Abstract. We study the measurability of sets of pairs of straight lines with respect to the group of motions in the simply isotropic space $I_3^{(1)}$ by solving PDEs. Also some Crofton type formulas are obtained for the corresponding densities.

1. Introduction. The simply isotropic space $I_3^{(1)}$ is defined (see [7]) as a projective space $P_3(R)$ in which the absolute consists of a plane $\omega$ (the absolute plane) and two complex conjugate straight lines $f_1, f_2$ (the absolute lines) in $\omega$ with a real intersection point $F$ (the absolute point). In homogeneous coordinates $(x_0, x_1, x_2, x_3)$ we can take the plane $x_0 = 0$ as the plane $\omega$, the line $x_0 = 0, x_1 + ix_2 = 0$ as the line $f_1$, the line $x_0 = 0, x_1 - ix_2 = 0$ as the line $f_2$ and the point $(0,0,0,1)$ as the point $F$. The 6-parameter group $B_6^{(1)}$ of transformations (in affine coordinates $(x,y,z)$)

$$
\begin{align*}
x' &= c_1 + x \cos \varphi - y \sin \varphi, \\
y' &= c_2 + x \sin \varphi + y \cos \varphi, \\
z' &= c_3 + c_4 x + c_5 y + z,
\end{align*}
$$

where $c_1,c_2,c_3,c_4,c_5,\varphi \in R$, is called the group of simply isotropic motions in $I_3^{(1)}$. 

2000 Mathematics Subject Classification: Primary 53C65.

Key words and phrases: measurability, density, simply isotropic space.

The paper is in final form and no version of it will be published elsewhere.
We emphasize that much of the material which is common in the geometry of the simply isotropic space $I^{(1)}_3$ can be found in [7]. Using some basic concepts of integral geometry in the sense of M. I. Stoka [9], [10], G. I. Drinfel’d and A. V. Lucenko [4], [5], [6], we study the measurability of sets of pairs of straight lines in $I^{(3)}_1$. Analogous problems for points and planes in $I^{(1)}_3$ have been treated in [2].

2. Measurability of a set of pairs of skew straight lines. A straight line is said to be (completely) isotropic if its infinite point coincides with the absolute point $F$; otherwise the straight line is said to be nonisotropic [7; p. 5].

Let $G_1$ and $G_2$ be two nonisotropic straight lines and denote by $U_1$ and $U_2$ their infinite points, respectively. Then $G_1$ and $G_2$ are said to be of type $\alpha$ or of type $\beta$ if the points $U_1$, $U_2$ and $F$ are noncollinear or collinear, respectively [7; p. 45].

2.1. Density of pairs of skew nonisotropic straight lines of type $\alpha$. Let $(G_1, G_2)$ be a pair of skew nonisotropic straight lines of type $\alpha$ determined by the equations

$$G_i : \quad x = a_i z + p_i, \quad y = b_i z + q_i, \quad i = 1, 2,$$

where

$$a_2 - a_1 (q_2 - q_1) - (b_2 - b_1) (p_2 - p_1) \neq 0,$$

$$a_1 b_2 - a_2 b_1 \neq 0.$$

Under the action of (1) the pair $(G_1, G_2)(a_1, b_1, p_1, q_1, a_2, b_2, p_2, q_2)$ is transformed into the pair $(G'_1, G'_2)(a'_1, b'_1, p'_1, q'_1, a'_2, b'_2, p'_2, q'_2)$ as

$$a'_i = K_i (a_i \cos \varphi - b_i \sin \varphi),$$

$$b'_i = K_i (a_i \sin \varphi + b_i \cos \varphi),$$

$$p'_i = K_i \left( [-c_3 a_i + p_i + c_5 (b_i p_i - a_i q_i)] \cos \varphi + [c_3 b_i - q_i + c_4 (b_i p_i - a_i q_i)] \sin \varphi \right) + c_1,$n

$$q'_i = K_i \left( [-c_3 a_i + p_i + c_5 (b_i p_i - a_i q_i)] \sin \varphi - [c_3 b_i - q_i + c_4 (b_i p_i - a_i q_i)] \cos \varphi \right) + c_2,$n

where $K_i = (1 + a_i c_4 + b_i c_5)^{-1}, i = 1, 2$.

The transformations (3) form the associated group $\overline{B}_6^{(1)}$ of $B_6^{(1)}$ [9; p. 34], [10; p. 17]. $\overline{B}_6^{(1)}$ is isomorphic to $B_6^{(1)}$ and the invariant density with respect to $B_6^{(1)}$ of the pairs of lines $(G_1, G_2)$, if it exists, coincides with the invariant density with respect to $\overline{B}_6^{(1)}$ of the points $(a_1, b_1, p_1, q_1, a_2, b_2, p_2, q_2)$ in the set of parameters. The associated group $\overline{B}_6^{(1)}$ has the infinitesimal operators

$$Y_1 = \frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2}, \quad Y_2 = \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}, \quad Y_3 = a_1 \frac{\partial}{\partial p_1} + b_1 \frac{\partial}{\partial q_1} + a_2 \frac{\partial}{\partial p_2} + b_2 \frac{\partial}{\partial q_2},$$

$$Y_4 = b_1 \frac{\partial}{\partial a_1} - a_1 \frac{\partial}{\partial b_1} + q_1 \frac{\partial}{\partial p_1} - p_1 \frac{\partial}{\partial q_1} + b_2 \frac{\partial}{\partial a_2} - a_2 \frac{\partial}{\partial b_2} + q_2 \frac{\partial}{\partial p_2} - p_2 \frac{\partial}{\partial q_2},$$

$$Y_5 = a_1^2 \frac{\partial}{\partial a_1} + a_1 b_1 \frac{\partial}{\partial b_1} + a_1 p_1 \frac{\partial}{\partial p_1} + b_1 p_1 \frac{\partial}{\partial q_1} + a_2 \frac{\partial}{\partial a_2} + a_2 b_2 \frac{\partial}{\partial b_2} + a_2 p_2 \frac{\partial}{\partial p_2} + b_2 q_2 \frac{\partial}{\partial q_2}, \quad Y_6 = a_1 b_1 \frac{\partial}{\partial a_1} + b_1^2 \frac{\partial}{\partial b_1} + a_1 q_1 \frac{\partial}{\partial p_1} + b_1 q_1 \frac{\partial}{\partial q_1} + a_2 b_2 \frac{\partial}{\partial a_2} + b_2^2 \frac{\partial}{\partial b_2} + a_2 q_2 \frac{\partial}{\partial p_2} + b_2 q_2 \frac{\partial}{\partial q_2}.$$
The associated group $B_{i}^{(1)}$ acts intransitively on the set of pairs $(G_{1},G_{2})$ and therefore the pairs $(G_{1},G_{2})$ do not have invariant density under $B_{i}^{(1)}$. The system $Y_{i}(f) = 0$, $i = 1, \ldots, 6$ has two independent integrals

