ALGEBRA, GEOMETRY AND MATHEMATICAL PHYSICS
BANACH CENTER PUBLICATIONS, VOLUME 93
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2011

NONLINEAR EVOLUTION EQUATIONS WITH EXPONENTIAL NONLINEARITIES: CONDITIONAL SYMMETRIES AND EXACT SOLUTIONS

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Abstract. New Q-conditional symmetries for a class of reaction-diffusion-convection equations with exponential diffusivities are derived. It is shown that the known results for reaction-diffusion equations with exponential diffusivities follow as particular cases from those obtained here but not vice versa. The symmetries obtained are applied to construct exact solutions of the relevant nonlinear equations. An application of exact solutions to solving a boundary-value problem with constant Dirichlet conditions is presented.

1. Introduction. In this paper we deal with equations of the form

$$u_t = (e^{nu}u_x)_x + \lambda e^{mu}u_x + C(u),\ \lambda \neq 0, \tag{1}$$

which form a special subclass of the general reaction-diffusion-convection (RDC) equation. Hereafter $u = u(t,x)$ is the unknown function, $C(u)$ is a given smooth function, n and m are arbitrary constants with the restriction $m \neq 0$ (otherwise the convective term is removable) while the subscripts t and x denote differentiation with respect to these variables. The main purpose of the paper is to investigate nonlinear equations of the

2010 *Mathematics Subject Classification*: Primary 35Q92; Secondary 22E70.
Key words and phrases: conditional symmetry, reaction-diffusion-convection equation, exact solution.
The paper is in final form and no version of it will be published elsewhere.

form (1), which involve three transport mechanisms: diffusion, convection and reaction (dissipation). Thus, we consider the cases when these equations describe such complicated processes that all the transport mechanisms must be taken into account, for example, in population dynamics (see examples in [16, 17]). On the other hand, one may consider this paper as a continuation of our work [7] (see its extended version in three preprints at ArXiv.org), where the RDC equations of the form

$$u_t = (u^n u_x)_x + \lambda u^m u_x + C(u) \tag{2}$$

were examined.

At the present time, all Lie symmetries of the general RDC are completely described [8, 9, 10] and the relevant Lie solutions are constructed for many equations which arise in applications (see, for example, [10] and the papers cited therein). The time is therefore ripe for a complete description of non-Lie symmetries. However, this seems to be an extremely difficult task because, firstly, several definitions of non-Lie symmetries have been introduced (nonclassical symmetry [3], conditional symmetry [12, Section 5.7], generalized conditional symmetry [14, 15] etc.), secondly, a complete description of non-Lie symmetries needs solving the corresponding systems of determining equations, which are *nonlinear* and can be fully solved only in exceptional cases.

The most common and widely used symmetry among non-Lie symmetries is nonclassical, which we will call Q-conditional symmetry following [12, Section 5.7]. It is well known that the notion of Q-conditional symmetry plays an important role in investigation of the nonlinear RDC equations since, having such symmetries in the explicit form, one may construct new exact solutions, which are not obtainable by the classical Lie machinery. A number of papers have been devoted to this topic during the last 20 years (see, e.g., [21, 13, 12, 18, 1, 11, 2, 19, 20]). However, a few of them only were devoted to the equations involving three transport mechanisms mentioned above [8, 5, 6, 7].

The paper is organized as follows. In the second section, we present the theorem giving a complete description of Q-conditional symmetries of the form

$$Q = \partial_t + \xi(u)\partial_x + \eta(u)\partial_u, \tag{3}$$

where $\xi(u)$ and $\eta(u)$ are yet-to-be determined functions, for the class of the nonlinear RDC equations (1). In the third section, we apply the Q-conditional symmetries obtained to construct new exact solutions of the relevant nonlinear RDC equations. It should be stressed that exact solutions in explicit form have been constructed for all the operators obtained. An application of the exact solution obtained to solve a boundary-value problem with the constant Dirichlet conditions is presented and its biological interpretation is discussed. The main results of the paper are summarized in the last section.

2. Q-conditional symmetry operators of the class of equations (1). Here we present the main result of the paper. Note that we search for Q-conditional symmetry operators which cannot be reduced to Lie symmetry operators described completely in [9].

