ALGEBRA, GEOMETRY AND MATHEMATICAL PHYSICS
BANACH CENTER PUBLICATIONS, VOLUME 93
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2011

ARTIN-SCHELTER REGULAR ALGEBRAS OF DIMENSION FIVE

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Abstract. We show that there are exactly three types of Hilbert series of Artin-Schelter regular algebras of dimension five with two generators. One of these cases (the most extreme) may not be realized by an enveloping algebra of a graded Lie algebra. This is a new phenomenon compared to lower dimensions, where all resolution types may be realized by such enveloping algebras.

1. Introduction. Artin-Schelter (AS) regular algebras is a class of graded algebras which may be thought of as homogeneous coordinate rings of non-commutative spaces. They were introduced by Artin and Schelter [1], who classified such algebras of dimension up to three which are generated in degree one. Since then these algebras of dimension three and four, and their module theory, have been intensively studied, see [2], [3]. Ideal theory (see [12], [7], [8]) and deformations (see [9]) have also been studied in recent years. This paper is concerned with basic questions for AS-regular algebras of dimension five. (We shall always assume our algebras to be generated in degree one.)

Fundamental invariants of a connected graded algebra are its Hilbert series, and, more refined, the graded betti numbers in a resolution of the residue field k. Unlike the polynomial ring, AS-algebras might not be defined by quadratic relations. For AS-algebras the graded betti numbers may thus be distinct from that of the polynomial ring. For instance in dimension three there are two types of resolutions for the residue field [1]:

2010 *Mathematics Subject Classification*: Primary 16S38; Secondary 16E05, 14A22.
Key words and phrases: Artin-Schelter regular, enveloping algebra, graded Lie algebra, Hilbert series.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc93-0-2

$$A \leftarrow A(-1)^3 \leftarrow A(-2)^3 \leftarrow A(-3),$$
$$A \leftarrow A(-1)^2 \leftarrow A(-3)^2 \leftarrow A(-4).$$

A general class of examples giving AS-algebras of arbitrarily large dimension may be obtained as follows. Let $\mathfrak{g} = \oplus_{i=1}^n \mathfrak{g}_i$ be a finite dimensional graded Lie algebra, generated by $\mathfrak{g}_1$. Then the enveloping algebra $U(\mathfrak{g})$ is AS-regular. The two resolutions above may both be realized by such algebras. The first by the polynomial ring, which is the enveloping algebra of the abelian Lie algebra, while the second resolution occurs for the enveloping algebra of the three-dimensional Heisenberg Lie algebra.

In [11] Lu, Palmieri, Wu, and Zhang investigated four-dimensional AS-algebras and in particular established the possible types of resolutions of the residue field when A is a domain. All these types may be realized by enveloping algebra of Lie algebras. This naturally raises the following question.

- May all resolution types or at least Hilbert series types of AS-algebras be realized by enveloping algebras of graded Lie algebras?

We show that this is not so. Our first main result is the construction of a five-dimensional AS-algebra with Hilbert series not among those of any enveloping Lie algebra generated in degree one.

When studying AS-algebras of dimension four much work has been concerned with AS-algebras which are defined by quadratic relations. In particular the Sklyanin algebra has been intensively studied [16], [17]. In the paper [11] the authors focus on the algebras which are the least like polynomial rings. They have two generators, and one relation of degree three and one of degree four. The authors obtain a classification of such algebras under certain genericity assumptions.

Here we consider AS-algebras of dimension five with two generators, the ones at the opposite extreme compared to polynomial rings. The algebra we exhibit in our first main result is of this kind. Our second main result shows that there are three Hilbert series of such algebras under the natural condition that it is a domain.

It is intriguing that several natural questions concerning AS-algebras are not known.

- Artin and Schelter in [1] conjecture that they are noetherian and domains. Levasseur [10] shows this in small dimensions.
- Is the Hilbert series of an Artin-Schelter regular algebra equal to the Hilbert series of a commutative graded polynomial ring? I.e. is there a finite set of positive integers n_i such that it is
$$\prod_i \frac{1}{(1-t^{n_i})}?$$
Polishchuk and Positselski [14] mention this as a conjecture in Remark 3, p. 135, and attribute it to Artin and Schelter.
- In [11, Question 1.6] they ask whether the minimal number of generators of an AS-regular algebra is always less than or equal to its global dimension.

For enveloping algebras of Lie algebras, this is true by the Poincaré-Birkhoff-Witt theorem. Our example of an AS-algebra that has a Hilbert series different from Hilbert series of enveloping algebras (generated in degree one) still has this property; its Hilbert series equals the Hilbert series of a polynomial ring with generators of degree $1, 1, 2, 3, 5$.

In the resolution of the residue field k the last term will be $A(-l)$ for some integer l (in the two resolutions above for algebras of dimension three, l is 3 and 4). It is natural to expect that the largest l occurs when the algebra is at the opposite extreme of the polynomial ring, when the algebra has two generators.

- What is the largest l such that $A(-l)$ is the last term in the resolution of k, for a given dimension of A?

For polynomial rings of dimension d then $l = d$. For enveloping algebras of Lie algebras the largest l that can occur for dimension d is $1 + \binom{d}{2}$. However the algebra we construct has $l = 12$, while the maximum for enveloping algebras is 11.

A last question to which we do not know a counterexample is the following.

- Are the global dimension and the Gelfand-Kirillov dimension of an AS-regular algebra equal?

The paper is organized as follows. In Section 2 we recall the definition of AS-regular algebras and recall some basic facts concerning their classification and concerning enveloping algebras of graded Lie algebras. In Section 3 we show that enveloping algebras of graded Lie algebras are Artin-Schelter regular. In Section 4 we classify the Hilbert series of enveloping algebras of graded Lie algebras of dimension five. In Section 5 we give an AS-regular algebra of dimension five whose Hilbert series is not that of any enveloping algebra generated in degree one. In Section 6 we classify the Hilbert series of AS-regular algebras of dimension five with two generators.

2. Preliminaries. In this section we first recall the definition of AS-regular algebras. Then we recall basic classification results concerning these in dimension three and four. Lastly we consider enveloping algebras of graded Lie algebras.

Definition 2.1 (AS-algebras). An algebra $A = k \oplus A_1 \oplus A_2 \oplus \cdots$ is called Artin-Schelter regular of dimension d if

(i) A has finite global dimension d.
(ii) A has finite Gelfand-Kirillov-dimension (so the Hilbert function of A is bounded by a polynomial).
(iii) A is Gorenstein; we have

$$\operatorname{Ext}^i_A(k, A) = \begin{cases} 0 & i \neq d \\ k(l) & i = d \end{cases}$$

(here $k(l)$ is the module k in degree $-l$).

Note. We shall in this paper only be concerned with algebras generated in degree one.