$$(5) \quad f' = \frac{a_{1}b_{2} - a_{2}b_{1}}{\sqrt{a_{1}^{2} + b_{1}^{2}} \sqrt{a_{2}^{2} + b_{2}^{2}}} , \quad f'' = \frac{(a_{2} - a_{1})(q_{2} - q_{1}) - (b_{2} - b_{1})(p_{2} - p_{1})}{a_{1}b_{2} - a_{2}b_{1}}$$

and (5) are absolute invariants of $B_{i}^{(1)}$. It follows that we can define the density for the set of the pairs $(G_{1},G_{2})$ $(a_{1},b_{1},p_{1},q_{1},a_{2},b_{2},p_{2},q_{2})$ of skew nonisotropic straight lines of type $\alpha$ by the equality

$$(6) \quad d(G_{1},G_{2}) = \left| \frac{(a_{2} - a_{1})(q_{2} - q_{1}) - (b_{2} - b_{1})(p_{2} - p_{1})}{\sqrt{a_{1}^{2} + b_{1}^{2}} \sqrt{a_{2}^{2} + b_{2}^{2}}} \right| \times \text{da}_{1} \wedge \text{db}_{1} \wedge \text{dp}_{1} \wedge \text{dq}_{1} \wedge \text{da}_{2} \wedge \text{db}_{2} \wedge \text{dp}_{2} \wedge \text{dq}_{2}.$$ 

**Remark 1.** We note that [7; p. 45]

$$(7) \quad \sin \psi = \frac{a_{1}b_{2} - a_{2}b_{1}}{\sqrt{a_{1}^{2} + b_{1}^{2}} \sqrt{a_{2}^{2} + b_{2}^{2}}} , \quad \delta(G_{1},G_{2}) = \frac{(a_{2} - a_{1})(q_{2} - q_{1}) - (b_{2} - b_{1})(p_{2} - p_{1})}{a_{1}b_{2} - a_{2}b_{1}},$$

where $\psi$ and $\delta(G_{1},G_{2})$ are the angle and the distance from $G_{1}$ to $G_{2}$, respectively.

Denote $\delta(G_{1},G_{2}) = \delta$ and replacing (7) into (6) we find another expression for density:

$$(8) \quad d(G_{1},G_{2}) = |\delta| \sin \psi \text{da}_{1} \wedge \text{db}_{1} \wedge \text{dp}_{1} \wedge \text{dq}_{1} \wedge \text{da}_{2} \wedge \text{db}_{2} \wedge \text{dp}_{2} \wedge \text{dq}_{2}. $$

On the other hand, the set of nonisotropic straight lines $G_{i}(a_{i},b_{i},p_{i},q_{i})$ is measurable with respect to the group $B_{i}^{(1)}$ and has the invariant density [2]

$$(9) \quad dG_{i} = \frac{\text{da}_{i} \wedge \text{db}_{i} \wedge \text{dp}_{i} \wedge \text{dq}_{i}}{(a_{i}^{2} + b_{i}^{2})^{2}}.$$ 

Then putting (9) in (8) we obtain

$$(10) \quad d(G_{1},G_{2}) = |\delta| \sin \psi (a_{1}^{2} + b_{1}^{2})(a_{2}^{2} + b_{2}^{2})dG_{1} \wedge dG_{2}.$$ 

Assume that the straight line $G_{i}$ has the angle $\varphi_{i}$ with the horizontal plane $Oxy$ and $\vec{G}_{i}$ denotes the orthogonal projection of $G_{i}$ on $Oxy$. Then [7; p. 48]

$$(11) \quad \varphi_{i} = \frac{1}{\sqrt{a_{i}^{2} + b_{i}^{2}}} , \quad \vec{G}_{i} : \quad b_{i}x - a_{i}y + a_{i}q_{i} - b_{i}p_{i} = 0, \quad z = 0$$

and [2]

$$(12) \quad dG_{i} = \frac{1}{|a_{i}|} d\vec{G}_{i} \wedge d\varphi_{i} \wedge dp_{i} = \frac{1}{|b_{i}|} \text{d}\vec{G}_{i} \wedge d\varphi_{i} \wedge dq_{i},$$

where

$$(12') \quad d\vec{G}_{i} = \frac{1}{(a_{i}^{2} + b_{i}^{2})^{2}} (b_{i}^{2} \text{da}_{i} \wedge dp_{i} - a_{i}b_{i} \text{da}_{i} \wedge dq_{i} - a_{i}b_{i} \text{db}_{i} \wedge dp_{i} + a_{i}^{2} \text{db}_{i} \wedge dq_{i})$$

is the density for $\vec{G}_{i}$ in $Oxy$. Note that the plane $Oxy$ is Euclidean and $d\vec{G}_{i}$ is the metric density for the straight lines in $Oxy$ [8; p. 29], [9; p. 66].

Applying (11) and (12) to (10), we get

$$(13) \quad d(G_{1},G_{2}) = \frac{|\delta| \sin \psi}{\varphi_{1}^{4} \varphi_{2}^{4}} |dG_{1} \wedge dG_{2}|.$$
By differentiation of (7) and (11) and by exterior multiplication of (15) we get

\[ d(G_1, G_2) = \frac{\delta \sin \psi}{a_1 a_2 \varphi_1 \varphi_2^4} \left( d\tilde{G}_1 \wedge d\tilde{G}_2 \wedge d\varphi_1 \wedge d\varphi_2 \wedge dp_1 \wedge dp_2 \right) \]
\[ = \frac{\delta \sin \psi}{a_1 b_2 \varphi_1 \varphi_2^4} d\tilde{G}_1 \wedge d\tilde{G}_2 \wedge d\varphi_1 \wedge d\varphi_2 \wedge dp_1 \wedge dq_2 \]
\[ = \frac{\delta \sin \psi}{b_1 a_2 \varphi_1 \varphi_2^4} d\tilde{G}_1 \wedge d\tilde{G}_2 \wedge d\varphi_1 \wedge d\varphi_2 \wedge dq_1 \wedge dp_2 \]
\[ = \frac{\delta \sin \psi}{b_1 b_2 \varphi_1 \varphi_2^4} d\tilde{G}_1 \wedge d\tilde{G}_2 \wedge d\varphi_1 \wedge d\varphi_2 \wedge dq_1 \wedge dq_2. \]

