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EXPLICIT REPRESENTATIONS OF CLASSICAL LIE SUPERALGEBRAS IN A GELFAND-ZETLIN BASIS

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Abstract. An explicit construction of all finite-dimensional irreducible representations of classical Lie algebras is a solved problem and a Gelfand-Zetlin type basis is known. However the latter lacks the orthogonality property or does not consist of weight vectors for $\mathfrak{so}(n)$ and $\mathfrak{sp}(2n)$. In case of Lie superalgebras all finite-dimensional irreducible representations are constructed explicitly only for $\mathfrak{gl}(1|n)$, $\mathfrak{gl}(2|2)$, $\mathfrak{osp}(3|2)$ and for the so called essentially typical representations of $\mathfrak{gl}(m|n)$. In the present paper we introduce an orthogonal basis of weight vectors for a class of infinite-dimensional representations of the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2n)$ and for all irreducible covariant tensor representations of the general linear Lie superalgebra $\mathfrak{gl}(m|n)$. Expressions for the transformation of the basis under the action of algebra generators are given. The results are a step towards the explicit construction of the parastatistics Fock space.

1. Introduction. The representation theory of simple Lie (super)algebras and in particular of classical Lie algebras A_n, B_n, C_n, D_n and basic classical Lie superalgebras $A(m|n), B(m|n), C(n), D(m|n)$ is a central topic in mathematics and physics. A lot is known about finite-dimensional irreducible representations of Lie algebras. The characters and dimensions are given by the Weyl formula. As regards the explicit construction of the representations the first results were given by Gelfand and Zetlin in two short papers [3, 4]. They solved the problem for the general linear Lie algebra $\mathfrak{gl}(n)$ [3] ($\equiv$ for the algebras of class A_n) and the orthogonal Lie algebras $B_n \equiv \mathfrak{so}(2n+1)$ and $D_n \equiv \mathfrak{so}(2n)$ [4]. The possibility to introduce a basis in any finite-dimensional irreducible $\mathfrak{gl}(n)$ module V stems from the fact that each such module V is a direct sum of irreducible $\mathfrak{gl}(n-1)$ modules $V = \sum_i \oplus V_i$, where the decomposition is multiplicity free. Consider

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now the chain of subalgebras

$$\mathfrak{gl}(n) \supset \mathfrak{gl}(n-1) \supset \ldots \supset \mathfrak{gl}(1)$$

and let

$$V \equiv V(n) \supset V(n-1) \supset \ldots \supset V(1) \tag{1}$$

be a flag of subspaces of the $\mathfrak{gl}(k)$ finite-dimensional irreducible modules, for each $k = 1, \ldots, n$. Since any irreducible $\mathfrak{gl}(1)$ module $V(1)$ is a one dimensional space the flag (1) determines a one dimensional subspace in V. Denote an arbitrary vector in this subspace by $|m)$ and let $\Lambda(k) = [m_{1k}, m_{2k}, \ldots, m_{kk}]$ be the highest weight of $V(k)$. The highest weights $\Lambda(n), \Lambda(n-1), \ldots, \Lambda(1)$ determine the vector $|m)$

$$|m) \equiv \left| \begin{array}{lllll} m_{1n} & \cdots & \cdots & m_{n-1,n} & m_{nn} \\ m_{1,n-1} & \cdots & \cdots & m_{n-1,n-1} & \\ \vdots & \cdot^{\cdot^{\cdot}} & & & \\ m_{11} & & & & \end{array} \right). \tag{2}$$

The vectors (2) corresponding to all possible flags (1), constitute a basis in V, which was introduced by Gelfand and Zetlin [3] and is now called a Gelfand-Zetlin (GZ) basis in the $\mathfrak{gl}(n)$ module V.

In a similar way one can introduce a basis in each finite-dimensional $\mathfrak{so}(n)$ module [4] considering the chain of subalgebras

$$\mathfrak{so}(n) \supset \mathfrak{so}(n-1) \supset \ldots \supset \mathfrak{so}(2).$$

In contrast to $\mathfrak{gl}(n)$, where the basis consists of orthonormal weight vectors, the GZ-basis vectors for $\mathfrak{so}(n)$ [4] are not eigenvectors for the Cartan subalgebra.

This approach does not work for the symplectic Lie algebras $C_n \equiv \mathfrak{sp}(2n)$ since the restriction $\mathfrak{sp}(2n) \downarrow \mathfrak{sp}(2n-2)$ is not multiplicity free. Since the two short papers of Gelfand and Zetlin [3, 4] were published in 1950, many different methods have been developed to construct bases in the modules of the classical Lie algebras (see for instance the review paper [10]). A complete solution of the problem for the $\mathfrak{sp}(2n)$ modules was eventually given by Molev [11] in 1999. He used finite-dimensional irreducible representations of the so called twisted Yangians. The paper of Molev [11] together with the papers of Gelfand and Zetlin [3, 4] provide explicit realizations of all finite-dimensional irreducible representations of the classical Lie algebras. Molev applied the approach based on the twisted Yangians also to the orthogonal Lie algebras [12, 13]. The new basis consists of weight vectors but in turn lacks the orthogonality property of the GZ basis. In such a way the problem to construct a natural basis for the Lie algebras $\mathfrak{so}(n)$ and $\mathfrak{sp}(2n)$, which accomodate the two properties, remains open.

