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ON THE MEASURABILITY OF SETS OF PAIRS OF STRAIGHT LINES IN THE SIMPLY ISOTROPIC SPACE

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Abstract. We study the measurability of sets of pairs of straight lines with respect to the group of motions in the simply isotropic space $I_3^{(1)}$ by solving PDEs. Also some Crofton type formulas are obtained for the corresponding densities.

1. Introduction. The simply isotropic space $I_3^{(1)}$ is defined (see [7]) as a projective space $P_3(R)$ in which the absolute consists of a plane ω (the absolute plane) and two complex conjugate straight lines f_1, f_2 (the absolute lines) in ω with a real intersection point F (the absolute point). In homogeneous coordinates (x_0, x_1, x_2, x_3) we can take the plane $x_0 = 0$ as the plane ω, the line $x_0 = 0,\ x_1 + ix_2 = 0$ as the line f_1, the line $x_0 = 0,\ x_1 - ix_2 = 0$ as the line f_2 and the point $(0, 0, 0, 1)$ as the point F. The 6-parameter group $B_6^{(1)}$ of transformations (in affine coordinates (x, y, z))

$$
\begin{aligned}
x' &= c_1 + x\cos\varphi - y\sin\varphi,\\
y' &= c_2 + x\sin\varphi + y\cos\varphi,\\
z' &= c_3 + c_4 x + c_5 y + z,
\end{aligned}
\tag{1}
$$

where $c_1, c_2, c_3, c_4, c_5, \varphi \in R$, is called the group of simply isotropic motions in $I_3^{(1)}$.

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We emphasize that much of the material which is common in the geometry of the simply isotropic space $I_3^{(1)}$ can be found in [7]. Using some basic concepts of integral geometry in the sense of M. I. Stoka [9], [10], G. I. Drinfel'd and A. V. Lucenko [4], [5], [6], we study the measurability of sets of pairs of straight lines in $I_1^{(3)}$. Analogous problems for points and planes in $I_3^{(1)}$ have been treated in [2].

2. Measurability of a set of pairs of skew straight lines. A straight line is said to be (completely) isotropic if its infinite point coincides with the absolute point F; otherwise the straight line is said to be nonisotropic [7; p. 5].

Let G_1 and G_2 be two nonisotropic straight lines and denote by U_1 and U_2 their infinite points, respectively. Then G_1 and G_2 are said to be of type α or of type β if the points U_1, U_2 and F are noncollinear or collinear, respectively [7; p. 45].

2.1. *Density of pairs of skew nonisotropic straight lines of type α.* Let (G_1, G_2) be a pair of skew nonisotropic straight lines of type α determined by the equations

$$G_i: \quad x = a_i z + p_i, \quad y = b_i z + q_i, \quad i = 1, 2, \tag{2}$$

where

$$(a_2 - a_1)(q_2 - q_1) - (b_2 - b_1)(p_2 - p_1) \neq 0, \tag{2'}$$

$$a_1 b_2 - a_2 b_1 \neq 0. \tag{2''}$$

Under the action of (1) the pair $(G_1, G_2)(a_1, b_1, p_1, q_1, a_2, b_2, p_2, q_2)$ is transformed into the pair (G_1', G_2') $(a_1', b_1', p_1', q_1', a_2', b_2', p_2', q_2')$ as

$$\begin{aligned}
&a_i' = K_i(a_i \cos\varphi - b_i \sin\varphi),\\
&b_i' = K_i(a_i \sin\varphi + b_i \cos\varphi),\\
&p_i' = K_i\{[-c_3 a_i + p_i + c_5(b_i p_i - a_i q_i)]\cos\varphi + [c_3 b_i - q_i + c_4(b_i p_i - a_i q_i)]\sin\varphi\} + c_1,\\
&q_i' = K_i\{[-c_3 a_i + p_i + c_5(b_i p_i - a_i q_i)]\sin\varphi - [c_3 b_i - q_i + c_4(b_i p_i - a_i q_i)]\cos\varphi\} + c_2,
\end{aligned} \tag{3}$$

where $K_i = (1 + a_i c_4 + b_i c_5)^{-1}, i = 1, 2$.

The transformations (3) form the associated group $\overline{B}_6^{(1)}$ of $B_6^{(1)}$ [9; p. 34], [10; p. 17]. $\overline{B}_6^{(1)}$ is isomorphic to $B_6^{(1)}$ and the invariant density with respect to $B_6^{(1)}$ of the pairs of lines (G_1, G_2), if it exists, coincides with the invariant density with respect to $\overline{B}_6^{(1)}$ of the points $(a_1, b_1, p_1, q_1, a_2, b_2, p_2, q_2)$ in the set of parameters. The associated group $\overline{B}_6^{(1)}$ has the infinitesimal operators

$$\begin{aligned}
&Y_1 = \frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2}, \quad Y_2 = \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}, \quad Y_3 = a_1\frac{\partial}{\partial p_1} + b_1\frac{\partial}{\partial q_1} + a_2\frac{\partial}{\partial p_2} + b_2\frac{\partial}{\partial q_2},\\
&Y_4 = b_1\frac{\partial}{\partial a_1} - a_1\frac{\partial}{\partial b_1} + q_1\frac{\partial}{\partial p_1} - p_1\frac{\partial}{\partial q_1} + b_2\frac{\partial}{\partial a_2} - a_2\frac{\partial}{\partial b_2} + q_2\frac{\partial}{\partial p_2} - p_2\frac{\partial}{\partial q_2},\\
&Y_5 = a_1^2\frac{\partial}{\partial a_1} + a_1 b_1\frac{\partial}{\partial b_1} + a_1 p_1\frac{\partial}{\partial p_1} + b_1 p_1\frac{\partial}{\partial q_1} + a_2^2\frac{\partial}{\partial a_2} + a_2 b_2\frac{\partial}{\partial b_2}\\
&\qquad + a_2 p_2\frac{\partial}{\partial p_2} + b_2 p_2\frac{\partial}{\partial q_2}, \quad Y_6 = a_1 b_1\frac{\partial}{\partial a_1} + b_1^2\frac{\partial}{\partial b_1} + a_1 q_1\frac{\partial}{\partial p_1} + b_1 q_1\frac{\partial}{\partial q_1}\\
&\qquad + a_2 b_2\frac{\partial}{\partial a_2} + b_2^2\frac{\partial}{\partial b_2} + a_2 q_2\frac{\partial}{\partial p_2} + b_2 q_2\frac{\partial}{\partial q_2}.
\end{aligned} \tag{4}$$