If we denote \( P_1 = G_1 \cap Oxy \), \( P_2 = G_2 \cap Oxy \), then on the plane \( Oxy \) we have

\[ dP_1 = dp_1 \wedge dq_1, \quad dP_2 = dp_2 \wedge dq_2. \]

By differentiation of (7) and (11) and by exterior multiplication of (15) we get

\[ d\delta \wedge d\psi \wedge d\varphi_1 \wedge d\varphi_2 \wedge dP_1 \wedge dP_2 \]
\[ = \frac{\sin \psi[(a_2 - a_1)(p_2 - p_1) + (b_2 - b_1)(q_2 - q_1)]}{\varphi_1^2 \varphi_2^2} da_1 \wedge db_1 \wedge dp_1 \wedge dq_1 \wedge da_2 \wedge db_2 \wedge dp_2 \wedge dq_2. \]

Inserting (16) into (8) we obtain

\[ d(G_1, G_2) = \frac{\delta \varphi_1 \varphi_2^2}{(a_2 - a_1)(p_2 - p_1) + (b_2 - b_1)(q_2 - q_1)} d\delta \wedge d\psi \wedge d\varphi_1 \wedge d\varphi_2 \wedge dP_1 \wedge dP_2. \]

We summarize the foregoing results in the following

**Theorem 1.** The density for the pairs \((G_1, G_2)\) of skew nonisotropic straight lines of type \(\alpha\), determined by (2), (2') and (2'') satisfies the relations (8), (10), (13), (14) and (17).

**2.2. Density of pairs of skew nonisotropic straight lines of type \(\beta\).** Let \((G_1, G_2)\) be a pair of skew nonisotropic straight lines of type \(\beta\) determined by (2), (2') and the equality

\[ a_1 b_2 - a_2 b_1 = 0. \]

Without loss of generality we can assume that \(a_1 \neq 0\) and then we have

\[ b_2 = \frac{a_2}{a_1} b_1. \]

From (2) and (18) it follows that the pairs \((G_1, G_2)\) are determined by the equations

\[ G_1 : \quad x = a_1 z + p_1, \quad y = b_1 z + q_1, \quad a_1 \neq 0, \]
\[ G_2 : \quad x = a_2 z + p_2, \quad y = \frac{a_2}{a_1} b_1 z + q_2, \quad a_2 \neq 0. \]

Now, under the action of (1), the pair \((G_1, G_2)(a_1, b_1, p_1, q_1, a_2, p_2, q_2)\) is transformed into the pair \((G'_1, G'_2)(a'_1, b'_1, p'_1, q'_1, a'_2, p'_2, q'_2)\) and the corresponding associated group \(B^{(1)}_6\)
has the infinitesimal operators (see (4))

$$Z_1 = \frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2}, \quad Z_2 = \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}, \quad Z_3 = a_1 \frac{\partial}{\partial p_1} + b_1 \frac{\partial}{\partial q_1} + a_2 \frac{\partial}{\partial p_2} + \frac{a_2 b_1}{a_1} \frac{\partial}{\partial q_2},$$

$$Z_4 = b_1 \frac{\partial}{\partial a_1} - a_1 \frac{\partial}{\partial b_1} + q_1 \frac{\partial}{\partial p_1} - p_1 \frac{\partial}{\partial q_1} + \frac{a_2 b_1}{a_1} \frac{\partial}{\partial a_2} + q_2 \frac{\partial}{\partial p_2} - p_2 \frac{\partial}{\partial q_2},$$

$$Z_5 = a_1^2 \frac{\partial}{\partial a_1} + a_1 b_1 \frac{\partial}{\partial b_1} + a_1 p_1 \frac{\partial}{\partial p_1} + b_1 p_1 \frac{\partial}{\partial q_1} + a_2^2 \frac{\partial}{\partial a_2} + a_2 p_2 \frac{\partial}{\partial p_2} + \frac{a_2}{a_1} b_2 p_2 \frac{\partial}{\partial q_2},$$

$$Z_6 = a_1 b_1 \frac{\partial}{\partial a_1} + b_2^2 \frac{\partial}{\partial b_1} + a_1 q_1 \frac{\partial}{\partial p_1} + b_1 q_1 \frac{\partial}{\partial q_1} + \frac{a_2^2}{a_1} b_1 \frac{\partial}{\partial a_2} + a_2 q_2 \frac{\partial}{\partial p_2} + \frac{a_2}{a_1} b_1 q_2 \frac{\partial}{\partial q_2}.$$ 

Since $B_6^{(1)}$ acts intransitively on the set of pairs $(G_1, G_2)$ it follows that the pairs $(G_1, G_2)$ do not have invariant density with respect to $B_6^{(1)}$. The system $Z_i(f) = 0, \ i = 1, \ldots, 6$ has two independent integrals

$$f' = \frac{b_1 (p_2 - p_1) - a_1 (q_2 - q_1)}{\sqrt{a_1^2 + b_1^2}}, \quad f'' = \frac{a_1 - a_2}{a_2 \sqrt{a_1^2 + b_1^2}},$$

that are absolute invariants of $B_6^{(1)}$.

Now we can define the density for the pairs $(G_1, G_2)$ of skew nonisotropic straight lines of type $\beta$ by the equality

$$d(G_1, G_2) = \left| (a_1 - a_2) \left[ b_1 (p_2 - p_1) - a_1 (q_2 - q_1) \right] \right| \left| da_1 \wedge db_1 \wedge dp_1 \wedge dq_1 \wedge da_2 \wedge dp_2 \wedge dq_2.\right|$$

**Remark 2.** We note that [7; p. 45–46]

$$a = \frac{b_1 (p_2 - p_1) - a_1 (q_2 - q_1)}{\sqrt{a_1^2 + b_1^2}}, \quad s = \frac{a_1 - a_2}{a_2 \sqrt{a_1^2 + b_1^2}}$$

are the distance and the angle from $G_1$ to $G_2$, respectively. It follows that

$$d(G_1, G_2) = |as| \left| da_1 \wedge db_1 \wedge dp_1 \wedge dq_1 \wedge da_2 \wedge dp_2 \wedge dq_2.\right|$$

By differentiation of (11) and (20) and by exterior multiplication of (15) we find

$$da \wedge ds \wedge d\varphi_1 \wedge dP_1 \wedge dP_2 \wedge d\varphi_2$$

$$= \left| \frac{s^2 a_1 [a_1 (p_2 - p_1) + b_1 (q_2 - q_1)]}{(a_1^2 - a_2^2)^2 \varphi_1^3} \right| \left| da_1 \wedge db_1 \wedge dp_1 \wedge dq_1 \wedge da_2 \wedge dp_2 \wedge dq_2.\right|$$

Putting (22) into (21) we obtain

$$d(G_1, G_2) = \left| \frac{a (a_1 - a_2)^2 \varphi_1^3}{sa_1 [a_1 (p_2 - p_1) + b_1 (q_2 - q_1)]} \right| \left| da \wedge ds \wedge d\varphi_1 \wedge dP_1 \wedge dP_2.\right|$$

So we can state:

**Theorem 2.** The density for the pairs $(G_1, G_2)$ of skew nonisotropic straight lines of type $\beta$, determined by (19), satisfies the relations (21) and (23).