2.1. Algebras of dimension three and four. From the original article of Artin and Schelter [1] the classification of these algebras in dimension ≤ 3 is known, see also Artin, Tate and Van den Bergh [2]. For two-dimensional algebras the only possible type of resolution of the residue field is the same as that of the polynomial ring

$$A \leftarrow A(-1)^2 \leftarrow A(-2). \tag{1}$$

For three-dimensional algebras there are two resolution types:

$$A \leftarrow A(-1)^3 \leftarrow A(-2)^3 \leftarrow A(-3), \tag{2}$$

$$A \leftarrow A(-1)^2 \leftarrow A(-3)^2 \leftarrow A(-4). \tag{3}$$

The Hilbert series of the algebras in (1),(2),and (3) are respectively:

$$\frac{1}{(1-t)^2}, \quad \frac{1}{(1-t)^3}, \quad \frac{1}{(1-t)^2(1-t^2)}.$$

In [11] Lu, Palmieri, Wu, and Zhang consider four-dimension algebras. Under the natural hypothesis that A is a domain they show that there are three possible resolution types:

$$A \leftarrow A(-1)^4 \leftarrow \quad A(-2)^6 \quad \leftarrow A(-3)^4 \leftarrow A(-4), \tag{4}$$

$$A \leftarrow A(-1)^3 \leftarrow A(-2)^2 \oplus A(-3)^2 \leftarrow A(-4)^3 \leftarrow A(-5), \tag{5}$$

$$A \leftarrow A(-1)^2 \leftarrow A(-3) \oplus A(-4) \leftarrow A(-6)^2 \leftarrow A(-7). \tag{6}$$

with Hilbert series, respectively:

$$\frac{1}{(1-t)^4}, \quad \frac{1}{(1-t)^3(1-t^2)}, \quad \frac{1}{(1-t)^2(1-t^2)(1-t^3)}.$$

The algebras (1), (2), and (4) are Koszul. The polynomial ring is the basic example for these types. The algebra (3) is 3-Koszul (see [4]). The algebras (5) are of a type studied in [6].

2.2. Enveloping algebras. It is an interesting observation that all these types of resolutions may be realized by enveloping algebras of graded Lie algebras. In fact such algebras are always AS-regular. This is known to experts, but an explicit reference seems hard to come by. Since we consider this such an important class of examples, we have included a proof of this fact in the next section.

For enveloping algebras the form of a minimal resolution may often easily be deduced from the Chevalley-Eilenberg complex which gives a resolution by left modules of the residue field k of $U = U(\mathfrak{g})$. Explicitly the Chevalley-Eilenberg resolution $C.$ has terms $C_p = U \otimes_k \wedge^p \mathfrak{g}$ and differential $d : C_p \to C_{p-1}$ where the image of $u \otimes x_1 \wedge \cdots \wedge x_p$ is given by

$$\sum_{l=1}^{p} (-1)^{l+1} u x_l \otimes x_1 \wedge \cdots \wedge \hat{x}_l \wedge \cdots \wedge x_p$$
$$+ \sum_{1 \le l < m \le p} (-1)^{l+m} u \otimes [x_l, x_m] \wedge x_1 \wedge \cdots \wedge \hat{x}_l \wedge \cdots \wedge \hat{x}_m \wedge \cdots \wedge x_p. \tag{7}$$

There is also a right Chevalley-Eilenberg resolution $C'.$ of k by right modules. It has terms $C'_p = \wedge^p \mathfrak{g} \otimes_k U$ and differential $d : C'_p \to C'_{p-1}$ where the image of $x_1 \wedge \cdots \wedge x_p \otimes_k U$ is given by

$$\sum_{l=1}^{p} (-1)^{p-l} x_1 \wedge \cdots \wedge \hat{x}_l \wedge \cdots \wedge x_p \otimes_k x_l u$$
$$+ \sum_{1 \le l < m \le p} (-1)^{l+m} x_1 \wedge \cdots \wedge \hat{x}_l \wedge \cdots \wedge \hat{x_m} \wedge \cdots \wedge x_p \wedge [x_l, x_m] \otimes_k u. \tag{8}$$

2.3. Minimal resolutions of enveloping algebras. If X is a finite set, we let $\mathcal{L}ie(X)$ be the free Lie algebra with generating set X, and $T(X)$ the free associative algebra (tensor algebra). It is the enveloping algebra of the free Lie algebra on X.

Now if $\mathfrak{h}$ is an ideal in a Lie algebra $\mathfrak{g}$, then $U(\mathfrak{g}/\mathfrak{h})$ is equal to $U(\mathfrak{g})/(\mathfrak{h})$. Hence if I is an ideal in $\mathcal{L}ie(X)$, the enveloping algebra of $\mathcal{L}ie(X)/I$ is $T(X)/(I)$.

The resolutions (1), (2), and (4) are realized by the polynomial ring, the enveloping algebra of the abelian Lie algebra. The resolution (3) may be realized by the enveloping algebra of the graded Heisenberg Lie algebra which is $\langle x, y\rangle \oplus \langle [x, y]\rangle$. The resolution (5) may be realized by the enveloping algebra of

$$\mathcal{L}ie(x, y, z)/([z, x], [z, y], [x, [x, y]], [y, [x, y]])$$

which as a graded Lie algebra may be written in terms of basis elements as

$$\langle x, y, z\rangle \oplus \langle [x, y]\rangle.$$

The resolution (6) may be realized by the enveloping algebra of

$$\mathcal{L}ie(x, y)/([x[x[xy]]], [[xy]y])$$

which as a graded Lie algebra may be written in terms of basis elements as

$$\langle x, y\rangle \oplus \langle [x, y]\rangle \oplus \langle [x, [x, y]]\rangle.$$

That the resolutions of these algebras are as stated, is easily worked out by writing the Chevalley-Eilenberg complex and figuring out which adjacent free terms may be cancelled to give a minimal resolution.

For a graded Lie algebra $\mathfrak{g} = \oplus \mathfrak{g}_i$ let $h_{\mathfrak{g}}(i) = \dim_k \mathfrak{g}_i$ be its Hilbert function. It is an easy consequence of the Poincaré-Birkhoff-Witt theorem that its Hilbert series is

$$\prod_i \frac{1}{(1-t^i)^{h_{\mathfrak{g}}(i)}}. \tag{9}$$

If $\mathfrak{g}$ is finite dimensional of small dimension, we shall display the Hilbert function by the sequence $h_{\mathfrak{g}}(1), h_{\mathfrak{g}}(2), \ldots, h_{\mathfrak{g}}(n)$ where n is the largest argument for which the value of the Hilbert function is nonzero.

3. Enveloping algebras are Artin-Schelter regular. This section contains a proof of the following.

Theorem 3.1. *Let $\mathfrak{g}$ be a finite dimensional positively graded Lie algebra. Then the enveloping algebra $U(\mathfrak{g})$ is an Artin-Schelter regular algebra. Its global dimension and Gelfand-Kirillov dimension are both equal to the vector space dimension of $\mathfrak{g}$.*

As said, this is known to experts, but we include a proof since we consider this an important class of AS-algebras, and a reference is hard to come by.

Proof of Theorem 3.1. There are three conditions for an algebra to be Artin-Schelter regular.

(i.) The global dimension must be finite. For an enveloping algebra, the global dimension is equal to the dimension of the Lie algebra, see Exercise 7.7.2 of [19].

(ii.) The Gelfand-Kirillov dimension must be finite. This follows by the Poincaré--Birkhoff-Witt theorem, see (9).

(iii.) The algebra must have the Gorenstein property. This follows by Proposition 3.5 below. ∎

REMARK 3.2. One may patch together an argument for the above theorem from sources in the litterature as follows. If A is an Auslander regular algebra (see [13], 3.2.4), then an Ore extension $A[x;\sigma,\delta]$ is also Auslander regular, [13], 3.2.16.4. But the enveloping algebra of a graded Lie algebra is an iterated Ore extension, as we note below, and so is Auslander regular. By [13], p. 127, an Auslander regular algebra is AS-regular.

To see that enveloping algebras of graded Lie algebras are iterated Ore extensions, let $\mathfrak{h} \subseteq \mathfrak{g}$ be an ideal in a Lie algebra with one-dimensional quotient $\mathfrak{g}/\mathfrak{h}$ and x an element of $\mathfrak{g}$ generating this quotient. Then the map $[x,-] : \mathfrak{h} \to \mathfrak{h}$ is a derivation. It is easily seen that a derivation on the Lie algebra extends to a derivation of the enveloping algebra. Then $U(\mathfrak{g})$ becomes an Ore extension $U(\mathfrak{h})[x;\sigma,\delta]$ with $\sigma = \mathrm{id}$ and $\delta = [x,-]$.

