{x}^\natural = B\mathfrak{x}$ and let $\mathfrak{y}^\natural = B\mathfrak{y}^{(a)}$. As in the previous proof, we ask for $\mathfrak{y}^\natural = \mathfrak{y}^{(a)\natural}$. As before we get $S^{e\natural} = c^{-\frac{n}{n+2}}S^e$, thus $H_n{}^{e\natural} = c^{-\frac{n^2}{n+2}}H_n{}^e$. We obtain $\mathfrak{Y}^{(a)\natural} = c^{\frac{n(1+an)}{n+2}}D^{*-1}\mathfrak{Y}^{(a)}$. Finally, $\langle\mathfrak{Y}^{(a)\natural},\mathfrak{y}^\natural\rangle = c^{\frac{an^2+2n+2}{n+2}}$ shows that either $c = 1$ or $a = -\frac{2(n+1)}{n^2}$. This shows that the normals are invariant with respect to the given groups. The maximality follows from Corollary 3 (ii). ∎

For hypersurfaces with non-singular Blaschke shape operator, there is a relative normalization which is invariant with respect to the full affine group.

COROLLARY 6. *Consider a one-parameter family* $\mathfrak{Y}^{(a)} = |H_n|^{-a}\mathfrak{Y}$ *with* $H_n \neq 0$. *Then* $I^{(0)} \subseteq I^{(a)}$ *for all* $a \in \mathbb{R}$.

4. First variation of area functionals. To do variational calculus we follow the notation of [12]. A *relative deformation* of a hypersurface $\mathfrak{x}$ with relative normal $\mathfrak{y}$ is a C^∞-family $(\mathfrak{x}^t,\mathfrak{y}^t)$ of non-degenerate relative hypersurfaces such that $\mathfrak{x}^0 = \mathfrak{x}$ and $\mathfrak{y}^0 = \mathfrak{y}$. We describe an infinitesimal deformation of $\mathfrak{x}$ by the pair (ψ,ϕ) defined by

$$\mathfrak{x}' = d\mathfrak{x}(\psi) + \phi\mathfrak{y}, \quad (.)' := \frac{\partial}{\partial t}(.)|_{t=0}.$$

We will use the following formulas from [12], Lemma 3.4 and (6.1.3), which hold for any relative deformation.

(2) $$\langle\mathfrak{Y}',d\mathfrak{x}\rangle = -\psi^\flat - d\phi,$$

$$(\log\omega)' = \langle\mathfrak{Y},\mathfrak{y}'\rangle - nH\phi + \operatorname{div}\psi, \tag{3}$$

$$(\log\hat\omega)' = \tfrac{1}{2}(-n\langle\mathfrak{Y},\mathfrak{y}'\rangle + \Box^*\phi - nH\phi + 2\hat{\operatorname{div}}\psi), \tag{4}$$

$$\rho' = -\rho\langle\mathfrak{Y},\mathfrak{y}'\rangle + h(\operatorname{grad}\rho,\psi+\operatorname{grad}\phi) - \phi. \tag{5}$$

Moreover, for a Euclidean deformation (i.e. $\mathfrak{y}^t$ is the Euclidean normal of $\mathfrak{x}^t$) with infinitesimal representation $(\tilde\psi,\tilde\phi)$ we obtain (cf. [12], (4.1.2b))

$$(\log H_n^E)' = \Box^E\tilde\phi + nH^E\tilde\phi - 2nT^{\flat E}(\tilde\psi + \operatorname{grad}(\mathrm{I\!I})\tilde\phi). \tag{6}$$

Proposition 7. *Let $\mathfrak{x}: M^n \to A^{n+1}$ be a non-degenerate hypersurface and $a,b\in\mathbb{R}$. Then*

(i) $\mathfrak{x}$ is $A^{(a,b)}$-critical if and only if

$$(a-1)nH^{(a,b)} - \frac{b}{\rho^{(a,b)}} = 0.$$

(ii) $\mathfrak{x}$ is $\hat A^{(a,b)}$-critical if and only if

$$(1+an)(\operatorname{div}(\hat\nabla^{(a,b)}T^{(a,b)}) - H^{(a,b)}) + \frac{b}{\rho^{(a,b)}} = 0.$$

Proof. Fix (a,b) and assume that for each deformed hypersurface $\mathfrak{x}^t$ the deformed normal is $\mathfrak{y}^{(a,b)t}$ from the two-parameter family. Then

$$d\mathfrak{x}(\psi) + \phi\mathfrak{y}^{(a,b)} = \mathfrak{x}' = d\mathfrak{x}(\tilde\psi) + \tilde\phi\mu$$

links the two representations. We will first compute the unknown part $\langle\mathfrak{Y}^{(a,b)},\mathfrak{y}^{(a,b)\prime}\rangle$ in the formulas above.