THEOREM 2.1. *A RDC equation from the class (1) is Q-conditional invariant under operator (3) if and only if it and the relevant operator (up to its equivalent representations)* *have the following forms:*

$$(i) \quad u_t = (e^u u_x)_x + \lambda e^u u_x + \lambda_0 + \lambda_1 e^u + \lambda_2 e^{-u} \tag{4}$$

$$Q_{1,2} = \partial_t + \Big(\frac{\lambda_0 \pm \sqrt{D}}{2} + \lambda_2 e^{-u}\Big)\partial_u, \; D = \lambda_0^2 - 4\lambda_1\lambda_2; \tag{5}$$

$$Q_{3,4} = \partial_t + \frac{-\lambda \pm \sqrt{P}}{2} e^u \partial_x + \left(\lambda_0 + \lambda_1 e^u + \lambda_2 e^{-u}\right)\partial_u, \; P = \lambda^2 - 4\lambda_1; \tag{6}$$

$$(ii) \quad u_t = (e^u u_x)_x + \lambda e^{2u} u_x + \frac{1}{9}\lambda^2 e^{3u} + \lambda_0 + \lambda_1 e^u + \lambda_2 e^{-u}, \tag{7}$$

$$Q_a = \partial_t + a e^u \partial_x + \left(-\frac{\lambda a}{3} e^{2u} - a^2 e^u + \Big(\frac{\lambda\lambda_2}{3a} + \lambda_0\Big) + \lambda_2 e^{-u}\right)\partial_u, \; a \neq 0, \tag{8}$$

where the parameter a is a root of the algebraic equation

$$9a^4 + 9\lambda_1 a^2 + 3\lambda\lambda_0 a + \lambda^2\lambda_2 = 0. \tag{9}$$

Proof. The proof is based on the known algorithm for finding Q-conditional symmetry operators (see, e.g., [12, 8, 7]). Firstly, we apply simple local substitutions to simplify equation (1). In the case $n = 0$, the substitution

$$u = \frac{1}{m} v \tag{10}$$

reduces this equation to the form

$$v_{xx} = v_t - \lambda e^v v_x + F(v), \tag{11}$$

where $F(v) = -mC(\frac{1}{m}u)$ while in the case $n \neq 0$, the substitution

$$u = \frac{1}{n}\ln v \tag{12}$$

reduces one to the form

$$v_{xx} = v^{-1} v_t - \lambda v^q v_x + F(v), \tag{13}$$

where $q = \frac{m}{n} - 1 \neq -1$, $F(v) = -nC(u)$. Obviously, both substitutions do not change the structure of operator (3) therefore it takes the form

$$Q = \partial_t + \xi(v)\partial_x + \eta(v)\partial_v. \tag{14}$$

The system of determining equations to find Q-conditional symmetry operators of the most general form

$$Q = \partial_t + \xi(t,x,v)\partial_x + \eta(t,x,v)\partial_v \tag{15}$$

for the general RDC equation

$$V_{xx} = F_0(V)V_t + F_1(V)V_x + F_2(V),$$

$F_i(V), i = 1,2,3$ being arbitrary functions, has been obtained in [8] (see p. 535).

In the case of equations (11) and (13), this system takes the forms

$$\begin{aligned}
&\xi_{vv} = 0,\\
&\eta_{vv} = -2\xi_v(\lambda e^v + \xi) + 2\xi_{xv},\\
&3\xi_v F - \lambda(\eta + \xi_x)e^v + \xi_x\eta - \xi_t - 2\xi\xi_x + 3\xi_v\eta - 2\eta_{xv} + \xi_{xx} = 0,\\
&\eta F_v + (2\xi_x - \eta_v)F - \lambda\eta_x e^v - \eta_{xx} + \eta_t + 2\xi_x\eta = 0,
\end{aligned} \tag{16}$$

and

$$\begin{aligned}&\xi_{vv} = 0,\\ &\eta_{vv} = -2\xi_v(\lambda v^q + \xi v^{-1}) + 2\xi_{xv},\\ &3\xi_v F - \lambda\xi_x v^q - \lambda q\eta v^{q-1} - (\xi_t + 2\xi\xi_x - 2\xi_v\eta)v^{-1} + \xi\eta v^{-2} - 2\eta_{xv} + \xi_{xx} = 0,\\ &\eta F_v + (2\xi_x - \eta_v)F - \lambda\eta_x v^q - \eta^2 v^{-2} + (\eta_t + 2\xi_x\eta)v^{-1} - \eta_{xx} = 0,\end{aligned} \tag{17}$$

respectively. It can be noted that systems (16) and (17) are nonlinear and their general solutions cannot be derived in a simple way. However, they are integrable in the case when operator (15) reduces to the form (14), i.e., $\xi = \xi(v),\ \eta = \eta(v)$.

