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SOME SPECIAL FUNCTIONS RELATED TO UNIMODULAR PSEUDO-ORTHOGONAL GROUPS

ALEXANDER NIZHNIKOV
Moscow Pedagogical State University
M. Pirogovskaya 1, 119991 Moscow, Russia

ILYA SHILIN
Sholokhov Moscow State University for the Humanities
V. Radishevskaya 16–18, 109240 Moscow, Russia, and
Moscow Aviation Institute (University of Aerospace Technology)
Volokolamskoe shosse 4, 125993 Moscow, Russia
E-mail: ilyashilin@li.ru

Abstract. We obtain some matrix elements of basis transformations in a representation space of the unimodular pseudo-orthogonal group. Using these elements, we derive some formulas for special functions.

1. Introduction. Throughout this paper, special functions occur as matrix elements of basis transformations and representation operators. We construct some elements of the matrices which connect different bases for class 1 representations of the unimodular pseudo-orthogonal group. These matrix elements are expressed in terms of Whittaker functions for the case $SO(2,2)$ and in terms of Vilenkin function for the general case. In this way, some integral relations are obtained for Gauss, Whittaker and Macdonald functions.

Let us assume that the linear space $\mathbb{R}^{p+q}$ is endowed with the form

$$\vartheta(x) := x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2.$$

We denote by X the cone $\vartheta(x) = 0$ without the origin. By definition, the unimodular pseudo-orthogonal group $SO(p,q)$ consists of all linear transformations of $\mathbb{R}^{p+q}$ preserving $\vartheta(x)$. In the case $p = 1$ or $q = 1$, we have the special Lorentz group. If $g \in SO(p,q)$, then

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$\det g = 1$ and the equation

$$ge_{p,q}g^{\mathrm{T}} = e_{p,q} \tag{1}$$

holds, where $e_{p,q} := \operatorname{diag}(1, \ldots, 1 - 1, \ldots, -1)$ and tr $e_{p,q} = p - q$. The group $SO(p, q)$ has 2 connected components. One of them consists of the matrices

$$g \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{2}$$

where A is a $p \times p$ matrix such that sign det A = sign det $D = 1$. This coset contains the identity element and will be under our consideration further. We denote this subgroup by G. The fixed points g under the Cartan involution $G \to G$, $g \mapsto e_{p,q}ge_{p,q}$ form a maximal compact subgroup $K \backsimeq SO(p) \times SO(q)$.

Let us consider the K-orbit of the point $(1, 0, \ldots, 0, 1)$ on X. It is a direct product of two spheres; we denote it as Γ_K. The measure $dx := \prod_{i=1}^{p+q} dx_i$ in $\mathbb{R}^{p+q}$ is invariant with respect to $SL(p + q, \mathbb{R})$, the generalized function

$$\delta\left(\sqrt{x_1^2 + \ldots + x_p^2} - 1\right) \cdot \delta\left(\sqrt{x_{p+1}^2 + \ldots + x_{p+q}^2} - 1\right)$$

is invariant with respect to K, and the polynomials $\sum_{i=1}^{p} x_i^2$ and $\sum_{i=p+1}^{p+q} x_i^2$ are both symmetric. This leads to the following K-invariant measure on Γ_K:

$$(\mathrm{d}x)_K := \frac{\mathrm{d}_{\theta(1)} \ldots \mathrm{d}_{\theta(p-1)} \mathrm{d}_{\vartheta(p+1)} \ldots \mathrm{d}_{\vartheta(p+q-1)}}{|x_{\theta(p)}|\ |x_{\vartheta(p+q)}|},$$

where θ is any permutation of the set $\{1, \ldots, p\}$ and ϑ is any permutation of the set $\{p + 1, \ldots, p + q\}$. In the spherical coordinate system

$$\begin{cases} x_1 = \sin\phi_1 \ldots \sin\phi_{p-1}, \\ x_2 = \sin\phi_1 \ldots \sin\phi_{p-2}\cos\phi_{p-1}, \\ \ldots \\ x_{p-1} = \sin\phi_1 \cos\phi_2, \\ x_p = \cos\phi_1, \\ x_{p+1} = \sin\psi_1 \ldots \sin\psi_{q-1}, \\ x_{p+2} = \sin\psi_1 \ldots \sin\psi_{q-2}\cos\psi_{q-1}, \\ \ldots \\ x_{p+q-1} = \sin\psi_1 \cos\psi_2, \\ x_{p+q} = \cos\psi_1, \end{cases}$$

we obtain

$$(\mathrm{d}x)_K = \prod_{i=1}^{p-2} \sin^{p-i-1}\phi_i\, \mathrm{d}\phi_i \cdot \prod_{i=1}^{q-2} \sin^{q-i-1}\phi_i\, \mathrm{d}\psi_i \cdot \mathrm{d}\phi_{p-1}\, \mathrm{d}\psi_{q-1},$$

if $p \geq 2, q \geq 2$. Here $\phi_1, \psi_1 \in [0; 2\pi)$ and $\phi_2, \psi_2, \phi_3, \psi_3, \ldots \in [0; \pi)$. So we have the following corollary for the G-invariant measure on X:

$$\mathrm{d}x := \frac{\mathrm{d}x_{\zeta(1)} \ldots \mathrm{d}x_{\zeta(p+q-1)}}{|x_{\zeta(p+q)}|}, \tag{3}$$

where $\zeta \in \mathbf{S}_{p+q}$.