$$\begin{aligned}\langle\mathfrak{Y}^{(a,b)},\mathfrak{y}^{(a,b)\prime}\rangle &= -\langle(\rho^{E-b}|H_n^E|^{-a}\mu)',\rho^{Eb}|H_n^E|^a(\mu + d\mathfrak{x}(\operatorname{grad}\log(\rho^{E-b}|H_n^E|^{-a})))\rangle\\ &= aH_n^{E-1}(H_n^{E\prime} - \mathrm{I\!I}(\operatorname{grad}(\mathrm{I\!I})H_n^E,\tilde\psi + \operatorname{grad}(\mathrm{I\!I})\tilde\phi))\\ &\quad + b\rho^{E-1}(\rho^{E\prime} - \mathrm{I\!I}(\operatorname{grad}(\mathrm{I\!I})\rho^E,\tilde\psi+\operatorname{grad}(\mathrm{I\!I})\tilde\phi))\\ &= a(\Box^{(a,b)}\phi + nH^{(a,b)}\phi) - b\rho^{(a,b)-1}\phi.\end{aligned} \tag{7}$$

We used the fact $\tilde\phi = \rho^{Eb}|H_n^E|^a\phi$. For the first part of the assertion we compute, using (3) and (7),

$$\begin{aligned}(A^{(a,b)})' &= \int\omega^{(a,b)\prime} = \int(\langle\mathfrak{Y}^{(a,b)},\mathfrak{y}^{(a,b)\prime}\rangle - nH^{(a,b)}\phi + \operatorname{div}(\nabla^{(a,b)})\psi)\omega^{(a,b)}\\ &= \int(a\Box^{(a,b)}\phi + (a-1)nH^{(a,b)}\phi - b\rho^{(a,b)-1}\phi)\omega^{(a,b)}.\end{aligned}$$

Now (i) follows by the fundamental theorem since $\int\Box^{(a,b)}(\cdot)\omega^{(a,b)} = 0$. For the second part (ii),

$$\begin{aligned}(\hat A^{(a,b)})' &= \int\hat\omega^{(a,b)\prime} = \frac{1}{2}\int(-n\langle\mathfrak{Y}^{(a,b)},\mathfrak{y}^{(a,b)\prime}\rangle + \Box^{*(a,b)}\phi - nH^{(a,b)}\phi)\hat\omega^{(a,b)}\\ &= \frac{1}{2}\int((\Box^{*(a,b)} - an\Box^{(a,b)})\phi - (1+an)nH^{(a,b)}\phi + nb\rho^{(a,b)-1}\phi)\hat\omega^{(a,b)}\\ &= \frac{n}{2}\int((1+an)\hat{\operatorname{div}}^{(a,b)}T^{(a,b)} - (1+an)H^{(a,b)} + b\rho^{(a,b)-1})\phi\hat\omega^{(a,b)},\end{aligned}$$

where we have used (4), (7), the definitions of $\square$, $\square^*$ and the identity

$$\int T^\flat(\operatorname{grad}\phi)\omega = -\int h\Big(\operatorname{grad}\phi, \operatorname{grad}\log\Big|\frac{\omega}{\hat{\omega}}\Big|\Big)\Big)\omega = \int(\hat{\operatorname{div}}T)\phi\omega. \quad \blacksquare$$

We will now prove Theorem 1. Observe that by applying the first Minkowski integral formula

$$0 = \int(1-\rho^{(a,b)}H^{(a,b)})\omega^{(a,b)} = \frac{n(a-1)-b}{n(a-1)}\int\omega^{(a,b)}$$

we get $b = n(a-1)$. Proposition 7 (i) states that there is the relation $H = f(\rho) = \frac{1}{\rho}$ for some function f on the real line. The assertion follows from the following theorem of U. Simon for the first relative curvature function, which is the mean curvature.

THEOREM 8 ([8], Theorem 6.1). *Let $\mathfrak{x} : M^n \to A^{n+1}$ be a closed locally strongly convex C^5-hypersurface with a relative normal $\mathfrak{y}$. Suppose $H > 0$. Assume that there exists a C^1-function f such that $H = f(\rho)$ and $f' \leq 0$. Then $\mathfrak{x}(M)$ is a sphere.*

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