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TERNARY ALGEBRAS AND CALCULUS OF CUBIC MATRICES

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Abstract. We study associative ternary algebras and describe a general approach which allows us to construct various classes of ternary algebras. Applying this approach to a central bimodule with a covariant derivative we construct a ternary algebra whose ternary multiplication is closely related to the curvature of the covariant derivative. We also apply our approach to a bimodule over two associative (binary) algebras in order to construct a ternary algebra which we use to produce a large class of Lie algebras. We study the calculus of cubic matrices and use this calculus to construct a matrix ternary algebra with associativity of second kind.

1. Introduction. In this paper we consider a concept of ternary algebra taking as underlying space one of the following structures: a complex vector space, a bimodule over a commutative unital ring, a central bimodule over an associative unital algebra and a bimodule over two associative unital algebras. We propose a general scheme to construct a class of ternary algebras and apply this scheme to construct a ternary algebra based on a covariant derivative on a central bimodule and its curvature. Bianchi's identities for curvature give us the properties of this ternary algebra. Applying the same general scheme we construct a ternary algebra by means of a bimodule over two associative algebras and we use this ternary algebra to construct a class of Lie algebras. It is well known that a large class of associative binary algebras can be constructed by means of square matrices and their multiplication. In section 3 we develop a calculus of space or cubic matrices and use this calculus to construct ternary algebras of cubic matrices. We find four different totally associative ternary multiplications of second kind of cubic matrices and prove that these are the only totally associative ternary multiplications of second kind in the case of cubic matrices. It is worth mentioning that our search for associative

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ternary multiplications of cubic matrices has shown that there is no totally associative ternary multiplication of first kind in the case of cubic matrices.

2. Ternary algebras and noncommutative geometry. In this section we consider a notion of ternary algebra taking as underlying structure one of the following: a complex vector space, a bimodule over a commutative unital ring and a central bimodule over an associative unital complex algebra. We begin this section by reminding reader a concept of associative ternary algebra and describing a general structure which allows us to construct a class of ternary algebras. This class of ternary algebras contains the ternary algebra of vector fields on a smooth finite dimensional manifold and a ternary algebra of rectangular matrices. We show that the curvature of an affine connection induces the structure of a ternary algebra on the module of vector fields on a smooth manifold. We also propose a generalization of the approach given in [2, 3] for constructing ternary algebra based on two modules over the algebras with involutions. Then this ternary algebra is used to construct a Lie algebra.

Let us begin with a notion of a ternary algebra in the case where the underlying structure is a complex vector space. A pair $(\mathfrak{V},\tau)$ is said to be a ternary $\mathbb{C}$-algebra or a triple $\mathbb{C}$-system if $\mathfrak{V}$ is a complex vector space, and $\tau:\mathfrak{V}\times\mathfrak{V}\times\mathfrak{V}\to\mathfrak{V}$ is a $\mathfrak{V}$-valued $\mathbb{C}$-trilinear form which will be referred to as a ternary law of composition or ternary multiplication. A ternary $\mathbb{C}$-algebra $(\mathfrak{V},\tau)$ is said to be `lr`-partially associative if its ternary law of multiplication satisfies

$$\tau(\tau(v,w,s),r,t)=\tau(v,w,\tau(s,r,t)), \tag{1}$$

where $v,w,s,r,t\in\mathfrak{V}$. A ternary $\mathbb{C}$-algebra $(\mathfrak{V},\tau)$ is said to be `lc`-partially associative of first kind if

$$\tau(\tau(v,w,s),r,t)=\tau(v,\tau(w,s,r),t), \tag{2}$$

and `lc`-partially associative of second kind if

$$\tau(\tau(v,w,s),r,t)=\tau(v,\tau(r,s,w),t). \tag{3}$$

A `lr`-partially associative ternary $\mathbb{C}$-algebra $(\mathfrak{V},\tau)$ will be referred to as a totally associative ternary $\mathbb{C}$-algebra of first kind if it is `lc`-partially associative of first kind, i.e. its ternary multiplication τ satisfies the relations (1) and (2). Similarly a `lr`-partially associative ternary $\mathbb{C}$-algebra $(\mathfrak{V},\tau)$ will be referred to as a totally associative of second kind if it is `lc`-partially associative of second kind, i.e. its ternary multiplication satisfies (1) and (3).