The associated group $\overline{B}_6^{(1)}$ acts intransitively on the set of pairs (G_1, G_2) and therefore the pairs (G_1, G_2) do not have invariant density under $B_6^{(1)}$. The system $Y_i(f) = 0$, $i = 1, \ldots, 6$ has two independent integrals

$$(5) \qquad f' = \frac{a_1 b_2 - a_2 b_1}{\sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2}}, \quad f'' = \frac{(a_2 - a_1)(q_2 - q_1) - (b_2 - b_1)(p_2 - p_1)}{a_1 b_2 - a_2 b_1}$$

and (5) are absolute invariants of $\overline{B}_6^{(1)}$. It follows that we can define the density for the set of the pairs (G_1, G_2) $(a_1, b_1, p_1, q_1, a_2, b_2, p_2, q_2)$ of skew nonisotropic straight lines of type α by the equality

$$(6) \qquad d(G_1, G_2) = \left| \frac{(a_2 - a_1)(q_2 - q_1) - (b_2 - b_1)(p_2 - p_1)}{\sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2}} \right| \times da_1 \wedge db_1 \wedge dp_1 \wedge dq_1 \wedge da_2 \wedge db_2 \wedge dp_2 \wedge dq_2.$$

REMARK 1. We note that [7; p. 45]

$$(7) \quad \sin\psi = \frac{a_1 b_2 - a_2 b_1}{\sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2}}, \quad \delta(G_1, G_2) = \frac{(a_2 - a_1)(q_2 - q_1) - (b_2 - b_1)(p_2 - p_1)}{a_1 b_2 - a_2 b_1},$$

where ψ and $\delta(G_1, G_2)$ are the angle and the distance from G_1 to G_2, respectively.

Denote $\delta(G_1, G_2) = \delta$ and replacing (7) into (6) we find another expression for density:

$$(8) \qquad d(G_1, G_2) = |\delta \sin\psi| da_1 \wedge db_1 \wedge dp_1 \wedge dq_1 \wedge da_2 \wedge db_2 \wedge dp_2 \wedge dq_2.$$

On the other hand, the set of nonisotropic straight lines $G_i(a_i, b_i, p_i, q_i)$ is measurable with respect to the group $B_6^{(1)}$ and has the invariant density [2]

$$(9) \qquad dG_i = \frac{da_i \wedge db_i \wedge dp_i \wedge dq_i}{(a_i^2 + b_i^2)^2}.$$

Then putting (9) in (8) we obtain

$$(10) \qquad d(G_1, G_2) = |\delta \sin\psi|(a_1^2 + b_1^2)(a_2^2 + b_2^2) dG_1 \wedge dG_2.$$

Assume that the straight line G_i has the angle φ_i with the horizontal plane Oxy and $\widetilde{G_i}$ denotes the orthogonal projection of G_i on Oxy. Then [7; p. 48]

$$(11) \qquad \varphi_i = \frac{1}{\sqrt{a_i^2 + b_i^2}}, \quad \widetilde{G_i}: \quad b_i x - a_i y + a_i q_i - b_i p_i = 0, \quad z = 0$$

and [2]

$$(12) \qquad dG_i = \frac{1}{|a_i|} d\widetilde{G_i} \wedge d\varphi_i \wedge dp_i = \frac{1}{|b_i|} d\widetilde{G_i} \wedge d\varphi_i \wedge dq_i,$$

where

$$(12') \qquad d\widetilde{G_i} = \frac{1}{(a_i^2 + b_i^2)^{\frac{3}{2}}} (b_i^2 da_i \wedge dp_i - a_i b_i da_i \wedge dq_i - a_i b_i db_i \wedge dp_i + a_i^2 db_i \wedge dq_i)$$

is the density for $\widetilde{G_i}$ in Oxy. Note that the plane Oxy is Euclidean and $d\widetilde{G_i}$ is the metric density for the straight lines in Oxy [8; p. 29], [9; p. 66].

Applying (11) and (12) to (10), we get

$$(13) \qquad d(G_1, G_2) = \left| \frac{\delta \sin\psi}{\varphi_1{}^4 \varphi_2{}^4} \right| dG_1 \wedge dG_2$$

and

$$d(G_1,G_2) = \left|\frac{\delta\sin\psi}{a_1a_2{\varphi_1}^4{\varphi_2}^4}\right| d\widetilde{G_1}\wedge d\widetilde{G_2}\wedge d\varphi_1\wedge d\varphi_2\wedge dp_1\wedge dp_2 \tag{14}$$
$$= \left|\frac{\delta\sin\psi}{a_1b_2{\varphi_1}^4{\varphi_2}^4}\right| d\widetilde{G_1}\wedge d\widetilde{G_2}\wedge d\varphi_1\wedge d\varphi_2\wedge dp_1\wedge dq_2$$
$$= \left|\frac{\delta\sin\psi}{b_1a_2{\varphi_1}^4{\varphi_2}^4}\right| d\widetilde{G_1}\wedge d\widetilde{G_2}\wedge d\varphi_1\wedge d\varphi_2\wedge dq_1\wedge dp_2$$
$$= \left|\frac{\delta\sin\psi}{b_1b_2{\varphi_1}^4{\varphi_2}^4}\right| d\widetilde{G_1}\wedge d\widetilde{G_2}\wedge d\varphi_1\wedge d\varphi_2\wedge dq_1\wedge dq_2.$$

If we denote $\overline{P_1} = G_1 \cap Oxy$, $\overline{P_2} = G_2 \cap Oxy$, then on the plane Oxy we have

$$d\overline{P}_1 = dp_1\wedge dq_1, \quad d\overline{P}_2 = dp_2\wedge dq_2. \tag{15}$$

By differentiation of (7) and (11) and by exterior multiplication of (15) we get

$$d\delta\wedge d\psi\wedge d\varphi_1\wedge d\varphi_2\wedge d\overline{P}_1\wedge d\overline{P}_2 \tag{16}$$
$$= \frac{\sin\psi[(a_2-a_1)(p_2-p_1)+(b_2-b_1)(q_2-q_1)]}{\varphi_1^2\varphi_2^2} da_1\wedge db_1\wedge dp_1\wedge dq_1\wedge da_2\wedge db_2\wedge dp_2\wedge dq_2.$$