Let us introduce the binary relations Ω_1 on the set $\{1,\dots,p\}$ and Ω_2 on the set $\{p+1,\dots,p+q\}$ by the condition that $(i,j)\in\Omega_l$ is equivalent to $i<j$. Let us consider now a rotation $r(i,j,t)$ through angle t on the (x_i,x_j) plane, where the condition $(i,j)\in\Omega_1$ or $(i,j)\in\Omega_2$ holds. Let $k(i,j)$ be the infinitesimal matrix of $r(i,j,t)$. Then its matrix elements are

$$k(i,j)_{st}=\begin{cases} 1, \text{ if } (s,t)=(i,j),\\ -1, \text{ if } (s,t)=(j,i),\\ 0, \text{ if } (s,t)\neq(i,j) \text{ and } (s,t)\neq(j,i).\end{cases}$$

The system of vectors $k(i,j)$, $(i,j)\in\Omega_1\cup\Omega_2$, is linearly independent; the dimension of their linear span $\mathfrak{k}$ is equal to $\frac{p^2+q^2-p-q}{2}$. Moreover, $\mathfrak{k}$ is a Lie algebra because

$$[k(i,j),k(\tilde{i},\tilde{j})]=\begin{cases} 0, \text{ if } \delta_{i\tilde{i}}\delta_{j\tilde{j}}=1,\\ 0, \text{ if } (i-\tilde{i})(j-\tilde{i}(j-\tilde{j})\neq 0,\\ k(j,\tilde{j}), \text{ if } i=\tilde{i},\\ k(i,\tilde{i}), \text{ if } j=\tilde{j},\\ -k(i,\tilde{j}), \text{ if } j=\tilde{i}.\end{cases}$$

Let $\hat{r}(i,j,t)$ be a hyperbolic rotation through angle t on the (x_i,x_j) plane, where $(i,j)\in\{1,\dots,p\}\times\{p+1,\dots,p+q\}$. If the infinitesimal matrix $h(i,j)$ corresponds to the rotation $\hat{r}(i,j,t)$, then its matrix elements are

$$h(i,j)_{st}=\begin{cases} 1, \text{ if } (i,j)=(s,t),\\ 1, \text{ if } (i,j)=(t,s),\\ 0, \text{ if } (i,j)\neq(s,t) \text{ and } (i,j)\neq(t,s).\end{cases}$$

These infinitesimal matrices are linearly independent and generate a linear space $\mathfrak{h}$, and $\dim\mathfrak{h}=pq$.

It is not hard to prove that $\det A\neq 0$ for A in (2). From (1) and (2), we have $B=(A^{-1})^{\mathrm{T}}C^{\mathrm{T}}D$. This means that any matrix $g\in G$ depends on $\binom{p+q}{2}=\frac{(p+q)(p+q-1)}{2}$ parameters. Therefore, $\mathfrak{k}\oplus\mathfrak{h}$ is the tangent space of G. It is easy to verify that $[\mathfrak{k},\mathfrak{h}]\subset\mathfrak{h}$ and $[\mathfrak{h},\mathfrak{h}]\subset\mathfrak{k}$. We denote the group $\exp\mathfrak{h}$ by H.

It is clear that K acts transitively on Γ_K. Let us note that, first, G is generated by the subgroups K and H [M] and, second, $\hat{r}^{-1}(i,j,t)(1,0,\dots,0,1)=(e^{-t},0,\dots,0,e^{-t})$. This means that the action $x\mapsto g^{-1}x$ of the group G is transitive on X. Let $\sigma\in\mathbb{C}$ and D_σ be a linear subspace in $C^\infty(X)$ consisting of σ-homogeneous functions. We define the representation T_σ in D_σ by left shifts: $T_\sigma(g)[f(x)]:=f(g^{-1}x)$. This representation is irreducible if $\sigma\notin\mathbb{Z}$ [M], [Ve].

For any pair $(D_\sigma,D_{\tilde{\sigma}})$, we define the bilinear functional

$$\mathsf{F}:\ (D_\sigma,D_{\tilde{\sigma}})\to\mathbb{C},\ (f_1,f_2)\mapsto\frac{\Gamma\left(\frac{p+q-1}{2}\right)}{2\pi^{\frac{p+q-1}{2}}}\int_\Gamma f_1(x)f_2(x)\,\mathrm{d}x_\Gamma.$$

Here Γ is a variety of X intersecting all or almost all generatrices. The words *almost all* mean here all generatrices except one of them. Every point $x\in\Gamma$ can be represented as $\{x_i=F_i(\xi_1,\dots,\xi_{p+q-2}),\quad i=1,\dots,p+q$. So we can write every point $x\in X$ as

$$\{x_i=tF_i(\xi_1,\dots,\xi_{p+q-2}),\quad i=1,\dots,p+q, \tag{4}$$

and, consequently,

$$\mathrm{d}x = t^{p+q-3}\,\mathrm{d}t\,\mathrm{d}\xi, \tag{5}$$

where $\tilde{G}$ is a subgroup of G, which acts transitively on Γ, and $\mathrm{d}\xi$ is the $\tilde{G}$-invariant measure on Γ.