Several important examples of ternary $\mathbb{C}$-algebras are based on one and the same structure which can be described as follows: suppose we are given a pair $(\mathfrak{V},T)$, where $\mathfrak{V}$ is a $\mathbb{C}$-vector space, T is a $\mathbb{C}$-multilinear mapping $T:\mathfrak{V}\times\mathfrak{V}\times\mathfrak{V}\times\mathfrak{V}^*\to\mathbb{C}$ and $\mathfrak{V}^*$ is the dual space. We construct a ternary $\mathbb{C}$-algebra $(\mathfrak{V},\tau_T)$ by defining a ternary multiplication τ_T as follows:

$$\theta(\tau_T(v,w,s))=T(v,w,s,\theta), \tag{4}$$

where $v,w,s\in\mathfrak{V},\ \theta\in\mathfrak{V}^*$. This rather general scheme for constructing ternary $\mathbb{C}$-algebras has a few particular cases. One of them is a pair $(\mathfrak{V},R)$, where $\mathfrak{V}$ is a $\mathbb{C}$-vector space, R is a $\mathbb{C}$-bilinear mapping $R : \mathfrak{V}\times\mathfrak{V} \to \mathrm{Lin}_{\mathbb{C}}(\mathfrak{V})$ and $\mathrm{Lin}_{\mathbb{C}}(\mathfrak{V})$ is the algebra of endomorphisms of a vector space $\mathfrak{V}$. Indeed if we are given a pair $(\mathfrak{V}, R)$ then we define

$$T_R(v, w, s, \theta) = \theta(R(v, w) \cdot s), \tag{5}$$

and making use of (4) we get the ternary $\mathbb{C}$-algebra $(\mathfrak{V}, \tau_R)$ whose ternary multiplication τ_R can be described implicitly by the formula

$$\tau_R(v, w, s) = R(v, w) \cdot s, \quad v, w, s \in \mathfrak{V}. \tag{6}$$

The second particular case is a triple $(\mathfrak{V}, L, p)$, where $\mathfrak{V}$ is a complex vector space, L is a $\mathbb{C}$-linear mapping $L : v \in \mathfrak{V} \to L_v \in \mathrm{Lin}_{\mathbb{C}}(\mathfrak{V})$, and p is a finite polynomial in two variables with complex coefficients, i.e.

$$p(x, y) = \sum_{m,n} \lambda_{mn} x^m\, y^n, \quad \lambda_{mn} \in \mathbb{C}.$$

In this case we define

$$R(v, w) = p\,(L_v, L_w) = \sum_{m,n} \lambda_{mn} L_v^m\, L_w^n$$

and construct the ternary multiplication by means of (6).

Now we consider a notion of ternary algebra in the case where its underlying structure is a bimodule over a commutative unital ring. Let K be a commutative unital ring and $\mathfrak{M}$ be a K-bimodule where we assume that the left K-module structure of $\mathfrak{M}$ coincides with its right K-module structure, i.e. $a\,m = m\,a$ for any $a \in K$ and $m \in \mathfrak{M}$. In this case a ternary K-algebra or a triple K-system is a pair $(\mathfrak{M}, \tau)$, where $\tau : \mathfrak{M} \otimes_K \mathfrak{M} \otimes_K \mathfrak{M} \to \mathfrak{M}$ is a homomorphism of K-bimodules. Analogously with the case of a ternary algebra with underlying vector space we can construct a ternary K-algebra by means of (4) or (6). Indeed if $\mathfrak{M}$ is a K-bimodule, $\mathfrak{M}^*$ is its dual module and $T : \mathfrak{M} \otimes_K \mathfrak{M} \otimes_K \mathfrak{M} \otimes_K \mathfrak{M}^* \to K$ then $(\mathfrak{M}, \tau_T)$ is the ternary algebra with the ternary multiplication defined by (4). Similarly given a K-bimodule $\mathfrak{M}$ and a mapping $L : \mathfrak{M} \otimes_K \mathfrak{M} \to \mathrm{End}_K(\mathfrak{M})$, we obtain the ternary algebra $(\mathfrak{M}, \tau_L)$ with the ternary multiplication τ_L defined by (6). This construction can be applied in differential geometry to construct a ternary algebra on a smooth manifold M whose underlying structure is the bimodule of vector fields of M over the commutative unital ring of smooth functions. Indeed identifying K with the commutative unital ring of smooth functions $C^\infty(M)$, $\mathfrak{M}$ with the bimodule of vector fields $\mathscr{D}(M)$, and assuming we are given a (1,3)-type tensor field T (in local coordinates of M its components are $T^l_{ijk}(x), x \in U \subset M$), we construct the ternary $C^\infty(M)$-algebra of vector fields $(\mathscr{D}, \tau_T)$ where the ternary product of vector fields is defined by

