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GEOMETRY OF NONCOMMUTATIVE ALGEBRAS

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Abstract. There has been several attempts to generalize commutative algebraic geometry to the noncommutative situation. Localizations with good properties rarely exist for noncommutative algebras, and this makes a direct generalization difficult. Our point of view, following Laudal, is that the points of the noncommutative geometry should be represented as simple modules, and that noncommutative deformations should be used to obtain a suitable localization in the noncommutative situation.

Let A be an algebra over an algebraically closed field k. If A is commutative and finitely generated over k, then any simple A-module has the form $M = A/\mathfrak{m}$, the residue field, for a maximal ideal $\mathfrak{m} \subseteq A$, and the commutative deformation functor Def_M has formal moduli $\hat{A}_\mathfrak{m}$. In the general case, we may replace the A-module $A/\mathfrak{m}$ with the simple A-module M, and use the formal moduli of the commutative deformation functor Def_M as a replacement for the complete local ring $\hat{A}_\mathfrak{m}$. We recall the construction of the commutative scheme $\mathrm{simp}(A)$, with points in bijective correspondence with the simple A-modules of finite dimension over k, and with complete local ring at a point M isomorphic to the formal moduli of the corresponding simple module M.

The scheme $\mathrm{simp}(A)$ has good properties, in particular when there are no infinitesimal relations between different points, i.e. when $\mathrm{Ext}^1_A(M, M') = 0$ for all pairs of non-isomorphic simple A-modules M, M'. It does not, however, characterize A. We use noncommutative deformation theory to define localizations, in general.

We consider the quantum plane, given by $A = k\langle x, y\rangle/(xy - qyx)$, as an example. This is an Artin-Schelter algebra of dimension two.

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1. Noncommutative deformations of modules. Let k be a field. For any integer $r \geq 1$, we consider the category a_r of r-pointed Artinian k-algebras. We recall that an object in a_r is an Artinian ring R, together with a pair of structural ring homomorphisms $f : k^r \to R$ and $g : R \to k^r$ with $g \circ f = \mathrm{id}$, such that the radical $I(R) = \ker(g)$ is nilpotent. The morphisms of a_r are the ring homomorphisms that commute with the structural morphisms. It follows from this definition that $I(R)$ is the Jacobson radical of R, and therefore that the simple right R-modules are the projections $\{k_1, \dots, k_r\}$ of k^r.

Let A be an associative k-algebra. For any family $\mathcal{M} = \{M_1, \dots, M_r\}$ of right A-modules, there is a noncommutative deformation functor $\mathsf{Def}_{\mathcal{M}} : \mathsf{a}_r \to \mathsf{Sets}$, introduced in Laudal [4]; see also Eriksen [2]. For an algebra R in a_r, we recall that a deformation of $\mathcal{M}$ over R is a pair $(M_R, \{\tau_i\}_{1 \leq i \leq r})$, where M_R is an R-A bimodule (on which k acts centrally) that is R-flat, and $\tau_i : k_i \otimes_R M_R \to M_i$ is an isomorphism of right A-modules for $1 \leq i \leq r$. Moreover, $(M_R, \{\tau_i\})$ and $(M'_R, \{\tau'_i\})$ are equivalent deformations over R if there is an isomorphism $\eta : M_R \to M'_R$ of R-A bimodules such that $\tau_i = \tau'_i \circ (1 \otimes \eta)$ for $1 \leq i \leq r$. One may prove that M_R is R-flat if and only if

$$M_R \cong (R_{ij} \otimes_k M_j) = \begin{pmatrix} R_{11} \otimes_k M_1 & R_{12} \otimes_k M_2 & \dots & R_{1r} \otimes_k M_r \\ R_{21} \otimes_k M_1 & R_{22} \otimes_k M_2 & \dots & R_{2r} \otimes_k M_r \\ \vdots & \vdots & \ddots & \dots \\ R_{r1} \otimes_k M_1 & R_{r2} \otimes_k M_2 & \dots & R_{rr} \otimes_k M_r \end{pmatrix}$$

considered as a left R-module, and a deformation in $\mathsf{Def}_{\mathcal{M}}(R)$ may be thought of as a right multiplication $A \to \mathrm{End}_R(M_R)$ of A on the left R-module M_R that lifts the multiplication $\rho : A \to \oplus_i \mathrm{End}_k(M_i)$ of A on the family $\mathcal{M}$.

Let us assume that $\mathcal{M}$ is a *swarm*, i.e. that $\mathrm{Ext}^1_A(M_i, M_j)$ has finite dimension over k for $1 \leq i, j \leq r$. Then $\mathsf{Def}_{\mathcal{M}}$ has a pro-representing hull or a formal moduli (H, M_H), see Laudal [4], Theorem 3.1. This means that H is a complete r-pointed k-algebra in the pro-category $\hat{\mathsf{a}}_r$, and that $M_H \in \mathsf{Def}_{\mathcal{M}}(H)$ is a family defined over H with the following versal property: For any algebra R in a_r and any deformation $M_R \in \mathsf{Def}_{\mathcal{M}}(R)$, there is a morphism $\phi : H \to R$ in $\hat{\mathsf{a}}_r$ such that $\mathsf{Def}_{\mathcal{M}}(\phi)(M_H) = M_R$. The formal moduli (H, M_H) is unique up to non-canonical isomorphism. However, the morphism ϕ is not uniquely determined by (R, M_R).

When $\mathcal{M}$ is a swarm with formal moduli (H, M_H), right multiplication on the H-A bimodule M_H by elements in A determines an algebra homomorphism

$$\eta : A \to \mathrm{End}_H(M_H)$$

We write $\mathcal{O}^A(\mathcal{M}) = \mathrm{End}_H(M_H)$ and call it the *algebra of observables*. Since M_H is H-flat, we have that $\mathrm{End}_H(M_H) \cong (H_{ij} \widehat{\otimes}_k \mathrm{Hom}_k(M_i, M_j))$, and it follows that $\mathcal{O}^A(\mathcal{M})$ is explicitly given as the matrix algebra

$$\begin{pmatrix} H_{11} \widehat{\otimes}_k \mathrm{End}_k(M_1) & H_{12} \widehat{\otimes}_k \mathrm{Hom}_k(M_1, M_2) & \dots & H_{1r} \widehat{\otimes}_k \mathrm{Hom}_k(M_1, M_r) \\ H_{21} \widehat{\otimes}_k \mathrm{Hom}_k(M_2, M_1) & H_{22} \widehat{\otimes}_k \mathrm{End}_k(M_2) & \dots & H_{2r} \widehat{\otimes}_k \mathrm{Hom}_k(M_2, M_r) \\ \vdots & \vdots & \ddots & \dots \\ H_{r1} \widehat{\otimes}_k \mathrm{Hom}_k(M_r, M_1) & H_{r2} \widehat{\otimes}_k \mathrm{Hom}_k(M_r, M_2) & \dots & H_{rr} \widehat{\otimes}_k \mathrm{End}_k(M_r) \end{pmatrix}$$