**2.3. Density of pairs of skew nonisotropic and isotropic straight lines.** Let $(G_1, G_2)$ be a pair of the skew nonisotropic straight line $G_1 : x = a_1 z + p_1, y = b_1 z + q_1$ and the isotropic straight line $G_2 : x = p_2, y = q_2$, where $b_1 (p_2 - p_1) - a_1 (q_2 - q_1) \neq 0.$ The
corresponding associated group \( B_6^{(1)} \) has the infinitesimal operators

\[
U_1 = \frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2}, \quad U_2 = \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}, \quad U_3 = a_1 \frac{\partial}{\partial p_1} + b_1 \frac{\partial}{\partial q_1},
\]

\[
U_4 = b_1 \frac{\partial}{\partial a_1} - a_1 \frac{\partial}{\partial b_1} + q_1 \frac{\partial}{\partial p_1} - p_1 \frac{\partial}{\partial q_1} + q_2 \frac{\partial}{\partial p_2} - p_2 \frac{\partial}{\partial q_2},
\]

\[
U_5 = a_2 \frac{\partial}{\partial a_1} + a_1 b_1 \frac{\partial}{\partial b_1} + a_1 p_1 \frac{\partial}{\partial p_1} + b_1 p_1 \frac{\partial}{\partial q_1},
\]

\[
U_6 = a_1 b_1 \frac{\partial}{\partial a_1} + b_1^2 \frac{\partial}{\partial b_1} + a_1 q_1 \frac{\partial}{\partial p_1} + b_1 q_1 \frac{\partial}{\partial q_1}.
\]

It is easy to verify that \( B_6^{(1)} \) acts intransitively on the set of pairs \((G_1, G_2)\) and therefore the pairs \((G_1, G_2)\) do not have invariant density under \( B_6^{(1)} \). The system \( U_i(f) = 0, i = 1, \ldots, 6 \) has the solution

\[
f = \frac{b_1(p_2 - p_1) - a_1(q_2 - q_1)}{\sqrt{a_1^2 + b_1^2}},
\]

that is an absolute invariant of \( B_6^{(1)} \).

We define the density of the pairs \((G_1, G_2)(a_1, b_1, p_1, q_1, p_2, q_2)\) by the equality

\[
d(G_1, G_2) = \left| \frac{b_1(p_2 - p_1) - a_1(q_2 - q_1)}{\sqrt{a_1^2 + b_1^2}} \right| da_1 \wedge db_1 \wedge dp_1 \wedge dq_1 \wedge dp_2 \wedge dq_2.
\]

**Remark 3.** We note that [7; p. 46]

\[
l = \frac{b_1(p_2 - p_1) - a_1(q_2 - q_1)}{\sqrt{a_1^2 + b_1^2}}
\]

is the distance from \( G_1 \) to \( G_2 \) and consequently (24) can be written in the form

\[
d(G_1, G_2) = |l| da_1 \wedge db_1 \wedge dp_1 \wedge dq_1 \wedge dp_2 \wedge dq_2.
\]

On the other hand, differentiating (11) (for \( i = 1 \)) and (25) and by exterior multiplication of (15) we obtain

\[
dl \wedge d\varphi_1 \wedge dP_1 \wedge dP_2 = \frac{a_1(p_2 - p_1) + b_1(q_2 - q_1)}{(a_1^2 + b_1^2)^2} da_1 \wedge db_1 \wedge dp_1 \wedge dq_1 \wedge dp_2 \wedge dq_2.
\]

Substituting (27) into (26), we find

\[
d(G_1, G_2) = \left| \frac{l_\varphi_1^4}{a_1(p_2 - p_1) + b_1(q_2 - q_1)} \right| dl \wedge d\varphi_1 \wedge dP_1 \wedge dP_2.
\]

Thus the following theorem is true:

**Theorem 3.** The density for the pairs \((G_1, G_2)\) of the skew nonisotropic straight line \( G_1 : x = a_1 z + p_1, y = b_1 z + q_1 \) and the isotropic straight line \( G_2 : x = p_2, y = q_2 \) satisfies the relations (26) and (28).