$$\omega(\tau_T(X, Y, Z)) = T(X, Y, Z, \omega), \quad X, Y, Z \in \mathscr{D}(M), \tag{7}$$

and ω is a differential 1-form on M. The formula (7) written in local coordinates takes the form

$$\tau^l_T(X, Y, Z) = T^l_{ijk} X^i Y^j Z^k.$$

A notion of ternary algebra in the case where the underlying structure is a central bimodule over an associative unital complex algebra can be defined as follows: Let $\mathscr{A}$ be an associative unital complex algebra with the unit $\mathbb{1}$, $Z(\mathscr{A})$ is the center of $\mathscr{A}$, $\mathfrak{M}$ be a central $\mathscr{A}$-bimodule, i.e. $\mathfrak{M}$ is a $\mathscr{A}$-bimodule satisfying $z\,m = m\,z$ for any $z \in Z(\mathscr{A}), m \in \mathfrak{M}$, then a pair $(\mathfrak{M}, \tau)$ is said to be a ternary $Z(\mathscr{A})$-algebra if $\tau : \mathfrak{M} \otimes_{\mathscr{A}} \mathfrak{M} \otimes_{\mathscr{A}} \mathfrak{M} \to \mathfrak{M}$ is a homomorphism of $Z(\mathscr{A})$-bimodules. It is worth mentioning that $\mathscr{A}$ is a complex vector space containing complex numbers $\{\alpha \cdot \mathbb{1} : \alpha \in \mathbb{C}\}$ as the subalgebra. Hence this subalgebra induces a complex vector space structure on $\mathfrak{M}$ and under this structure of complex vector space we can consider the algebra of linear endomorphisms $\mathrm{End}_{\mathbb{C}}(\mathfrak{M})$. Consequently in this case a ternary $\mathscr{A}$-algebra $(\mathfrak{M}, \tau)$ is also a ternary $\mathbb{C}$-algebra since τ is a $\mathbb{C}$-trilinear mapping. Making use of (4) and (6) we can construct several examples of ternary algebras with underlying central bimodule. One of these examples is very important because it is closely related to the noncommutative geometry of modules. Let $\mathrm{Der}(\mathscr{A})$ be the left $\mathscr{A}$-module of derivations of $\mathscr{A}$. A connection or covariant derivative on a central $\mathscr{A}$-bimodule $\mathfrak{M}$ [4] is a homomorphism of left $Z(\mathscr{A})$-modules $\nabla : X \in \mathrm{Der}(\mathscr{A}) \to \nabla_X \in \mathrm{End}_{\mathbb{C}}(\mathfrak{M})$, i.e. $\nabla_{zX}(m) = z\,\nabla_X(m)$ for any $z \in Z(\mathscr{A}), X \in \mathrm{Der}(\mathscr{A})$, satisfying

$$\nabla_X(a\,m\,b) = a\,\nabla_X(m)\,b + X(a)\,m\,b + a\,m\,X(b), \quad a, b \in \mathscr{A}.$$

The curvature of the connection ∇ is the mapping $R : \mathrm{Der}(\mathscr{A}) \otimes_{\mathbb{C}} \mathrm{Der}(\mathscr{A}) \to \mathrm{End}_{\mathbb{C}}(\mathfrak{M})$ defined by

$$R_{X,Y} = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]},$$

where $[X, Y]$ is the commutator of two derivations of $\mathscr{A}$ equipping it with the structure of Lie algebra. It can be proved that the curvature R is a $Z(\mathscr{A})$-bilinear mapping from $\mathrm{Der}(\mathscr{A}) \times \mathrm{Der}(\mathscr{A})$ to the $Z(\mathscr{A})$-module $\mathrm{End}_{\mathscr{A}}(\mathfrak{M})$ of $\mathscr{A}$-endomorphisms of the bimodule $\mathfrak{M}$.

PROPOSITION 2.1. *The pair $(Der(\mathscr{A}), \mathcal{T})$ is the ternary $Z(\mathscr{A})$-algebra with the ternary multiplication $\mathcal{T}(X, Y, Z) = R_{X,Y}(Z)$. The ternary multiplication of this $Z(\mathscr{A})$-algebra satisfies*

$$\mathcal{T}(X, Y, Z) + \mathcal{T}(Y, Z, X) + \mathcal{T}(Z, X, Y) = 0, \tag{8}$$

$$\begin{aligned} &\nabla_X \mathcal{T}(Y, Z, W) + \nabla_Y \mathcal{T}(Z, X, W) + \nabla_Z \mathcal{T}(X, Y, W) \\ &\qquad = \mathcal{T}(Y, Z, \nabla_X W) + \mathcal{T}(Z, X, \nabla_Y W) + \mathcal{T}(X, Y, \nabla_Z W) \\ &\qquad\quad + \mathcal{T}([X, Y], Z, W) + \mathcal{T}([Y, Z], X, W) + \mathcal{T}([Z, X], Y, W). \end{aligned} \tag{9}$$