Let us write $\rho_i : A \to \mathrm{End}_k(M_i)$ for the structural algebra homomorphism defining the right A-module structure on M_i for $1 \leq i \leq r$, and

$$\rho : A \to \bigoplus_{1 \leq i \leq r} \mathrm{End}_k(M_i)$$

for their direct sum. Since H is a complete r-pointed algebra in $\hat{\mathsf{a}}_r$, there is a natural morphism $H \to k^r$, inducing an algebra homomorphism

$$\pi : \mathcal{O}^A(\mathcal{M}) \to \bigoplus_{1 \leq i \leq r} \mathrm{End}_k(M_i)$$

By construction, there is a right action of $\mathcal{O}^A(\mathcal{M})$ on the family $\mathcal{M}$ extending the right action of A, in the sense that the diagram

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[dr, "\rho"'] & \mathcal{O}^A(\mathcal{M}) \arrow[d, "\pi"] \\ & \bigoplus_{1 \leq i \leq r} \mathrm{End}_k(M_i) \end{tikzcd}$$

commutes.

Lemma 1.1. *Let $f : A \to B$ be an algebra homomorphism, and let $\mathcal{M}$ be a swarm of right B-modules. If $\mathcal{M}$ is a swarm of right A-modules via f, then then there is a natural algebra homomorphism $\mathcal{O}^A(\mathcal{M}) \to \mathcal{O}^B(\mathcal{M})$ such that the diagram*

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & \mathcal{O}^A(\mathcal{M}) \arrow[d] \\ B \arrow[r, "\eta_B"'] & \mathcal{O}^B(\mathcal{M}) \end{tikzcd}$$

commutes.

Proof. Let (H^A, M_{H^A}) be the formal moduli of $\mathsf{Def}^A_{\mathcal{M}}$, the noncommutative deformation functor of $\mathcal{M}$ considered as a family of right A-modules, and let (H^B, M_{H^B}) be the formal moduli of $\mathsf{Def}^B_{\mathcal{M}}$, the noncommutative deformation functor of $\mathcal{M}$ considered as a family of right B-modules. Since $M_{H^B} \in \mathsf{Def}^A_{\mathcal{M}}(H^B)$ is also a lifting of A-modules to H^B, there is a natural morphism $H^A \to H^B$ by the versal property of H^A, and hence a natural morphism $\mathcal{O}^A(\mathcal{M}) \to \mathcal{O}^B(\mathcal{M})$. ∎

2. Laudal's Generalized Burnside Theorem. Let A be a finite-dimensional algebra over a field k. Then the simple right modules over A are the simple right modules over the semi-simple quotient algebra $A/\operatorname{rad}(A)$, where $\operatorname{rad}(A)$ is the Jacobson radical of A. By the classification theory for semi-simple algebras, it follows that there are finitely many non-isomorphic simple right A-modules.

We consider the noncommutative deformation functor $\mathsf{Def}_{\mathcal{M}} : \mathsf{a}_r \to \mathsf{Sets}$ of the family $\mathcal{M} = \{M_1, M_2, \dots, M_r\}$ of simple right A-modules. Clearly, $\mathcal{M}$ is a swarm, hence $\mathsf{Def}_{\mathcal{M}}$

has a formal moduli (H, M_H), and we consider the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[dr, "\rho"'] & \mathcal{O}^A(\mathcal{M}) \arrow[d, "\pi"] \\ & \bigoplus_{1 \le i \le r} \operatorname{End}_k(M_i) \end{tikzcd}$$

By a classical result, due to Burnside, the algebra homomorphism ρ is surjective when k is algebraically closed. This result may be stated in the following form:

Theorem 2.1 (Burnside's Theorem). *If $\operatorname{End}_A(M_i) = k$ for $1 \le i \le r$, then ρ is surjective. In particular, ρ is surjective when k is algebraically closed.*

Proof. Consider the factorization $A \to A/\operatorname{rad}(A) \to \oplus_i \operatorname{End}_k(M_i)$ of ρ. If $\operatorname{End}_A(M_i) = k$ for $1 \le i \le r$, then $A/\operatorname{rad}(A) \to \oplus_i \operatorname{End}_k(M_i)$ is an isomorphism by the classification theory for semi-simple algebras. Since $\operatorname{End}_A(M_i)$ is a division ring of finite dimension over k, it is clear that $\operatorname{End}_A(M_i) = k$ whenever k is algebraically closed. ∎

Let us write $\overline{\rho} : A/\operatorname{rad} A \to \oplus_i \operatorname{End}_k(M_i)$ for the algebra homomorphism induced by ρ. We observe that ρ is surjective if and only if $\overline{\rho}$ is an isomorphism. Moreover, let us write $J = \operatorname{rad}(\mathcal{O}^A(\mathcal{M}))$ for the Jacobson radical of $\mathcal{O}^A(\mathcal{M})$. Then we see that

$$J = (\operatorname{rad}(H)_{ij} \widehat{\otimes}_k \operatorname{Hom}_k(M_i, M_j)) = \ker(\pi)$$

Since $\rho(\operatorname{rad} A) = 0$ by definition, it follows that $\eta(\operatorname{rad} A) \subseteq J$. Hence there are induced morphisms

$$\operatorname{gr}(\eta)_q : \operatorname{rad}(A)^q / \operatorname{rad}(A)^{q+1} \to J^q / J^{q+1}$$

for all $q \ge 0$. We may identify $\operatorname{gr}(\eta)_0$ with $\overline{\rho}$, since $\mathcal{O}^A(\mathcal{M})/J \cong \oplus_i \operatorname{End}_k(M_i)$. The conclusion in Burnside's Theorem is therefore equivalent to the statement that $\operatorname{gr}(\eta)_0$ is an isomorphism.