LEMMA 1.1. *If $\tilde{\sigma} = -\sigma - p - q + 2$, then F does not depend on Γ.*

Proof. It follows from homogeneity of the functions f_1 and f_2 and formulas (5) and (3). ∎

2. Bases transforms and Vilenkin function. In addition to Γ_K, let us introduce some other contours on X intersecting almost all generatrices.

We now denote as Γ_1 the intersection of the cone X and the the cylinder $x_1^2 + \ldots + x_{p-1}^2 = 1$. The subgroup $H_1 \simeq SO(p-1) \times SO(1,q)$ acts transitively on Γ_1. The H_1-invariant measure

$$(\mathrm{d}x)_{H_1} = \frac{\mathrm{d}x_1 \ldots \mathrm{d}x_{p-2}}{|x_{p-1}|} \cdot \frac{\mathrm{d}x_p \ldots \mathrm{d}x_{p+q-1}}{|x_{p+q}|}$$

on Γ_1 follows from (5). In the cylinder coordinate system

$$\begin{cases} x_1 = \sin\phi_1 \ldots \sin\phi_{p-3}\sin\phi_{p-2}, \\ x_2 = \sin\phi_1 \ldots \sin\phi_{p-3}\cos\phi_{p-2}, \\ \ldots \\ x_{p-2} = \sin\phi_1\cos\phi_2, \\ x_{p-1} = \cos\phi_1, \\ x_p = \sinh\alpha, \\ x_{p+1} = \cosh\alpha\sin\psi_1 \ldots \sin\psi_{q-1}, \\ x_{p+2} = \cosh\alpha\sin\psi_1 \ldots \cos\psi_{q-1}, \\ \ldots \\ x_{p+q} = \cosh\alpha\cos\psi_1, \end{cases}$$

we have

$$(\mathrm{d}x)_{H_1} = \prod_{i=1}^{p-3} \sin^{p-i-2}\phi_i\,\mathrm{d}\phi_i \cdot \prod_{i=1}^{q-2} \sin^{p-i-1}\psi_i\,\mathrm{d}\psi_i \cdot \sinh^{q-2}\alpha\,\mathrm{d}\phi_{p-2}\,\mathrm{d}\psi_{q-1}\,\mathrm{d}\alpha.$$

Here $\alpha \in [0; +\infty)$, $\phi_1, \psi_1 \in [0; 2\pi)$ and $\phi_2, \psi_2, \phi_3, \psi_3, \ldots \in [0; \pi)$.

Let Γ_2 be the intersection of X and the cylinder $x_1^2 + x_{p-1}^2 - x_{p+1}^2 = 1$. We denote by H_2 the subgroup acting transitively on Γ_2. Thus, $H_2 \simeq SO(p-1,1) \times SO(1,q-1)$. From (5), we have

$$(\mathrm{d}x)_{H_2} = \frac{\mathrm{d}x_1 \ldots \mathrm{d}x_{p-1}}{|x_{p+1}|} \cdot \frac{\mathrm{d}x_{p+2} \ldots \mathrm{d}x_{p+q}}{|x_p|}.$$

In the cylinder coordinate system

$$\begin{cases} x_1 = \cosh\alpha \sin\phi_1 \dots \sin\phi_{p-3} \sin\phi_{p-2}, \\ x_2 = \cosh\alpha \sin\phi_1 \dots \sin\phi_{p-3} \cos\phi_{p-2}, \\ \dots \\ x_{p-2} = \cosh\alpha \sin\phi_1 \cos\phi_2, \\ x_{p-1} = \cosh\alpha \cos\phi_1, \\ x_p = \sinh\beta, \\ x_{p+1} = \sinh\alpha, \\ x_{p+2} = \cosh\beta \sin\psi_1 \dots \cos\psi_{q-2}, \\ \dots \\ x_{p+q} = \cosh\beta \cos\psi_1, \end{cases}$$

we have

$$(\mathrm{d}x)_{H_2} = \prod_{i=1}^{p-3} \sin^{p-i-2}\phi_i \,\mathrm{d}\phi_i \cdot \prod_{i=1}^{q-3} \sin^{p-i-2}\psi_i \,\mathrm{d}\psi_i \cdot \sinh^{p-2}\alpha \,\sinh^{q-2}\beta \,\mathrm{d}\phi_{p-2}\,\mathrm{d}\psi_{q-2}\,\mathrm{d}\alpha\,\mathrm{d}\beta.$$

Here $\alpha, \beta \in [0; +\infty)$, $\phi_1, \psi_1 \in [0; 2\pi)$ and $\phi_2, \psi_2, \phi_3, \psi_3, \ldots \in [0; \pi)$.

We now define three bases in the space D_σ. These bases consist of continuations of basis functions on Γ_K, Γ_1 and Γ_2.

We denote as $\{f_V^\sigma\}$ the basis in D_σ related to reduction $G \supset K$, where

$$f_V^\sigma(x) = (x_1^2 + \ldots + x_p^2)^{\frac{\sigma - v_1 - v_{p+1}}{2}} \,\Xi^p_{V^*}(x^*)\,\Xi^q_{V^{**}}(x^{**})$$

and the function

$$\Xi^l_{(y_1,\ldots,\pm y_{l-1})}(x) = \prod_{i=1}^{l-2} (x_1^2 + \ldots + x_{l-1}^2)^{y_i - y_{i+1}} \cdot C^{\frac{l-i}{2} + y_{i+1}}_{y_i - y_{i+1}} \left(\frac{x_{l-i}}{x_1^2 + \ldots + x_{l-1}^2} \right) (x_2 + \mathbf{i}x_1)^{y_l - 2}$$

was defined in [Vi, 9.3.6.2].