The first property (8) of ternary multiplication $\mathcal{T}$ follows from the first Bianchi's identity for curvature and the second (9) from the second Bianchi's identity. It should be mentioned that the second property of ternary multiplication can be viewed as an analog of Leibniz rule for ternary multiplication. An important example of ternary algebra described in Proposition 2.1 can be constructed within the framework of classical (commutative) differential geometry as follows: Let $\pi : E \to M$ be a vector bundle over a smooth finite dimensional manifold M, $\mathscr{A} = C^\infty(M)$ be the algebra of smooth functions on a smooth manifold M, and $\mathfrak{M} = \Gamma(E)$ be the module of smooth sections of E. Given a $C^\infty(M)$-multilinear mapping $T : \Gamma(E) \times \Gamma(E) \times \Gamma(E) \times \Gamma(E^*) \to C^\infty(M)$, where E^* is the dual bundle, we obtain the ternary algebra $(\Gamma(E), \tau)$ of smooth sections of a vector bundle E with the ternary multiplication $\theta(\tau(\xi, \eta, \chi)) = T(\xi, \eta, \chi, \theta)$, where ξ, η, χ are sections of E, and $\theta \in \Gamma(E^*)$. Particularly if M is a smooth manifold, $E = TM$ is the tangent bundle, $E^* = T^*M$ is the cotangent bundle, $\Gamma(E) = \mathfrak{D}(M)$ is the module of vector fields, $\Gamma(E^*) = \Omega^1(M)$ is the module of 1-forms, ∇ is an affine connection on M, and

$$R_{X,Y} = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}, \quad X, Y \in \mathfrak{D}(M)$$

is the curvature of ∇ then we have the $C^\infty(M)$-multilinear mapping

$$T : \mathfrak{D}(M) \times \mathfrak{D}(M) \times \mathfrak{D}(M) \times \Omega^1(M) \to C^\infty(M),$$

induced by the curvature

$$T(X, Y, Z, \omega) = \omega\,(R_{X,Y}(Z)),$$

and this mapping induces the structure of the ternary algebra $(\mathfrak{D}(M), \tau)$ on the module of vector fields with the ternary multiplication

$$\mathcal{T}(X, Y, Z) = R_{X,Y}(Z).$$

Our next aim in this section is to construct a ternary algebra in a more general situation when we have a bimodule whose left module structure is determined by one associative unital algebra and the right module structure by another. Let $\mathscr{A}, \mathscr{B}$ be unital associative algebras over $\mathbb{C}$ with involutions respectively $a \to a^\star$ and $b \to b^*$, where $a \in \mathscr{A}, b \in \mathscr{B}$. Suppose we are given two bimodules $\mathfrak{M}, \overline{\mathfrak{M}}$ where $\mathfrak{M}$ is an $(\mathscr{A}, \mathscr{B})$-bimodule and $\overline{\mathfrak{M}}$ is a $(\mathscr{B}, \mathscr{A})$-bimodule. We also assume that these bimodules are isomorphic as Abelian groups, i.e. there is an isomorphism $m \in \mathfrak{M} \to \overline{m} \in \overline{\mathfrak{M}}$ satisfying

$$\overline{a\,m} = \overline{m}\,a^\star, \quad \overline{m\,b} = b^*\,\overline{m}, \quad \forall m \in \mathfrak{M},\ a \in \mathscr{A},\ b \in \mathscr{B}.$$

In these settings a pair $(\mathfrak{M}, \tau)$ is said to be a ternary $(\mathscr{A}, \mathscr{B})$-algebra with ternary multiplication τ if $\tau : \mathfrak{M} \times \mathfrak{M} \times \mathfrak{M} \to \mathfrak{M}$ defined by $\tau(m, n, p) = \rho(m, \overline{n}, p)$, where $\rho : \mathfrak{M} \otimes_{\mathscr{B}} \overline{\mathfrak{M}} \otimes_{\mathscr{A}} \mathfrak{M} \to \mathfrak{M}$, is a homomorphism of $(\mathscr{A}, \mathscr{B})$-bimodules. Let us mention that τ is a $\mathbb{C}$-trilinear mapping with regard to vector space structures of $\mathfrak{M}, \overline{\mathfrak{M}}, \mathscr{A}, \mathscr{B}$ hence considering $\mathfrak{M}$ as a complex vector space we have the ternary $\mathbb{C}$-algebra $(\mathfrak{M}, \tau)$.