Theorem 2.2 (Laudal's Generalized Burnside Theorem). *Let A be a finite-dimensional algebra over a field k, and let $\mathcal{M} = \{M_1, M_2, \dots, M_r\}$ be the family of simple right A-modules. If $\operatorname{End}_A(M_i) = k$ for $1 \le i \le r$, then $\eta : A \to \mathcal{O}^A(\mathcal{M})$ is an isomorphism. In particular, η is an isomorphism when k is algebraically closed.*

Proof. Since A and $\mathcal{O}^A(\mathcal{M})$ are complete in the $\operatorname{rad}(A)$-adic and J-adic topologies, it follows that η is surjective if $A \to \mathcal{O}^A(\mathcal{M})/J^2$ is surjective. It is therefore enough to prove that η is injective and that $\operatorname{gr}(\eta)_q$ is an isomorphism for $q = 0$ and $q = 1$. By Burnside's Theorem, we know that $\operatorname{gr}(\eta)_0$ is an isomorphism. To prove that η is injective, let us consider the kernel $\ker(\eta) \subseteq A$. It is determined by the obstruction calculus of $\mathsf{Def}_{\mathcal{M}}$; see Laudal [4], Theorem 3.2 for details. When A is finite-dimensional, the right regular A-module A_A has a decomposition series

$$0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = A_A$$

with F_p/F_{p-1} a simple right A-module for $1 \le p \le n$. That is, A_A is an *iterated extension* of the modules in $\mathcal{M}$. This implies that η is injective; see Laudal [4], Corollary 3.1. Finally, we must prove that $\operatorname{gr}(\eta)_1 : \operatorname{rad}(A)/\operatorname{rad}(A)^2 \to J/J^2$ is an isomorphism. This follows from the Wedderburn-Malcev Theorem; see Laudal [4], Theorem 3.4 for details. ∎

3. Properties of the algebra of observables. Let A be a finitely generated algebra over a field k, and let $\mathcal{M} = \{M_1, \dots, M_r\}$ be any family of right A-modules of finite dimension over k. Even though A may be non-Noetherian, and it may be difficult to compute $\mathrm{Ext}^1_A(M_i, M_j)$ using free resolutions, we may show the following result:

LEMMA 3.1. *If A is a finitely generated k-algebra and $\mathcal{M}$ is a family of finite-dimensional right A-modules, then $\mathcal{M}$ is a swarm.*

Proof. We have that $\mathrm{Ext}^1_A(M_i, M_j) \cong \mathsf{HH}^1(A, \mathrm{Hom}_k(M_i, M_j))$ for $1 \le i, j \le r$. The Hochschild cohomology group is given by

$$\mathsf{HH}^1(A, \mathrm{Hom}_k(M_i, M_j)) = \mathrm{Der}_k(A, \mathrm{Hom}_k(M_i, M_j))/\,\mathrm{Inner}_k(A, \mathrm{Hom}_k(M_i, M_j))$$

where we write $\mathrm{Inner}_k(A, \mathrm{Hom}_k(M_i, M_j))$ for the inner derivations of A with values in $\mathrm{Hom}_k(M_i, M_j)$. Since a derivation is determined by its values on a set algebra generators and $\dim_k \mathrm{Hom}_k(M_i, M_j) < \infty$, it follows that $\mathrm{Ext}^1_A(M_i, M_j)$ has finite dimension over k for $1 \le i, j \le r$. ∎

Hence $\mathsf{Def}_{\mathcal{M}}$ has a formal moduli (M, M_H), and hence we may consider the algebra $B = \mathcal{O}^A(\mathcal{M})$ of observables. It is clear that

$$B/\,\mathrm{rad}(B) \cong \bigoplus_i \mathrm{End}_k(M_i)$$

is semi-simple, with $\mathcal{M}$ as the set of simple modules, so $\mathcal{M}$ is the family of simple right B-modules. In fact, it follows from the proof of Lemma 3.1 that $\mathcal{M}$ is a swarm of B-modules, since a derivation on a power series algebra in a finite number of variables $\{x_1, \dots, x_m\}$ is determined by its values on x_i for $1 \le i \le m$.

PROPOSITION 3.2. *If k is an algebraically closed field, then the algebra homomorphism $\eta_B : B \to \mathcal{O}^B(\mathcal{M})$ is an isomorphism.*

Proof. Since $\mathcal{M}$ is a swarm of A-modules and of B-modules, we may consider the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\eta^A"] \arrow[d, "\eta_A"'] & \mathcal{O}^A(\mathcal{M}) = B \arrow[d, "\eta_B"] \\
B \arrow[r, "\eta_B"'] \arrow[d] & \mathcal{O}^B(\mathcal{M}) = C \arrow[d] \\
B_n \arrow[r, "\eta_{B_n}"'] & \mathcal{O}^{B_n}(\mathcal{M})
\end{tikzcd}$$

for all $n \ge 1$, where $B_n = B/(\mathrm{rad}\, B)^n$. By Laudal's Generalized Burnside Theorem, η_{B_n} is an isomorphism for all $n \ge 1$. Since any deformation of the family $\mathcal{M}$, as B-module is also a deformation as B_n-module, for some n, it follows that C is the projective limit of $\mathcal{O}^{B_n}(\mathcal{M})$, hence the algebra homomorphism η_B is also an isomorphism. ∎

In particular, the proposition implies that the assignment $(A, \mathcal{M}) \mapsto (B, \mathcal{M})$ is a closure operation when k is algebraically closed. In other words, the algebra $B = \mathcal{O}^A(\mathcal{M})$ has the following properties:

1. The family $\mathcal{M}$ is the family of simple B-modules.
2. The family $\mathcal{M}$ has the same module-theoretic properties, in terms of extensions, higher extensions and Massey products, considered as a family of modules over B as over A.

Moreover, these properties characterize the algebra $B = \mathcal{O}^A(\mathcal{M})$, and make it natural to call it the algebra of observables.

4. Finite dimensional simple representations. Let k be an algebraically closed field, and let A be a finitely generated k-algebra. We denote by $\operatorname{simp}_n(A)$ the set of isomorphism classes of simple right A-modules M of dimension $\dim_k M = n$, and by

$$\operatorname{simp}(A) = \bigcup_{n\geq 1} \operatorname{simp}_n(A)$$

the set of isomorphism classes of simple right A-modules of finite dimension over k.