3. **Measurability of a set of pairs of intersecting straight lines**

3.1. **Density of pairs of intersecting nonisotropic straight lines of type \( \alpha \).** Let \((G_1, G_2)\) be a pair of intersecting nonisotropic straight lines of type \( \alpha \) given by
and from (31) and (32) it follows that
\[ G_1 : \quad x = a_1 z + p - a_1 r, \quad y = b_1 z + q - b_1 r, \]
\[ G_2 : \quad x = a_2 z + p - a_2 r, \quad y = b_2 z + q - b_2 r, \]
i.e. \( G_1 \cap G_2 = P(p, q, r) \) and (2") is true. The corresponding associated group \( \overline{B}_6^{(1)} \) consists of the transformations
\[
a'_i = (1 + a_i c_4 + b_i c_5)^{-1}(a_i \cos \varphi - b_i \sin \varphi),
\]
\[
b'_i = (1 + a_i c_4 + b_i c_5)^{-1}(a_i \sin \varphi + b_i \cos \varphi), \quad i = 1, 2,
\]
\[
p' = c_1 + p \cos \varphi - q \sin \varphi,
\]
\[
q' = c_2 + p \sin \varphi + q \cos \varphi,
\]
\[
r' = c_3 + c_4 p + c_5 q + r
\]
and has the infinitesimal operators
\[
Y_1 = \frac{\partial}{\partial p}, \quad Y_2 = \frac{\partial}{\partial q}, \quad Y_3 = \frac{\partial}{\partial r}, \quad Y_4 = b_1 \frac{\partial}{\partial a_1} - a_1 \frac{\partial}{\partial b_1} + b_2 \frac{\partial}{\partial a_2} - a_2 \frac{\partial}{\partial b_2} + q \frac{\partial}{\partial p} - p \frac{\partial}{\partial q},
\]
\[
Y_5 = a_1^2 \frac{\partial}{\partial a_1} + a_1 b_1 \frac{\partial}{\partial b_1} + a_2^2 \frac{\partial}{\partial a_2} + a_2 b_2 \frac{\partial}{\partial b_2} - p \frac{\partial}{\partial r},
\]
\[
Y_6 = a_1 b_1 \frac{\partial}{\partial a_1} + b_1^2 \frac{\partial}{\partial b_1} + a_2 b_2 \frac{\partial}{\partial a_2} + b_2^2 \frac{\partial}{\partial b_2} - q \frac{\partial}{\partial r}.
\]
The group \( \overline{B}_6^{(1)} \) is intransitive and therefore the set of pairs of intersecting straight lines (29) is not measurable with respect to \( B_6^{(1)} \). But the value
\[
f = \frac{a_1 b_2 - a_2 b_1}{\sqrt{a_1^2 + b_1^2} \sqrt{a_2^2 + b_2^2}}
\]
is an absolute invariant of \( \overline{B}_6^{(1)} \) and so we can define the density for the pairs \((G_1, G_2)\) \((a_1, b_1, a_2, b_2, p, q, r)\) by the equality
\[
d(G_1, G_2) = \left| \frac{a_1 b_2 - a_2 b_1}{\sqrt{a_1^2 + b_1^2} \sqrt{a_2^2 + b_2^2}} \right| da_1 \wedge db_1 \wedge da_2 \wedge db_2 \wedge dp \wedge dq \wedge dr.
\]
Replacing (7) to (30) we obtain
\[
d(G_1, G_2) = | \sin \psi | da_1 \wedge db_1 \wedge da_2 \wedge db_2 \wedge dp \wedge dq \wedge dr.
\]
Denote by \((x_1, y_1, 0)\) and \((x_2, y_2, 0)\) the coordinates of the points \( \overline{P}_1 = G_1 \cap Oxy \) and \( \overline{P}_2 = G_2 \cap Oxy \), respectively. Since
\[
x_1 = p - a_1 r, \quad y_1 = q - b_1 r, \quad i = 1, 2,
\]
then
\[
d\overline{P}_1 \wedge d\overline{P}_2 \wedge d\overline{P} = dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge dp \wedge dq \wedge dr
\]
\[
= r^4 da_1 \wedge db_1 \wedge da_2 \wedge db_2 \wedge dp \wedge dq \wedge dr
\]
and from (31) and (32) it follows that
\[
d(G_1, G_2) = \left| \frac{\sin \psi}{r^4} \right| d\overline{P}_1 \wedge d\overline{P}_2 \wedge d\overline{P}.
\]
On the other hand, on the plane $Oxy$ we have the classical Blaschke formula [1], [8; p. 59]

\begin{equation}
(34) \quad dP_1 \wedge dP_2 \wedge d\tilde{P} = \tilde{D}^3 d\tilde{G}_1 \wedge d\tilde{G}_2 \wedge d\tilde{G},
\end{equation}

where $\tilde{P}$ is the orthogonal projection of the point $P$ on the coordinate plane $Oxy$, $\tilde{G}_1 = P_1 \tilde{P}$, $\tilde{G}_2 = P_2 \tilde{P}$, $\tilde{G} = P_1 P_2$ and $\tilde{D}$ is the diameter of the circumscribed circle of the triangle $P_1 P_2 \tilde{P}$. Since $dP = d\tilde{P} \wedge dr$, by (33) and (34) we get

\begin{equation}
(35) \quad d(G_1, G_2) = \left| \frac{\tilde{D}^3 \sin \psi}{r^4} \right| d\tilde{G}_1 \wedge d\tilde{G}_2 \wedge d\tilde{G} \wedge dr.
\end{equation}

Further, we have

\begin{equation}
G_i: \quad b_i x - a_i y + a_i q - b_i p = 0, \quad z = 0
\end{equation}

and according to (12') we can write

\begin{equation}
(36) \quad d\tilde{G}_i = \frac{1}{(a_i^2 + b_i^2)^{3/2}} (b_i^2 da_i \wedge dp - a_i b_i da_i \wedge dq - a_i b_i db_i \wedge dp + a_i^2 db_i \wedge dq).
\end{equation}

From (11) we compute

\begin{equation}
(37) \quad d\varphi_i = -\frac{1}{(a_i^2 + b_i^2)^{3/2}} (a_i da_i + b_i db_i).
\end{equation}

By exterior multiplication of the forms of (36) and (37) for $i = 1, 2$, we find

\begin{equation}
\begin{aligned}
&d\varphi_1 \wedge d\tilde{G}_1 \wedge d\varphi_2 \wedge d\tilde{G}_2 = \sin \psi \frac{a_1^3 a_2^3}{\varphi_1^3 \varphi_2^3} da_1 \wedge db_1 \wedge da_2 \wedge db_2 \wedge dp \wedge dq,
\end{aligned}
\end{equation}

and comparing with (31), we obtain

\begin{equation}
(38) \quad d(G_1, G_2) = \phi_1^3 \phi_2^3 d\varphi_1 \wedge d\tilde{G}_1 \wedge d\varphi_2 \wedge d\tilde{G}_2 \wedge dr.
\end{equation}

Thus we are ready to state the following

**Theorem 4.** The density for the pairs $(G_1, G_2)$ of intersecting nonisotropic straight lines of type $\alpha$, determined by (29), satisfies the relations (31), (33), (35) and (38).

**3.2. Density of pairs of intersecting nonisotropic straight lines of type $\beta$.** Let $(G_1, G_2)$ be a pair of intersecting nonisotropic straight lines of type $\beta$. Without loss of generality, we can assume that $G_1$ and $G_2$ have equations of the form

\begin{equation}
\begin{aligned}
G_1: \quad x &= a_1 z + p - a_1 r, \quad y = b_1 z + q - b_1 r, \quad a_1 \neq 0, \\
G_2: \quad x &= a_2 z + p - a_2 r, \quad y = \frac{a_2}{a_1} b_1 z + q - \frac{a_2}{a_1} b_1 r, \quad a_2 \neq 0.
\end{aligned}
\end{equation}