Now our aim is to construct an important example of a ternary $(\mathscr{A}, \mathscr{B})$-algebra which can be used to construct a class of Lie algebras. For this purpose we assume that there are two homomorphisms

$$m \otimes \overline{n} \in \mathfrak{M} \otimes_{\mathscr{B}} \overline{\mathfrak{M}} \to \langle m, \overline{n} \rangle \in \mathscr{A}, \quad \overline{m} \otimes n \in \overline{\mathfrak{M}} \otimes_{\mathscr{A}} \mathfrak{M} \to \langle \overline{m}, n \rangle \in \mathscr{B}$$

respectively of $\mathscr{A}$-bimodules and $\mathscr{B}$-bimodules which satisfy

$$\langle m, \overline{n} \rangle^\star = \langle n, \overline{m} \rangle, \ \ \langle \overline{m}, n \rangle^* = \langle \overline{n}, m \rangle, \ \ \langle m, \overline{n} \rangle p = m \langle \overline{n}, p \rangle,$$

where $m, n, p \in \mathfrak{M}$.

Proposition 2.2. *The pair* $(\mathfrak{M}, \tau)$*, where*

$$\tau(m, n, p) = \langle m, \overline{n} \rangle p, \quad m, n, p \in \mathfrak{M}, \tag{10}$$

is an `lr`*-partially associative ternary* $(\mathscr{A}, \mathscr{B})$*-algebra.*

Proof. Indeed for any five elements m, n, p, q, r of $\mathfrak{M}$ we have

$$\begin{aligned}\tau(\tau(m, n, p), q.r) &= \tau(\langle m, \overline{n} \rangle p, q, r) \\ &= \langle \langle m, \overline{n} \rangle p, \overline{q} \rangle r = \langle m, \overline{n} \rangle \langle p, \overline{q} \rangle r.\end{aligned}$$

On the other hand

$$\tau(m, n, \tau(p, q, r)) = \langle m, \overline{n}\rangle\langle p, \overline{q}\rangle r,$$

and this ends the proof. ∎

Note that the ternary product (10) follows the general scheme based on (6). Making use of ternary multiplication τ we can construct a new ternary multiplication by setting $\sigma(m, n, p) = \langle m, \overline{n}\rangle p + p\langle\overline{n}, m\rangle$, where $m, n, p \in \mathfrak{M}$.

Proposition 2.3. *The ternary $\mathscr{A}, \mathscr{B}$-algebra $(\mathfrak{M}, \sigma)$ is the ternary algebra of Jordan type, i.e. $\sigma(m, n, p) = \sigma(p, n, m)$, and the ternary multiplication σ satisfies the identity*

$$\sigma(m, n, \sigma(p, q, r)) - \sigma(p, q, \sigma(m, n, r)) + \sigma(\sigma(p, q, m), n, r) - \sigma(m, \sigma(q, p, n), r) = 0,$$

where $m, n, p, q, r \in \mathfrak{M}$.

As mentioned before, the ternary $(\mathscr{A}, \mathscr{B})$-algebra $(\mathfrak{M}, \sigma)$ is an important example of ternary $(\mathscr{A}, \mathscr{B})$-algebras because it may be used to construct a class of Lie algebras as follows: Given elements $m, n \in \mathfrak{M}, a \in \mathscr{A}, b \in \mathscr{B}$ we form the square matrix

$$A = \begin{pmatrix} a & m \\ \bar{n} & b \end{pmatrix}.$$

and denote the vector space of all such matrices by $\mathscr{M}$. Given two matrices of this kind we define their product by

$$\begin{pmatrix} a & m \\ \overline{n} & b \end{pmatrix}\begin{pmatrix} a' & m' \\ \overline{n}' & b' \end{pmatrix} = \begin{pmatrix} a\,a' + \langle m, \overline{n}'\rangle & a\,m' + m\,b' \\ \overline{n}\,a' + b\,\overline{n}' & b\,b' + \langle\overline{n}, m'\rangle \end{pmatrix}. \tag{11}$$

Proposition 2.4. *The vector space $\mathscr{M}$ endowed with the product (11) is a complex unital associative algebra with the unity element*

$$E = \begin{pmatrix} e & 0 \\ \bar{0} & e' \end{pmatrix},$$

where e is the unity element of $\mathscr{A}$ and e′ is the unity element of $\mathscr{B}$.