3.1. Some basic facts on modules and their duals. Before proving Proposition 3.5 below, we recall some general facts about modules over (non-commutative) rings. If A and B are left modules over a ring R, denote by $\mathrm{Hom}_R(A,B)$ the group of left homomorphisms. If A and B are R-bimodules, denote by $\mathrm{Hom}_{R-R}(A,B)$ the group of bimodule homomorphisms. For a left R-module A we write the dual $A^* = \mathrm{Hom}_R(A,R)$, which is a right R-module. If A is a bimodule, this dual is also naturally a left R-module.

LEMMA 3.3. *Let R be a ring, B a left R-module and A an R-bimodule.*

a. $Hom_R(B, A^*) \cong Hom_R(A \otimes_R B, R)$.

b. If B is also an R-bimodule, there is a natural isomorphism of bimodule homomorphisms

$$Hom_{R-R}(B, A^*) \cong Hom_{R-R}(A \otimes_R B, R).$$

c. In particular if $B = A^*$ we get a natural pairing

$$A \otimes_R A^* \to R.$$

Proof. a. This is just the standard adjunction between $\mathrm{Hom}_R(A,-)$ and $A \otimes_R -$ for a bimodule A.

b. Let $B \xrightarrow{\phi} A^*$ be a left R-module homomorphism. It corresponds to the pairing $A \otimes_R B \to R$ given by $a \otimes b \mapsto \phi(b)(a)$. That this pairing is a bi-module homomorphism means that

$$\phi(br)(a) = \phi(b)(a) \cdot r.$$

That ϕ is a bi-module homomorphism means that $\phi(br) = \phi(b) \cdot r$ which again says the same as the equation above.

c. This natural pairing corresponds to the identity $A^* \to A^*$. ∎

PROPOSITION 3.4. *Let R be a ring and let A and B be R-bimodules. Given a homomorphism $A \xrightarrow{\alpha} B$ of left modules and $B^* \xrightarrow{\beta} A^*$ of right modules. Then β is dual to α iff the natural pairings*

$$A \otimes_R A^* \to R, \quad B \otimes_R B^* \to R$$

fulfill the following for a in A and b' in B^*:

$$\langle a, \beta(b') \rangle = \langle \alpha(a), b' \rangle.$$

Proof. The left side of the equation above is equal to $\beta(b')(a)$ while the right side is equal to $b'(\alpha(a)) = b' \circ \alpha(a)$. So the equation above says $\beta(b') = b' \circ \alpha$, which means that β is dual to α. ∎

3.2. Duality between the left and right Chevalley-Eilenberg complex. For a finite-dimensional Lie algebra $\mathfrak{g}$ with basis $\{x_1, x_2, \cdots, x_{\dim_k \mathfrak{g}}\}$, we consider the form

$$\Delta(\mathfrak{g}) = \sum_{1 \leq i < j \leq \dim_k \mathfrak{g}} (-1)^{i+j} [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x_i} \wedge \cdots \wedge \hat{x_j} \wedge \cdots \wedge x_{\dim_k \mathfrak{g}}.$$

This form lies in $\wedge^{\dim_k \mathfrak{g} - 1} \mathfrak{g}$ and is uniquely determined up to a scalar, since it is easily seen that it is invariant under substitutions $x_i \mapsto x_i + \alpha x_j$ with $j \neq i$. Note that this form occurs in one of the parts of the initial differential of the Chevalley-Eilenberg complex. In particular the vanishing of this form is equivalent to the top Lie algebra homology $H_{\dim_k \mathfrak{g}}(\mathfrak{g}, k)$ (defined as $H_{\dim_k \mathfrak{g}}(k \otimes_U C.)$) being nonzero, isomorphic to k.

Proposition 3.5. *If the form $\Delta(\mathfrak{g})$ vanishes, the dual of the left Chevalley-Eilenberg complex is isomorphic to the right Chevalley-Eilenberg complex. In particular this holds for positively graded Lie algebras.*

Proof. Note that the form vanishes if each term $[x_i, x_j]$ is contained in the linear span of the other x_k. This is certainly true for a positively graded Lie algebra, since the degree of the bracket will be larger than both the degree of x_i and of x_j.

Now let n be $\dim_k \mathfrak{g}$. There is a natural perfect pairing

$$\wedge^p \mathfrak{g} \otimes_k \wedge^{n-p} \mathfrak{g} \longrightarrow \wedge^n \mathfrak{g}$$

giving an isomorphism

$$\wedge^p \mathfrak{g} \to \mathrm{Hom}_k(\wedge^{n-p} \mathfrak{g}, \wedge^n \mathfrak{g}).$$

We get an induced pairing

$$(U \otimes_k \wedge^p \mathfrak{g}) \otimes_U (\wedge^{n-p} \mathfrak{g} \otimes_k U) \xrightarrow{\langle , \rangle} U \otimes_k \wedge^n \mathfrak{g}$$

which by Lemma 3.3.b corresponds to the isomorphism of U-bimodules

$$\wedge^{n-p} \mathfrak{g} \otimes_k U \to \mathrm{Hom}_U(U \otimes_k \wedge^p \mathfrak{g}, U \otimes_k \wedge^n \mathfrak{g}).$$

For the left Chevalley-Eilenberg complex $C.$ of (7), we shall show that $\mathrm{Hom}_U(C., U \otimes_k \wedge^n \mathfrak{g})$ is isomorphic to the right Chevalley-Eilenberg complex $C'.$ of (8), but equipped with differential $(-1)^{n-1} d'$. According to Proposition 3.4 we must then show that for $u \otimes \mathbf{x}$ in $U \otimes_k \wedge^p \mathfrak{g}$ and $\mathbf{y} \otimes v$ in $\wedge^{n-p+1} \mathfrak{g} \otimes_k U$ we have

$$\langle d(u \otimes \mathbf{x}), \mathbf{y} \otimes v \rangle = (-1)^{n-1} \langle u \otimes \mathbf{x}, d'(\mathbf{y} \otimes v) \rangle. \tag{10}$$

We may as well assume that $\mathbf{x} = x_1 \wedge \cdots \wedge x_p$ and $\mathbf{y} = x_q \wedge \ldots \wedge x_{q+n-p}$ where $q \leq p$. We divide into three cases.

1. $q \leq p - 2$. Then $\mathbf{x}$ and $\mathbf{y}$ have at least three overlapping x_i's, and we see that both expressions in (10) are zero.

2. $q = p-1$. Then $\mathbf{x}$ and $\mathbf{y}$ have two overlapping x_i's. The left side of (10) is then

$$(-1)^{(2p-1)} uv \otimes_k [x_{p-1}, x_p] \wedge x_1 \wedge \cdots \wedge x_{n-1}$$

and this is equal to the right side which is

$$(-1)^{n-1} \cdot (-1)^{(2p-1)} uv \otimes_k x_1 \wedge \cdots \wedge x_{n-1} \wedge [x_{p-1}, x_p].$$

3. $q = p$, so $\mathbf{x}$ and $\mathbf{y}$ have one overlapping x_i. The left side of (10) is then

$$\begin{aligned} &(-1)^{p+1} u x_p v \otimes_k x_1 \wedge \cdots \wedge x_n \\ &+ \sum_{i<p} (-1)^{i+p} uv \otimes_k [x_i, x_p] \wedge x_1 \wedge \cdots \hat{x_i} \cdots \wedge x_n, \end{aligned}$$

while the right hand side is

$$\begin{aligned} &(-1)^{n-1} [(-1)^{n-p} u x_p v \otimes_k x_1 \wedge \cdots \wedge x_n \\ &+ \sum_{p<j} (-1)^{p+j}\ uv \otimes_k x_1 \wedge \cdots \hat{x_j} \cdots \wedge x_n \wedge [x_p, x_j]]. \end{aligned}$$

These two expressions are equal provided

$$\begin{aligned} &\sum_{i<p} (-1)^{i+p} [x_i, x_p] \wedge x_1 \wedge \cdots \hat{x_i} \cdots \wedge x_n \\ &- \sum_{p<j} (-1)^{p+j} [x_p, x_j] \wedge x_1 \wedge \cdots \hat{x_j} \cdots \wedge x_n \end{aligned}$$

is zero. But this expression is just $(-1)^{n-p} \Delta(\mathfrak{g}) \wedge x_p$ and hence is zero. ■

REMARK 3.6. One may show that the form $\Delta(\mathfrak{g})$ vanishes for nilpotent and semi-simple Lie algebras. In general it does not however vanish.