In contrast to the classical Lie algebras, the finite-dimensional irreducible modules over any basic Lie superalgebra are fully classified [6] but so far it is not known how to construct explicitly all such modules and especially the so called indecomposible modules. Some steps towards a generalization of the concept of the GZ basis have been done (see [15, 16, 17]). Irrespective of the progress, there is still much to be done in order to complete the representation theory of the basic classical Lie superalgebras. In the present paper we make a step further in this respect not for finite-dimensional but for a class of infinite-dimensional irreducible representations of the Lie superalgebra $B(0|n) \equiv \mathfrak{osp}(1|2n)$ and also for a class of finite-dimensional irreducible representations of the general linear Lie superalgebra $\mathfrak{gl}(m|n)$, the so called covariant tensor $\mathfrak{gl}(m|n)$ representations. In both cases the corresponding basis vectors are orthonormal and weight vectors.

2. Explicit representations of the Lie superalgebra $\mathfrak{osp}(1|2n)$. The orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2n)$ [6] consists of matrices of the form

$$\begin{pmatrix} 0 & a & a_1 \\ a_1^t & b & c \\ -a^t & d & -b^t \end{pmatrix},$$

where a and a_1 are $(1 \times n)$-matrices, b is any $(n \times n)$-matrix, and c and d are symmetric $(n \times n)$-matrices. The even elements have $a = a_1 = 0$ and the odd elements are those with $b = c = d = 0$. Denote the row and column indices running from 0 to $2n$ and by e_{ij} the matrix with zeros everywhere except a 1 in position (i, j). Then as a basis in the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{osp}(1|2n)$ consider

$$h_j = e_{jj} - e_{n+j,n+j} \qquad (j = 1, \ldots, n).$$

In terms of the dual basis δ_j of $\mathfrak{h}^*$, the root vectors and corresponding roots of $\mathfrak{osp}(1|2n)$ are given by:

$$\begin{aligned}
&e_{0,k} - e_{n+k,0} \leftrightarrow -\delta_k, && e_{0,n+k} + e_{k,0} \leftrightarrow \delta_k, \quad k = 1, \ldots, n, \text{ odd},\\
&e_{j,n+k} + e_{k,n+j} \leftrightarrow \delta_j + \delta_k, && e_{n+j,k} + e_{n+k,j} \leftrightarrow -\delta_j - \delta_k, \ j \le k = 1, \ldots, n, \text{ even},\\
&e_{j,k} - e_{n+k,n+j} \leftrightarrow \delta_j - \delta_k, && j \ne k = 1, \ldots, n, \text{ even}.
\end{aligned}$$

Introduce the following multiples of the odd root vectors

$$b_k^+ = \sqrt{2}(e_{0,n+k} + e_{k,0}), \qquad b_k^- = \sqrt{2}(e_{0,k} - e_{n+k,0}) \qquad (k = 1, \ldots, n).$$

Then the following holds [2]:

THEOREM 2.1. *As a Lie superalgebra defined by generators and relations, $\mathfrak{osp}(1|2n)$ is generated by $2n$ odd elements $b_k^\pm$ subject to the following relations*

$$[\{b_j^\xi, b_k^\eta\}, b_l^\epsilon] = (\epsilon - \xi)\delta_{jl} b_k^\eta + (\epsilon - \eta)\delta_{kl} b_j^\xi. \tag{3}$$

We can construct representations of $\mathfrak{osp}(1|2n)$ using an induced module construction with an appropriate chain of subalgebras (see [7] for more details).

PROPOSITION 2.2. *A basis for the even subalgebra $\mathfrak{sp}(2n)$ of $\mathfrak{osp}(1|2n)$ is given by the $2n^2 + n$ elements $\{b_j^\pm, b_k^\pm\}, 1 \le j \le k \le n$, $\{b_j^+, b_k^-\}, 1 \le j, k \le n$. The n^2 elements $\{b_j^+, b_k^-\}, j, k = 1, \ldots, n$ are a basis for the $\mathfrak{sp}(2n)$ subalgebra $\mathfrak{gl}(n)$.*

Define a one-dimensional $\mathfrak{gl}(n)$ module $V(p)$, spanned on a vector $|0\rangle$, setting $\{b_j^-, b_k^+\}|0\rangle = p\,\delta_{jk}\,|0\rangle$, $j, k = 1, \ldots, n$, and p an arbitrary parameter. The subalgebra $\mathfrak{gl}(n)$ can be extended to a parabolic subalgebra $\mathcal{P} = \text{span}\{\{b_j^+, b_k^-\}, b_j^-, \{b_j^-, b_k^-\}, j,k = 1,\ldots,n\}$ of $\mathfrak{osp}(1|2n)$ and $V(p)$ to a $\mathcal{P}$ module requiring $b_j^-|0\rangle = 0,\ j = 1,\ldots,n$. Now we define the induced $\mathfrak{osp}(1|2n)$ module $\overline{W}(p)$: $\overline{W}(p) = \text{Ind}_{\mathcal{P}}^{\mathfrak{osp}(1|2n)} V(p)$.