Introducing notations

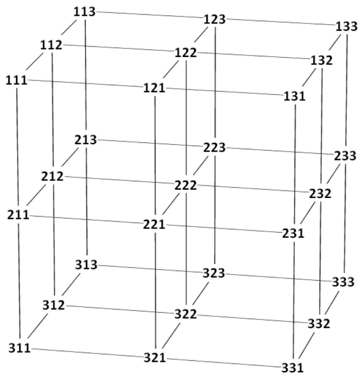
$$L_m = \begin{pmatrix} 0 & m \\ \bar{0} & 0, \end{pmatrix}, \quad \overline{L}_n = \begin{pmatrix} 0 & 0 \\ \overline{n} & 0 \end{pmatrix}, \quad K_{mn} = \begin{pmatrix} \langle m, \overline{n}\rangle & 0 \\ \bar{0} & -\langle\overline{n}, m\rangle \end{pmatrix},$$

and denoting by $\mathscr{L} \subset \mathscr{M}$ the vector subspace generated by the matrices $L_m, \overline{L}_n, K_{mn}$ we prove

Proposition 2.5. *$\mathscr{L}$ is the Lie algebra with commutation relations*

$$[L_m, \overline{L}_n] = K_{mn}, \qquad [K_{mn}, L_p] = L_{\sigma(m,n,p)},$$
$$[K_{mn}, \overline{L}_p] = -\overline{L}_{\sigma(n,m,p)}, \quad [K_{mn}, K_{pq}] = K_{\sigma(m,n,p)\,q} - K_{p\,\sigma(n,m,q)}.$$

3. Calculus of cubic matrices. In this section we consider a vector space of cubic matrices, where by cubic matrix we mean a quantity $A = (A_{ijk})$ with three subscripts i, j, k each running over some set of integers. We use this vector space to construct a ternary algebra by means of a triple product of cubic matrices. A triple product or ternary multiplication of cubic matrices is constructed in analogy with the classical product of



two rectangular matrices by means of summation which is taken over a certain system of subscripts of three cubic matrices.

Our aim is to find all totally associative ternary multiplications of first or second kind, and we prove that there are four ternary multiplications of cubic matrices each yielding a totally associative ternary algebra of second kind. We also mention a ternary multiplication of cubic matrices, which is neither partially nor totally associative and can be used to construct a ternary analog of the algebra of Pauli matrices [1, 5, 6, 7].

Let $A = (A_{ijk})$ be a quantity with three subscripts i, j, k, where $A_{ijk} \in \mathbb{C}$ and i, j, k are integers running from 1 to N. We will call A a space matrix of order N or cubic N-matrix provided that its entries A_{ijk} are arranged in the vertices of a 3–dimensional lattice. Particularly the structure of a cubic matrix of order three is shown in the figure. Let us denote the set of cubic N-matrices by $\mathrm{CMat}_N(\mathbb{C})$, i.e.

$$\mathrm{CMat}_N(\mathbb{C}) = \{A = (A_{ijk}) : A_{ijk} \in \mathbb{C},\ i, j, k = 1, 2, \ldots, N\}.$$

The set of cubic N-matrices $\mathrm{CMat}_N(\mathbb{C})$ is the vector space if we define the addition of cubic matrices and multiplication by complex numbers as usual

$$A + B = (A_{ijk} + B_{ijk}), \quad \lambda\, A = (\lambda\, A_{ijk}), \quad \lambda \in \mathbb{C}. \tag{12}$$

The above figure shows that we can slice a cubic matrix of third order by fixing a value of one of subscripts i, j, k. For instance, if we fix a value of subscript k letting i, j to range from 1 to 3 then the obtained slice of cubic matrix A is a square matrix of order 3. Hence in this case we have three slices of cubic matrix A which are square matrices of order 3. Thus a cubic matrix of third order can be represented as a set of square matrices of order 3 as follows

$$A = (A_{ijk}) = \left\| \begin{array}{ccc|ccc|ccc} a_{111} & a_{121} & a_{131} & a_{112} & a_{122} & a_{132} & a_{113} & a_{123} & a_{133} \\ a_{211} & a_{221} & a_{231} & a_{212} & a_{222} & a_{232} & a_{213} & a_{223} & a_{233} \\ a_{311} & a_{321} & a_{331} & a_{312} & a_{322} & a_{332} & a_{313} & a_{323} & a_{333} \end{array} \right\|.$$

We can construct various subspaces of the vector space $\mathrm{CMat}_N(\mathbb{C})$ by means of cubic N-matrices with symmetries based on the representation of $\mathbb{Z}_2$ by $(-1, 1)$ or based on the representation of $\mathbb{Z}_3$ by $(1, j, j^2)$, where j is a cubic root of unity. The former gives a notion of a skew-symmetric (with respect to a pair of subscripts) cubic N-matrix