4. Hilbert series of enveloping algebras of dimension five. In this section we give the Hilbert series of enveloping algebras of five-dimensional graded Lie algebras. The classification of resolutions is just slightly more refined. In all cases save one there is one resolution type for each Hilbert series.

PROPOSITION 4.1. *The following are the Hilbert functions of graded Lie algebras of dimension five which are generated in degree one:*

$$a)\ 5 \quad b)\ 4,1 \quad c)\ 3,2 \quad d)\ 3,1,1 \quad e)\ 2,1,2 \quad f)\ 2,1,1,1.$$

Proof. Denote by $\mathcal{L}_n$ the degree n piece of a free Lie algebra $\mathcal{L}ie(X)$. The cases above may then be realized as follows.

Case	Lie algebra	Basis
a)	Abelian Lie algebra	x, y, z, w, t
b)	$\mathcal{L}ie(x,y,z,w)/(([z,-],[w,-]) + \mathcal{L}_3)$	$x, y, z, w, [x,y]$
c)	$\mathcal{L}ie(x,y,z)/(([y,z]) + \mathcal{L}_3)$	$x, y, z, [x,y], [x,z]$
d)	$\mathcal{L}ie(x,y,z)/(([z,-],[x,[x,y]]) + \mathcal{L}_4)$	$x, y, z, [x,y], [[x,y],y]$
e)	$\mathcal{L}ie(x,y)/(\mathcal{L}_4)$	$x, y, [x,y], [x,[x,y]], [[x,y],y]$
f)	$\mathcal{L}ie(x,y)/(([x,[x,y]]) + \mathcal{L}_5)$	$x, y, [x,y], [[x,y],y], [[[x,y],y],y]$

That there are no more possible Hilbert functions for Lie algebras generated in degree one, is trivial to verify. ∎

REMARK 4.2. In all these cases save one there is only one possible resolution type of enveloping algebras associated to graded Lie algebras with this Hilbert function. The exception is for the Hilbert function $4, 1$. Consider the Lie algebra

$$\mathfrak{g} = \mathcal{L}ie(x, y, z, w)/(([z, -], [w, -]) + \mathcal{L}_3)$$

from the proof of case b) above, and

$$\mathfrak{h} = \mathcal{L}ie(x, y, z, w)/([x, y] - [z, w], [x, z], [x, w], [y, z], [y, w]).$$

It is clear that $\mathfrak{g}$ has two (necessary) relations in degree three, killing $[x, [x, y]]$ and $[[x, y], y]$. For $\mathfrak{h}$ all cubic relations are consequences of the quadratic relations. For instance,

$$[x, [x, y]] = [x, [z, w]] = [[w, x], z] + [[x, z], w] = 0.$$

With U the enveloping algebra of $\mathfrak{g}$ and V the enveloping algebra of $\mathfrak{h}$, the minimal resolutions of k are respectively

$$\begin{aligned} U &\leftarrow U(-1)^4 \leftarrow U(-2)^5 \oplus U(-3)^2 \\ &\qquad \leftarrow U(-3)^2 \oplus U(-4)^5 \leftarrow U(-5)^4 \leftarrow U(-6), \\ V &\leftarrow V(-1)^4 \leftarrow V(-2)^5 \leftarrow V(-4)^5 \leftarrow V(-5)^4 \hookleftarrow V(-6). \end{aligned}$$

Thus the two Artin-Schelter regular algebras U and V have the same Hilbert series, but different Betti numbers.

This result can also be obtained easily by considering the rank of the relevant differential in the Chevalley-Eilenberg resolution (the part from $U(-3)^4 \leftarrow U(-3)^4$). In our examples, this map has rank two or four. The eager reader is encouraged to check that there are also examples with rank three.

REMARK 4.3. We shall be concerned with the classification of Hilbert series of algebras of dimension five generated by two variables. This occurs in cases e) and f) As mentioned above there is only one type of resolution of enveloping algebras for each Hilbert function. They are in case e) and f) respectively

$$\begin{aligned} A &\leftarrow A(-1)^2 \leftarrow A(-4)^3 \leftarrow A(-6)^3 \leftarrow A(-9)^2 \leftarrow A(-10), \\ A &\leftarrow A(-1)^2 \leftarrow A(-3) \oplus A(-5)^2 \\ &\qquad \leftarrow A(-6)^2 \oplus A(-8) \leftarrow A(-10)^2 \leftarrow A(-11). \end{aligned}$$

In the introduction we asked the question of how large l could be in the last term $A(-l)$ in the minimal resolution of k, for a given global dimension. By the remark above we see that $l = 11$ may occur for global dimension five. For enveloping algebras this is the largest l as the following shows.

PROPOSITION 4.4. *For an enveloping algebra of a graded Lie algebra of dimension d, the highest possible twist l of the last term in a minimal resolution of k is $1 + \binom{d}{2}$.*

Proof. The highest possible twist is the degree of $\wedge^{\dim \mathfrak{g}}\mathfrak{g}$ as we see from the Chevalley-Eilenberg complex. If $h_{\mathfrak{g}}$ is the Hilbert function of $\mathfrak{g}$, then this is $\sum i \cdot h_{\mathfrak{g}}(i)$. Since $\mathfrak{g}$ is generated in degree one, clearly $h_{\mathfrak{g}}(1) \geq 2$ and if $h_{\mathfrak{g}}(i) = 0$ for some i, it is zero for every successive argument. This gives that

$$l \leq 1 \cdot 2 + 2 \cdot 1 + 3 \cdot 1 + \cdots + (d-1) \cdot 1 = 1 + \binom{d}{2}.$$

On the other hand there does actually exist an (infinite dimensional) graded Lie algebra with Hilbert function values

$$2, 1, 1, \cdots, 1, \cdots.$$

It is the quotient of $\mathcal{L}ie(x,y)$ by the bigraded ideal generated by all Lie monomials of bidegree (a,b) with $a \geq 2$. The quotient Lie algebra $\hat{\mathfrak{g}}$ has a standard basis consisting of y and the Lie monomials L_i of bidegree $(1, i-1)$ for $i \geq 1$ defined inductively by $L_1 = x$ and $L_i = [L_{i-1}, y]$.

The quotient of $\hat{\mathfrak{g}}$ by L_d will then be a finite dimensional Lie algebra with $l = 1 + \binom{d}{2}$. ■

We shall see in the next section that for global dimension five, $l = 11$ is not the largest twist. In fact we exhibit an algebra where $l = 12$.

5. An extremal algebra of dimension five. We now give our first main result, namely an AS-regular algebra of dimension five which has a Hilbert series not occurring for enveloping algebras generated in degree one. This shows that the numerical classes of AS-regular algebras generated in degree one extend beyond that of enveloping algebras.

DEFINITION 5.1. Let $\mathcal{A}$ be the quotient algebra of the tensor algebra $k\langle x, y\rangle$ by the ideal generated by the commutator relations

$$[x^2, y], \quad [x, y^3], \quad [x, yRy], \tag{11}$$

where R is $yxyx + xy^2x + xyxy$.