Then the corresponding associated group $B_6^{(1)}$ has the infinitesimal operators

\begin{equation}
\begin{aligned}
Z_1 &= \frac{\partial}{\partial p}, \quad Z_2 = \frac{\partial}{\partial q}, \quad Z_3 = \frac{\partial}{\partial r}, \quad Z_4 = b_1 \frac{\partial}{\partial a_1} - a_1 \frac{\partial}{\partial b_1} + \frac{a_2}{a_1} b_1 \frac{\partial}{\partial a_2} + q \frac{\partial}{\partial p} - p \frac{\partial}{\partial q}, \\
Z_5 &= a_1^2 \frac{\partial}{\partial a_1} + a_1 b_1 \frac{\partial}{\partial b_1} + a_2^2 \frac{\partial}{\partial a_2} - p \frac{\partial}{\partial r}, \quad Z_6 = a_1 b_1 \frac{\partial}{\partial a_1} + b_1^2 \frac{\partial}{\partial b_1} + \frac{a_2}{a_1} b_1 \frac{\partial}{\partial a_2} - q \frac{\partial}{\partial r},
\end{aligned}
\end{equation}

and it is intransitive. Consequently the set of pairs of intersecting straight lines of type $\beta$ is not measurable under $B_6^{(1)}$. 
The system $Z_i(f) = 0$, $i = 1, \ldots, 6$, has an independent straight lines

$$f = \frac{a_1 - a_2}{a_2 \sqrt{a_1^2 + b_1^2}}$$

and it is an absolute invariant of $B_6^{(1)}$. We define the invariant density of the pairs $(G_1, G_2)(a_1, b_1, a_2, p, q, r)$ of type $\beta$ by the equality

$$d(G_1, G_2) = \left|\frac{a_1 - a_2}{a_2 \sqrt{a_1^2 + b_1^2}}\right| da_1 \land db_1 \land da_2 \land dp \land dq \land dr.$$  \hspace{1cm} (40)

In view of (20), (40) yields

$$d(G_1, G_2) = |s| da_1 \land db_1 \land da_2 \land dp \land dq \land dr.$$  \hspace{1cm} (41)

In this case, we find

$$da_1 \land db_1 \land dp \land dq \land dr = \frac{1}{r^2} d\overline{P}_1 \land dP,$$

where $\overline{P}_1 = G_1 \cap Oxy$, $P = G_1 \cap G_2$. But

$$ds = \frac{a_1 a_2 + b_1^2}{a_2(a_1^2 + b_1^2)^{3/2}} da_1 - \frac{(a_1 - a_2)b_1}{a_2(a_1^2 + b_1^2)^{3/2}} db_1 - \frac{a_1}{a_2^2(a_1^2 + b_1^2)^{3/2}} da_2$$  \hspace{1cm} (43)

and from (41), (42) and (43) we deduce

$$d(G_1, G_2) = \left|\frac{a_2(a_1 - a_2)}{a_1 r^2}\right| ds \land d\overline{P}_1 \land dP.$$  \hspace{1cm} (44)

We note that the straight lines $G_1$ and $G_2$ lie in the isotropic plane $\iota : b_1 x - a_1 y + a_2 q - b_1 p = 0$ and hence their orthogonal projections $\widetilde{G}_1$ and $\widetilde{G}_2$ coincide on $Oxy$, i.e. $\widetilde{G}_1 \equiv \widetilde{G}_2 \equiv \widetilde{G}$. Then (12'), (37) and (43) imply

$$ds \land d\varphi_1 \land d\widetilde{G} = \frac{a_1}{(a_1^2 + b_1^2)^{3/2}} (b_1 da_1 \land db_1 \land da_2 \land dp - a_1 da_1 \land db_1 \land da_2 \land dq).$$  \hspace{1cm} (45)

From (40) and (45) we obtain

$$d(G_1, G_2) = \left|\frac{s \varphi_1^5}{a_1 b_1}\right| ds \land d\varphi_1 \land d\widetilde{G} \land dq \land dr = \left|\frac{s \varphi_1^5}{a_1^2}\right| ds \land d\varphi_1 \land d\widetilde{G} \land dp \land dr.$$  \hspace{1cm} (46)

Therefore we have:

**THEOREM 5.** The density for the pairs $(G_1, G_2)$ of intersecting nonisotropic straight lines of type $\beta$, determined by (39), satisfies the relations (41), (44) and (46).

**3.3. Density of pairs of intersecting nonisotropic and isotropic straight lines.** Let $(G_1, G_2)$ be a pair of intersecting straight lines

$$G_1 : \quad x = a_1 z + p - a_1 r, \quad y = b_1 z + q - b_1 r,$$  \hspace{1cm} (47)

$$G_2 : \quad x = p, \quad y = q,$$
i.e. \( G_1 \) is nonisotropic and \( G_2 \) is isotropic. Now the corresponding associated group \( \overline{B}_6^{(1)} \) has the infinitesimal operators

\[
\begin{align*}
U_1 &= \frac{\partial}{\partial p}, \quad U_2 = \frac{\partial}{\partial q}, \quad U_3 = \frac{\partial}{\partial r}, \quad U_4 = b_1 \frac{\partial}{\partial a_1} - a_1 \frac{\partial}{\partial b_1} + q \frac{\partial}{\partial p} - p \frac{\partial}{\partial q}, \\
U_5 &= a_1^2 \frac{\partial}{\partial a_1} + a_1 b_1 \frac{\partial}{\partial b_1} - p \frac{\partial}{\partial r}, \quad U_6 = a_1 b_1 \frac{\partial}{\partial a_1} + b_2^2 \frac{\partial}{\partial b_1} - q \frac{\partial}{\partial r}
\end{align*}
\]

and it is transitive. Then the integral invariant function \( f = f(a_1, b_1, p, q, r) \) of the group \( B_6^{(1)} \) satisfies the system of R. Deltheil [3; p. 28], [9; p. 11], namely
\[
\begin{align*}
U_1(f) &= 0, \quad U_2(f) = 0, \quad U_3(f) = 0, \quad U_4(f) = 0, \quad U_5(f) + 3a_1 f = 0, \quad U_6(f) + 3b_1 f = 0
\end{align*}
\]
and has the solution

\[
f = \frac{h}{(a_1 + b_1)^\frac{3}{2}},
\]
where \( h = \text{const} \). Thus we have

**Theorem 6.** The set of the pairs \((G_1, G_2)\) of intersecting straight lines (47) is measurable with respect to \( B_6^{(1)} \) and the corresponding invariant density is

\[
d(G_1, G_2) = \frac{1}{(a_1^2 + b_1^2)^\frac{3}{2}} da_1 \wedge db_1 \wedge dp \wedge dq \wedge dr.
\]

From (11) and (48) it follows immediately that

\[
d(G_1, G_2) = \varphi_1^3 da_1 \wedge db_1 \wedge dp \wedge dq \wedge dr.
\]

On the other hand, by direct computation we obtain

\[
da_1 \wedge db_1 \wedge dp \wedge dq = \frac{1}{r^2} d\tilde{P} \wedge d\overline{P},
\]

where \( \tilde{P} = G_1 \cap Oxy \) and \( \overline{P} = G_2 \cap Oxy \). Applying (50) to (49) we get

\[
d(G_1, G_2) = \varphi_1^3 \frac{1}{r^2} dr \wedge d\tilde{P} \wedge d\overline{P}.
\]

Similarly, from (49) and

\[
da_1 \wedge db_1 \wedge dp \wedge dq \wedge dr = -\frac{1}{\varphi_1 r^3} d\varphi_1 \wedge d\tilde{P} \wedge d\overline{P}
\]

we find

\[
d(G_1, G_2) = \varphi_1^2 \frac{1}{r^3} d\varphi_1 \wedge d\tilde{P} \wedge d\overline{P}.
\]

We establish the following result:

**Theorem 7.** The density for the pairs \((G_1, G_2)\) of intersecting straight lines (47) satisfies the relations (49), (51) and (52).