Inserting (16) into (8) we obtain

$$d(G_1,G_2) \tag{17}$$
$$= \left|\frac{\delta\varphi_1^2\varphi_2^2}{(a_2-a_1)(p_2-p_1)+(b_2-b_1)(q_2-q_1)}\right| d\delta\wedge d\psi\wedge d\varphi_1\wedge d\varphi_2\wedge d\overline{P}_1\wedge d\overline{P}_2.$$

We summarize the foregoing results in the following

THEOREM 1. *The density for the pairs (G_1, G_2) of skew nonisotropic straight lines of type α, determined by (2), (2′) and (2″) satisfies the relations (8), (10), (13), (14) and (17).*

2.2. *Density of pairs of skew nonisotropic straight lines of type β.* Let (G_1, G_2) be a pair of skew nonisotropic straight lines of type β determined by (2), (2′) and the equality

$$a_1b_2 - a_2b_1 = 0.$$

Without loss of generality we can assume that $a_1 \neq 0$ and then we have

$$b_2 = \frac{a_2}{a_1}b_1. \tag{18}$$

From (2) and (18) it follows that the pairs (G_1, G_2) are determined by the equations

$$\begin{aligned} G_1:&\quad x = a_1z + p_1, \quad y = b_1z + q_1, \qquad a_1 \neq 0,\\ G_2:&\quad x = a_2z + p_2, \quad y = \frac{a_2}{a_1}b_1z + q_2, \quad a_2 \neq 0. \end{aligned} \tag{19}$$

Now, under the action of (1), the pair $(G_1, G_2)(a_1, b_1, p_1, q_1, a_2, p_2, q_2)$ is transformed into the pair $(G_1', G_2')(a_1', b_1', p_1', q_1', a_2', p_2', q_2')$ and the corresponding associated group $\overline{B}_6^{(1)}$

has the infinitesimal operators (see (4))

$$Z_1 = \frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2}, \quad Z_2 = \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}, \quad Z_3 = a_1\frac{\partial}{\partial p_1} + b_1\frac{\partial}{\partial q_1} + a_2\frac{\partial}{\partial p_2} + \frac{a_2}{a_1}b_1\frac{\partial}{\partial q_2},$$
$$Z_4 = b_1\frac{\partial}{\partial a_1} - a_1\frac{\partial}{\partial b_1} + q_1\frac{\partial}{\partial p_1} - p_1\frac{\partial}{\partial q_1} + \frac{a_2}{a_1}b_1\frac{\partial}{\partial a_2} + q_2\frac{\partial}{\partial p_2} - p_2\frac{\partial}{\partial q_2},$$
$$Z_5 = a_1^2\frac{\partial}{\partial a_1} + a_1b_1\frac{\partial}{\partial b_1} + a_1p_1\frac{\partial}{\partial p_1} + b_1p_1\frac{\partial}{\partial q_1} + a_2^2\frac{\partial}{\partial a_2} + a_2p_2\frac{\partial}{\partial p_2} + \frac{a_2}{a_1}b_1p_2\frac{\partial}{\partial q_2},$$
$$Z_6 = a_1b_1\frac{\partial}{\partial a_1} + b_1^2\frac{\partial}{\partial b_1} + a_1q_1\frac{\partial}{\partial p_1} + b_1q_1\frac{\partial}{\partial q_1} + \frac{a_2^2}{a_1}b_1\frac{\partial}{\partial a_2} + a_2q_2\frac{\partial}{\partial p_2} + \frac{a_2}{a_1}b_1q_2\frac{\partial}{\partial q_2}.$$

Since $\overline{B}_6^{(1)}$ acts intransitively on the set of pairs (G_1, G_2) it follows that the pairs (G_1, G_2) do not have invariant density with respect to $B_6^{(1)}$. The system $Z_i(f) = 0$, $i = 1, \dots, 6$ has two independent integrals

$$f' = \frac{b_1(p_2 - p_1) - a_1(q_2 - q_1)}{\sqrt{a_1^2 + b_1^2}}, \qquad f'' = \frac{a_1 - a_2}{a_2\sqrt{a_1^2 + b_1^2}},$$

that are absolute invariants of $\overline{B}_6^{(1)}$.

Now we can define the density for the pairs $(G_1, G_2)(a_1, b_1, p_1, q_1, a_2, p_2, q_2)$ of skew nonisotropic straight lines of type β by the equality

$$d(G_1, G_2) = \left|\frac{(a_1 - a_2)[b_1(p_2 - p_1) - a_1(q_2 - q_1)]}{a_2(a_1^2 + b_1^2)}\right| da_1 \wedge db_1 \wedge dp_1 \wedge dq_1 \wedge da_2 \wedge dp_2 \wedge dq_2.$$

Remark 2. We note that [7; p. 45-46]

$$a = \frac{b_1(p_2 - p_1) - a_1(q_2 - q_1)}{\sqrt{a_1^2 + b_1^2}}, \qquad s = \frac{a_1 - a_2}{a_2\sqrt{a_1^2 + b_1^2}} \tag{20}$$

are the distance and the angle from G_1 to G_2, respectively. It follows that

$$d(G_1, G_2) = |as| da_1 \wedge db_1 \wedge dp_1 \wedge dq_1 \wedge da_2 \wedge dp_2 \wedge dq_2. \tag{21}$$

By differentiation of (11) and (20) and by exterior multiplication of (15) we find

$$\begin{aligned} &da \wedge ds \wedge d\varphi_1 \wedge d\overline{P}_1 \wedge d\overline{P}_2 \\ &= \left|\frac{s^2a_1[a_1(p_2 - p_1) + b_1(q_2 - q_1)]}{(a_1 - a_2)^2\varphi_1^3}\right| da_1 \wedge db_1 \wedge dp_1 \wedge dq_1 \wedge da_2 \wedge dp_2 \wedge dq_2. \end{aligned} \tag{22}$$

Putting (22) into (21) we obtain

$$d(G_1, G_2) = \left|\frac{a(a_1 - a_2)^2\varphi_1^3}{sa_1[a_1(p_2 - p_1) + b_1(q_2 - q_1)]}\right| da \wedge ds \wedge d\varphi_1 \wedge d\overline{P}_1 \wedge d\overline{P}_2. \tag{23}$$

So we can state:

Theorem 2. *The density for the pairs (G_1, G_2) of skew nonisotropic straight lines of type β, determined by (19), satisfies the relations (21) and (23).*