Consider system (16). The first and second equations of this system don't contain the function F therefore they can be easily integrated:

$$\xi = av + c_1, \tag{18}$$

$$\eta = -2\lambda a e^v - \frac{1}{3}av^3 - ac_1v^2 + c_2v + c_3, \tag{19}$$

where $c_i,\ i = 1,2,3$ and a are arbitrary constants. Now one easily checks that the 3-rd and 4-th equations of system (16) are incompatible if $a \neq 0$. So, we should set $a = 0$. Substituting (18) and (19) with $a = 0$ into the 3-rd equation of (16), we obtain $\lambda(c_2v+c_3)e^v = 0$ so that $c_2 = c_3 = 0$ (we recall the restriction $\lambda \neq 0$). It means that operator (14) takes the form

$$Q = \partial_t + c_1\partial_x, \tag{20}$$

but it is nothing other than a Lie symmetry operator for arbitrary RDC of the form (11). Thus, system (16) does not lead to any Q-conditional symmetry operators (14).

In the case of system (17), the result is different. The first equation of system (17) leads again to (18), while the second one produces the formula

$$\eta = 2\lambda a\ln(v) - 2ac_1v(\ln(v) - 1) - a^2v^2 + c_2v + c_3$$

if $q = -2$ and the formula

$$\eta = -\frac{2\lambda a}{(q+1)(q+2)}v^{q+2} - 2ac_1v(\ln(v) - 1) - a^2v^2 + c_2v + c_3$$

if $q \neq -2$. Thus, we need to examine separately two subcases: $a = 0$ and $a \neq 0$.

In the subcase $a = 0$, the general solutions of two equations from (17) have the form (18) and (19) with $a = 0$. Substituting these expressions into the third equations of (17), we obtain the algebraic condition $(c_2v + c_3)(\lambda qv^{q-1} - c_1v^{-2}) = 0$. If $c_2 = c_3 = 0$ then we immediately arrive at the Lie symmetry operator (20). If $q = c_1 = 0$ then we obtain the equation

$$v_{xx} = v^{-1}v_t - \lambda v_x + (c_2v + c_3)(\mu - v^{-1}), \tag{21}$$

which is Q-conditionally invariant under the operator

$$Q = \partial_t + (c_2v + c_3)\partial_v. \tag{22}$$

Applying substitution (12) and the notations $\lambda_0 = c_2 - \mu c_3,\ \lambda_1 = -\mu c_2,\ \lambda_2 = c_3$, we arrive at the RDC equation (4) and the corresponding Q-conditional symmetry operator (5).

Let us consider the subcase $a \neq 0$. Then the 3-rd equation of system (17) is equivalent to

$$F = \frac{\eta}{3a}(\lambda q v^{q-1} - 3av^{-1} - c_1 v^{-2}). \tag{23}$$

Substituting (23) into the last equation of (17), we arrive at the expression

$$\eta^2(\lambda q(q-1)v^{q-2} + 2c_1 v^{-3}) = 0.$$

Since $\eta \neq 0$ (otherwise $\lambda a = 0$) we obtain $c_1 = 0,\ q(q-1) = 0, q \neq -1$. If $c_1 = q = 0$ then the equation

$$v_{xx} = v^{-1}v_t - \lambda v_x + a(a+\lambda)v - c_2 - c_3 v^{-1} \tag{24}$$

and the corresponding Q-conditional symmetry operator

$$Q = \partial_t + av\partial_x - (a(a+\lambda)v^2 - c_2 v - c_3)\partial_v \tag{25}$$

are obtained. If $c_1 = 0,\ q = 1$ then we find the equation

$$v_{xx} = v^{-1}v_t - \lambda v v_x - \left(\frac{\lambda}{3a} - v^{-1}\right)\left(\frac{1}{3}\lambda a v^3 + a^2 v^2 - c_2 v - c_3\right) \tag{26}$$

and the operator

$$Q = \partial_t + av\partial_x - \left(\frac{1}{3}\lambda a v^3 + a^2 v^2 - c_2 v - c_3\right)\partial_v. \tag{27}$$