THEOREM 5.2. *The algebra $\mathcal{A}$ is AS-regular. Its resolution is*

$$\mathcal{A} \xleftarrow{d_1} \mathcal{A}(-1)^2 \xleftarrow{d_2} \mathcal{A}(-3) \oplus \mathcal{A}(-4) \oplus \mathcal{A}(-7)$$
$$\xleftarrow{d_3} \mathcal{A}(-5) \oplus \mathcal{A}(-8) \oplus \mathcal{A}(-9) \xleftarrow{d_4} \mathcal{A}(-11)^2 \xleftarrow{d_5} \mathcal{A}(-12)$$

where the differentials are

$$d_1 = \begin{bmatrix} x & y \end{bmatrix},$$

$$d_2 = \begin{bmatrix} xy & y^3 & yRy \\ -x^2 & -y^2x & -Ryx \end{bmatrix},$$

$$d_3 = \begin{bmatrix} y^2 & Ry & 0 \\ -x & 0 & -yR \\ 0 & -x & y^2 \end{bmatrix},$$

$$d_4 = \begin{bmatrix} -yRy & xyR \\ y^3 & -xy^2 \\ yx & -x^2 \end{bmatrix},$$

$$d_5 = \begin{bmatrix} x \\ y \end{bmatrix}.$$

The Hilbert series of $\mathcal{A}$ is

$$\frac{1}{(1-t)^2(1-t^2)(1-t^3)(1-t^5)}.$$

Remark 5.3. The algebra is bigraded. If we list the bidegrees of the generators the resolution takes the following form.

$$\begin{array}{ccccccccccc} \mathcal{A} & \leftarrow & \mathcal{A}^2 & \leftarrow & \mathcal{A}^3 & \leftarrow & \mathcal{A}^3 & \leftarrow & \mathcal{A}^2 & \leftarrow & \mathcal{A} \\ (0,0) & & (1,0) & & (2,1) & & (2,3) & & (4,7) & & (5,7) \\ & & (0,1) & & (1,3) & & (4,4) & & (5,6) & & \\ & & & & (3,4) & & (3,6) & & & & \end{array}.$$

The two-variable Hilbert series of the algebra is

$$\frac{1}{(1-t)(1-u)(1-tu)(1-tu^2)(1-t^2u^3)}.$$

Remark 5.4. The algebra may be deformed by letting the third relation be

$$[x, yRy] + t[x, y^2x^2y^2]. \tag{12}$$

This will again give an AS-regular algebra with the same resolution type. In addition we may deform the commutator relations

$$[x^2, y] \rightsquigarrow x^2y - pyx^2, \quad [x, y^3] \rightsquigarrow xy^3 - qy^3x$$

which must then be accompanied by a suitable deformation of the relation (12) above. According to our computations these deformations give all algebras giving a bigraded resolution of the form in Remark 5.3.

To prove the form of the Hilbert series and that the complex above gives a resolution of $\mathcal{A}$, we invoke Bergman's diamond lemma [5].

Diamond lemma, specialized to two variables. S is a set of pairs $\sigma = (W_\sigma,\ f_\sigma)$, where W_σ is a monomial and f_σ a polynomial in $k\langle x, y\rangle$. A *reduction* based on S consists of exchanging the monomial W_σ with the polynomial f_σ. If we have two pairs σ, τ such that $W_\sigma = AB$ and $W_\tau = BC$, there is a choice of reducing the monomial ABC starting with σ or τ. This is called an overlap ambiguity. The similar case of inclusion ambiguity will not play any role for the application we have in mind. The ambiguity is resolvable if there are further reductions of the results of these two choices, giving a common answer. The elements of $k\langle x, y\rangle$ that cannot be reduced by S are called irreducible. An element is called uniquely reducible if it can be reduced, in a finite number of steps, to an irreducible element, and this irreducible element is unique. We also need a partial order on monomials such that i) $B < B'$ implies $ABC < AB'C$ for all monomials A, C, ii) any monomial appearing with a nonzero coefficient in f_σ is $< W_\sigma$, and iii) the ordering has the descending chain condition.

THEOREM 5.5 (Bergman's diamond lemma). *Under the assumptions above, the following are equivalent:*

a) *All ambiguities of S are resolvable.*
b) *All elements of $k\langle x, y\rangle$ are uniquely reducible under S.*
c) *The irreducible elements form a set of representatives for $k\langle x, y\rangle/(W_\sigma - f_\sigma)_{\sigma \in S}$.*

Under these conditions, products in $\mathcal{A}$ can be formed by multiplying the corresponding irreducible representatives, and then reducing the answer.

Using this theorem, we will say that an element of $\mathcal{A}$ is written in *standard form* if it is written as an irreducible element in $k\langle x, y\rangle$.

Proof of Theorem 5.2. That the differentials give a complex is straightforward to see except perhaps for the product of the first row in d_3 and second row in d_4. This product is

$$-y^2xyR + Ryxy^2.$$

But this becomes zero in $\mathcal{A}$ because it may be verified to be equal to

$$[x, yRy]y + y[x, yRy].$$

We choose a monomial ordering as follows. First, if $\deg m_1 < \deg m_2$ then $m_1 < m_2$. If $\deg m_1 = \deg m_2$, write $m_1 = z_1 z_2 \cdots z_n$, $m_2 = w_1 w_2 \cdots w_n$ where each z_i, w_j is either x or y. Let $l_i(m_1)$ count the number of ys among $z_1 \cdots z_i$, and similarly for $l_i(m_2)$. If $l_i(m_1) \geq l_i(m_2)$ for each i, then $m_1 \leq m_2$. The needed properties are easily verified.

Now let

$$A = xy, \quad B = xy^2, \quad C = AB = xyxy^2.$$

With this monomial ordering, the three relations (11) for the algebra must be divided by choosing

$$\begin{aligned} &W_1 = x^2y, \quad W_2 = xy^3, \quad W_3 = A^2B, \\ &f_1 = yx^2, \quad\ \ f_2 = y^3x, \quad\ \ f_3 = -(ABA + BA^2 - yCx - yBAx - y^2A^2x). \end{aligned}$$

The overlap ambiguities can be resolved, except for one. Consider for instance the overlap $x^2y^3 = W_1y^2 = xW_2$. By first replacing W_1 by yx^2 we get yx^2y^2. It is easy to see that this can be reduced further, by replacing W_1 by yx^2 twice, to y^3x^2. If we on the other hand start by replacing W_2 by y^3x we get xy^3x. Replacing W_2 by y^3x once more, we obtain the same irreducible polynomial as before. Thus this ambiguity is resolved. The only ambiguity that cannot be resolved is

$$xyxyxy^3 = W_3y = xyxyW_2.$$

By reducing W_3y and $xyxyW_2$ as much as possible, we come to an equation expressing $xyxy^2xy^2$ in terms of smaller monomials (in terms of the chosen ordering). Therefore we must introduce a fourth reduction, with $W_4 = CB = AB^2$. This introduces further ambiguities, but a routine computation shows that they are all resolvable. By using the reductions by W_1 and W_2 the words can be reduced to the form

$$y^{n_y}Mx^{n_x}$$

where M is a tensor monomial in A and B. By also using the reductions by W_3 and W_4, we see that the standard monomials become

$$y^{n_y} B^{n_B} C^{n_C} A^{n_A} x^{n_x}.$$

Immediate consequences are the following.

- $h_{\mathcal{A}}$ The Hilbert series of $\mathcal{A}$ is as stated in the theorem.
- x, y Multiplication by x (from the left or from the right) is injective: since x^2 is central and the normal form shows that multiplication by it is injective, so is multiplication by x. Similarly for y since y^3 is central.
- d_5 The last map in the alleged resolution is injective, since each of its two terms is injective.