2.3. *Density of pairs of skew nonisotropic and isotropic straight lines.* Let (G_1, G_2) be a pair of the skew nonisotropic straight line $G_1 : x = a_1z + p_1, y = b_1z + q_1$ and the isotropic straight line $G_2 : x = p_2, y = q_2$, where $b_1(p_2 - p_1) - a_1(q_2 - q_1) \neq 0$. The corresponding associated group $\overline{B}_6^{(1)}$ has the infinitesimal operators

$$\begin{aligned}
U_1 &= \frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2}, \quad U_2 = \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}, \quad U_3 = a_1\frac{\partial}{\partial p_1} + b_1\frac{\partial}{\partial q_1},\\
U_4 &= b_1\frac{\partial}{\partial a_1} - a_1\frac{\partial}{\partial b_1} + q_1\frac{\partial}{\partial p_1} - p_1\frac{\partial}{\partial q_1} + q_2\frac{\partial}{\partial p_2} - p_2\frac{\partial}{\partial q_2},\\
U_5 &= a_1^2\frac{\partial}{\partial a_1} + a_1b_1\frac{\partial}{\partial b_1} + a_1p_1\frac{\partial}{\partial p_1} + b_1p_1\frac{\partial}{\partial q_1},\\
U_6 &= a_1b_1\frac{\partial}{\partial a_1} + b_1^2\frac{\partial}{\partial b_1} + a_1q_1\frac{\partial}{\partial p_1} + b_1q_1\frac{\partial}{\partial q_1}.
\end{aligned}$$

It is easy to verify that $\overline{B}_6^{(1)}$ acts intransitively on the set of pairs (G_1, G_2) and therefore the pairs (G_1, G_2) do not have invariant density under $B_6^{(1)}$. The system $U_i(f) = 0$, $i = 1, \dots, 6$ has the solution

$$f = \frac{b_1(p_2 - p_1) - a_1(q_2 - q_1)}{\sqrt{a_1^2 + b_1^2}},$$

that is an absolute invariant of $\overline{B}_6^{(1)}$.

We define the density of the pairs $(G_1, G_2)(a_1, b_1, p_1, q_1, p_2, q_2)$ by the equality

$$d(G_1, G_2) = \left|\frac{b_1(p_2 - p_1) - a_1(q_2 - q_1)}{\sqrt{a_1^2 + b_1^2}}\right| da_1 \wedge db_1 \wedge dp_1 \wedge dq_1 \wedge dp_2 \wedge dq_2. \tag{24}$$

Remark 3. We note that [7; p. 46]

$$l = \frac{b_1(p_2 - p_1) - a_1(q_2 - q_1)}{\sqrt{a_1^2 + b_1^2}} \tag{25}$$

is the distance from G_1 to G_2 and consequently (24) can be written in the form

$$d(G_1, G_2) = |l| da_1 \wedge db_1 \wedge dp_1 \wedge dq_1 \wedge dp_2 \wedge dq_2. \tag{26}$$

On the other hand, differentiating (11) (for $i = 1$) and (25) and by exterior multiplication of (15) we obtain

$$dl \wedge d\varphi_1 \wedge d\overline{P}_1 \wedge d\overline{P}_2 = \frac{a_1(p_2 - p_1) + b_1(q_2 - q_1)}{(a_1^2 + b_1^2)^2} da_1 \wedge db_1 \wedge dp_1 \wedge dq_1 \wedge dp_2 \wedge dq_2. \tag{27}$$

Substituting (27) into (26), we find

$$d(G_1, G_2) = \left|\frac{l\varphi_1^4}{a_1(p_2 - p_1) + b_1(q_2 - q_1)}\right| dl \wedge d\varphi_1 \wedge d\overline{P}_1 \wedge d\overline{P}_2. \tag{28}$$

Thus the following theorem is true:

Theorem 3. *The density for the pairs (G_1, G_2) of the skew nonisotropic straight line $G_1 : x = a_1z + p_1, y = b_1z + q_1$ and the isotropic straight line $G_2 : x = p_2, y = q_2$ satisfies the relations (26) and (28).*

3. Measurability of a set of pairs of intersecting straight lines

3.1. *Density of pairs of intersecting nonisotropic straight lines of type α.* Let (G_1, G_2) be a pair of intersecting nonisotropic straight lines of type α given by

$$\begin{aligned} G_1: &\quad x = a_1 z + p - a_1 r, \quad y = b_1 z + q - b_1 r, \\ G_2: &\quad x = a_2 z + p - a_2 r, \quad y = b_2 z + q - b_2 r, \end{aligned} \tag{29}$$

i.e. $G_1 \cap G_2 = P(p,q,r)$ and $(2'')$ is true. The corresponding associated group $\overline{B}_6^{(1)}$ consists of the transformations

$$\begin{aligned} a_i' &= (1 + a_i c_4 + b_i c_5)^{-1}(a_i \cos\varphi - b_i \sin\varphi), \\ b_i' &= (1 + a_i c_4 + b_i c_5)^{-1}(a_i \sin\varphi + b_i \cos\varphi), \qquad i = 1, 2, \\ p' &= c_1 + p\cos\varphi - q\sin\varphi, \\ q' &= c_2 + p\sin\varphi + q\cos\varphi, \\ r' &= c_3 + c_4 p + c_5 q + r \end{aligned}$$

and has the infinitesimal operators

$$\begin{aligned} &Y_1 = \frac{\partial}{\partial p}, \quad Y_2 = \frac{\partial}{\partial q}, \quad Y_3 = \frac{\partial}{\partial r}, \quad Y_4 = b_1\frac{\partial}{\partial a_1} - a_1\frac{\partial}{\partial b_1} + b_2\frac{\partial}{\partial a_2} - a_2\frac{\partial}{\partial b_2} + q\frac{\partial}{\partial p} - p\frac{\partial}{\partial q}, \\ &Y_5 = a_1^2\frac{\partial}{\partial a_1} + a_1 b_1\frac{\partial}{\partial b_1} + a_2^2\frac{\partial}{\partial a_2} + a_2 b_2\frac{\partial}{\partial b_2} - p\frac{\partial}{\partial r}, \\ &Y_6 = a_1 b_1\frac{\partial}{\partial a_1} + b_1^2\frac{\partial}{\partial b_1} + a_2 b_2\frac{\partial}{\partial a_2} + b_2^2\frac{\partial}{\partial b_2} - q\frac{\partial}{\partial r}. \end{aligned}$$