Finally, taking into account substitution (12) and using the notations $\lambda_0 = c_2,\ \lambda_1 = -a(a+\lambda),\ \lambda_2 = c_3$, equation (24) and operator (25) are reduced to those of the form (4) and (6), respectively. Dealing in the same way with equation (26) and operator (27) and using the notations $\lambda_0 = -\frac{\lambda}{3a}c_3 + c_2,\ \lambda_1 = -\frac{\lambda}{3a}c_2 - a^2,\ \lambda_2 = c_3$, equation (7), operator (8) and condition (9) are obtained.

The proof is now completed. ■

3. Exact solutions of RDC equations with exponential nonlinearities. Here we construct exact solutions of equations (4) and (7) using the Q-conditional symmetry operators found above and show that they are non-Lie solutions, i.e. cannot be obtained using Lie symmetry operators. As it follows from the proof presented in section 2, the Q-conditional symmetry operators have essentially simpler structure if one uses substitution (12). So we will map equations (4) and (7) and the corresponding operators to the simpler forms using (12) and construct exact solutions for the equations obtained.

Substitution (12) reduces equation (4) and operator (5) to the forms

$$v_{xx} = v^{-1}v_t - \lambda v_x - \lambda_0 - \lambda_1 v - \lambda_2 v^{-1} \tag{28}$$

and

$$Q_{1,2} = \partial_t + \left(\frac{\lambda_0 \pm \sqrt{D}}{2}v + \lambda_2\right)\partial_v, \tag{29}$$

respectively. To construct the corresponding solutions one needs to solve the overdetermined system consisting of (28) and

$$Q_{1,2}(v) \equiv v_t - \frac{\lambda_0 \pm \sqrt{D}}{2}v - \lambda_2 = 0, \tag{30}$$

which is compatible because $Q_{1,2}$ are the operators of Q-conditional symmetries. So, extracting v_t from (30) and substituting into (28), we arrive at the linear ODE (with respect to the variable x)

$$v_{xx} + \lambda v_x + \lambda_1 v = \frac{-\lambda_0 \pm \sqrt{D}}{2}, \tag{31}$$

which possesses the general solution

$$v(t,x) = \begin{cases} \alpha_1(t)e^{-\lambda x} - \frac{\lambda_0 \mp |\lambda_0|}{2\lambda}x + \alpha_2(t), \ \lambda_1 = 0, \\ e^{-\frac{\lambda}{2}x}\Big(\alpha_1(t) + x\alpha_2(t)\Big) - \frac{2(\lambda_0 \mp \sqrt{D})}{\lambda^2}, \ \lambda_1 \neq 0, \ P = \lambda^2 - 4\lambda_1 = 0, \\ e^{-\frac{\lambda}{2}x}\Big(\alpha_1(t)e^{\frac{\sqrt{P}}{2}x} + \alpha_2(t)e^{-\frac{\sqrt{P}}{2}x}\Big) - \frac{\lambda_0 \mp \sqrt{D}}{\lambda_1}, \ \lambda_1 \neq 0, \ P > 0, \\ e^{-\frac{\lambda}{2}x}\Big(\alpha_1(t)\cos(\frac{\sqrt{-P}}{2}x) + \alpha_2(t)\sin(-\frac{\sqrt{P}}{2}x)\Big) - \frac{\lambda_0 \mp \sqrt{D}}{\lambda_1}, \ \lambda_1 \neq 0, \ P < 0, \end{cases}$$

where $\alpha_i(t), i = 1,2$ are arbitrary smooth functions at the moment. Substituting the expressions obtained above into (30), we obtain four systems of first-order ODEs to find the functions $\alpha_i(t), i = 1,2$ depending on λ_1 and P. These systems are integrable therefore their solutions $v(t,x)$ have been found in explicit form. Applying substitution (12) to them the following exact solutions of equation (4) have been constructed.