By the Poincaré-Birkhoff-Witt theorem [6], it is easy to give a basis for $\overline{W}(p)$:

$$(b_1^+)^{k_1}\cdots(b_n^+)^{k_n}(\{b_1^+,b_2^+\})^{k_{12}}(\{b_1^+,b_3^+\})^{k_{13}}\cdots(\{b_{n-1}^+,b_n^+\})^{k_{n-1,n}}|0\rangle,$$

where $k_1,\ldots,k_n,k_{12},k_{13},\ldots,k_{n-1,n} \in \mathbb{Z}_+$. However in general $\overline{W}(p)$ is not an irreducible module and let $M(p)$ be the maximal nontrivial submodule of $\overline{W}(p)$. The purpose is now to determine the vectors belonging to $M(p)$ and also to find explicit matrix elements of the $\mathfrak{osp}(1|2n)$ generators $b_j^\pm$ in an appropriate basis of $W(p) = \overline{W}(p)/M(p)$.

From the basis in $\overline{W}(p)$, it is easy to write down the character of $\overline{W}(p)$:

$$\text{char}\,\overline{W}(p) = \frac{(x_1\cdots x_n)^{p/2}}{\prod_{i=1}^n(1-x_i)\prod_{1\le j<k\le n}(1-x_jx_k)}.$$

Such expressions have an interesting expansion in terms of Schur functions.

PROPOSITION 2.3 (Cauchy, Littlewood). *Let $x_1,\ldots,x_n$ be a set of n variables. Then [8]*

$$\frac{1}{\prod_{i=1}^n(1-x_i)\prod_{1\le j<k\le n}(1-x_jx_k)} = \sum_\lambda s_\lambda(x_1,\ldots,x_n) = \sum_\lambda s_\lambda(x).$$

Herein the sum is over all partitions λ and $s_\lambda(x)$ is the Schur symmetric function [9].

The characters of finite dimensional $\mathfrak{gl}(n)$ representations are given by such Schur functions $s_\lambda(x)$. Therefore we can use the $\mathfrak{gl}(n)$ GZ basis vectors as our new basis for $\overline{W}(p)$. Thus the new basis of $\overline{W}(p)$ consists of vectors of the form

$$|m) \equiv |m)^n \equiv \left|\begin{array}{lllll} m_{1n} & \cdots & \cdots & m_{n-1,n} & m_{nn} \\ m_{1,n-1} & \cdots & \cdots & m_{n-1,n-1} & \\ \vdots & \iddots & & & \\ m_{11} & & & & \end{array}\right) = \left|\begin{array}{l} [m]^n \\ |m)^{n-1} \end{array}\right),$$

where the top line of the pattern, also denoted by the n-tuple $[m]^n$, is any partition λ (consisting of non increasing nonnegative numbers) with length $\ell(\lambda) \le n$. The label p itself is dropped in the notation of $|m)$. The remaining $n-1$ lines of the pattern will sometimes be denoted by $|m)^{n-1}$. So all m_{ij} in the above GZ pattern are nonnegative integers, satisfying the *betweenness conditions* $m_{i,j+1} \ge m_{ij} \ge m_{i+1,j+1}, 1 \le i \le j \le n-1$. The task is now to give the explicit action of the generating elements $b_i^\pm$ of $\mathfrak{osp}(1|2n)$ (18). For this purpose, we introduce the following notations : $l_{ij} = m_{ij} - i$ for $i = 1,\ldots,n$, and let each symbol $\pm i_k, k$ attached as a subscript to $|m)$ indicate a replacement $m_{i_k,k} \to m_{i_k,k} \pm 1$. Then this action is given by:

$$b_j^+|m) =$$
$$\sum_{i_n=1}^{n}\sum_{i_{n-1}=1}^{n-1}\ldots\sum_{i_j=1}^{j} S(i_n,i_{n-1})S(i_{n-1},i_{n-2})\ldots S(i_{j+1},i_j)\left(\frac{\prod_{k=1}^{j-1}(l_{k,j-1}-l_{i_j,j}-1)}{\prod_{k\neq i_j=1}^{j}(l_{kj}-l_{i_j,j})}\right)^{1/2}$$
$$\times\prod_{r=1}^{n-j}\left(\frac{\prod_{k\neq i_{n-r}=1}^{n-r}(l_{k,n-r}-l_{i_{n-r+1},n-r+1}-1)\prod_{k\neq i_{n-r+1}=1}^{n-r+1}(l_{k,n-r+1}-l_{i_{n-r},n-r})}{\prod_{k\neq i_{n-r+1}=1}^{n-r+1}(l_{k,n-r+1}-l_{i_{n-r+1},n-r+1})\prod_{k\neq i_{n-r}=1}^{n-r}(l_{k,n-r}-l_{i_{n-r},n-r}-1)}\right)^{1/2}$$
$$\times F_{i_n}(m_{1n},m_{2n},\ldots,m_{nn})\;|m)_{+i_n,n;+i_{n-1},n-1;\ldots;+i_j,j}; \qquad (4)$$

$$b_j^-|m) =$$
$$\sum_{i_n=1}^{n}\sum_{i_{n-1}=1}^{n-1}\ldots\sum_{i_j=1}^{j} S(i_n,i_{n-1})S(i_{n-1},i_{n-2})\ldots S(i_{j+1},i_j)\left(\frac{\prod_{k=1}^{j-1}(l_{k,j-1}-l_{i_j,j})}{\prod_{k\neq i_j=1}^{j}(l_{kj}-l_{i_j,j}+1)}\right)^{1/2}$$
$$\times\prod_{r=1}^{n-j}\left(\frac{\prod_{k\neq i_{n-r}=1}^{n-r}(l_{k,n-r}-l_{i_{n-r+1},n-r+1})\prod_{k\neq i_{n-r+1}=1}^{n-r+1}(l_{k,n-r+1}-l_{i_{n-r},n-r}+1)}{\prod_{k\neq i_{n-r+1}=1}^{n-r+1}(l_{k,n-r+1}-l_{i_{n-r+1},n-r+1}+1)\prod_{k\neq i_{n-r}=1}^{n-r}(l_{k,n-r}-l_{i_{n-r},n-r})}\right)^{1/2}$$
$$\times F_{i_n}(m_{1n},\ldots,m_{i_n,n}-1,\ldots,m_{nn})\;|m)_{-i_n,n;-i_{n-1},n-1;\ldots;-i_j,j}, \qquad (5)$$

where

$$F_k(m_{1n},m_{2n},\ldots,m_{nn}) =(-1)^{m_{k+1,n}+\cdots+m_{nn}}(m_{kn}+n+1-k+\mathcal{E}_{m_{kn}}(p-n))^{1/2}$$
$$\times\prod_{j\neq k=1}^{n}\left(\frac{m_{jn}-m_{kn}-j+k}{m_{jn}-m_{kn}-j+k-\mathcal{O}_{m_{jn}-m_{kn}}}\right)^{1/2}, \qquad (6)$$

with

$$\mathcal{E}_j = 1 \text{ if } j \text{ is even and } 0 \text{ otherwise},$$
$$\mathcal{O}_j = 1 \text{ if } j \text{ is odd and } 0 \text{ otherwise},$$

and

$$S(k,r) = \begin{cases} \;\;1 & \text{for} \quad k\leq r \\ -1 & \text{for} \quad k>r. \end{cases}$$

Note that, $\{b_j^-,b_j^+\} = 2h_j, j = 1,\ldots,n$ and

$$h_j|m) = \left(\frac{p}{2}+\sum_{k=1}^{j}m_{kj}-\sum_{k=1}^{j-1}m_{k,j-1}\right)|m).$$

The proof that (4)-(5) do give a representation of $\mathfrak{osp}(1|2n)$ consists of verifying that all triple relations (3) hold when acting on any vector $|m)$. Each such verification leads to an algebraic identity in n variables $m_{1n},\ldots,m_{nn}$. Consider now the factor $(m_{kn}+n+1-k+\mathcal{E}_{m_{kn}}(p-n))$ in the expression of $F_k([m]^n)$ (6). This is the only factor in the right hand side of (6) that may become zero. If this factor is zero or negative, the assigned vector $|m')$ belongs to $M(p)$. Recall that the integers m_{jn} satisfy $m_{1n}\geq m_{2n}\geq\cdots\geq m_{nn}\geq 0$. If $m_{kn}=0$ (its smallest possible value), then this factor in F_k takes the value $(p-k+1)$. So the p-values $1,2,\ldots,n-1$ play a special role leading to the following result:

THEOREM 2.4. *The $\mathfrak{osp}(1|2n)$ representation $W(p)$ is an irreducible representation if and only if $p \in \{1, 2, \ldots, n-1\}$ or $p > n-1$. For $p > n-1$, $W(p) = \overline{W}(p)$ and for $p \in \{1, 2, \ldots, n-1\}$, $W(p) = \overline{W}(p)/M(p)$ with $M(p) \neq 0$. The explicit action of the $\mathfrak{osp}(1|2n)$ generators in $W(p)$ is given by (4)-(5), and the basis vectors are orthogonal weight vectors. For $p \in \{1, 2, \ldots, n-1\}$ this action remains valid, provided one keeps in mind that all vectors with $m_{p+1,n} \neq 0$ must vanish.*