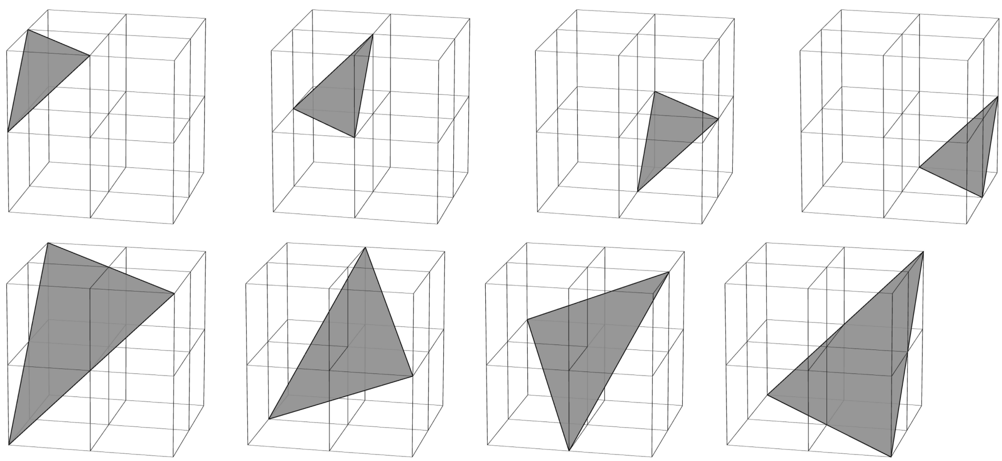
[8], and this kind of cubic N-matrices may be viewed as an analog of skew-symmetric square matrices. However, in the case of cubic matrices we have one more less classical possibility to construct matrices with certain symmetries and these symmetries are based on the representation of the group $\mathbb{Z}_3$ by $(1, j, j^2)$. Indeed a cubic matrix $A = (A_{ikl})$ has three subscripts and the permutations of these subscripts generated by action of $\mathbb{Z}_3$ and accompanied by multiplication of a corresponding factor give us various symmetries of the cubic matrix. Making use of this peculiar property of cubic matrices we define a j-skew-symmetric cubic N-matrix $A = (A_{ikl})$ by

$$A_{ikl} = j\, A_{kli} = j^2\, A_{lik}.$$

Similarly a cubic matrix $\overline{A} = (\overline{A}_{ikl})$ of order N is said to be j^2-skew-symmetric if its entries satisfy

$$\overline{A}_{ikl} = j\overline{A}_{kli} = j^2\overline{A}_{lik}.$$

The subspaces of j-skew-symmetric or j^2-skew-symmetric cubic N-matrices are spanned by the cubic matrices shown in the following figure.

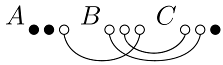


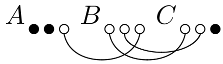
Now our aim is to construct an associative ternary algebra by equipping the vector space of cubic N-matrices with an associative ternary multiplication. Given three cubic matrices of order N we have in total nine subscripts, and because the number of possible combinations of subscripts is finite we can use the methods of computer algebra to find all associative ternary multiplications either of first or second kind. Summarizing the results obtained with the help of computer algebra we can state that there is no totally associative ternary multiplication of first kind of cubic N-matrices which means that it is not possible to construct a totally associative ternary algebra of first kind by means of the vector space of cubic N-matrices. As to a totally associative ternary multiplication of second kind we have found four different multiplications of this kind of cubic N-matrices and the methods of computer algebra we have used guarantee that these are the only ternary multiplications satisfying the requirement of total associativity of second kind.

THEOREM 3.1. *There are only four different triple products of complex cubic matrices of order N which obey the totally ternary associativity of second kind. These are*

1) $(A \odot B \odot C)_{ijk} = \sum_{l,m,n} A_{ilm} B_{nlm} C_{njk}$, $\quad A \odot B \odot C \to$ 

2) $(A \odot B \odot C)_{ijk} = \sum_{l,m,n} A_{ilm} B_{nml} C_{njk}$, $\quad A \odot B \odot C \to$ 

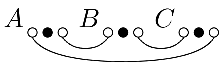
3) $(A \odot B \odot C)_{ijk} = \sum_{l,m,n} A_{ijl} B_{nml} C_{mnk}$, $\quad A \odot B \odot C \to$ 

4) $(A \odot B \odot C)_{ijk} = \sum_{l,m,n} A_{ijl} B_{mnl} C_{mnk}$, $\quad A \odot B \odot C \to$ 

where we use the diagrams containing $\bullet$ which stands for free subscript (no summation) and pairs of $\circ$ (joint by an arc) which stand for contraction with respect to corresponding subscripts. Thus there are four different totally associative ternary algebras of second kind which can be constructed by means of the vector space of cubic N-matrices.