If $\lambda_1 = 0$ then the exact solutions are

$$u = \ln\left(e^{\lambda_0 t}(C_1 e^{-\lambda x} + C_2) - \frac{\lambda_2}{\lambda_0}\right), \ \lambda_0 \neq 0.$$

$$u = \ln\left(C_1 e^{-\lambda x} - \frac{\lambda_0}{\lambda}x + \lambda_2 t + C_2\right);$$

if $\lambda_1 \neq 0$ then the exact solutions are

$$u = \ln\left(b(x)\exp\Big(\frac{\lambda_0 \pm \sqrt{D}}{2}t\Big) + \frac{-\lambda_0 \pm \sqrt{D}}{2\lambda_1}\right), \tag{32}$$

where

$$b(x) = \begin{cases} e^{-\frac{\lambda}{2}x}\left(C_1 + C_2 x\right), \ P = 0, \\ e^{-\frac{\lambda}{2}x}\left(C_1\exp\Big(\frac{\sqrt{P}}{2}x\Big) + C_2\exp\Big(-\frac{\sqrt{P}}{2}x\Big)\right), \ P > 0, \\ e^{-\frac{\lambda}{2}x}\left(C_1\cos\Big(\frac{\sqrt{-P}}{2}x\Big) + C_2\sin\Big(\frac{\sqrt{-P}}{2}x\Big)\right), \ P < 0. \end{cases} \tag{33}$$

In a quite similar way exact solutions of equation (4) were constructed using operator (6). Finally, the exact solutions

$$u = \ln\left(\frac{\lambda_2 t(\frac{t}{2} + C_1) - \frac{x}{\lambda}}{t + C_1}\right), \ \lambda_0 = 0$$

and

$$u = \ln\left(\frac{\lambda_0 x - C_1\lambda_2 e^{-\lambda_0 t} - \lambda\lambda_2 t}{C_1\lambda_0 e^{-\lambda_0 t} - \lambda}\right), \ \lambda_0 \neq 0,$$

were found if $\lambda_1 = 0$. If $\lambda_1 \neq 0$ then the corresponding exact solutions are

$$u = \ln\left(\frac{C_2}{t + C_1}\exp\Big(\frac{1}{2}(\lambda_0 t - (\lambda \pm \sqrt{P})x)\Big) - \frac{1}{\lambda_1(t + C_1)} - \frac{\lambda_0}{2\lambda_1}\right), \ D = 0;$$

$$u = \ln\left(\frac{C_1 \exp\Big(\frac{1}{2}((\lambda_0 \pm \sqrt{D})t - (\lambda \pm \sqrt{P})x)\Big)}{1 - \exp(\pm\sqrt{D}t)} \pm \frac{\sqrt{D}(1+\exp(\pm\sqrt{D}t))}{2\lambda_1(1-\exp(\pm\sqrt{D}t))} - \frac{\lambda_0}{2\lambda_1}\right), \ D>0;$$

$$u = \ln\left(C_1 \sec(\frac{\sqrt{-D}}{2}t) \exp\Big(\frac{1}{2}(\lambda_0 t - (\lambda \pm \sqrt{P})x)\Big) + \frac{\sqrt{-D}}{2\lambda_1}\tan(\frac{\sqrt{-D}}{2}t) - \frac{\lambda_0}{2\lambda_1}\right), \ D<0.$$

The most cumbersome structure of the conditional symmetry operator occurs in case (ii) of Theorem 2.1. As a consequence, essential difficulties arise if one applies operator (8) to find exact solutions of equation (7). We again use substitution (12) to simplify calculations. The corresponding overdetermined system takes the form

$$v_{xx} = v^{-1}v_t - \lambda v v_x - \frac{1}{9}\lambda^2 v^3 - \lambda_0 - \lambda_1 v - \lambda_2 v^{-1}, \tag{34}$$

$$Q_a(v) \equiv v_t + a v v_x + \frac{\lambda}{3} a v^3 + a^2 v^2 - \left(\frac{\lambda\lambda_2}{3a} + \lambda_0\right) v - \lambda_2 = 0. \tag{35}$$

Extracting v_t from (35) and substituting into (34), we arrive at a nonlinear ODE (with respect to the variable x), which reduces to the form

$$v^*_{yy} + 3v^* v^*_y + (v^*)^3 + 3pv^* + 2q = 0 \tag{36}$$

by the substitution $v = v^* - \frac{a}{\lambda}$, $y = \frac{\lambda}{3}x$ (hereafter $p = \frac{1}{\lambda^2}(2a^2 + 3\lambda_1)$, $q = -\frac{1}{2\lambda^3 a}(7a^4 + 9\lambda_1 a^2 + 3\lambda_2\lambda^2)$).