3. Explicit representations of the Lie superalgebra $\mathfrak{gl}(m|n)$. The Lie superalgebra $\mathfrak{gl}(m|n)$ can be defined [6] through its natural matrix realization

$$\mathfrak{gl}(m|n) = \left\{ x = \begin{pmatrix} A & B \\ C & D \end{pmatrix} | A \in M_{m\times m}, B \in M_{m\times n}, C \in M_{n\times m}, D \in M_{n\times n} \right\},$$

where $M_{p\times q}$ is the space of all $p \times q$ complex matrices. The even subalgebra $\mathfrak{gl}(m|n)_{\bar{0}}$ has $B = 0$ and $C = 0$; the odd subspace $\mathfrak{gl}(m|n)_{\bar{1}}$ has $A = 0$ and $D = 0$. A basis for $\mathfrak{gl}(m|n)$ consists of matrices e_{ij} $(i, j = 1, 2, \ldots, r \equiv m+n)$ with entry 1 at position (i, j) and 0 elsewhere. A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{gl}(m|n)$ is spanned by the elements e_{jj} $(j = 1, 2, \ldots, r)$, and a set of generators of $\mathfrak{gl}(m|n)$ is given by the Chevalley generators e_{jj} $(j = 1, \ldots, r)$, $e_{i,i+1}$ and $e_{i+1,i}$ $(i = 1, \ldots, r-1)$. Then $\mathfrak{gl}(m|n)$ can be defined as the free associative superalgebra over $\mathbb{C}$ and generators e_{jj}, $(j = 1, 2, \ldots, r \equiv n+m)$ and $e_{i,i+1}$, $e_{i+1,i}$ $(i = 1, 2, \ldots, r-1)$ subject to the following relations (unless stated otherwise, the indices below run over all possible values):

- The Cartan-Kac relations:

$$[e_{ii}, e_{jj}] = 0; \tag{7}$$
$$[e_{ii}, e_{j,j+1}] = (\delta_{ij} - \delta_{i,j+1})e_{j,j+1}, \tag{8}$$
$$[e_{ii}, e_{j+1,j}] = -(\delta_{ij} - \delta_{i,j+1})e_{j+1,j}; \tag{9}$$
$$[e_{i,i+1}, e_{j+1,j}] = 0 \ \text{ if } i \neq j; \tag{10}$$
$$[e_{i,i+1}, e_{i+1,i}] = e_{ii} - e_{i+1,i+1} \ \text{ if } i \neq m; \tag{11}$$
$$\{e_{m,m+1}, e_{m+1,m}\} = e_{mm} + e_{m+1,m+1}; \tag{12}$$

- The Serre relations for the $e_{i,i+1}$:

$$e_{i,i+1}e_{j,j+1} = e_{j,j+1}e_{i,i+1} \text{ if } |i-j| \neq 1; \qquad e_{m,m+1}^2 = 0; \tag{13}$$
$$e_{i,i+1}^2 e_{i+1,i+2} - 2e_{i,i+1}e_{i+1,i+2}e_{i,i+1} + e_{i+1,i+2}e_{i,i+1}^2 = 0,$$
$$\text{for } i \in \{1, \ldots, m-1\} \cup \{m+1, \ldots, n+m-2\}; \tag{14}$$
$$e_{i+1,i+2}^2 e_{i,i+1} - 2e_{i+1,i+2}e_{i,i+1}e_{i+1,i+2} + e_{i,i+1}e_{i+1,i+2}^2 = 0,$$
$$\text{for } i \in \{1, \ldots, m-2\} \cup \{m, \ldots, n+m-2\}; \tag{15}$$
$$e_{m,m+1}e_{m-1,m}e_{m,m+1}e_{m+1,m+2} + e_{m-1,m}e_{m,m+1}e_{m+1,m+2}e_{m,m+1} +$$
$$e_{m,m+1}e_{m+1,m+2}e_{m,m+1}e_{m-1,m} + e_{m+1,m+2}e_{m,m+1}e_{m-1,m}e_{m,m+1}$$
$$-2e_{m,m+1}e_{m-1,m}e_{m+1,m+2}e_{m,m+1} = 0; \tag{16}$$

- The relations obtained from (13)-(16) by replacing every $e_{i,i+1}$ by $e_{i+1,i}$.