Let $\rho : A \to \operatorname{End}_k(M)$ be the structure morphism of a simple module $M \in \operatorname{simp}_n(A)$, and let $\mathfrak{m}_M = \ker(\rho)$ be the corresponding primitive ideal. It follows from Burnside's Theorem that $\mathfrak{m}_M \subseteq A$ is a maximal ideal. We define the radical

$$\operatorname{rad}(A)^\infty = \bigcap_{\substack{M\in\operatorname{simp}(A)\\ m\geq 1}} \mathfrak{m}_M^m$$

and say that A is *geometric* if $\operatorname{rad}(A)^\infty = 0$.

Example 4.1. The free associative k-algebra $A = k\langle x_1, x_2, \dots, x_d\rangle$ is geometric. On the other hand, a simple k-algebra A is geometric only if it has finite dimension over k. In particular, the first Weyl algebra $A_1(k)$ is not geometric.

5. The commutative scheme structure. Let k be an algebraically closed field, and let A be a finitely generated geometric k-algebra. In this section, we shall discuss the commutative scheme structure of $\operatorname{simp}_n(A)$.

For any integer $n \geq 1$, let $I(n) \subseteq A$ be the ideal generated by the n-commutators $\{[a_1, a_2, \dots, a_{2n}] : a_1, a_2, \dots, a_{2n} \in A\}$. We recall that that n-commutators are given by

$$[a_1, a_2, \dots, a_{2n}] = \sum_{\sigma\in S_{2n}} \operatorname{sgn}(\sigma)\cdot a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(2n)}$$

for any sequence $a_1, a_2, \dots, a_{2n} \in A$. We define $A(n) = A/I(n)$ to be the corresponding factor algebra. For any $M \in \operatorname{simp}_n(A)$, there is a factorization $A \to A(n) \to \operatorname{End}_k(M)$ of the structure morphism; see Formanek [3]. Hence any $M \in \operatorname{simp}_n(A)$ can be considered as a simple right $A(n)$-module in a natural way.

Lemma 5.1. *Let $\mathcal{M}$ be a finite subset of* simp A*, let $\mathfrak{r} = \cap_{M\in\mathcal{M}}\, m_M$, and write J for the Jacobson radical of $\mathcal{O}^A(\mathcal{M})$. Then the algebra homomorphism*

$$A/\mathfrak{r}^m \to \mathcal{O}^A(\mathcal{M})/J^m$$

induced by $\eta : A \to \mathcal{O}^A(\mathcal{M})$ is surjective for all $m \geq 2$.

Proof. Since $\mathcal{M}$ is a swarm, we may consider the algebra homomorphism $\eta : A \to \mathcal{O}^A(\mathcal{M})$ and the induced homomorphism $\overline{\eta} : A/\mathfrak{r}^m \to \mathcal{O}^A(\mathcal{M})/J^m$. We let $B = A/\mathfrak{r}^m$, which is a finite dimensional k-algebra, and consider the natural commutative diagram

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[d] & \mathcal{O}^A(\mathcal{M}) \arrow[d] \arrow[dr] & \\ B \arrow[r, "\cong"] \arrow[rr, bend right, "\overline{\eta}"'] & \mathcal{O}^B(\mathcal{M}) \arrow[r, dotted, "\alpha"] & \mathcal{O}^A(\mathcal{M})/J^m \end{tikzcd}$$

Since $\mathcal{M}$ is the family of simple right B-modules, it follows from Laudal's Generalized Burnside Theorem that $\eta_B : B \to \mathcal{O}^B(\mathcal{M})$ is an isomorphism. The obvious homomorphism, $\overline{\eta}$ induces an algebra morphism $\alpha : \mathcal{O}^B(\mathcal{M}) \to \mathcal{O}^A(\mathcal{M})/J^m$ that commutes with $\mathcal{O}^A(\mathcal{M}) \to \mathcal{O}^A(\mathcal{M})/J^m$, and it follows that α is surjective. ∎

PROPOSITION 5.2. *Let $M, N \in \operatorname{simp}(A)$ be non-isomorphic simple left modules, and let $\mathfrak{r} = \mathfrak{m}_M \cap \mathfrak{m}_N$. Then we have*

1. $\operatorname{Ext}^1_A(M,N) \cong \operatorname{Ext}^1_{A/\mathfrak{r}^m}(M,N)$ *for all* $m \geq 1$,
2. $\operatorname{Ext}^1_A(M,M) \cong \operatorname{Ext}^1_{A(n)}(M,M)$ *with* $n = \dim_k M$.

Proof. The first part follows from Lemma 5.1 applied to $\mathcal{M} = \{M, N\}$ for $m = 2$. For the second part, notice that any derivation maps a standard n-commutator into a sum of standard n-commutators. Hence any derivation of A with values in $\operatorname{End}_k(M)$ factors through $A(n)$. ∎

EXAMPLE 5.3. We remark that we may well have that $\operatorname{Ext}^1_A(M,N)$ and $\operatorname{Ext}^1_{A(n)}(M,N)$ are non-isomorphic when M, N in $\operatorname{simp}_n(A)$ are non-isomorphic simple modules. Consider for example the algebra A with quotient $A(1)$, given by

$$A = \begin{pmatrix} k[x] & k[x] \\ 0 & k[x] \end{pmatrix} \quad \text{and} \quad A(1) = \begin{pmatrix} k[x] & 0 \\ 0 & k[x] \end{pmatrix}$$

We see that $\operatorname{Ext}^1_A(M,N) \cong k$ and $\operatorname{Ext}^1_{A(1)}(M,N) = 0$ when $M = k[x]/(x) \oplus 0$ and $N = 0 \oplus k[x]/(x)$. On the other hand, we have that $\operatorname{Ext}^1_A(M,M) \cong \operatorname{Ext}^1_{A(1)}(M,M) \cong k$ and $\operatorname{Ext}^1_A(N,N) \cong \operatorname{Ext}^1_{A(1)}(N,N) \cong k$.