Following [6], we use now the known substitution $v^* = \frac{w_y}{w}$ to linearize equation (36):

$$w_{yyy} + 3pw_y + 2qw = 0. \tag{37}$$

Equation (37) is a linear third order ODE, hence its general solution can be easily constructed. Four different cases occur depending on p and q. Corresponding calculations are rather cumbersome but very similar to those presented in [7] (see P.10063-7). Here we present only the final results.

Case 1: $p = q = 0$. The exact solutions of equation (7) are

$$u = \ln\left(\frac{3}{a^2 t + \lambda x + C_1} - \frac{a}{\lambda}\right) \tag{38}$$

and

$$u = \ln\left(\frac{6(a^2 t + \lambda x + C_1)}{(a^2 t + \lambda x + C_1)^2 + 6a\lambda t + C_2} - \frac{a}{\lambda}\right),$$

where the coefficient restrictions

$$\lambda_0 = -\frac{8a^3}{9\lambda}, \ \lambda_1 = -\frac{2a^2}{3}, \ \lambda_2 = -\frac{a^4}{3\lambda^2}$$

are assumed.

Case 2, $p^3 = -q^2 \neq 0$, leads to the exact solutions

$$u = \ln\left(\frac{\alpha(-2C_1 e^{\beta t - \alpha x} + 1)}{\lambda(C_1 e^{\beta t - \alpha x} + 1)} - \frac{a}{\lambda}\right), \tag{39}$$

$$u = \ln\left(\frac{-2\alpha C_1 e^{\beta t - \alpha x} + \alpha(\gamma t + C_2 + \lambda x) + 3\lambda}{\lambda(C_1 e^{\beta t - \alpha x} + \gamma t + C_2 + \lambda x)} - \frac{a}{\lambda}\right),$$

where $\alpha = \pm\sqrt{-2a^2 - 3\lambda_1}$, $\beta = \frac{a\alpha}{\lambda}(\alpha \mp a)$, $\gamma = a(a \pm 2\alpha)$.

In this case the coefficient of equation (7) must satisfy the restrictions: $2a^2+3\lambda_1<0$, $\lambda_2=-\frac{3}{\lambda^2}(3a^4+3\lambda_1a^2+\lambda_0\lambda a)$ and $\Delta\equiv 4(2a^2+3\lambda_1)^3+(20a^3+9\lambda\lambda_0+18\lambda_1a)^2=0$.

Case 3: $p^3+q^2<0$. The corresponding exact solution involves three different exponents and has the form:

$$u=\ln\left(\frac{\alpha_1C_1e^{\beta_1t+\gamma_1x}+\alpha_2C_2e^{\beta_2t+\gamma_2x}+\alpha_3e^{\beta_3t+\gamma_3x}}{C_1e^{\beta_1t+\gamma_1x}+C_2e^{\beta_2t+\gamma_2x}+e^{\beta_3t+\gamma_3x}}-\frac{a}{\lambda}\right),$$

where $\beta_i=\frac{a\alpha_i}{3}(a+\lambda(\alpha_1+\alpha_3))$, $\gamma_i=\frac{\alpha_i\lambda}{3}$, and the parameters $\alpha_i, i=1,2,3$ are calculated by the known Cardano formulae:

$$\alpha_1=-2\sqrt{-p}\cos\left(\tfrac{1}{3}\arctan\left(\frac{\sqrt{-p^3-q^2}}{q}\right)\right),$$
$$\alpha_{2,3}=2\sqrt{-p}\cos\left(\tfrac{1}{3}\arctan\left(\frac{\sqrt{-p^3-q^2}}{q}\right)\pm\tfrac{\pi}{3}\right),$$

if $q>0$;

$$\alpha_1=2\sqrt{-p}\cos\left(\tfrac{1}{3}\arctan\left(\frac{\sqrt{-p^3-q^2}}{q}\right)\right),$$
$$\alpha_{2,3}=-2\sqrt{-p}\cos\left(\tfrac{1}{3}\arctan\left(\frac{\sqrt{-p^3-q^2}}{q}\right)\pm\tfrac{\pi}{3}\right),$$

if $q<0$, and

$$\alpha_1=0,$$
$$\alpha_{2,3}=\pm\sqrt{-3p}$$

if $q=0$.