In addition, let us introduce two bases $\{f_L^\sigma\}$ and $\{f_M^\sigma\}$, where

$$f_L^\sigma(x) = (x_1^2 + \ldots + x_{p-1}^2)^{\frac{\sigma - l_1 + q/2 - 1}{2}} \,\Xi^{p-1}_{L^*}(x^*)\,\Xi^{q+1}_{L^{**},\lambda}(x^{**}),$$

$$f_M^\sigma(x) = (x_1^2 + \ldots + x_{p-1}^2 - x_{p+1}^2)^{\frac{\sigma + p + q}{2} - 3} \,\Xi^p_{M^*,\mu^*}(x^*)\,\Xi^q_{M^{**},\mu^{**}}(x^{**}),$$

where

$$\Xi^m_{Y,\upsilon}(\xi) = (\xi_1^2 + \ldots + \xi_{m-1}^2)^{\frac{3-p}{4} - \frac{y_1}{2}} \,P^{\frac{3-p}{2} - y_1}_{-\frac{1}{2} + \mathbf{i}\upsilon}(\xi_1)\,\Xi^{m-1}_Y(\xi).$$

It is possible to unify the results about matrix elements c^σ_{VL} and c^σ_{LM} of basis transformations. In order to do it, we introduce the function $V(z_1, z_2, z_3, z_4, z_5, z_6)$. Let us call it the Vilenkin function.

$$
\begin{aligned}
V(z_1,z_2,z_3,z_4,z_5,z_6) := {} & 2^{-1}\sqrt{\pi^{-(z_5+z_6)}}\,\mathbf{i}^{\frac{z_6}{2}+\sigma-2z_3-z_2-2}\,\Gamma\left(\frac{z_5+z_6-1}{2}\right)\\
& \cdot \Gamma^{-1}\left(\frac{z_5}{2}+z_2-1\right)\left|\Gamma\left(\frac{z_6-1}{2}+z_3+\mathbf{i}z_4\right)\right|\\
& \cdot \sqrt{\frac{z_1!\,\Gamma(z_5-2)(2z_1+z_5-2)z_4\cosh(\pi z_4)}{(z_5-2)\Gamma(z_5+z_1-2)}}\\
& \cdot \sum_{s=0}^{\left[\frac{z_1-z_2}{2}\right]}\sum_{t=0}^{\infty}\frac{(-1)^s 2^{z_1-z_2-2s}}{s!\,t!(z_4-z_5-2s)!}\,\Gamma\left(\frac{z_5-2}{2}+z_1-s\right)\\
& \cdot \Gamma\left(\frac{\sigma-3z_3-z_1}{2}+s+t\right)\Gamma^{-1}(z_1-z_2-s)\,\Gamma^{-1}\left(\frac{\sigma-3z_3-z_1}{2}+s\right)\\
& \cdot \left[\sin\left[\left(-\frac{1}{2}+\mathbf{i}z_4\right)\pi\right]\Gamma\left(\frac{\sigma+z_1+z_6-z_3}{2}-z_2-s-t\right)\right.\\
& \cdot \Gamma\left(\frac{z_3-\sigma-z_1-z_6+1}{2}+z_2+s+t-\mathbf{i}z_4\right)\\
& \cdot \Gamma\left(\frac{z_3-\sigma-z_1-z_6+1}{2}-z_2+s+t+\mathbf{i}z_4\right)\\
& \cdot 2^{\frac{\sigma+z_1-3z_3}{2}-z_2-s-t+1}\,\Gamma^{-1}\left(\frac{3z_3-\sigma-z_1}{2}+s+t+z_2\right)\\
& \cdot {}_3F_2\left(z_2-z_1+2s+t,\ \frac{z_3-\sigma-z_1-z_6+1}{2}+z_2+s+t-\mathbf{i}z_4,\right.\\
& \frac{z_3-\sigma-z_1-z_6+1}{2}-z_2+s+t+\mathbf{i}z_4;\ \frac{3z_3-\sigma-z_1}{2}+s+t+z_2,\\
& \left.\frac{z_3-\sigma-z_1-z_6+1}{2}-z_2+s+t+1;\ \frac{1}{2}\right)\\
& +\Gamma\left(\frac{\sigma+z_6-z_1-z_3}{2}+s\right)\Gamma^{-1}\left(\frac{z_2+z_3-z_1-\sigma-z_6}{2}+s+t\right)\\
& \cdot \Gamma^{-1}(2s+t+z_2-z_1)\,\Gamma^{-1}\left(\frac{z_6}{2}+z_3\right){}_3F_2\left(\frac{\sigma+z_6-z_1-z_3}{2}+s,\right.\\
& \left.\left.\frac{1}{2}-\mathbf{i}z_4,\ \frac{1}{2}+\mathbf{i}z_4;\ 1-s-t-\frac{z_2+z_3-z_1-\sigma-z_6}{2},\ \frac{z_6}{2}+z_3;\ \frac{1}{2}\right)\right].
\end{aligned}
$$

Here z_1 and z_2 are both positive integers. Quadratic brackets mean the integer part of a number.