LEMMA 5.4. *Let R be any k-algebra. If $R \otimes_k \operatorname{End}_k(V)$ satisfy the standard n-commutator relations for a vector space V of dimension n, then R is commutative.*

Proof. Let $r_1, r_2 \in R$, and consider the element given by

$$([r_1, r_2]\, e_{11})\, e_{12}\, e_{22}\, e_{23}\, e_{33} \ldots e_{n-1,n}\, e_{n,n}$$

Since this is a standard n-commutator in $R \otimes_k \operatorname{End}_k(V)$, it follows that R is commutative if all n-commutators vanish. ∎

LEMMA 5.5. *Let $\mathcal{M} = \{M_1, \ldots, M_r\}$ be a finite subset of $\operatorname{simp}_n(A)$. We may consider $\mathcal{M}$ as a family of simple $A(n)$-modules, and we have that*

1. $\operatorname{Ext}^1_{A(n)}(M_i, M_j) = 0$ *for* $1 \leq i \neq j \leq r$
2. $H^{A(n)}(M_i) \cong H^A(M_i)^{comm} = H^A(M_i)(1)$ *for* $1 \leq i \leq r$

In particular, the pro-representing hull

$$H^{A(n)}(\mathcal{M}) = \begin{pmatrix} H^A(M_1)^{comm} & 0 & \dots & 0 \\ 0 & H^A(M_2)^{comm} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & H^A(M_r)^{comm} \end{pmatrix}$$

is commutative.

Proof. Let s_n be a standard n-commutator in $\mathcal{O}^{A(n)}(\mathcal{M}) \cong M_n(H^{A(n)}(\mathcal{M}))$. The algebra homomorphism $\eta_m : A(n)/\mathfrak{r}^m \to \mathcal{O}^{A(n)}(\mathcal{M})/J^m$ induced by $\eta : A(n) \to \mathcal{O}^{A(n)}(\mathcal{M})$ is surjective for all $m \geq 2$ by Lemma 5.1. This implies that $s_n = 0 \mod J^m$ for all $m \geq 2$, hence $s_n = 0$, and it follows from Lemma 5.4 that $H^{A(n)}$ is commutative. To prove that the commutativization $H^A(M_i)^{\mathrm{comm}} \cong H^{A(n)}(M_i)$, we consider the commutative diagram

$$\begin{tikzcd}
 & A \arrow[d] \arrow[dr] & \\
Z(A(n)) \arrow[r] \arrow[d] & A(n) \arrow[d, "\alpha"] & H^A(M) \otimes_k \mathrm{End}_k(M) \arrow[dl] \\
H^A(M)^{\mathrm{comm}} \arrow[r] & H^A(M)^{\mathrm{comm}} \otimes_k \mathrm{End}_k(M) &
\end{tikzcd}$$

where $M = M_i$ and $Z(A(n))$ is the center of $A(n)$. The existence of α is a consequence of the fact that $I(n) \subseteq A$ maps to zero in $H^A(M)^{\mathrm{comm}} \otimes_k \mathrm{End}_k(M) \cong \mathrm{M}_n(H^A(M)^{\mathrm{comm}})$. We note that there are natural morphisms of formal moduli

$$H^A(M) \to H^{A(n)}(M) \to H^A(M)^{\mathrm{comm}} \to H^{A(n)}(M)^{\mathrm{comm}}$$

Since $H^{A(n)}(M)$ is commutative, the composition

$$H^{A(n)}(M) \to H^A(M)^{\mathrm{comm}} \to H^{A(n)}(M)^{\mathrm{comm}}$$

must be an isomorphism. By Proposition 5.2, the tangent spaces of $H^{A(n)}(M)$ and $H^A(M)$ are isomorphic, and this proves that $H^A(M)^{\mathrm{comm}} \cong H^{A(n)}(M)$ ∎

Corollary 5.6. *Let $A = k\langle x_1, x_2, \dots, x_d\rangle$, and let $M \in \mathrm{simp}_n(A)$. Then*

$$H^A(M)^{comm} \cong H^{A(n)}(M) \cong k[[t_1, t_2, \dots, t_{(d-1)n^2+1}]]$$

Proof. See Corollary 5.11 in Procesi [6]. ∎

In general, the family $\mathrm{simp}_n(A)$ is, of course, not finite, and we consider the projective limit $\eta(n)$ of the algebra homomorphisms $\eta(\mathcal{M}) : A(n) \to \mathcal{O}^{A(n)}(\mathcal{M})$ for all finite subfamilies $\mathcal{M} \subseteq \mathrm{simp}_n(A)$. By Lemma 5.5, $\eta(n)$ is given by

$$\eta(n) : A(n) \to \prod_{M \in \mathrm{simp}_n(A)} H^{A(n)}(M) \otimes_k \mathrm{End}_k(M)$$

We shall write $O(n) = \mathrm{Im}\,\eta(n)$ for the image of $\eta(n)$. We remark that $\eta(n)$ is not injective in general; see Example 11 in Laudal [5] for a counter-example.

Let us choose a k-linear base of M for any $M \in \mathrm{simp}_n(A)$, and use this base to obtain an identification $\mathrm{End}_k(M) \cong \mathrm{M}_n(k)$. The codomain of $\eta(n)$ is given by

$$\prod_{M \in \mathrm{simp}_n(A)} H^{A(n)}(M) \otimes_k \mathrm{End}_k(M) \cong \mathrm{M}_n(B(n))$$

where $B(n)$ is the commutative k-algebra

$$B(n) = \prod_{M \in \mathrm{simp}_n(A)} H^{A(n)}(M)$$

Let $\{x_1, x_2, \dots, x_d\}$ be a set of generators of A as a k-algebra, and consider their images $\eta(n)(x_i) = (x_i^{p,q}) \in \mathrm{M}_n(B(n))$ for $1 \leq i \leq d$. Since $B(n)$ is commutative, the subalgebra $C(n) \subseteq B(n)$ generated by the elements $\{x_i^{p,q} : 1 \leq i \leq d,\ 1 \leq p, q \leq n\} \subseteq B(n)$ is commutative. We notice that there are natural inclusions $O(n) \subseteq \mathrm{M}_n(C(n)) \subseteq \mathrm{M}_n(B(n))$. Hence there is a natural composition of k-algebra homomorphisms

$$\alpha_M : \mathrm{M}_n(C(n)) \to \mathrm{M}_n(H^{A(n)}(M)) \to \mathrm{M}_n(k)$$

for any $M \in \mathrm{simp}_n(A)$, and therefore an induced composition of algebra homomorphisms of the centers

$$Z(\alpha_M) : C(n) \to H^{A(n)}(M) \to k$$

It follows that there is a natural set-theoretic injective map $t : \mathrm{simp}_n(A) \to \mathrm{Max}(C(n))$, defined by $t(M) = \ker(Z(\alpha_M))$, where $\mathrm{Max}(C(n))$ denotes the set of maximal ideals of $C(n)$.