In this case the coefficient of equation (7) must satisfy the restrictions: $\lambda_2=-\frac{3}{\lambda^2}(3a^4+3\lambda_1a^2+\lambda_0\lambda a)$ and $\Delta<0$.

Finally, *Case 4*, $p^3+q^2>0$, leads to the exact solutions

$$u=\ln\left(\frac{C_1ae^{-\lambda_0t+\frac{a}{2}x}-a\cos(\alpha x)-6\alpha\sin(\alpha x)}{\lambda(C_1e^{-\lambda_0t+\frac{a}{2}x}+2\cos(\alpha x))}-\frac{a}{\lambda}\right),$$

where $\alpha=\pm\frac{\sqrt{3a^2+4\lambda_1}}{2}$, $3a^2+4\lambda_1>0$, $\lambda_0=-\frac{3a(a^2+\lambda_1)}{\lambda}$, and

$$u=\ln\left(\frac{-2\alpha C_1e^{\gamma t-\alpha\lambda x}+\alpha\sin(\lambda\beta x+\delta t)+3\beta\cos(\lambda\beta x+\delta t)}{C_1e^{\gamma t-\alpha\lambda x}+\sin(\lambda\beta x+\delta t)}-\frac{a}{\lambda}\right),$$

where $\alpha=-\frac{1}{2}\left(\sqrt[3]{-q+\sqrt{p^3+q^2}}-\sqrt[3]{q+\sqrt{p^3+q^2}}\right)$, $\gamma=a(\lambda(\alpha^2+3\beta^2)-\alpha a)$, $\beta=\frac{1}{2\sqrt{3}}(\sqrt[3]{-q+\sqrt{p^3+q^2}}+\sqrt[3]{q+\sqrt{p^3+q^2}})$, $\delta=a\beta(a+2\alpha\lambda)$.

The restrictions on the coefficients are following: $2a^2+3\lambda_1<0$, $\lambda_2=-\frac{3}{\lambda^2}(3a^4+3\lambda_1a^2+\lambda_0\lambda a)$ and $\Delta>0$.

It should be noted that the solutions found are not obtainable by using Lie symmetries, excepting the cases when they are plane wave solutions (see (38) and (39)). We have also checked that solution (32)–(33) with $\lambda=0$ produces the solutions obtained in [2] (see formulae (3.11a), (3.11b), (3.11c) therein) for the reaction-diffusion equation, which is a particular case of (4). Finally, we show how to apply the exact solution (32) with $P>0$ to solve a boundary-value problem.

EXAMPLE 3.1. Let us consider the following reaction-diffusion equation with exponential nonlinearities:

$$u_t = (e^{mu}u_x)_x + \lambda e^{mu}u_x + \lambda_0 + \lambda_1 e^{mu} + \lambda_2 e^{-mu}, \quad m \neq 0 \tag{40}$$

This equation can be applied to describe processes in population dynamics, when the diffusion and convection coefficients and the reaction term exponentially depend on the population density. One notes that this equation reduces to the form

$$u_t = (e^u u_x)_x + \lambda e^u u_x + \lambda_0 + \lambda_1 e^u + \lambda_2 e^{-u} \tag{41}$$

by renaming $mu \to u, \lambda_i \to \lambda_i/m, i = 0, 1, 2$. In the general case, equation (41) possesses two steady-state points. If one of them is $u_0 = 0$ then the second point can be put $u_1 = 1$ without losing generality. This assumption leads to the conditions $\lambda_0 + \lambda_1 + \lambda_2 = 0$ and $\lambda_2 = e\lambda_1$. Hence we consider the nonlinear RDC equation

$$u_t = (e^u u_x)_x + \lambda e^u u_x + \lambda_1(e^u + e^{1-u} - (1+e)), \quad \lambda_1 \neq 0, \tag{42}$$

possessing the steady-state points $u_0 = 0, u_1 = 1$. Now one notes that solutions (32) with $P > 0$ and the coefficients arising in equation (42) take the forms

$$u = \ln\left(\left(C_1 e^{\frac{-\lambda+\sqrt{P}}{2}x} + C_2 e^{\frac{-\lambda-\sqrt{P}}{2}x}\right)e^{-\lambda_1 t} + e\right), \tag{43}$$

and

$$u = \ln\left(\left(C_1 e^{\frac{-\lambda+\sqrt{P}}{2}x} + C_2 e^{\frac{-\lambda-\sqrt{P}}{2}x}\right)e^{-\lambda_1 e t} + 1\right). \tag{44}$$