The ternary multiplication of cubic N-matrices defined by the formula

$$(A \circledcirc B \circledcirc C)_{ikl} = \sum_{p,q,r} A_{piq} B_{qkr} C_{rlp}, \quad A \circledcirc B \circledcirc C \to \tag{13}$$



where A, B, C are cubic matrices of order N, has been studied in the papers [1, 5, 6, 7], where it is shown that this multiplication can be used to construct a ternary analog of Pauli matrices by means of the ternary j-commutator defined by

$$[A, B, C] = A \circledcirc B \circledcirc C + j\, B \circledcirc C \circledcirc A + j^2\, C \circledcirc A \circledcirc B.$$

It can be checked that ternary multiplication (13) is neither partially nor totally associative. It is worth mentioning that the ternary product (13) has certain symmetric properties with respect to cyclic permutations of its factors. Indeed if we perform a cyclic permutation of cubic N-matrices A, B, C and the same cyclic permutation of subscripts i, k, l in (13) then the product does not change, i.e.

$$(A \circledcirc B \circledcirc C)_{ikl} = (B \circledcirc C \circledcirc A)_{kli} = (C \circledcirc A \circledcirc B)_{lik}. \tag{14}$$

It is well known that the notion of trace of a square matrix plays an important role in the matrix calculus. We end this section by giving the definition of trace of a cubic N-matrix and describing some of its properties with respect to ternary multiplication (13) and ternary multiplications given in Theorem 3.1. In analogy with the trace of a square matrix we define the trace of a cubic N-matrix $A = (A_{ijk})$ as the sum of entries of its main diagonal, i.e.

$$\mathrm{Tr}\,(A) = \sum_{i}^{N} A_{iii} = A_{111} + A_{222} + \ldots + A_{NNN}.$$

It is easy to check that the trace of a cubic N-matrix is a linear function with respect to addition of cubic matrices and multiplication by scalars (12), i.e. for any two cubic N-matrices A, B and a complex number λ we have

$$\mathrm{Tr}\,(A + B) = \mathrm{Tr}\,(A) + \mathrm{Tr}\,(B), \quad \mathrm{Tr}\,(\lambda \cdot A) = \lambda \cdot \mathrm{Tr}\,(A).$$

The trace of the product of two square matrices does not change if we rearrange the matrices. The trace of a cubic N-matrix has the same property in the case of non-associative ternary multiplication (13). Indeed it immediately follows from (14) that the trace of the product of three cubic N-matrices (13) is invariant under a cyclic permutation of matrices, i.e. for any three cubic N-matrices we have

$$\mathrm{Tr}\,(A \circledcirc B \circledcirc C) = \mathrm{Tr}\,(B \circledcirc C \circledcirc A) = \mathrm{Tr}\,(C \circledcirc A \circledcirc B).$$

PROPOSITION 3.2. *For any five cubic N-matrices A, B, C, D, E and any totally associative ter-nary multiplication of second kind given in Theorem 3.1 the trace satisfies the relations*

$$\mathrm{Tr}\,((\mathrm{A} \odot \mathrm{B} \odot \mathrm{C}) \odot \mathrm{D} \odot \mathrm{E}) = \mathrm{Tr}\,(\mathrm{A} \odot (\mathrm{D} \odot \mathrm{C} \odot \mathrm{B}) \odot \mathrm{E}) = \mathrm{Tr}\,(\mathrm{A} \odot \mathrm{B} \odot (\mathrm{C} \odot \mathrm{D} \odot \mathrm{E})).$$

For any five cubic N-matrices A, B, C, D, E and non-associative ternary multiplication defined by (13) the trace satisfies

$$\mathrm{Tr}\,((\mathrm{A} \circledcirc \mathrm{B} \circledcirc \mathrm{C}) \circledcirc \mathrm{D} \circledcirc \mathrm{E}) = \mathrm{Tr}\,(\mathrm{E} \circledcirc (\mathrm{A} \circledcirc \mathrm{B} \circledcirc \mathrm{C}) \circledcirc \mathrm{D}) = \mathrm{Tr}\,(\mathrm{D} \circledcirc \mathrm{E} \circledcirc (\mathrm{A} \circledcirc \mathrm{B} \circledcirc \mathrm{C})).$$

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