THEOREM 2.1. $c^{\sigma}_{VL} = V(v_1, v_2, v_{p+1}, \lambda, p, q)$.

Proof. Since

$$c^{\sigma}_{VL} = \frac{\Gamma\left(\frac{p+q-1}{2}\right)}{2\pi^{\frac{p+q}{2}-1}}\int_{\Gamma_1} f^{\sigma}_V(x) f^{-\sigma-p-q+2}_L(x)\,(\mathrm{d}x)_{H_1}$$

and

$$\int_{\xi_1^2+\ldots+\xi_{p-2}^2=1} \Xi^{p-1}_{(l_1,\ldots,l_{p-2})}(x)\,\Xi^{p-1}_{(v_2,\ldots,v_{p-1})}(x)\,(\mathrm{d}x)_K \neq 0$$

only for $l_i = v_{i+1}$, then (see [Vi])

$$c^{\sigma}_{VL} \sim \int_0^{+\infty} (\mathbf{i}\sinh\alpha)^{\frac{q}{2}+\sigma-v_2-2v_{p+1}} C^{\frac{p}{2}+v_2-1}_{v_1-v_2}(\coth\alpha) P^{1-\frac{q}{2}}_{-\frac{1}{2}+\mathbf{i}\lambda}(\coth\alpha)\,\mathrm{d}\alpha.$$

Now we can use the formulas

$$C^b_a(x) = \Gamma^{-1}(b) \sum_{j=0}^{[\frac{a}{2}]} \frac{(-1)^j \Gamma(a+b-j)}{j!\,(a-2j)!\,\Gamma(a-j)}\,(2x)^{a-2j}$$

and [E, 18.2.10]. ∎

THEOREM 2.2. $c^{\sigma}_{LM} = V(l_1,\mu^*,l_{p+1},l_{p+2},p-1,q+1)$.

The formulas for c^{σ}_{VL} and c^{σ}_{LM} lead to some relations for special functions. For example, we derive the representation of the Gauss hypergeometric function.

THEOREM 2.3. *If* $2-p-q < \operatorname{re}\sigma < 0$ *and* $\alpha \neq 0$ *then*

$$\begin{aligned} {}_2F_1\left(-\sigma-\frac{1}{2},\sigma+\frac{3}{2};\frac{1}{2}+l;\frac{1-\cosh\alpha}{2}\right) &= (-1)^{l-1}\,2^{-\sigma-\frac{5}{2}}\,\pi^{-\frac{1}{2}}\,e^{-\alpha}\,\sinh\alpha\,\sin(-\pi\sigma) \\ \cdot\left(\frac{\cosh\alpha+1}{\cosh\alpha-1}\right)^{\frac{l}{2}+\frac{1}{4}} &\Gamma(\sigma+1-l)\,\Gamma\left(l-\frac{3}{2}\right)\int_0^{\infty}\rho^{-\sigma-1}\,K_{\sigma+1}(\rho e^{-\alpha}) \\ &\cdot\sum_{s=0}^{\infty}(-1)^n\,\Gamma^{-2}(s+1)\,\Gamma^{-1}(s-\sigma)\,G^{21}_{13}\left(\frac{\rho^2}{4}\,\middle|\,\begin{matrix}-s\\ -\sigma-1,\ 0\end{matrix}\right)\mathrm{d}\rho. \end{aligned}$$

Proof. Let $q=1$ and let $\tilde{x}$ belong to the hyperboloid $\vartheta(\tilde{x})=1$. Let $\Phi(x) = (x_1\tilde{x}_1+\ldots+x_p\tilde{x}_p - x_{p+1}\tilde{x}_{p+1} - \ldots x_{p+q}\tilde{x}_{p+q})^{\sigma}$. Then

$$f_L^{-\sigma-p-q+2}(x) = \sum_V c_{LV}^{-\sigma-p-q+2} f_V^{-\sigma-p-q+2}(x).$$

The equality

$$\int_{\Gamma_1} f_L^{-\sigma-p-q+2}(x)\,\Phi(x)\,(\mathrm{d}x)_{H_1} = \int_{\Gamma_K}\sum_V c_{LV}^{-\sigma-p-q+2} f_V^{-\sigma-p-q+2}(x)\,\Phi(x)\,(\mathrm{d}x)_K$$

leads to our formula. ∎

3. A formula for Meijer and Legendre functions. Since the groups $SO(p,q)$ and $SO(q,p)$ are isomorphic, we can assume that $p \geq q$. Let us consider all $p-1$ partitions Ω of the direct product $\{1,\ldots,p\}\times\{p+1,\ldots,p+q\}$ into p classes

$$\Lambda_{\Omega,i} = \{(x_{i1},p+1)\},\ldots,(x_{iq},p+q)\ :\ \text{if}\, s\neq t\,\text{then}\, x_{is}\neq x_{it}\}.$$

The set of matrices $h(m,n)$, where $(i,j)\in\Lambda_{\Omega,i}$, generate a maximal $\mathbb{R}$-diagonalizable subalgebra $A_{\Omega,i}$ in $\mathfrak{h}$. For different $\Lambda_{\Omega,i}$ and $\Lambda_{\hat{\Omega},\hat{i}}$, such subalgebras are conjugate under Cartan involution. For any $g\in G$, we have $g=g_1g_2g_3$, where $g_1,g_3\in K$ and $g_2\in H_{\Omega,i} = \exp A_{\Omega,i}$.