Proposition 5.7. *For all $M \in \mathrm{simp}_n(A)$, there is a natural isomorphism*

$$\widehat{C}(n)_{t(M)} \cong H^{A(n)}(M)$$

Proof. The algebra homomorphism $\eta_M : A(n) \to H^{A(n)}(M) \otimes_k \mathrm{End}_k(M)$ is topologically surjective for any $M \in \mathrm{simp}_n(A)$ by Lemma 5.1. This means that we have a surjective homomorphism

$$\widehat{C}(n)_{t(M)} \to H^{A(n)}(M)$$

By the versal property of $H^{A(n)}$, there is a homomorphism $H^{A(n)}(M) \to \widehat{C}(n)_{t(M)}$ that composed with the former gives an automorphism of $H^{A(n)}(M)$, and this implies that $H^{A(n)}(M) \to \widehat{C}(n)_{t(M)}$ is injective. Let $\mathfrak{m}_M = t(M) \in \mathrm{Max}(C(n))$. Since

$$\mathrm{M}_n(C(n)) \subseteq \prod_{M \in \mathrm{simp}_n(A)} H^{A(n)}(M) \otimes_k \mathrm{End}_k(V)$$

it follows that the finite dimensional k-algebra $\mathrm{M}_n(C(n)/\mathfrak{m}_V^2)$ sits in a finite dimensional quotient of some

$$\prod_{M \in \mathcal{M}} H^{A(n)}(M) \otimes_k \mathrm{End}_k(M)$$

where $\mathcal{M} \subseteq \mathrm{simp}_n(A)$ is a finite subset. However, the homomorphism

$$A(n) \to \prod_{M \in \mathcal{M}} H^{A(n)}(M) \otimes_k \mathrm{End}_k(M)$$

is topologically surjective by Lemma 5.1. Hence the morphism $A(n) \to \mathrm{M}_n(C(n)/\mathfrak{m}_M^2)$ is surjective, and this implies that $H^{A(n)}(M) \to \widehat{C}(n)_{t(M)}$ is surjective. ∎

THEOREM 5.8. *For any $M \in \operatorname{simp}_n(A)$, there exists a Zariski-open neighbourhood U_M of $t(M)$ in $\operatorname{Max}(C(n))$ such that any maximal ideal $\mathfrak{m} \in U_M$ is the image $\mathfrak{m} = t(N) = \mathfrak{m}_N$ of a unique simple module $N \in \operatorname{simp}_n(A)$. Let $U(n) \subseteq \operatorname{Max}(C(n))$ be the open subscheme*

$$U(n) = \bigcup_{M \in \operatorname{simp}_n(A)} U_M$$

Then $O(n)$ defines a noncommutative structure sheaf $\mathcal{O}(n) = \mathcal{O}_{\operatorname{simp}_n(A)}$ of Azumaya algebras on the topological space $\operatorname{simp}_n(A)$ with the Jacobson topology. The center $\mathcal{S}(n)$ of $\mathcal{O}(n)$ defines a scheme structure on $\operatorname{simp}_n(A)$, and there is a morphism of schemes

$$\kappa : U(n) \to \operatorname{simp}_n(A)$$

such that $\widehat{\mathcal{S}}(n)_{\kappa(M)} \cong H^{A(n)}(M)$ for all $M \in \operatorname{simp}_n(A)$.

Proof. Let $\rho : A \to \mathrm{M}_n(k)$ be the structure homomorphism of $M \in \operatorname{simp}_n(A)$. We write $e_{ij} \in \mathrm{M}_n(k)$ for the elementary matrices, and pick $y_{ij} \in A$ such that $\rho(y_{ij}) = e_{ij}$ for $1 \leq i, j \leq n$. Let σ denote a cyclic permutation of the integers $\{1, 2, \ldots, n\}$ of order n, and define

$$s_k = [y_{\sigma^k(1),\sigma^k(2)},\ y_{\sigma^k(2),\sigma^k(2)},\ y_{\sigma^k(2),\sigma^k(3)},\ \ldots,\ y_{\sigma^k(n),\sigma^k(n)}]$$

for $0 \leq k \leq n-1$. Moreover, let $s = s_0 + s_1 + \cdots + s_{n-1} \in I(n-1) \subseteq A$. We see that $\rho(s) \in \mathrm{M}_n(k)$ is a matrix with entry 1 in position $(\sigma^k(1), \sigma^k(n))$ for $k = 0, 1, \ldots, n-1$ and 0 in all other positions. In particular, $\det \rho(s) = \pm 1$, hence $\det(s) \in C(n)$ is non-zero at the point $\mathfrak{m}_M \in \operatorname{Max}(C(n))$ corresponding to M. Put $U_M = D(\det(s)) \subseteq \operatorname{Max}(C(n))$, and consider the localization $O(n)_{\{s\}} \subseteq \mathrm{M}_n(C(n)_{\{\det(s)\}})$, where the inclusion follows from general properties of localization. Any closed point $\mathfrak{m}'_M \in U_M$ corresponds to an n-dimensional representation of A for which the element $s \in I(n-1)$ is invertible. This representation can not have a m-dimensional quotient with $m < n$, so it must be simple.

Since $s \in I(n-1)$, the localized k-algebra $O(n)_{\{s\}}$ does not have any simple modules of dimension other than n. In fact, for any finite dimensional $O(n)_{\{s\}}$-module M of dimension m, the image $\hat{s}$ of s in $\operatorname{End}_k(M)$ must be invertible. However, the inverse $\hat{s}^{-1}$ must be the image of a polynomial of degree $m-1$ in s. Therefore, if M is simple over $O(n)_{\{s\}}$, that is if the homomorphism $O(n)_{\{s\}} \to \operatorname{End}_k(M)$ is surjective, M must also be simple over A. Since $s \in I(n-1)$ it follows that $m \geq n$. If $m > n$, we may construct in the same way as above an element in $I(n)$ mapping into a nonzero element of $\operatorname{End}_k(M)$. Since $0 = I(n) \subseteq A(n)$, and therefore $I(n) = 0$ in $O(n)_{\{s\}}$, we have proved that $m = n$. By a theorem of M. Artin, $O(n)_{\{s\}}$ must be an Azumaya algebra with center $S(n)_{\{s\}} = Z(O(n)_{\{s\}})$; see Artin [1]. Therefore, $O(n)$ defines a presheaf $\mathcal{O}(n)$ of Azumaya algebras on $\operatorname{simp}_n(A)$, with center $\mathcal{S}(n) = Z(\mathcal{O}(n))$. Any $M \in \operatorname{simp}_n(A)$ corresponding to $\mathfrak{m}_M \in \operatorname{Max}(C(n))$ maps to a point $\kappa(M) \in \operatorname{simp}_n(A)$. Let $\mathfrak{m}_{\kappa(M)}$ be the corresponding maximal ideal of $\mathcal{S}(n)$. Since $O(n)$ is locally Azumaya, it follows that

$$\widehat{\mathcal{S}}(n)_{\mathfrak{m}_{\kappa(M)}} \cong H^{O(n)}(M) \cong H^{A(n)}(M) \ \blacksquare$$