The group $\overline{B}_6^{(1)}$ is intransitive and therefore the set of pairs of intersecting straight lines (29) is not measurable with respect to $B_6^{(1)}$. But the value

$$f = \frac{a_1 b_2 - a_2 b_1}{\sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2}}$$

is an absolute invariant of $\overline{B}_6^{(1)}$ and so we can define the density for the pairs (G_1, G_2) $(a_1, b_1, a_2, b_2, p, q, r)$ by the equality

$$d(G_1, G_2) = \left|\frac{a_1 b_2 - a_2 b_1}{\sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2}}\right| da_1 \wedge db_1 \wedge da_2 \wedge db_2 \wedge dp \wedge dq \wedge dr. \tag{30}$$

Replacing (7) to (30) we obtain

$$d(G_1, G_2) = |\sin\psi| da_1 \wedge db_1 \wedge da_2 \wedge db_2 \wedge dp \wedge dq \wedge dr. \tag{31}$$

Denote by $(x_1, y_1, 0)$ and $(x_2, y_2, 0)$ the coordinates of the points $\overline{P}_1 = G_1 \cap Oxy$ and $\overline{P}_2 = G_2 \cap Oxy$, respectively. Since

$$x_i = p - a_i r, \qquad y_i = q - b_i r, \qquad i = 1, 2,$$

then

$$\begin{aligned} d\overline{P}_1 \wedge d\overline{P}_2 \wedge dP &= dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge dp \wedge dq \wedge dr \\ &= r^4 da_1 \wedge db_1 \wedge da_2 \wedge db_2 \wedge dp \wedge dq \wedge dr \end{aligned} \tag{32}$$

and from (31) and (32) it follows that

$$d(G_1, G_2) = \left|\frac{\sin\psi}{r^4}\right| d\overline{P}_1 \wedge d\overline{P}_2 \wedge dP. \tag{33}$$

On the other hand, on the plane Oxy we have the classical Blaschke formula [1], [8; p. 59]

$$dP_1 \wedge d\overline{P}_2 \wedge d\widetilde{P} = \widetilde{D}^3 d\widetilde{G}_1 \wedge d\widetilde{G}_2 \wedge d\overline{G}, \tag{34}$$

where $\widetilde{P}$ is the orthogonal projection of the point P on the coordinate plane Oxy, $\widetilde{G}_1 = \overline{P}_1\widetilde{P}$, $\widetilde{G}_2 = \overline{P}_2\widetilde{P}$, $\overline{G} = \overline{P}_1\overline{P}_2$ and $\widetilde{D}$ is the diameter of the circumscribed circle of the triangle $\overline{P}_1\overline{P}_2\widetilde{P}$. Since $dP = d\widetilde{P} \wedge dr$, by (33) and (34) we get

$$d(G_1, G_2) = \left|\frac{\widetilde{D}^3 \sin\psi}{r^4}\right| d\widetilde{G}_1 \wedge d\widetilde{G}_2 \wedge d\overline{G} \wedge dr. \tag{35}$$

Further, we have

$$\widetilde{G}_i : \quad b_i x - a_i y + a_i q - b_i p = 0, \quad z = 0$$

and according to (12′) we can write

$$d\widetilde{G}_i = \frac{1}{(a_i^2 + b_i^2)^{\frac{3}{2}}}(b_i^2 da_i \wedge dp - a_i b_i da_i \wedge dq - a_i b_i db_i \wedge dp + a_i^2 db_i \wedge dq). \tag{36}$$

From (11) we compute

$$d\varphi_i = -\frac{1}{(a_i^2 + b_i^2)^{\frac{3}{2}}}(a_i da_i + b_i db_i). \tag{37}$$

By exterior multiplication of the forms of (36) and (37) for $i = 1, 2$, we find

$$d\varphi_1 \wedge d\widetilde{G}_1 \wedge d\varphi_2 \wedge d\widetilde{G}_2 = \frac{\sin\psi}{\varphi_1^3 \varphi_2^3} da_1 \wedge db_1 \wedge da_2 \wedge db_2 \wedge dp \wedge dq$$

and comparing with (31), we obtain

$$d(G_1, G_2) = \varphi_1^3 \varphi_2^3 d\varphi_1 \wedge d\widetilde{G}_1 \wedge d\varphi_2 \wedge d\widetilde{G}_2 \wedge dr. \tag{38}$$

Thus we are ready to state the following

THEOREM 4. *The density for the pairs (G_1, G_2) of intersecting nonisotropic straight lines of type α, determined by (29), satisfies the relations (31), (33), (35) and (38).*

3.2. *Density of pairs of intersecting nonisotropic straight lines of type β.* Let (G_1, G_2) be a pair of intersecting nonisotropic straight lines of type β. Without loss of generality, we can assume that G_1 and G_2 have equations of the form

$$\begin{aligned} &G_1: \quad x = a_1 z + p - a_1 r, \quad y = b_1 z + q - b_1 r, \quad a_1 \neq 0, \\ &G_2: \quad x = a_2 z + p - a_2 r, \quad y = \frac{a_2}{a_1} b_1 z + q - \frac{a_2}{a_1} b_1 r, \quad a_2 \neq 0. \end{aligned} \tag{39}$$

Then the corresponding associated group $\overline{B}_6^{(1)}$ has the infinitesimal operators

$$Z_1 = \frac{\partial}{\partial p}, \quad Z_2 = \frac{\partial}{\partial q}, \quad Z_3 = \frac{\partial}{\partial r}, \quad Z_4 = b_1 \frac{\partial}{\partial a_1} - a_1 \frac{\partial}{\partial b_1} + \frac{a_2}{a_1} b_1 \frac{\partial}{\partial a_2} + q \frac{\partial}{\partial p} - p \frac{\partial}{\partial q},$$
$$Z_5 = a_1^2 \frac{\partial}{\partial a_1} + a_1 b_1 \frac{\partial}{\partial b_1} + a_2^2 \frac{\partial}{\partial a_2} - p \frac{\partial}{\partial r}, \quad Z_6 = a_1 b_1 \frac{\partial}{\partial a_1} + b_1^2 \frac{\partial}{\partial b_1} + \frac{a_2^2}{a_1} b_1 \frac{\partial}{\partial a_2} - q \frac{\partial}{\partial r}$$

and it is intransitive. Consequently the set of pairs of intersecting straight lines of type β is not measurable under $B_6^{(1)}$.