We will consider a class of irreducible finite-dimensional $\mathfrak{gl}(m|n)$ modules, namely the so called covariant simple modules $V([\mu]_r)$ [1, 18]. They are in one-to-one correspondence with the set of all nonnegative integer $r = m + n$ tuples

$$[\mu]_r = [\mu_{1r}, \mu_{2r}, \ldots, \mu_{rr}], \tag{17}$$

for which

$$\mu_{ir} - \mu_{i+1,r} \in \mathbb{Z}_+, \ \forall i \neq m, \ i = 1, \ldots, r-1$$

and

$$\mu_{mr} \geq \#\{i : \mu_{ir} > 0, \ m+1 \leq i \leq r\}.$$

Within a given $\mathfrak{gl}(m|n)$ module $V([\mu]_r)$ the numbers (17) are fixed. The possibility of introducing a Gelfand-Zetlin basis in any covariant simple $\mathfrak{gl}(m|n)$ module $V([\mu]_r)$ stems from the following propositions.

Proposition 3.1. *Consider the $\mathfrak{gl}(m|n)$ module $V([\mu]_r)$ as a $\mathfrak{gl}(m|n-1)$ module. Then $V([\mu]_r)$ can be represented as a direct sum of covariant simple $\mathfrak{gl}(m|n-1)$ modules,*

$$V([\mu]_r) = \bigoplus_j V_j([\mu]_{r-1}),$$

where

I. All $V_j([\mu]_{r-1})$ carry inequivalent representations of $\mathfrak{gl}(m|n-1)$

$$\begin{aligned}
&[\mu]_{r-1} = [\mu_{1,r-1}, \mu_{2,r-1}, \ldots, \mu_{r-1,r-1}], \\
&\mu_{i,r-1} - \mu_{i+1,r-1} \in \mathbb{Z}_+, \ \forall i \neq m, \ i = 1, \ldots, r-2, \\
&\mu_{m,r-1} \geq \#\{i : \mu_{i,r-1} > 0, \ m+1 \leq i \leq r-1\}.
\end{aligned}$$

II.

1. $\mu_{ir} - \mu_{i,r-1} = \theta_{i,r-1} \in \{0,1\}, \quad 1 \leq i \leq m,$
2. $\mu_{i,r} - \mu_{i,r-1} \in \mathbb{Z}_+$ *and* $\mu_{i,r-1} - \mu_{i+1,r} \in \mathbb{Z}_+, \quad m+1 \leq i \leq r-1,$

Proposition 3.2. *Consider a covariant $\mathfrak{gl}(m|1)$ module $V([\mu]_{m+1})$ as a $\mathfrak{gl}(m)$ module. Then $V([\mu]_{m+1})$ can be represented as a direct sum of simple $\mathfrak{gl}(m)$ modules,*

$$V([\mu]_{m+1}) = \bigoplus_j V_j([\mu]_m),$$

where

I. All $V_j([\mu]_m)$ carry inequivalent representations of $\mathfrak{gl}(m)$

$$[\mu]_m = [\mu_{1m}, \mu_{2m}, \ldots, \mu_{mm}], \ \mu_{im} - \mu_{i+1,m} \in \mathbb{Z}_+.$$

II.

1. $\mu_{i,m+1} - \mu_{im} = \theta_{im} \in \{0,1\}, \quad 1 \leq i \leq m,$
2. *if* $\mu_{m,m+1} = 0,$ *then* $\theta_{mm} = 0.$

Both propositions follow from the character formula for simple covariant $\mathfrak{gl}(m|n)$ modules [1, 18].

Using Proposition 1, Proposition 2 and the $\mathfrak{gl}(m)$ GZ-basis we have:

PROPOSITION 3.3. *The set of vectors*

$$
|\mu) = \left|\begin{array}{llllllll}
\mu_{1r} & \cdots & \mu_{m-1,r} & \mu_{mr} & \mu_{m+1,r} & \cdots & \mu_{r-1,r} & \mu_{rr} \\
\mu_{1,r-1} & \cdots & \mu_{m-1,r-1} & \mu_{m,r-1} & \mu_{m+1,r-1} & \cdots & \mu_{r-1,r-1} & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \iddots & & \\
\mu_{1,m+1} & \cdots & \mu_{m-1,m+1} & \mu_{m,m+1} & \mu_{m+1,m+1} & & & \\
\mu_{1m} & \cdots & \mu_{m-1,m} & \mu_{mm} & & & & \\
\mu_{1,m-1} & \cdots & \mu_{m-1,m-1} & & & & & \\
\vdots & \iddots & & & & & & \\
\mu_{11} & & & & & & &
\end{array}\right) \tag{18}
$$