So far we know that the Hilbert series of the complex equals the Hilbert series of the $\mathcal{A}$-module k, therefore it is enough to check exactness at all but one of the terms. We know 1. that d_5 is injective, and 2. the image of d_2 equals the kernel of d_1.

3. The image of d_5 equals the kernel of d_4: let an element in $\ker d_4$ be written as $[f,\, g]^T$, where all monomials appearing are in standard form. From d_5 we can alter g by any multiple $y \cdot -$. That is to say, we can assume that no monomial occurring in g starts with y. The second relation imposed by d_4 then gives the relation

$$y^3 f + Bg \equiv 0.$$

All monomials appearing are on standard form! Since all monomials in the first term start with y, and none in the second does, we see that $f = 0$, and hence $g = 0$ (modulo $\operatorname{im} d_5$). So $\operatorname{im} d_5 = \ker d_4$.

4. The image of d_4 equals the kernel of $\ker d_3$: an element in $\ker d_3$ can be written $[f,\, g,\, h]^T$, where all monomials appearing are in standard form. From d_4 we can alter g by any multiple $y^3 \cdot -$ and $B \cdot -$. In other words, we can assume that no monomial appearing in g starts with y^3 or B. The last condition imposed by d_3 is that $-xg + y^2 h = 0$. We will first rule out the possibility that g contains monomials starting with y^2. In that case, xg contains monomials starting with $xy^2 = B$, and these cannot be countered by terms in $y^2 h$, all of whose monomials start with y^2. Then we exclude the possibility that g contains monomials not starting with y. Since no monomials in g start with B, any such monomial would either start with xy or just be a power of x (the latter being trivially ruled out). If a monomial in g starts with xy, then xg would start with $x^2 y$. But x^2 is central, so can be moved to the right, and any such monomial would give a contribution of y times something not starting with y, and so cannot be countered by anything from $y^2 h$. The only case not excluded so far is $g = yg'$, where no monomial in g' starts with y.

Consider the first relation imposed by d_3:

$$0 = y^2 f + Ryg = y^2 f + yA^2 g + BAg + ABg. \tag{13}$$

Note that

$$ABg = xyxy^3 g' = y^3 xyg'$$

so when reduced to standard form, it will have all terms starting with y. But $BAg = B^2g'$ is on standard form and starts with x. Monomials here cannot be countered by any other terms in relation (13). Therefore g', and hence g, must be zero. The third condition imposed by d_3 then shows that h is zero, and hence also f. This concludes the proof that the alleged resolution is indeed the resolution of k as an $\mathcal{A}$-module. ∎

6. Necessary conditions. In this section we show that there are three possible Hilbert series of AS-regular algebras of global dimension five with two generators, under the natural extra conditions that the algebra is an integral domain and that its Gelfand-Kirillov dimension is greater or equal to 2. The arguments are of a numerical nature and concern the possible resolution types. There will be five possible resolutions but these give only three distinct Hilbert series.

LEMMA 6.1. *Let A be a regular algebra with Hilbert series $h_A(t)$. Suppose $h(t) = p(t) \cdot h_A(t)$ is a power series with non-negative coefficients. If $p(t) = (1-t)^r \cdot q(t)$ where $q(1) \neq 0$, then $q(1) > 0$.*

Proof. This follows exactly as in the proof of Proposition 2.21 in [3]. ∎

We now suppose that A is a regular algebra of global dimension 5 having two generators in degree one. The minimal resolution of k will then have length 5 and must have the form

$$A \xleftarrow{d_1} A(-1)^2 \xleftarrow{d_2} \oplus_{i=1}^n A(-a_i) \xleftarrow{d_3} \oplus_{i=1}^n A(a_i - l) \xleftarrow{d_4} A(-l+1)^2 \xleftarrow{d_5} A(-l) \qquad (14)$$

where we order $a_1 \leq a_2 \leq \cdots \leq a_n$.

THEOREM 6.2. *Let A be a regular algebra of global dimension five with resolution as above. Suppose A is an integral domain and has GKdim $A \geq 2$. Then $a_{i+1} + a_{n+1-i} < l$ for $i = 1, \ldots, n-1$.*

Proof. Suppose for some r that $a_{r+1} + a_{n+1-r} \geq l$. We may assume that $r+1 \leq n+1-r$ or $2r \leq n$. Let us suppose that r is chosen maximal for these conditions. We get $-a_{r+1} \leq a_{n+1-r} - l$. We then get a subcomplex of the resolution (14):

$$A \xleftarrow{\delta_1} A(-1)^2 \xleftarrow{\delta_2} \oplus_{i=1}^r A(-a_i) \xleftarrow{\delta_3} \oplus_{i=1}^r A(a_{n+1-i} - l). \qquad (15)$$

Now the power series in $\mathbf{Z}[[t, t^{-1}]]$ have a partial order defined by $h(t) \geq g(t)$ if each coefficient $h_i \geq g_i$. Consider now the map $A(-a_i) \to A(-1)^2$ coming from the map d_2. It cannot be zero since the resolution is minimal. For a general quotient $A(-1)^2 \to A(-1)$, the composition $A(-a_i) \to A(-1)$ is injective, since A is an integral domain. Hence $A(-a_1)$ maps injectively into $\operatorname{im}\delta_2$ and so

$$h_{\operatorname{im}\delta_2} \geq h_{A(-a_1)}.$$

Now there is a sequence (with possible cohomology in the middle):

$$0 \to \operatorname{im}\delta_3 \to \oplus_{i=1}^r A(-a_i) \to \operatorname{im}\delta_2 \to 0,$$

which gives

$$\sum_{i=1}^r h_{A(-a_i)} \geq h_{\operatorname{im}\delta_3} + h_{\operatorname{im}\delta_2} \geq h_{\operatorname{im}\delta_3} + h_{A(-a_1)}$$

or

$$\sum_{i=2}^{r} h_{A(-a_i)} \geq h_{\mathrm{im}\,\delta_3}.$$

The short exact sequence

$$0 \to \ker \delta_3 \to \oplus_{i=1}^{r} A(a_{n+1-i} - l) \to \mathrm{im}\,\delta_3 \to 0$$

then gives

$$h_{\ker \delta_3} \geq \sum_{i=1}^{r} h_{A(a_{n+1-i}-l)} - \sum_{i=2}^{r} h_{A(-a_i)}.$$

Since $\ker \delta_3 \subseteq \ker d_3 = \mathrm{im}\, d_4$ we get

$$2h_{A(-l+1)} - h_{A(-l)} \geq \sum_{i=1}^{r} h_{A(a_{n+1-i}-l)} - \sum_{i=2}^{r} h_{A(-a_i)},$$

which gives

$$\Big(-t^l + 2t^{l-1} - \sum_{i=1}^{r} t^{l-a_{n+1-i}} + \sum_{i=2}^{r} t^{a_i} \Big) \cdot h_A \geq 0.$$

According to Lemma 6.1 the derivative of the first expression will have a value at $t = 1$ which is zero or negative, so

$$-l + 2(l-1) - \sum_{i=1}^{r} (l - a_{n+1-i}) + \sum_{i=2}^{r} a_i \leq 0,$$

$$\sum_{i=1}^{r} a_i + \sum_{i=1}^{r} a_{n+1-i} \leq (r-1)l + 2 + a_1. \tag{16}$$

Since GKdim $A \geq 2$ we have by Lemma 6.4 that

$$\sum_{i=1}^{n} a_i = \frac{n-1}{2} \cdot l + 2. \tag{17}$$

Together with (16) above we get

$$\sum_{i=r+1}^{n-r} a_i \geq \frac{n-2r+1}{2} \cdot l - a_1.$$

Suppose first that n is even. Since

$$a_i + a_{n+1-i} \leq a_{i+1} + a_{n+1-i} \leq l - 1$$

for $r + 1 \leq i \leq n/2$ we get

$$\frac{n-2r}{2}(l-1) \geq \frac{n-2r+1}{2} l - a_1,$$

$$a_1 \geq l/2 + \frac{n-2r}{2}.$$

Since the a_i are nondecreasing we get by (17)

$$\frac{n-1}{2} l + 2 \geq \frac{l}{2} \cdot n.$$

This gives $l \leq 4$ which is impossible for a resolution of length 5.