LEMMA 3.1. *If $\hat{\sigma} = -\sigma - p - q + 2$, then the functional* F *is invariant under the pair* $(T_\sigma, T_{\hat{\sigma}})$*, i. e.* $\mathsf{F}(T_\sigma(g)(f_1), T_{-\sigma-p-q+2}(g)(f_2)) = \mathsf{F}(f_1, f_2)$.

Proof. Without loss of generality, let us prove this lemma for the simplest case $p = 2, q = 1$. In this case, we have the only partition Ω into two classes $\Lambda_{\Omega,1} = \{(1,3)\}$ and $\Lambda_{\Omega,2} = \{(2,3)\}$. We will deal with the class $\Lambda_{\Omega,1}$. It is sufficient to argue separately for restrictions of T_σ to K and $H_{\Omega,1}$. According to (4), we can write an arbitrary point $x \in X$ as $x = (t\cos\phi, t\sin\phi, t)$, where the point $(\cos\phi, \sin\phi, 1)$ belongs to Γ_K. If $g \in K$, then

$$g^{-1}(\alpha)x = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t\cos\phi \\ t\sin\phi \\ t \end{pmatrix} = \begin{pmatrix} t\cos(\phi-\alpha) \\ t\sin(\phi-\alpha) \\ t \end{pmatrix} \equiv \begin{pmatrix} t'\cos\phi' \\ t'\sin\phi' \\ t'. \end{pmatrix} \tag{6}$$

If $g \in H_{\Omega,1}$ then

$$g^{-1}(s)x = \begin{pmatrix} \cosh s & 0 & -\sinh s \\ 0 & 1 & 0 \\ -\sinh s & 0 & \cosh s \end{pmatrix} \begin{pmatrix} t\cos\phi \\ t\sin\phi \\ t \end{pmatrix}$$
$$= \begin{pmatrix} t\cosh s\cos\phi - t\sinh s \\ t\sin\phi \\ -t\sinh s\cos\phi + t\cosh s \end{pmatrix} \equiv \begin{pmatrix} t'\cos\phi' \\ t'\sin\phi' \\ t'. \end{pmatrix}. \tag{7}$$

We obtain immediately from (7) that

$$\cos\phi' = \frac{t\cosh s\cos\phi - t\sinh s}{t'}, \tag{8}$$
$$\sin\phi' = \frac{t\sin\phi}{t'}, \tag{9}$$
$$t' = -t\sinh s\cos\phi + t\cosh s. \tag{10}$$

Let us find the partial derivative of $\cos\phi'$ with respect to ϕ from (8):

$$-\sin\phi'\,\mathrm{d}\phi' = -\frac{t^2\sin\phi\,\mathrm{d}\phi}{t'^2}. \tag{11}$$

Formulas (9), (10) and (11) lead to

$$\mathrm{d}\phi = \frac{t'\,\mathrm{d}\phi'}{t}. \tag{12}$$

Therefore,

$$\begin{aligned} &\mathsf{F}(T_\sigma(g)(f_1), T_{-\sigma-1}(g)(f_2)) \\ &\quad = \int_{\Gamma_K} f_1(t'\cos\phi', t'\sin\phi', t')\,|_{t=1}\; f_2(t'\cos\phi', t'\sin\phi', t')\,|_{t=1}\; \mathrm{d}\phi \\ &\quad = \int_{\Gamma_K} t'^{\sigma} f_1(\cos\phi', \sin\phi', 1)\, t'^{-\sigma-1} f_2(\cos\phi', \sin\phi', 1)\, t'\,\mathrm{d}\phi' \\ &\quad = \int_{\Gamma_K} f_1(\cos\phi', \sin\phi', 1)\, f_2(\cos\phi', \sin\phi', 1)\,\mathrm{d}\phi' = \mathsf{F}(f_1, f_2). \end{aligned}$$

Formula (6) leads to $t' = t$, $dt' = dt$, $d\phi' = d\phi$. Thus,

$$\begin{aligned}
&\mathsf{F}(T_\sigma(g)(f_1), T_{-\sigma-1}(g)(f_2)) \\
&\quad = \int_{\Gamma_K} f_1(t'\cos\phi', t'\sin\phi', t')\,|_{t=1}\; f_2(t'\cos\phi', t'\sin\phi', t')\,|_{t=1}\; \mathrm{d}\phi \\
&\qquad = \int_{\Gamma_K} f_1(\cos\phi', \sin\phi', 1)\, f_2(\cos\phi', \sin\phi', 1)\, \mathrm{d}\phi' = \mathsf{F}(f_1, f_2). \quad \blacksquare
\end{aligned}$$