### 4. Measurability of a set of pairs of parallel straight lines

#### 4.1. Density of pairs of parallel nonisotropic straight lines of different isotropic planes.

Let \((G_1, G_2)\) be a pair of parallel nonisotropic straight lines
\[ (53) \]
\[
G_1 : \quad x = az + p_1, \quad y = bz + q_1,
\]
\[
G_2 : \quad x = az + p_2, \quad y = bz + q_2, \quad a^2 + b^2 \neq 0
\]

that lie in different isotropic planes, i.e.

\[ (53') \]
\[
a(q_2 - q_1) - b(p_2 - p_1) \neq 0.
\]

The corresponding associated group \( \mathcal{B}^{(1)}_6 \) has the infinitesimal operators

\[
Y_1 = \frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2}, Y_2 = \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}, Y_3 = aY_1 + bY_2,
\]
\[
Y_4 = -b \frac{\partial}{\partial a} + a \frac{\partial}{\partial b} - q_1 \frac{\partial}{\partial p_1} + p_1 \frac{\partial}{\partial q_1} - q_2 \frac{\partial}{\partial p_2} + p_2 \frac{\partial}{\partial q_2},
\]
\[
Y_5 = a^2 \frac{\partial}{\partial a} + ab \frac{\partial}{\partial b} + ap_1 \frac{\partial}{\partial p_1} + bp_1 \frac{\partial}{\partial q_1} + ap_2 \frac{\partial}{\partial p_2} + bp_2 \frac{\partial}{\partial q_2},
\]
\[
Y_6 = ab \frac{\partial}{\partial a} + b^2 \frac{\partial}{\partial b} + aq_1 \frac{\partial}{\partial p_1} + bq_1 \frac{\partial}{\partial q_1} + aq_2 \frac{\partial}{\partial p_2} + bq_2 \frac{\partial}{\partial q_2}
\]

and obviously it is intransitive. Hence the set of pairs of parallel nonisotropic straight lines \( (53), (53') \) is not measurable with respect to \( \mathcal{B}^{(1)}_6 \).

On the other hand, the value

\[
f = \frac{a(q_2 - q_1) - b(p_2 - p_1)}{\sqrt{a^2 + b^2}}
\]

is an absolute invariant of \( \mathcal{B}^{(1)}_6 \) and we define the density for the pairs \((G_1, G_2)\) \((a, b, p_1, q_1, p_2, q_2)\) by the equality

\[ (54) \]
\[
d(G_1, G_2) = \left| \frac{a(q_2 - q_1) - b(p_2 - p_1)}{\sqrt{a^2 + b^2}} \right| da \wedge db \wedge dp_1 \wedge dq_1 \wedge dp_2 \wedge dq_2.
\]

**Remark 4.** The parallel straight lines on the coordinate plane \( Oxy \)

\[
\tilde{G}_1 : \quad bx - ay + aq_1 - bp_1 = 0, \quad z = 0
\]

and

\[
\tilde{G}_2 : \quad bx - ay + aq_2 - bp_2 = 0, \quad z = 0
\]

are the orthogonal projections of the parallel straight lines \( G_1 \) and \( G_2 \), respectively. Then the Euclidean distance between \( \tilde{G}_1 \) and \( \tilde{G}_2 \) is

\[ (55) \]
\[
\delta(\tilde{G}_1, \tilde{G}_2) = \left| \frac{a(q_2 - q_1) - b(p_2 - p_1)}{\sqrt{a^2 + b^2}} \right|.
\]

Putting (55) into (54) we get

\[ (56) \]
\[
d(G_1, G_2) = \delta(\tilde{G}_1, \tilde{G}_2) da \wedge db \wedge dp_1 \wedge dq_1 \wedge dp_2 \wedge dq_2.
\]

A computation leads to

\[ (57) \]
\[
d\delta \wedge d\varphi \wedge d\overline{P}_1 \wedge d\overline{P}_2 = -\frac{a(p_2 - p_1) + b(q_2 - q_1)}{(a^2 + b^2)^2} da \wedge db \wedge dp_1 \wedge dq_1 \wedge dp_2 \wedge dq_2,
\]

where \( \delta = \delta(\tilde{G}_1, \tilde{G}_2), \varphi = \zeta(G_1, Oxy) = \zeta(G_2, Oxy), \overline{P}_1 = G_1 \cap Oxy, \overline{P}_2 = G_2 \cap Oxy.\)
In view of (57), (56) yields

\[ d(G_1, G_2) = \left| \frac{\delta}{\varphi^4[a(p_2 - p_1) + b(q_2 - q_1)]} \right| d\delta \wedge d\varphi \wedge d\overline{P}_1 \wedge d\overline{P}_2. \]

**Theorem 8.** *The density for the pairs \((G_1, G_2)\) of parallel nonisotropic straight lines (53), (53') satisfies the relations (56) and (58).*

### 4.2. Density of pairs of parallel nonisotropic straight lines of coinciding isotropic planes.

Let \((G_1, G_2)\) be a pair of parallel nonisotropic straight lines that lie in an isotropic plane. Assume that \(G_1\) and \(G_2\) have the equations

\[
G_1: \quad x = az + p_1, \quad y = bz + q_1, \quad a \neq 0, \\
G_2: \quad x = az + p_2, \quad y = bz + q_1 + \frac{b}{a}(p_2 - p_1), \quad b \neq 0.
\]