Suppose then $n = 2k+1$ is odd. Again since $a_i + a_{n+1-i} \leq l-1$ for $r+1 \leq i \leq n/2$ we get

$$\frac{n-2r-1}{2}(l-1) + a_{k+1} \geq \frac{n-2r+1}{2}l - a_1, \tag{18}$$

$$a_{k+1} + a_1 - \frac{n-2r-1}{2} \geq l. \tag{19}$$

Therefore

$$a_{k+i} + a_i \geq a_{k+1} + a_1 \geq l$$

for $i = 1, \ldots, k$. Together with (17) this gives

$$\frac{n-1}{2} \cdot l + 2 \geq a_{2k+1} + k \cdot l.$$

Therefore $a_{2k+1} \leq 2$ and so

$$l \leq a_1 + a_{k+1} \leq 2a_{2k+1} \leq 4$$

which is not possible. ∎

COROLLARY 6.3. *We have $a_i + a_{n+1-i} < l$ for $i = 1, \ldots, n-1$.*

Proof. Follows since $a_i \leq a_{i+1}$. ∎

LEMMA 6.4. *Let A be a regular algebra with resolution (14).*

1. *If GKdim $A \geq 2$ then*

$$2\sum_{i=1}^{n} a_i = (n-1)l + 4. \tag{20}$$

2. *If GKdim $A \geq 4$ then*

$$4\sum_{i=1}^{n} a_i^3 - 6l\sum_{i=1}^{n} a_i^2 + l^3(n-1) + 12l - 8 = 0. \tag{21}$$

Proof. By the resolution (14) we have

$$h_A(t) = 1/q(t)$$

where

$$q(t) = 1 - 2t + \sum_{i=1}^{n} t^{a_i} - \sum_{i=1}^{n} t^{l-a_{n+1-i}} + 2t^{l-1} - t^l.$$

By Stephenson-Zhang [18], the Gelfand-Kirillov dimension is the order of the pole of $h_A(t)$ at $t = 1$. That GKdim $A \geq 2$ is then equivalent to $q'(1) = 0$, giving 1. That GKdim $A \geq 4$ is equivalent to $q^{(3)}(1) = 0$ giving 2. ∎

REMARK 6.5. Note that given 1. it follows that $q''(1) = 0$ which gives GKdim $A \geq 3$, and given 2. it follows that $q^{(4)}(1) = 0$ which gives GKdim $A \geq 5$.

Now we come to the main result of this section.

THEOREM 6.6. *Let A be a regular algebra of global dimension 5, having the resolution (14). If A is an integral domain and GKdim $A \geq 4$, then either*

1. *$n = 3$ and (a_1, a_2, a_3) is $(3,5,5)$, $(4,4,4)$ or $(3,4,7)$,*
2. *$n = 4$ and (a_1, a_2, a_3, a_4) is $(4,4,4,5)$, or*
3. *$n = 5$ and $(a_1, a_2, a_3, a_4, a_5)$ is $(4,4,4,5,5)$.*

Proof. Let us analyse the expression (21) from Lemma 6.4. Given n and l we can vary the a_i's. We want to investigate when the expression on the left side of (21) is minimal. To do this let us first consider

$$4(a^3+b^3)-6l(a^2+b^2) \tag{22}$$

where we keep $a+b$ constant equal to, say $2s$. Let $a=s-\alpha$ and $b=s+\alpha$. Then the above expression becomes

$$4(2s^3+6s\alpha^2)-6l(2s^2+2\alpha^2)=8s^3-12ls^2+12\alpha^2(2s-l).$$

When $a+b=2s<l$ we see that (22) will decrease when a and b diverge. When $a+b=2s>l$, (22) will decrease when a and b converge. This observation motivates the strategy of proof.

We will now find the minimum of the expression (21) under suitable conditions. Suppose first that n is odd, equal to $2k+1\geq 5$, so $k\geq 2$. By (20)

$$\sum_{i=1}^{n} a_i = kl+2.$$

Since now $a_{i+1}+a_{n+1-i}\leq l-1$ for $i=1,\dots,k$ we get that

$$kl+2=a_1+\sum_{i=1}^{k}(a_{i+1}+a_{n+1-i})\leq a_1+k(l-1)$$

giving $a_1\geq 2+k$. We will now show that when the a_i's are integers and fulfil

1. $2+k\leq a_1\leq\cdots\leq a_n$,
2. $a_i+a_{n+1-i}\leq l-1$ for $i=1,\dots,k+1$,
3. $\sum_{i=1}^{n} a_i = kl+2$,

the expression (21) takes its minimal value when

$$a_1=\cdots=a_{k+1}=2+k,\quad a_{k+2}=\cdots=a_{2k+1}=l-k-3.$$

It is clear that only a finite number of integers a_i fulfil these conditions, and suppose they have values such that (21) has its minimum value.

a. Suppose $a_i=2+k$ for $i<n$. Let $l=2k+5+t$. Inserting this in condition 3. gives

$$a_n=k(t+1)+2,$$

so in particular $t\geq 0$. Now

$$(k+2)+k(t+1)+2=a_1+a_n\leq l-1=2k+4+t.$$

giving $kt\leq t$. Since $k\geq 2$ we must have $t=0$. But then $a_i=2+k=l-k-3$ for all i.

b. Otherwise let $j<n$ be minimal such that $a_j>2+k$. Suppose that $a_j+a_n\leq l-1$. Then we can decrease a_j by 1, increase a_n by 1, still have conditions 1, 2, and 3 fulfilled and (21) will decrease. This is against assumption.

c. Hence $a_j+a_n\geq l$. If $j\leq k+1$ we have $a_j+a_{n+1-j}\leq l-1$ where $n+1-j\geq j$. Let n' be maximal such that $a_j+a_{n'}\leq l-1$. Then we can decrease a_j by 1 and increase $a_{n'}$ by 1, keep the conditions 1, 2, and 3, and (21) will decrease. Against assumption.

d. Hence $j\geq k+2$. Now we have $a_1=\cdots=a_{k+1}=2+k$ and

$$a_i+a_{n+1-i}\leq l-1$$

for $i = 1, \ldots, k$. If one of these inequalities was strict we would by (20) get

$$kl + 2 < k(l-1) + (2+k)$$

which is equivalent to $2 < 2$. Hence each $a_i = l - 3 - k$ for $i \geq k+2$.

The value of the expression (21) then becomes

$$4[(k+1)(2+k)^3 + k(l-k-3)^3] - 6l[(k+1)(2+k)^2 + k(l-k-3)^2] + 2kl^3 + 12l - 8$$

which after some reductions becomes

$$(k+2)[6l(k-1) - 4(2k^2+5k-3)]. \tag{23}$$

Since $l - k - 3 = a_n \geq a_1 = 2 + k$ we get $l \geq 2k+5$, so the expression above is greater than or equal to

$$(k+2)[6(2k+5)(k-1) - 4(2k^2+5k-3)] = (k+2)[4k^2 - 2k - 18].$$

For $k \geq 3$ this is positive. For $k = 2$ the expression (23) becomes

$$4[6l - 60].$$

For $l \geq 11$ this gives that the minimum value of (21) under our conditions is positive. Since $l \geq 2k+5 = 9$ when $k = 2$, we must look at two values of l. When $l = 10$ conditions 1. and 3., give the possible values $a_1 = a_2 = a_3 = 4$ and $a_4 = a_5 = 5$ and this is in fact a solution to (21). When $l = 9$, all the a_i's would have to be 4 to fulfil 1. and 2. But this is not a solution to (21).