The system $Z_i(f) = 0$, $i = 1, \dots, 6$, has an independent integral

$$f = \frac{a_1 - a_2}{a_2\sqrt{a_1^2 + b_1^2}}$$

and it is an absolute invariant of $B_6^{(1)}$. We define the invariant density of the pairs $(G_1, G_2)(a_1, b_1, a_2, p, q, r)$ of type β by the equality

$$d(G_1, G_2) = \left|\frac{a_1 - a_2}{a_2\sqrt{a_1^2 + b_1^2}}\right| da_1 \wedge db_1 \wedge da_2 \wedge dp \wedge dq \wedge dr. \tag{40}$$

In view of (20), (40) yields

$$d(G_1, G_2) = |s| da_1 \wedge db_1 \wedge da_2 \wedge dp \wedge dq \wedge dr. \tag{41}$$

In this case, we find

$$da_1 \wedge db_1 \wedge dp \wedge dq \wedge dr = \frac{1}{r^2} d\overline{P}_1 \wedge dP, \tag{42}$$

where $\overline{P}_1 = G_1 \cap Oxy$, $P = G_1 \cap G_2$. But

$$ds = \frac{a_1a_2 + b_1^2}{a_2(a_1^2 + b_1^2)^{\frac{3}{2}}} da_1 - \frac{(a_1 - a_2)b_1}{a_2(a_1^2 + b_1^2)^{\frac{3}{2}}} db_1 - \frac{a_1}{a_2^2(a_1^2 + b_1^2)^{\frac{1}{2}}} da_2 \tag{43}$$

and from (41), (42) and (43) we deduce

$$d(G_1, G_2) = \left|\frac{a_2(a_1 - a_2)}{a_1 r^2}\right| ds \wedge d\overline{P}_1 \wedge dP. \tag{44}$$

We note that the straight lines G_1 and G_2 lie in the isotropic plane $\iota : b_1x - a_1y + a_1q - b_1p = 0$ and hence their orthogonal projections $\widetilde{G}_1$ and $\widetilde{G}_2$ coincide on Oxy, i.e. $\widetilde{G}_1 \equiv \widetilde{G}_2 \equiv \widetilde{G}$. Then (12′), (37) and (43) imply

$$ds \wedge d\varphi_1 \wedge d\widetilde{G} = \frac{a_1}{(a_1^2 + b_1^2)^{\frac{5}{2}}} (b_1 da_1 \wedge db_1 \wedge da_2 \wedge dp - a_1 da_1 \wedge db_1 \wedge da_2 \wedge dq). \tag{45}$$

From (40) and (45) we obtain

$$d(G_1, G_2) = \left|\frac{s\varphi_1^5}{a_1 b_1}\right| ds \wedge d\varphi_1 \wedge d\widetilde{G} \wedge dq \wedge dr = \left|\frac{s\varphi_1^5}{a_1^2}\right| ds \wedge d\varphi_1 \wedge d\widetilde{G} \wedge dp \wedge dr. \tag{46}$$

Therefore we have:

THEOREM 5. *The density for the pairs (G_1, G_2) of intersecting nonisotropic straight lines of type β, determined by (39), satisfies the relations (41), (44) and (46).*

3.3. *Density of pairs of intersecting nonisotropic and isotropic straight lines.* Let (G_1, G_2) be a pair of intersecting straight lines

$$\begin{aligned} G_1 :&\quad x = a_1 z + p - a_1 r, \quad y = b_1 z + q - b_1 r, \\ G_2 :&\quad x = p, \quad y = q, \end{aligned} \tag{47}$$

i.e. G_1 is nonisotropic and G_2 is isotropic. Now the corresponding associated group $\overline{B}_6^{(1)}$ has the infinitesimal operators

$$U_1=\frac{\partial}{\partial p},\quad U_2=\frac{\partial}{\partial q},\quad U_3=\frac{\partial}{\partial r},\quad U_4=b_1\frac{\partial}{\partial a_1}-a_1\frac{\partial}{\partial b_1}+q\frac{\partial}{\partial p}-p\frac{\partial}{\partial q},$$
$$U_5=a_1^2\frac{\partial}{\partial a_1}+a_1b_1\frac{\partial}{\partial b_1}-p\frac{\partial}{\partial r},\quad U_6=a_1b_1\frac{\partial}{\partial a_1}+b_1^2\frac{\partial}{\partial b_1}-q\frac{\partial}{\partial r}$$

and it is transitive. Then the integral invariant function $f=f(a_1,b_1,p,q,r)$ of the group $B_6^{(1)}$ satisfies the system of R. Deltheil [3; p. 28], [9; p. 11], namely

$$U_1(f)=0,\quad U_2(f)=0,\quad U_3(f)=0,\quad U_4(f)=0,\quad U_5(f)+3a_1f=0,\quad U_6(f)+3b_1f=0$$

and has the solution

$$f=\frac{h}{(a_1+b_1)^{\frac{3}{2}}},$$

where $h=const$. Thus we have

THEOREM 6. *The set of the pairs (G_1,G_2) of intersecting straight lines (47) is measurable with respect to $B_6^{(1)}$ and the corresponding invariant density is*

$$d(G_1,G_2)=\frac{1}{(a_1^2+b_1^2)^{\frac{3}{2}}}da_1\wedge db_1\wedge dp\wedge dq\wedge dr. \tag{48}$$

From (11) and (48) it follows immediately that

$$d(G_1,G_2)=\varphi_1^3da_1\wedge db_1\wedge dp\wedge dq\wedge dr. \tag{49}$$

On the other hand, by direct computation we obtain

$$da_1\wedge db_1\wedge dp\wedge dq=\frac{1}{r^2}d\widetilde{P}\wedge d\overline{P}, \tag{50}$$

where $\widetilde{P}=G_1\cap Oxy$ and $\overline{P}=G_2\cap Oxy$. Applying (50) to (49) we get

$$d(G_1,G_2)=\frac{\varphi_1^3}{r^2}dr\wedge d\widetilde{P}\wedge d\overline{P}. \tag{51}$$

Similarly, from (49) and

$$da_1\wedge db_1\wedge dp\wedge dq\wedge dr=-\frac{1}{\varphi_1r^3}d\varphi_1\wedge d\widetilde{P}\wedge d\overline{P}$$

we find

$$d(G_1,G_2)=\left|\frac{\varphi_1^2}{r^3}\right|d\varphi_1\wedge d\widetilde{P}\wedge d\overline{P}. \tag{52}$$

We establish the following result:

THEOREM 7. *The density for the pairs (G_1,G_2) of intersecting straight lines (47) satisfies the relations (49), (51) and (52).*

4. Measurability of a set of pairs of parallel straight lines

4.1. *Density of pairs of parallel nonisotropic straight lines of different isotropic planes.* Let (G_1,G_2) be a pair of parallel nonisotropic straight lines

$$G_1: \quad x = az + p_1, \quad y = bz + q_1, \qquad (53)$$
$$G_2: \quad x = az + p_2, \quad y = bz + q_2, \quad a^2 + b^2 \neq 0$$

that lie in different isotropic planes, i.e.

$$a(q_2 - q_1) - b(p_2 - p_1) \neq 0. \qquad (53')$$

The corresponding associated group $\overline{B}_6^{(1)}$ has the infinitesimal operators

$$\begin{aligned}
Y_1 &= \frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2}, Y_2 = \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}, Y_3 = aY_1 + bY_2,\\
Y_4 &= -b\frac{\partial}{\partial a} + a\frac{\partial}{\partial b} - q_1\frac{\partial}{\partial p_1} + p_1\frac{\partial}{\partial q_1} - q_2\frac{\partial}{\partial p_2} + p_2\frac{\partial}{\partial q_2},\\
Y_5 &= a^2\frac{\partial}{\partial a} + ab\frac{\partial}{\partial b} + ap_1\frac{\partial}{\partial p_1} + bp_1\frac{\partial}{\partial q_1} + ap_2\frac{\partial}{\partial p_2} + bp_2\frac{\partial}{\partial q_2},\\
Y_6 &= ab\frac{\partial}{\partial a} + b^2\frac{\partial}{\partial b} + aq_1\frac{\partial}{\partial p_1} + bq_1\frac{\partial}{\partial q_1} + aq_2\frac{\partial}{\partial p_2} + bq_2\frac{\partial}{\partial q_2}
\end{aligned}$$

and obviously it is intransitive. Hence the set of pairs of parallel nonisotropic straight lines (53), (53′) is not measurable with respect to $B_6^{(1)}$.

On the other hand, the value

$$f = \frac{a(q_2 - q_1) - b(p_2 - p_1)}{\sqrt{a^2 + b^2}}$$

is an absolute invariant of $\overline{B}_6^{(1)}$ and we define the density for the pairs (G_1, G_2) $(a, b, p_1, q_1, p_2, q_2)$ by the equality

$$d(G_1, G_2) = \left|\frac{a(q_2 - q_1) - b(p_2 - p_1)}{\sqrt{a^2 + b^2}}\right| da \wedge db \wedge dp_1 \wedge dq_1 \wedge dp_2 \wedge dq_2. \qquad (54)$$

Remark 4. The parallel straight lines on the coordinate plane Oxy

$$\widetilde{G}_1: \quad bx - ay + aq_1 - bp_1 = 0, \quad z = 0$$

and

$$\widetilde{G}_2: \quad bx - ay + aq_2 - bp_2 = 0, \quad z = 0$$

are the orthogonal projections of the parallel straight lines G_1 and G_2, respectively. Then the Euclidean distance between $\widetilde{G}_1$ and $\widetilde{G}_2$ is

$$\delta(\widetilde{G}_1, \widetilde{G}_2) = \left|\frac{a(q_2 - q_1) - b(p_2 - p_1)}{\sqrt{a^2 + b^2}}\right|. \qquad (55)$$

Putting (55) into (54) we get

$$d(G_1, G_2) = \delta(\widetilde{G}_1, \widetilde{G}_2) da \wedge db \wedge dp_1 \wedge dq_1 \wedge dp_2 \wedge dq_2. \qquad (56)$$

A computation leads to

$$d\delta \wedge d\varphi \wedge d\overline{P}_1 \wedge d\overline{P}_2 = -\frac{a(p_2 - p_1) + b(q_2 - q_1)}{(a^2 + b^2)^2} da \wedge db \wedge dp_1 \wedge dq_1 \wedge dp_2 \wedge dq_2, \qquad (57)$$

where $\delta = \delta(\widetilde{G}_1, \widetilde{G}_2)$, $\varphi = \angle(G_1, Oxy) = \angle(G_2, Oxy)$, $\overline{P}_1 = G_1 \cap Oxy$, $\overline{P}_2 = G_2 \cap Oxy$.

In view of (57), (56) yields

$$d(G_1, G_2) = \left|\frac{\delta}{\varphi^4[a(p_2 - p_1) + b(q_2 - q_1)]}\right| d\delta \wedge d\varphi \wedge d\overline{P}_1 \wedge d\overline{P}_2. \tag{58}$$

THEOREM 8. *The density for the pairs (G_1, G_2) of parallel nonisotropic straight lines (53), (53′) satisfies the relations (56) and (58).*

4.2. *Density of pairs of parallel nonisotropic straight lines of coinciding isotropic planes.* Let (G_1, G_2) be a pair of parallel nonisotropic straight lines that lie in an isotropic plane. Assume that G_1 and G_2 have the equations

$$\begin{aligned} G_1: &\quad x = az + p_1, \quad y = bz + q_1, \quad a \neq 0,\\ G_2: &\quad x = az + p_2, \quad y = bz + q_1 + \frac{b}{a}(p_2 - p_1), \quad b \neq 0. \end{aligned} \tag{59}$$

Then the corresponding associated group has the infinitesimal operators

$$\begin{aligned} Z_1 &= \frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2}, \quad Z_2 = \frac{\partial}{\partial q_1}, \quad Z_3 = aZ_1 + bZ_2,\\ Z_4 &= -b\frac{\partial}{\partial a} + a\frac{\partial}{\partial b} - q_1\frac{\partial}{\partial p_1} + p_1\frac{\partial}{\partial q_1} - \left[q_1 + \frac{b}{a}(p_2 - p_1)\right]\frac{\partial}{\partial p_2},\\ Z_5 &= a^2\frac{\partial}{\partial a} + ab\frac{\partial}{\partial b} + ap_1\frac{\partial}{\partial p_1} + bp_1\frac{\partial}{\partial q_1} + ap_2\frac{\partial}{\partial p_2},\\ Z_6 &= ab\frac{\partial}{\partial a} + b^2\frac{\partial}{\partial b} + aq_1\frac{\partial}{\partial p_1} + bq_1\frac{\partial}{\partial q_1} + a\left[q_1 + \frac{b}{a}(p_2 - p_1)\right]\frac{\partial}{\partial p_2} \end{aligned}$$

and it is intransitive. Therefore the set of pairs of parallel nonisotropic straight lines (59) is not measurable with respect to $B_6^{(1)}$. The system $Z_i(f) = 0$, $i = 1, \dots 6$, has the solution

$$f = \frac{p_2 - p_1}{a}$$

and it is an absolute invariant of $\overline{B}_6^{(1)}$. Then we define the density for the pairs (G_1, G_2) (a, b, p_1, q_1, p_2) by the equality

$$d(G_1, G_2) = \left|\frac{p_2 - p_1}{a}\right| da \wedge db \wedge dp_1 \wedge dq_1 \wedge dp_2. \tag{60}$$