THEOREM 3.2. *For* $-1 < \operatorname{re}\sigma < 0$,

$$\begin{aligned}
&\int_0^{+\infty} J_0(e^t\rho)\, G_{13}^{21}\left(\frac{\rho^2}{4}\,\middle|\,\begin{matrix}0\\ \sigma,0,0\end{matrix}\right) G_{13}^{21}\left(\frac{e^{2t}\rho^2}{4}\,\middle|\,\begin{matrix}0\\ -\sigma-1,0,0\end{matrix}\right) \mathrm{d}\rho \\
&\quad = -2^{\frac{5}{2}}\pi\sigma\sin^{-1}(-\pi\sigma)\sinh^{\frac{1}{2}}(-\beta)\,\Gamma(-\sigma-1) \\
&\qquad \cdot \sum_{s=0}^{\infty}(-1)^s\,\Gamma^{-1}(-\sigma-3-s)\,P_{-\sigma-\frac{3}{2}}^{-\frac{1}{2}-s}(\cosh\beta), \qquad (13)
\end{aligned}$$

where $\cosh\beta = \cosh t + \frac{d^2 e^t}{2}$.

Proof. Let $p = 1$ and $q = 3$. Then

$$\begin{aligned}
f_V^\sigma(x) &= x_1^{\sigma-v_2}\, C_{v_1-v_2}^{v_2+\frac{1}{2}}\left(\frac{x_4}{x_1}\right)(x_3 + \mathbf{i}x_2)^{v_2}, \\
f_L^\sigma(x) &= (x_1 + x_4)^\sigma\, \exp\frac{\mathbf{i}(l_1x_2 + l_2x_3)}{x_1 + x_4}.
\end{aligned}$$

Consider the restriction T_σ^* of representation T_σ to the subgroup $N\times\tilde{H}$. Here the subgroup N consists of the matrices

$$\begin{pmatrix}
1+\frac{d^2}{2} & d\cos\tau & d\sin\tau & \frac{d^2}{2} \\
d\cos\tau & 1 & 0 & d\cos\tau \\
d\sin\tau & 0 & 1 & d\sin\tau \\
-\frac{d^2}{2} & -d\cos\tau & -d\sin\tau & 1-\frac{d^2}{2}
\end{pmatrix}$$

and $\tilde{H}$ consists of the matrices $\hat{r}(1,4,t)$. According to lemma 3.1, we obtain the matrix elements $t_{L\hat{L}}^{*\sigma}(g)$ of T_σ^* in the following way:

$$t_{L\hat{L}}^{*\sigma}(n\hat{r}) = \mathsf{F}(T_\sigma(n\hat{r})(f_L^\sigma, f_{-\hat{L}}^{-\sigma-2}) = \mathsf{F}(T_\sigma(\hat{r})(f_L^\sigma), T_{-\sigma-2}(n^{-1})(f_{-\hat{L}}^{-\sigma-2})).$$

Let us integrate over Γ_N. Further, let us derive the matrix elements $t_{V\hat{V}}^\sigma 1(n\hat{r})$ and use relation between $t_{L\hat{L}}^\sigma(n\hat{r})$ and $t_{V\hat{V}}^\sigma(n\hat{r})$. In this way, we obtain the relation containing the Bessel function, the Legendre function, the Gauss hypergeometric function, and two Meijer functions. The simplest form of this formula corresponds to $V \equiv (v_1, v_2) = (0,0)$ and coincides with formula (13). $\blacksquare$

4. Two formulas for Gauss, Whittaker, and Macdonald functions. Let us consider the case $p = q = 2$. We put here

$$f_V^\sigma(x) = (x_1^2 + x_2^2)^{\frac{\sigma - v_1 - v_2}{2}} (x_2 + \mathbf{i}x_1)^{v_1} (x_4 + \mathbf{i}x_3)^{v_2},$$
$$f_L^\sigma(x) = (x_2 + x_4)^\sigma \exp \frac{\mathbf{i}(l_1 x_1 - l_2 x_3)}{x_2 + x_4}.$$

LEMMA 4.1. *If* $|l_1| < l_2$, *then*

$$c_{VL}^\sigma = 2^{-3}\,\pi^{-1}\,(l_2^2 - l_1^2)^{-\frac{\sigma}{2}-1}\,\Gamma^{-1}\left(\frac{v_1 + v_2 - \sigma}{2}\right)\Gamma^{-1}\left(\frac{v_2 - v_1 - \sigma}{2}\right) \cdot W_{\frac{v_2+v_1}{2},\,\frac{\sigma+1}{2}}(l_2 - l_1)\,W_{\frac{v_2-v_1}{2},\,\frac{\sigma+1}{2}}(l_2 + l_1).$$

If $|l_2| < l_1$, *then*

$$c_{VL}^\sigma = 2^{-3}\,\pi^{-1}\,(l_1^2 - l_2^2)^{-\frac{\sigma}{2}-1}\,\Gamma^{-1}\left(-\frac{v_1 + v_2 + \sigma}{2}\right)\Gamma^{-1}\left(\frac{v_2 - v_1 - \sigma}{2}\right) \cdot W_{-\frac{v_2+v_1}{2},\,\frac{\sigma+1}{2}}(l_1 - l_2)\,W_{\frac{v_2-v_1}{2},\,\frac{\sigma+1}{2}}(l_2 + l_1).$$