Thus, the exact solution of the boundary-value problem for the nonlinear equation (42) with the zero Dirichlet conditions

$$u(t, 0) = 0, \quad u(t, +\infty) = 0, \tag{45}$$

and the initial condition

$$u(0, x) = \ln\left(C_1 e^{\frac{-\lambda+\sqrt{P}}{2}x} - C_1 e^{\frac{-\lambda-\sqrt{P}}{2}x} + 1\right), \tag{46}$$

is given in the domain $(t, x) \in [0, +\infty) \times [0, +\infty)$ by the formula (43), where $C_2 = -C_1$ and $P = \lambda^2 - 4\lambda_1 > 0$.

The similar boundary-value problem, however, with the non-zero Dirichlet conditions

$$u(t, 0) = 1, \quad u(t, +\infty) = 1 \tag{47}$$

and the initial condition

$$u(0, x) = \ln\left(C_1 e^{\frac{-\lambda+\sqrt{P}}{2}x} - C_1 e^{\frac{-\lambda-\sqrt{P}}{2}x} + e\right), \tag{48}$$

possesses the exact solution (44) with the coefficient restrictions listed above. Moreover, both solutions are bounded and non-negative in the given domain if $C_1 > 0$, $\lambda_1 > 0$.

REMARK. Solutions (43)–(44) are not valid for $\lambda = 0$. The similar problem for equation (42) with $\lambda = 0$ has been solved in [4].

We remind the reader that (42) possesses two steady-state points $u_0 = 0$ and $u_1 = 1$ what is common for many equations arising in population dynamics, including the famous Fisher equation. Solutions (43)–(44) give the space-time distribution of population for the situation when the population density on the boundary is equal to the steady-state point. In the case $u_0 = 0$, we note that the solution is vanishing if $t \to +\infty$, therefore the population dies. In the case $u_1 = 1$, the solution tends to 1 and this predicts an optimistic scenario for the population.

4. Conclusions. In this paper, Theorem 2.1 giving the full description of the Q-conditional symmetry operators of the form (3) for the class of equations (1) is proved. It should be stressed that all the Q-conditional symmetry operators listed in Theorems 2.1 contain the same type of nonlinearities with respect to the dependent variable u as the relevant RDC equations. Analogous results were earlier obtained for single reaction-diffusion equations in [2]. We have checked that many Q-conditional (i.e. non-classical) symmetries obtained in [2] can be derived from those presented in Theorem 2.1 (note some symmetries, for instance cases 3 and 6 from table 1 [2], which are treated as new non-classical symmetries, are equivalent to the Lie symmetries). On the other hand, we point out that there is an essential difference between the class of equations (1) and the relevant reaction-diffusion equation investigated in [2]: there are two different RDC equations, (4) and (7), admitting different Q-conditional symmetries, while one reaction-diffusion equation only possesses this kind of symmetries (in fact, equations (4) and (7) with $\lambda = 0$ are identical).

All the Q-conditional symmetries presented in Theorem 2.1 have successfully been applied to construct new exact solutions in explicit form. However, we noted that some of them have been obtained earlier in [4] using the method of additional generating conditions. This is another confirmation of the known hypothesis that any exact solution can be obtained by the relevant Lie or conditional symmetry operator.

The solutions obtained above can be used to solve the relevant boundary-value problems. In the particular case, solution (32) was used to solve the boundary-value problem with the constant Dirichlet conditions for the nonlinear RDC equation (40) and a possible biological interpretation was presented.

The work is in progress to solve the overdetermined systems (16) and (17) in the general case. It should be stressed that this is a highly non-trivial problem because there are no general methods for solving such systems.

Acknowledgments. The authors are grateful to the unknown referee for the useful comments. R. Cherniha thanks the Organizing Committee of the 5th Baltic-Nordic Workshop 'Algebra, Geometry, and Mathematical Physics' for the partial financial support.

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Received February 1, 2010; Revised January 31, 2011