satisfying the conditions

1. $\mu_{ir} \in \mathbb{Z}_+$ *are fixed and* $\mu_{jr} - \mu_{j+1,r} \in \mathbb{Z}_+,\ j \neq m,\ 1 \leq j \leq r-1,$
   $\mu_{mr} \geq \#\{i : \mu_{ir} > 0,\ m+1 \leq i \leq r\};$
2. $\mu_{ip} - \mu_{i,p-1} \equiv \theta_{i,p-1} \in \{0,1\}, \quad 1 \leq i \leq m;\ m+1 \leq p \leq r;$
3. $\mu_{mp} \geq \#\{i : \mu_{ip} > 0,\ m+1 \leq i \leq p\}, \quad 1 \leq m+1 \leq p \leq r;$
4. *if* $\mu_{m,m+1} = 0,$ *then* $\theta_{mm} = 0;$
5. $\mu_{ip} - \mu_{i+1,p} \in \mathbb{Z}_+, \quad 1 \leq i \leq m-1;\ m+1 \leq p \leq r-1;$
6. $\mu_{i,j+1} - \mu_{ij} \in \mathbb{Z}_+$ *and* $\mu_{i,j} - \mu_{i+1,j+1} \in \mathbb{Z}_+,$
   $1 \leq i \leq j \leq m-1$ *or* $m+1 \leq i \leq j \leq r-1.$

constitute a basis in $V([\mu]_r)$.

The last condition corresponds to the in-betweenness condition and ensures that the triangular pattern to the right of the $n \times m$ rectangle μ_{ip} $(1 \leq i \leq m;\ m+1 \leq p \leq r)$ in (18) corresponds to a classical GZ pattern for $\mathfrak{gl}(n)$, and that the triangular pattern below this rectangle corresponds to a GZ pattern for $\mathfrak{gl}(m)$.

We shall refer to the basis (18) as the GZ basis for the covariant $\mathfrak{gl}(m|n)$ representations. The task is now to give the explicit action of the $\mathfrak{gl}(m|n)$ Chevalley generators on the basis vectors (18). For this purpose, we introduce the following notations : $l_{ij} = \mu_{ij} - i + m + 1$ for $1 \leq i \leq m$, $l_{pj} = -\mu_{pj} + p - m$ for $m+1 \leq p \leq r$, and $|\mu)_{\pm ij}$ is the pattern obtained from $|\mu)$ by replacing the entry μ_{ij} by $\mu_{ij} \pm 1$. Then this action is given by:

$$
e_{kk}|\mu) = \Big(\sum_{j=1}^{k} \mu_{jk} - \sum_{j=1}^{k-1} \mu_{j,k-1}\Big)|\mu), \quad 1 \leq k \leq r; \tag{19}
$$

$$
e_{k,k+1}|\mu) = \sum_{j=1}^{k} \left(-\frac{\prod_{i=1}^{k+1}(l_{i,k+1} - l_{jk}) \prod_{i=1}^{k-1}(l_{i,k-1} - l_{jk} - 1)}{\prod_{i\neq j=1}^{k}(l_{ik} - l_{jk})(l_{ik} - l_{jk} - 1)} \right)^{1/2} |\mu)_{jk},
$$
$$
1 \leq k \leq m-1; \tag{20}
$$

$$
e_{k+1,k}|\mu) = \sum_{j=1}^{k} \left(-\frac{\prod_{i=1}^{k+1}(l_{i,k+1} - l_{jk} + 1) \prod_{i=1}^{k-1}(l_{i,k-1} - l_{jk})}{\prod_{i\neq j=1}^{k}(l_{ik} - l_{jk} + 1)(l_{ik} - l_{jk})} \right)^{1/2} |\mu)_{-jk},
$$
$$
1 \leq k \leq m-1; \tag{21}
$$

$$e_{m,m+1}|\mu) = \sum_{i=1}^{m} \theta_{im}(-1)^{i-1}(-1)^{\theta_{1m}+\ldots+\theta_{i-1,m}}(l_{i,m+1}-l_{m+1,m+1})^{1/2}$$
$$\times \left(\frac{\prod_{k=1}^{m-1}(l_{k,m-1}-l_{i,m+1})}{\prod_{k\neq i=1}^{m}(l_{k,m+1}-l_{i,m+1})} \right)^{1/2} |\mu)_{im}; \tag{22}$$

$$e_{m+1,m}|\mu) = \sum_{i=1}^{m} (1-\theta_{im})(-1)^{i-1}(-1)^{\theta_{1m}+\ldots+\theta_{i-1,m}}(l_{i,m+1}-l_{m+1,m+1})^{1/2}$$
$$\times \left(\frac{\prod_{k=1}^{m-1}(l_{k,m-1}-l_{i,m+1})}{\prod_{k\neq i=1}^{m}(l_{k,m+1}-l_{i,m+1})} \right)^{1/2} |\mu)_{-im}; \tag{23}$$