If $l_1 < 0$ *and* $|l_2| < |l_1|$, *then*

$$c_{VL}^\sigma = 2^{-3}\,\pi^{-1}\,(l_1^2 - l_2^2)^{-\frac{\sigma}{2}-1}\,\Gamma^{-1}\left(\frac{v_1 + v_2 - \sigma}{2}\right)\Gamma^{-1}\left(\frac{v_1 - v_2 - \sigma}{2}\right) \cdot W_{\frac{v_2+v_1}{2},\,\frac{\sigma+1}{2}}(l_2 - l_1)\,W_{\frac{v_1-v_2}{2},\,\frac{\sigma+1}{2}}(|l_2 + l_1|).$$

If $l_2 < 0$ *and* $|l_1| < |l_2|$ *then*

$$c_{VL}^\sigma = 2^{-3}\,\pi^{-1}\,(l_2^2 - l_1^2)^{-\frac{\sigma}{2}-1}\,\Gamma^{-1}\left(-\frac{v_1 + v_2 + \sigma}{2}\right)\Gamma^{-1}\left(\frac{v_1 - v_2 - \sigma}{2}\right) \cdot W_{-\frac{v_2+v_1}{2},\,\frac{\sigma+1}{2}}(l_1 - l_2)\,W_{\frac{k_1-k_2}{2},\,\frac{\sigma+1}{2}}(|l_2 + l_1|).$$

Proof. Let us choose the integration contour $\Gamma = \Gamma_N$ in formula $c_{VL}^\sigma = \mathsf{F}(f_V^\sigma, f_L^{-\sigma-2})$, where the tangent space of the subgroup N is generated by the matrices

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Let $(t, \frac{1-t^2+s^2}{2}, s, \frac{1+t^2-s^2}{2})$ be a parametrization of Γ_N. We compute this integral by formula [E, 3.2.12]. ■

THEOREM 4.2. *If $\beta > 0$ and $l_2 > |l_1|$, then*

$$
\begin{aligned}
{}_2F_1\left(1, \sigma+2; 2; 1+\frac{l_1}{l_2}\tanh\beta\right) &= 2^{-1}\,\pi\,\sigma\,(\sigma+1)^{-1} \\
&\quad\cdot (\mathbf{i}l_2)^{\sigma+2}\,(l_2-l_1)^{-\frac{\sigma+2}{2}}\,\sin(\sigma+1)\,\sinh^{-1}\beta\,\tanh^{-1}\beta\,e^{-\beta} \\
&\quad\cdot \sum_{v_1=0}^{\infty}\sum_{v_2=-v_1}^{v_1}(v_1^2+v_2^2)^{-\frac{\sigma+1}{2}}\,\Gamma^{-1}\left(\frac{v_1+v_2-\sigma}{2}\right)\Gamma^{-1}\left(\frac{v_2-v_1-\sigma}{2}\right) \\
&\quad\cdot W_{\frac{v_2+v_1}{2},\,\frac{\sigma+1}{2}}(l_2-l_1)\,W_{\frac{v_2-v_1}{2},\,\frac{\sigma+1}{2}}(l_2+l_1)\,K_{\sigma+1}\left((v_1^2+v_2^2)^{\frac{1}{2}}\,e^{-\beta}\right).
\end{aligned}
$$

Proof. Let $u(x) := (x_1\cosh\beta - x_3\sinh\beta)^{\sigma}$. Then $\mathsf{F}(u, f_L^{-\sigma-2})_{\Gamma:=\Gamma_N} = \mathsf{F}(u, f_L^{-\sigma-2})_{\Gamma:=\Gamma_K}$. In order to compute the corresponding integral along the contour Γ_K, we use

$$
f_L^{-\sigma-2} = \sum_{v_1=0}^{\infty}\sum_{v_2=-v_1}^{v_1} c_{LV}^{-\sigma-2} f_V^{-\sigma-2} = \sum_{v_1=0}^{\infty}\sum_{v_2=-v_1}^{v_1} c_{VL}^{\sigma} f_V^{-\sigma-2}. \ \blacksquare
$$

In the same way, we derive

THEOREM 4.3. *If $\zeta > 1$ and $l_2 > |l_1|$, then*

$$
\begin{aligned}
&\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(\mathbf{i}l_2)^{\sigma}\,(l_2^2-l_1^2)^{-\frac{\sigma}{2}-1}\,W_{\frac{v_2+v_1}{2},\,\frac{\sigma+1}{2}}(l_2-l_1) \\
&\qquad\cdot W_{\frac{v_2-v_1}{2},\,\frac{\sigma+1}{2}}(l_2+l_1)\,{}_2F_1\left(1,-\sigma;2;1+\frac{l_1}{l_2}\cdot\frac{\zeta^2-1}{\zeta^2+1}\right)\mathrm{d}l_1\,\mathrm{d}l_2 \\
&= 32\,(v_1^2+v_2^2)^{\frac{\sigma+1}{2}}\,\zeta^{-1}\left(\frac{\zeta^2+1}{2\zeta}\right)^{\sigma+2}\left(\frac{\zeta^2-1}{\zeta^2+1}\right)^{\sigma+1}\Gamma\left(\frac{v_1+v_2-\sigma}{2}\right) \\
&\qquad\cdot\Gamma\left(\frac{v_2+v_1-\sigma}{2}\right)\Gamma^{-1}(\sigma+2)\,\Gamma^{-1}(-\sigma)\,K_{-\sigma-1}\left(\frac{1}{2}\sqrt{v_1^2+v_2^2}\right).
\end{aligned}
$$

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