Then the corresponding associated group has the infinitesimal operators

\[
Z_1 = \frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2}, \quad Z_2 = \frac{\partial}{\partial q_1}, \quad Z_3 = aZ_1 + bZ_2,
\]

\[
Z_4 = -b\frac{\partial}{\partial a} + a\frac{\partial}{\partial b} - q_1\frac{\partial}{\partial p_1} + p_1\frac{\partial}{\partial q_1} - \left[q_1 + \frac{b}{a}(p_2 - p_1)\right]\frac{\partial}{\partial p_2},
\]

\[
Z_5 = a^2\frac{\partial}{\partial a} + ab\frac{\partial}{\partial b} + ap_1\frac{\partial}{\partial p_1} + bp_1\frac{\partial}{\partial q_1} + ap_2\frac{\partial}{\partial p_2},
\]

\[
Z_6 = ab\frac{\partial}{\partial a} + b^2\frac{\partial}{\partial b} + aq_1\frac{\partial}{\partial p_1} + bq_1\frac{\partial}{\partial q_1} + a\left[q_1 + \frac{b}{a}(p_2 - p_1)\right]\frac{\partial}{\partial p_2}
\]

and it is intransitive. Therefore the set of pairs of parallel nonisotropic straight lines (59) is not measurable with respect to \(B_6^{(1)}\). The system \(Z_i(f) = 0, i = 1, \ldots, 6\), has the solution

\[ f = \frac{p_2 - p_1}{a} \]

and it is an absolute invariant of \(B_6^{(1)}\). Then we define the density for the pairs \((G_1, G_2)\) \((a, b, p_1, q_1, p_2)\) by the equality

\[ d(G_1, G_2) = \left| \frac{p_2 - p_1}{a} \right| da \wedge db \wedge dp_1 \wedge dq_1 \wedge dp_2. \]

**Remark 5.** The isotropic plane \(\iota: ax + by = 0\) intersects the straight lines \(G_1\) and \(G_2\) at the points

\[
P_1 \left( -\frac{b(aq_1 - bp_1)}{a^2 + b^2}, \frac{a(aq_1 - bp_1)}{a^2 + b^2}, \frac{ap_1 + bq_1}{a^2 + b^2} \right)
\]

and

\[
P_2 \left( -\frac{b(aq_1 - bp_1)}{a^2 + b^2}, \frac{a(aq_1 - bp_1)}{a^2 + b^2}, -\frac{(a^2 + b^2)p_2 + b(aq_1 - bp_1)}{a(a^2 + b^2)} \right),
\]

respectively. Obviously \(P_1\) and \(P_2\) are parallel points and then

\[ s(P_1, P_2) = \frac{p_1 - p_2}{a}, \]

i.e. (61) is the oriented \(s\)-distance from \(G_1\) to \(G_2\). Further we shall denote \(s(G_1, G_2) = s(P_1, P_2)\) by \(s\).
Hence by (60) and (61) we have
\[ d(G_1, G_2) = |s| \alpha \, ds \wedge \delta P_1 \wedge \delta P_2. \]  

We compute
\[ \alpha \, ds \wedge \delta P_1 \wedge \delta P_2 = \frac{a}{s^2} \, ds \wedge dP_1 \wedge dP_2, \]
and
\[ \alpha \, ds \wedge \delta P_1 \wedge \delta P_2 = -\frac{a}{s} \, d\varphi \wedge dP_1 \wedge dP_2, \]
where \( P_1 = G_1 \cap Oxy, P_2 = G_2 \cap Oxy, \varphi = \angle(G_1, Oxy) = \angle(G_2, Oxy). \) Substituting (63) and (64) into (62), we find
\[ d(G_1, G_2) = \left| \frac{a}{s} \right| \, ds \wedge \delta P_1 \wedge \delta P_2 \]
and
\[ d(G_1, G_2) = \left| \frac{a}{s} \right| \, d\varphi \wedge dP_1 \wedge dP_2, \]
respectively. Thus we have the following

**Theorem 9.** The density for the pairs \((G_1, G_2)\) of parallel nonisotropic straight lines (59) satisfies the relations (62), (65) and (66).

### 4.3. Density of pairs of isotropic straight lines.

Let \((G_1, G_2)\) be a pair of isotropic straight lines and
\[ G_1 : \quad x = p_1, \quad y = q_1, \quad G_2 : \quad x = p_2, \quad y = q_2, \]
where \((p_2 - p_1)^2 + (q_2 - q_1)^2 \neq 0.\) The corresponding associated group \(B_6^{(1)}\) has the infinitesimal operators
\[ U_1 = \frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2}, \quad U_2 = \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}, \quad U_3 = 0, \]
\[ U_4 = -q_1 \frac{\partial}{\partial p_1} + p_1 \frac{\partial}{\partial q_1} + q_2 \frac{\partial}{\partial p_2} - p_2 \frac{\partial}{\partial q_2}, \quad U_5 = 0, \quad U_6 = 0 \]
and it is intransitive. It follows that the set of pairs of isotropic straight lines is not measurable with respect to \(B_6^{(1)}.\) But
\[ f = (p_2 - p_1)^2 + (q_2 - q_1)^2 \]
is an absolute invariant of \(B_6^{(1)}\) and we can define the density for the pairs \((G_1, G_2)\) \((p_1, q_1, p_2, q_2)\) by the equality
\[ d(G_1, G_2) = \sqrt{(p_2 - p_1)^2 + (q_2 - q_1)^2} \, dp_1 \wedge dq_1 \wedge dp_2 \wedge dq_2. \]

**Remark 6.** We note that [7; p. 46]
\[ l^* = \sqrt{(p_2 - p_1)^2 + (q_2 - q_1)^2} \]
is the distance between \(G_1\) and \(G_2.\)
Inserting (15) and (69) into (68) we obtain

\[ d(G_1, G_2) = l^* d\overline{P}_1 \wedge d\overline{P}_2. \]  

On the other hand, we have

\[ d\overline{P}_1 \wedge d\overline{P}_2 = -\frac{1}{p_2 - p_1} dl^* \wedge dq_1 \wedge d\overline{P}_2 = -\frac{1}{q_2 - q_1} dp_1 \wedge dl^* \wedge d\overline{P}_2 \]

\[ = \frac{1}{p_2 - p_1} d\overline{P}_1 \wedge dl^* \wedge dq_2 = \frac{1}{q_2 - q_1} d\overline{P}_1 \wedge dp_2 \wedge dl^* \]

and therefore

\[ d(G_1, G_2) = \frac{l^*}{|p_2 - p_1|} dl^* \wedge dq_1 \wedge d\overline{P}_2 = \frac{l^*}{|q_2 - q_1|} dp_1 \wedge dl^* \wedge d\overline{P}_2 \]

\[ = \frac{l^*}{|p_2 - p_1|} d\overline{P}_1 \wedge dl^* \wedge dq_2 = \frac{l^*}{|q_2 - q_1|} d\overline{P}_1 \wedge dp_2 \wedge dl^*. \]  

Thus we have the following

**Theorem 10.** The density for the pairs \((G_1, G_2)\) of isotropic straight lines (67) satisfies the relations (70) and (71).

**References**