$$e_{p,p+1}|\mu) = \sum_{i=1}^{m} \theta_{ip}(-1)^{\theta_{1p}+\ldots+\theta_{i-1,p}+\theta_{i+1,p-1}+\ldots+\theta_{m,p-1}}(1-\theta_{i,p-1})$$
$$\times \prod_{k\neq i=1}^{m} \left(\frac{(l_{i,p+1}-l_{kp})(l_{i,p+1}-l_{kp}-1)}{(l_{i,p+1}-l_{k,p+1})(l_{i,p+1}-l_{k,p-1}-1)} \right)^{1/2}$$
$$\times \left(\frac{\prod_{q=m+1}^{p-1}(l_{i,p+1}-l_{q,p-1}-1)\prod_{q=m+1}^{p+1}(l_{i,p+1}-l_{q,p+1})}{\prod_{q=m+1}^{p}(l_{i,p+1}-l_{qp}-1)(l_{i,p+1}-l_{qp})} \right)^{1/2} |\mu)_{ip}$$
$$+ \sum_{s=m+1}^{p} \left(-\frac{\prod_{q=m+1}^{p-1}(l_{q,p-1}-l_{sp}+1)\prod_{q=m+1}^{p+1}(l_{q,p+1}-l_{sp})}{\prod_{q\neq s=m+1}^{p}(l_{qp}-l_{sp})(l_{qp}-l_{sp}+1)} \right)^{1/2}$$
$$\times \prod_{k=1}^{m} \left(\frac{(l_{kp}-l_{sp})(l_{kp}-l_{sp}+1)}{(l_{k,p+1}-l_{sp})(l_{k,p-1}-l_{sp}+1)} \right)^{1/2} |\mu)_{sp}, \quad m+1 \leq p \leq r-1; \tag{24}$$

$$e_{p+1,p}|\mu) = \sum_{i=1}^{m} \theta_{i,p-1}(-1)^{\theta_{1p}+\ldots+\theta_{i-1,p}+\theta_{i+1,p-1}+\ldots+\theta_{m,p-1}}(1-\theta_{ip})$$
$$\times \prod_{k\neq i=1}^{m} \left(\frac{(l_{i,p+1}-l_{kp})(l_{i,p+1}-l_{kp}-1)}{(l_{i,p+1}-l_{k,p+1})(l_{i,p+1}-l_{k,p-1}-1)} \right)^{1/2}$$
$$\times \left(\frac{\prod_{q=m+1}^{p-1}(l_{i,p+1}-l_{q,p-1}-1)\prod_{q=m+1}^{p+1}(l_{i,p+1}-l_{q,p+1})}{\prod_{q=m+1}^{p}(l_{i,p+1}-l_{qp}-1)(l_{i,p+1}-l_{qp})} \right)^{1/2} |\mu)_{-ip}$$
$$+ \sum_{s=m+1}^{p} \left(-\frac{\prod_{q=m+1}^{p-1}(l_{q,p-1}-l_{sp})\prod_{q=m+1}^{p+1}(l_{q,p+1}-l_{sp}-1)}{\prod_{q\neq s=m+1}^{p}(l_{qp}-l_{sp}-1)(l_{qp}-l_{sp})} \right)^{1/2}$$
$$\times \prod_{k=1}^{m} \left(\frac{(l_{kp}-l_{sp}-1)(l_{kp}-l_{sp})}{(l_{k,p+1}-l_{sp}-1)(l_{k,p-1}-l_{sp})} \right)^{1/2} |\mu)_{-sp}, \quad m+1 \leq p \leq r-1. \tag{25}$$

In the above expressions, $\sum_{k\neq i=1}^{m}$ or $\prod_{k\neq i=1}^{m}$ means that k takes all values from 1 to m with $k \neq i$. If a vector from the rhs of (19)-(25) does not belong to the module under consideration, then the corresponding term is zero even if the coefficient in front is undefined; if an equal number of factors in numerator and denominator are simultaneously equal to zero, they should be cancelled out.

To prove that the explicit actions (19)-(25) give a representation of $\mathfrak{gl}(m|n)$ we showed that (19)-(25) satisfy the relations (7)-(16). The irreducibility follows from the fact that for any nonzero vector $x \in V([\mu]_r)$ there exists a polynomial f of $\mathfrak{gl}(m|n)$ generators such that $fx = V([\mu]_r)$.

4. Comments. The motivation for the present work comes from some physical ideas. The constructed $\mathfrak{osp}(1|2n)$ modules in Section 2 are actually state spaces of a system of n pairs of paraboson operators $b_j^{\pm}$, $j = 1, \ldots, n$ [5]. The latter are generalizations of ordinary Bose statistics. Similarly Green [5] generalized Fermi statistics to parafermion statistics. On the other hand it is known that the Fock space corresponding to a system of m pairs of parafermions and n pairs of parabosons corresponds to an infinite-dimensional unitary representation of the Lie superalgebra $\mathfrak{osp}(2m+1|2n)$ [14]. In order to construct it explicitly the branching $\mathfrak{osp}(2m+1|2n) \supset \mathfrak{gl}(m|n)$ should be known as well as a basis description for the covariant tensor $\mathfrak{gl}(m|n)$ representations. In such a way the results of Section 3 are a first step for the explicit construction of the parastatistics Fock space.

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