REMARK 5. The isotropic plane $\iota : ax + by = 0$ intersects the straight lines G_1 and G_2 at the points

$$P_1\left(-\frac{b(aq_1 - bp_1)}{a^2 + b^2}, \frac{a(aq_1 - bp_1)}{a^2 + b^2}, -\frac{ap_1 + bq_1}{a^2 + b^2}\right)$$

and

$$P_2\left(-\frac{b(aq_1 - bp_1)}{a^2 + b^2}, \frac{a(aq_1 - bp_1)}{a^2 + b^2}, -\frac{(a^2 + b^2)p_2 + b(aq_1 - bp_1)}{a(a^2 + b^2)}\right),$$

respectively. Obviously P_1 and P_2 are parallel points and then

$$s(P_1, P_2) = \frac{p_1 - p_2}{a}, \tag{61}$$

i.e. (61) is the oriented s-distance from G_1 to G_2. Further we shall denote $s(G_1, G_2) = s(P_1, P_2)$ by s.

Hence by (60) and (61) we have

$$d(G_1, G_2) = |s| da \wedge db \wedge dp_1 \wedge dq_1 \wedge dp_2. \tag{62}$$

We compute

$$da \wedge db \wedge dp_1 \wedge dq_1 \wedge dp_2 = \frac{a}{s^2} ds \wedge d\overline{P}_1 \wedge d\overline{P}_2 \tag{63}$$

and

$$da \wedge db \wedge dp_1 \wedge dq_1 \wedge dp_2 = -\frac{a}{s\varphi} d\varphi \wedge d\overline{P}_1 \wedge d\overline{P}_2, \tag{64}$$

where $\overline{P}_1 = G_1 \cap Oxy$, $\overline{P}_2 = G_2 \cap Oxy$, $\varphi = \angle(G_1, Oxy) = \angle(G_2, Oxy)$. Substituting (63) and (64) into (62), we find

$$d(G_1, G_2) = \left|\frac{a}{s}\right| ds \wedge d\overline{P}_1 \wedge d\overline{P}_2 \tag{65}$$

and

$$d(G_1, G_2) = \left|\frac{a}{\varphi}\right| d\varphi \wedge d\overline{P}_1 \wedge d\overline{P}_2, \tag{66}$$

respectively. Thus we have the following

THEOREM 9. *The density for the pairs (G_1, G_2) of parallel nonisotropic straight lines (59) satisfies the relations (62), (65) and (66).*

4.3. *Density of pairs of isotropic straight lines.* Let (G_1, G_2) be a pair of isotropic straight lines and

$$G_1 : \quad x = p_1, \quad y = q_1, \quad G_2 : \quad x = p_2, \quad y = q_2, \tag{67}$$

where $(p_2 - p_1)^2 + (q_2 - q_1)^2 \neq 0$. The corresponding associated group $\overline{B}_6^{(1)}$ has the infinitesimal operators

$$U_1 = \frac{\partial}{\partial p_1} + \frac{\partial}{\partial p_2}, \quad U_2 = \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}, \quad U_3 = 0,$$

$$U_4 = -q_1 \frac{\partial}{\partial p_1} + p_1 \frac{\partial}{\partial q_1} + q_2 \frac{\partial}{\partial p_2} - p_2 \frac{\partial}{\partial q_2}, \quad U_5 = 0, \quad U_6 = 0$$

and it is intransitive. It follows that the set of pairs of isotropic straight lines is not measurable with respect to $B_6^{(1)}$. But

$$f = (p_2 - p_1)^2 + (q_2 - q_1)^2$$

is an absolute invariant of $\overline{B}_6^{(1)}$ and we can define the density for the pairs (G_1, G_2) (p_1, q_1, p_2, q_2) by the equality

$$d(G_1, G_2) = \sqrt{(p_2 - p_1)^2 + (q_2 - q_1)^2} dp_1 \wedge dq_1 \wedge dp_2 \wedge dq_2. \tag{68}$$

REMARK 6. We note that [7; p. 46]

$$l^\star = \sqrt{(p_2 - p_1)^2 + (q_2 - q_1)^2} \tag{69}$$

is the distance between G_1 and G_2.

Inserting (15) and (69) into (68) we obtain

$$d(G_1, G_2) = l^\star d\overline{P}_1 \wedge d\overline{P}_2. \tag{70}$$

On the other hand, we have

$$\begin{aligned} d\overline{P}_1 \wedge d\overline{P}_2 &= -\frac{1}{p_2 - p_1} dl^\star \wedge dq_1 \wedge d\overline{P}_2 = -\frac{1}{q_2 - q_1} dp_1 \wedge dl^\star \wedge d\overline{P}_2 \\ &= \frac{1}{p_2 - p_1} d\overline{P}_1 \wedge dl^\star \wedge dq_2 \quad = \frac{1}{q_2 - q_1} d\overline{P}_1 \wedge dp_2 \wedge dl^\star \end{aligned}$$

and therefore

$$\begin{aligned} d(G_1, G_2) &= \frac{l^\star}{|p_2 - p_1|} dl^\star \wedge dq_1 \wedge d\overline{P}_2 = \frac{l^\star}{|q_2 - q_1|} dp_1 \wedge dl^\star \wedge d\overline{P}_2 \\ &= \frac{l^\star}{|p_2 - p_1|} d\overline{P}_1 \wedge dl^\star \wedge dq_2 = \frac{l^\star}{|q_2 - q_1|} d\overline{P}_1 \wedge dp_2 \wedge dl^\star. \end{aligned} \tag{71}$$

Thus we have the following

THEOREM 10. *The density for the pairs* (G_1, G_2) *of isotropic straight lines (67) satisfies the relations (70) and (71).*

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