Suppose now that $n = 2k$ is even ≥ 4. The equation (20) then gives $l = 2u$ even. Since $a_{i+1} + a_{n+1-i} \leq l - 1$ for $i = 1, \ldots, k$ we get

$$u(n-1) + 2 \leq a_1 + \left(\frac{n-2}{2}\right)(l-1) + a_{k+1} \leq a_1 + \left(\frac{n-1}{2}\right)(l-1)$$

giving

$$a_1 \geq 2 + \frac{n-1}{2} = k + 3/2$$

which implies $a_i \geq 2 + k$. Since $a_1 + a_n \leq l - 1$ we get $l \geq 2k+5$ and so $l \geq 2k+6$ since l is even. Hence $u \geq k+3$.

We will now show that when the a_i are integers and fulfil

1. $2 + k \leq a_1 \leq \cdots \leq a_n$,
2. $a_i + a_{n+1-i} \leq l - 1$,
3. $2a_{k+1} \leq l - 1$,
4. $\sum_{i=1}^n a_i = (n-1)u + 2$,

then the expression (21) has its minimum value when

$$\begin{aligned} a_1 = \cdots = a_k &= 2 + k, \\ a_{k+1} &= u - 1, \\ a_{k+2} = \cdots = a_{2k} &= 2u - k - 3. \end{aligned}$$

a. Suppose $a_i = 2 + k$ for $i < n$. Let $u = k + 2 + t$ where $t \geq 1$. Inserting into condition 4. we get

$$a_n = (2k-1)t + 2.$$

This gives

$$(2+k)+(2k-1)t+2=a_1+a_n\leq l-1=2k+3+2t,$$

which reduces to $(2k-3)t \leq k-1$. Since $k\geq 2$ and $t\geq 1$ this has the only solution $k=2$ and $t=1$, and so $u=5$. Then $a_3=4=u-1$ and $a_4=5=2u-k-3$.

b. Otherwise let $j<n$ be minimal such that $a_j>2+k$. Suppose $a_j+a_n\leq l-1$. Then we can decrease a_j by 1 and increase a_n by 1, still have conditions 1.-4., and (21) will decrease, contrary to assumption.

c. So $a_j+a_n\geq l$. Suppose $j\leq k$. Since $a_j+a_{n+1-j}\leq l-1$ where now $n+1-j>j$, let n' be maximal such that $a_j+a_{n'}\leq l-1$. If $n'=k+1$ we must have $j=k$. This would violate Theorem 6.2 (with $i=k-1$). So $n'\geq k+2$. Then we may decrease a_j by 1, increase $a_{n'}$ by 1, still have conditions 1.-4., and (21) will decrease, contrary to assumption.

d. So $j\geq k+1$. We then have $a_1=\cdots=a_k=2+k$ and $a_i+a_{n+1-i}\leq l-1$ for $i=1,\ldots,k-1$, and also $a_{k+1}\leq u-1$. If one of these inequalities are strict, we get from condition 4. that

$$(2k-1)u+2<(k-1)(l-1)+(2+k)+(u-1) \tag{24}$$

which reduces to $2<2$. Hence $a_{k+1}=u-1$ and $a_i=l-3-k$ for $i\geq k+2$.

The minimal value of (21) is then

$$\begin{aligned}&4[(k-1)(2u-k-3)^3+(u-1)^3+k(2+k)^3]\\&-6\cdot 2u[(k-1)(2u-k-3)^2+(u-1)^2+k(2+k)^2]+(2k-1)l^3+12l-8\\&=4[3u^2+u(3k^2-3k-21)-(2k^3+6k^2-8k-24)].\end{aligned} \tag{25}$$

If we keep k fixed and take the derivative with respect to u we get $24u+4(3k^2-3k-21)$. For $k\geq 3$ this is positive for $u\geq 1$. Since $u\geq k+3$, the expression (25) is therefore greater than or equal to

$$4(k^3+3k^2-4k-12)=4(k+2)(k+3)(k-2)$$

which is positive when $k\geq 3$. When $k=2$ the expression (25) is $4(3u^2-15)$. For $u\geq 6$ this is positive. When $u=5$ this is 0. We get from conditions 1.-4. that $a_1=a_2=a_3=4$ and $a_4=5$ and this is also a solution to (21).

Suppose now that $n=2$. Again $l=2u$ must be even. The equations (20) and (21) (with $a=a_1$ and $b=a_2$) then become:

$$\begin{gathered}a+b=u+2,\\(a^3+b^3)-3u(a^2+b^2)+2u^3+6u-2=0.\end{gathered}$$

If we put $b=u+2-a$ the second equation becomes

$$3[u^2(a-2)-u(a^2-2)+2(a-1)^2]=0. \tag{26}$$

Now $2a\leq u+2$ or $u\geq 2a-2$. Taking the derivative of the above with respect to u we get

$$2u(a^2-2)-(a^2-2)\geq 4(a-1)(a^2-2)-(a^2-2),$$

which is positive for $a \geq 3$. Hence in this case (26) is (since $u \geq 2a-2$) greater than or equal to

$$3(a-1)[4(a-1)(a-2) - 2(a^2-2) + 2(a-1)]$$

which is easily seen to be positive for $a \geq 4$. Hence $a \leq 3$. If $a = 3$, (26) becomes $3(u^2 - 7u + 8)$ which does not have integer solutions. If $a = 2$ it becomes $3(-2u + 2)$ which is nonzero since $u \geq 2a - 2 = 2$.

Suppose now that $n = 3$. The equations (20) and (21) then become:

$$a_1 + a_2 + a_3 = l + 2,$$
$$2(a_1^3 + a_2^3 + a_3^3) - 3l(a_1^2 + a_2^2 + a_3^2) + l^3 + 6l - 4 = 0.$$

If we substitute the first expression for l into the second equation we get (assuming that not both $a_1 = 3$ and $a_2 = 3$, a case easily ruled out)

$$a_3 = 2 + \frac{a_1 + a_2 - 2}{(a_1 - 2)(a_2 - 2) - 1}.$$

We know that $a_2 + a_3 \leq l - 1$. Since the sum of the a_i is $l + 2$ we get $a_1 \geq 3$.

If $a_1 = 3$ then $\frac{a_2+1}{a_2-3}$ must be an integer. This gives (a_2, a_3) either $(4, 7)$ or $(5, 5)$ when $a_2 \leq a_3$. If $a_1 = 4$ then $\frac{a_2+2}{2a_2-5}$ is an integer which gives $a_2 = 4$ and $a_3 = 4$ when the sequence is nondecreasing. If $a_1 \geq 5$ and $a_2 \geq 5$, we get $a_3 < 5$ and do not have a nondecreasing sequence. ∎

We know that in all three cases when $n = 3$ there are algebras with these resolution types. However the cases when $n = 4$ and $n = 5$ are open.

Question. Is there an Artin-Schelter regular algebra A with minimal resolution

$$A \leftarrow A(-1)^2 \leftarrow A(-4)^3 \oplus A(-5) \leftarrow A(-5) \oplus A(-6)^3 \leftarrow A(-9)^2 \hookleftarrow A(-10)?$$

Is there one with an additional summand $A(-5)$ at steps two and three?

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Received January 30, 2010; Revised January 19, 2011