6. The noncommutative scheme structure. In this section, we shall use noncommutative deformations of modules to define localizations, and use this to construct a "structure" presheaf of noncommutative algebras on the Jacobson topology defined on the space simp(A) of finitely dimensional simple modules. Recall that the Jacobson topology is the topology defined, just as the Zariski topology in ordinary algebraic geometry, by considering for any $a \in A$ the subset $D(a) := \{V \in \text{simp}(A) | \det(\rho(a)) \neq 0\}$, and showing that this set forms a basis for a topology.

6.1. Geometric localizations. The localization L of a commutative k-algebra A in a maximal ideal $\mathfrak{m}$ is given by the following universal property: It is a k-algebra L together with a diagram

$$\begin{tikzcd} A \arrow[r, "\rho_L"] \arrow[dr, "\kappa^A"'] & L \arrow[d, "\kappa^L"] \\ & A/\mathfrak{m} \end{tikzcd}$$

such that $\rho_L(a)$ is a unit in L whenever $\kappa_A(a)$ is a unit in $A/\mathfrak{m}$ and such that for any other L' with this property, there exists a unique morphism $\phi : L \to L'$ such that $\rho_{L'} = \rho_L \circ \phi$.

In the following, A is a not necessarily commutative, associative k-algebra.

Definition 6.1. Let A be any k-algebra and $\mathcal{M} = \{M_1, \dots, M_n\}$ a family of right A-modules. Then L is called a *geometric localization of A in $\mathcal{M}$* if there exists a diagram

$$\begin{tikzcd} A \arrow[r, "\rho_L"] \arrow[dr, "\kappa^A"'] & L \arrow[d, "\kappa^L"] \\ & \prod_{i=1}^n \operatorname{End}_k(M_i) \end{tikzcd}$$

such that $\rho_L(a)$ is a unit in L whenever $\kappa^A(a)$ is a unit in $\prod_{i=1}^n \operatorname{End}_k(M_i)$, and if

$$\mathcal{O}^A(\mathcal{M}) \simeq \mathcal{O}^L(\mathcal{M}).$$

We write $L = A_{\mathcal{M}}$, and notice that geometric localizations might not be unique.

Lemma 6.2. *Assume that A is a geometric k-algebra, and that $\mathcal{M} = \{M_1, \dots, M_n\}$ is a family of finite dimensional, simple right A-modules. Then the geometric localization $A_{\mathcal{M}}$ of A in $\mathcal{M} = \{M_1, \dots, M_n\}$ exists, and* simp$(A_{\mathcal{M}}) = \mathcal{M}$.

Proof. We consider the structural morphism for the family $\mathcal{M}$;

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[dr, "\kappa"'] & \mathcal{O}^A(\mathcal{M}) \arrow[d, "\pi"] \\ & \prod_{i=1}^n \operatorname{End}_k(M_i) \end{tikzcd}$$

The subalgebra $A_{\mathcal{M}}$ of $\mathcal{O}^A(\mathcal{M})$ generated by the image $\eta(A)$ together with all inverses of elements in $\eta(A)$ mapping to units in $\prod_{i=1}^n \operatorname{End}_k(M_i)$ is a geometric localization of A in $\mathcal{M}$. If $a \in A$ maps to a unit in $\prod_{i=1}^n \operatorname{End}_k(M_i)$, there exists an element $b \in A$ such that ab maps to 1. Then $r := 1 - ab$ sits in the radical, and $ab = 1 - r$ is invertible in $O^A(\mathcal{M})$, since $\mathcal{O}^A(\mathcal{M}) = \varprojlim \mathcal{O}^A(\mathcal{M}) / \operatorname{rad}^n$, and so a is invertible, with a unique left=right inverse. To prove that $\mathcal{O}^A(\mathcal{M}) = \mathcal{O}^{A_{\mathcal{M}}}(\mathcal{M})$, we need only see that any deformation of $\mathcal{M}$ as an A-module is also a deformation as $A_{\mathcal{M}}$-module. ∎

For U any open set in the Jacobson topology, we *define*

$$\mathcal{O}(U) = \varprojlim_{\underline{c} \subseteq U} A_{\underline{c}} \subseteq \varprojlim_{\underline{c} \subseteq U} \mathcal{O}^A(\underline{c})$$

It is clear that this is a presheaf, the *structure presheaf*, $\mathcal{O}_{\mathrm{simp}(A)}$, on $X = \mathrm{simp}(A)$.

PROPOSITION 6.3. *If A is geometric, we have an injective homomorphism,*

$$A \subset \Gamma(\mathrm{simp}\, A, \mathcal{O}_{\mathrm{simp}(A)}) := \mathcal{A}.$$

If A is commutative, then $(\mathrm{simp}(A), \mathcal{O}_{\mathrm{simp}(A)}) \simeq (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$, *restricted to the closed points.*

Proof. The first statement follows by definition of geometric and the rest is clear. See also [5]. ■

DEFINITION 6.4. If $A = \mathcal{A}$ we call $\mathrm{simp}_{nc} A = (\mathrm{simp}(A), \mathcal{O}_{\mathrm{simp}(A)})$ an affine prescheme, and we say that the set of simple A-modules $|\,\mathrm{simp}(A)|$ is a prescheme for A. A not necessarily commutative prescheme is a topological space with a presheaf of rings that can be covered by affine preschemes.

Notice that if A has finite nilpotency, i.e. if the kernel of the morphism,

$$A \to \prod_{V \in simp(A)} \mathrm{End}_k(V)$$

is nilpotent, then $A = \mathcal{A}$, therefore $|\,\mathrm{simp}(A)|$ is a prescheme for A.

7. Example: The quantum plane $A_q = k\langle x, y\rangle/(xy - qyx)$. For $|q| \neq 1$ the finite dimensional simple modules, are the points of the two coordinate axes.

Let for $\alpha\beta = 0$,

$$V_{\alpha,\beta} = A_q/(x-\alpha, y-\beta) = k\langle x, y\rangle/(x-\alpha, y-\beta) \cong k.$$

Recall that for a general k-algebra A, and A-modules M, N,

$$\mathrm{Ext}^i_A(M, N) \cong \mathsf{HH}^i(A, \mathrm{Hom}_k(M, N))$$

where $HH^\cdot$ is the Hochshild cohomology. Also recall that $\mathrm{Hom}_k(M, N)$ is an A-bimodule by the action on $\phi \in \mathrm{Hom}_k(M, N)$ by $a \in A$ given by $(a\phi)(m) = \phi(ma)$ and $(\phi a)(m) = \phi(m)a$. Computing the Hochshild cohomology leads to,

$$\mathrm{Ext}^1_{A_q}(V_{\alpha,\beta}, V_{\alpha',\beta'}) \cong \mathrm{Der}_k(A_q, \mathrm{Hom}_k(V_{\alpha,\beta}, V_{\alpha',\beta'}))/\,\mathrm{Inner},$$

where $\mathrm{Der}_k(A_q, \mathrm{Hom}_k(V_{\alpha,\beta}, V_{\alpha',\beta'}))$ denotes the k-derivations from A_q to the bimodule $\mathrm{Hom}_k(V_{\alpha,\beta}, V_{\alpha',\beta'})$, and where Inner denotes the derivations on the form $\delta = \mathrm{ad}(\phi)$.

First of all, if $q = 1$ we have the ordinary commutative affine plane, and everything is known. The points are exactly the simple one-dimensional modules $V_{\alpha,\beta}$, $\dim_k \mathrm{Ext}^1_{A_1}(V_{\alpha,\beta}, V_{\alpha',\beta'}) = 1$ if $(\alpha, \beta) = (\alpha', \beta')$, otherwise 0, and the geometric localization at a point is the ordinary localization. In the rest of this section, we assume $|q| \neq 1$. The simple modules are then reduced to the modules, $X_\alpha := V_{\alpha,0}$ and $Y_\beta := V_{0,\beta}$, and to, $X_0 = Y_0 = V_{0,0}$.

An easy computation leads to the results,

- $\alpha = \beta = 0 \Rightarrow \mathrm{Ext}^1_{A_q}(X_\alpha, X_\beta) = \langle d_x, d_y\rangle$,

- $\alpha = \beta \neq 0 \Rightarrow \mathrm{Ext}^1_{A_q}(X_\alpha, X_\beta) = \langle d_x \rangle$,
- $\alpha = q\beta \neq 0 \Rightarrow \mathrm{Ext}^1_{A_q}(X_\alpha, X_\beta) = \langle d_y \rangle$,
- otherwise $\mathrm{Ext}^1_{A_q}(X_\alpha, X_\beta) = 0$,

where d_x (resp. d_y) is the class of the derivation for which, $d_x(x) = 1, d_x(y) = 0$, (resp. $d_y(x) = o, d_y(y) = 1$).

To compute $\mathrm{Ext}^1_{A_q}(X_\alpha, Y_\beta)$, for $\alpha{\cdot}\beta \neq 0$ we notice that the trivial derivations are given by $\mathrm{ad}_\phi(x) = \phi x - x\phi = \phi\alpha$, $\mathrm{ad}_\phi(y) = \phi y - y\phi = -\beta\phi$, or simply by, $\mathrm{ad}_\phi = \phi(\alpha d_x - \beta d_y)$. A general derivation $\delta : A_q \to \mathrm{Hom}_k(X_\alpha, Y_\beta) = k$ is given by the values, $\delta(x) = \eta$, $\delta(y) = \xi$. The relation in A_q, $xy - qyx = 0$ implies the equation,

$$0 = \delta(xy - qyx) = x\delta(y) + \delta(x)y - q(y\delta(x) + \delta(y)x) = \alpha\xi + \eta\beta.$$

So, if $\alpha \neq 0$, $\xi = -\frac{\beta}{\alpha}\eta$ and $\delta = \eta d_x - \frac{\beta}{\alpha}\eta d_y = \frac{\eta}{\alpha}(\alpha d_x - \beta d_y)$. Similarly, if $\beta \neq 0$, $\eta = -\frac{\alpha}{\beta}\xi$ and $\delta = -\frac{\alpha}{\beta}\xi d_x + \xi d_y = -\frac{\xi}{\beta}(\alpha d_x - \beta d_y)$. So, in both cases we have,

$$\mathrm{Ext}^1_{A_q}(X_\alpha, Y_\beta) = 0.$$

We notice that the remaining cases (Y_α, Y_β), (Y_α, X_β) follows by symmetry, except for the case $V := X_0 = Y_0$. Now $V = A_q/(x,y) \cong k$, and we easily prove that,

$$\mathrm{Ext}^1_{A_q}(V,V) \cong \mathsf{HH}^1(A_q, \mathrm{Hom}_k(V,V)) \cong \mathrm{Der}_k(A_q, \mathrm{End}_k(V))/\,\mathrm{Inner} = \langle d_x, d_y \rangle.$$

Following the algorithm of computing generalized Massey Products in e.g. [7], we find,

$$\hat{H}(V) \cong k\langle\langle x, y\rangle\rangle/(xy - qyx)$$

with the obvious algebraization $A_q = H(V) = k\langle x, y\rangle/(xy - qyx)$, and a canonical injection, $\rho_{\underline{c}} : A_q \hookrightarrow \hat{H}(V)$. Hence also an injection $\rho : A_q \hookrightarrow \Gamma(\mathrm{simp}(A_q), \mathcal{O}_{\mathrm{simp}(A_q)} = \mathcal{A})$. However the element $1 + xy$ in A is a unit at all points of $\mathrm{simp}(A)$, therefore the inverse is an element of $\mathcal{A}$, but not of A. This proves that A_q does not map isomorphically to $\Gamma(\mathrm{simp}\, A_q, \mathcal{O}_{\mathrm{simp}\, A_q})$ and so $|\,\mathrm{simp}\, A_q|$ is not a scheme for A_q.

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