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TWISTED SPECTRAL TRIPLES AND COVARIANT DIFFERENTIAL CALCULI

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Abstract. Connes and Moscovici recently studied "twisted" spectral triples (A, H, D) in which the commutators $[D, a]$ are replaced by $D \circ a - \sigma(a) \circ D$, where σ is a second representation of A on H. The aim of this note is to point out that this yields representations of arbitrary covariant differential calculi over Hopf algebras in the sense of Woronowicz. For compact quantum groups, H can be completed to a Hilbert space and the calculus is given by bounded operators. At the end, we discuss an explicit example of Heckenberger's 3-dimensional covariant differential calculi on quantum $SU(2)$.

1. Introduction. One of the basic invariants of Connes' spectral triples (A, H, D) is the operator-valued derivation $\mathrm{d} : a \mapsto [D, a]$. For Dirac-type operators on manifolds, $\mathrm{d}a$ is the differential of $a \in C^\infty(X)$ that acts by Clifford multiplication. Thus one can view $\Omega^1 := \{\sum_i a_i \mathrm{d}b_i \,|\, a_i, b_i \in A\}$ as an abstract space of 1-forms defined by the triple. Its elements also play an important role as "gauge transformations" of the triple: perturbing D to $D + \omega$, $\omega \in \Omega^1$, yields a new spectral triple with similar properties. In particular, the bimodule Ω^1 remains obviously the same as also does the class of d in $\mathrm{H}^1(A, \Omega^1)$ (the space of derivations $A \to \Omega^1$ modulo inner ones), and it is this class rather than d itself that enters the homological constructions in noncommutative geometry. Similarly, the more sophisticated invariants (Chern character, spectral action) are invariant under the transformation $D \mapsto D + \omega$.

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Given an abstract noncommutative algebra A, there is no longer a canonical pair (Ω^1, d) attached to A, and it becomes a rather subtle task to motivate a specific choice. One possibility suggested by Woronowicz is to take into account Hopf algebra symmetries of A. This leads to the concept of covariant differential calculi [Wo1, Wo2] that we will review in some detail in the next section.

Several authors have worked intensively on the classification of these structures. In particular, Heckenberger has classified the left-covariant calculi of rank 3 over the standard quantum $SU(2)$ group [He]. The result was that covariance alone does not fix a particular (Ω^1, d): even when requiring the universal differentially graded algebra generated by (Ω^1, d) to resemble the de Rham complex of $SU(2)$, the number of nonisomorphic calculi reduces only to seven. Furthermore, Schmüdgen has proved the nonexistence of spectral triples that lead to these calculi [Sch1, Sch2].

The aim of the present note is to remark that this problem naturally points towards twisted spectral triples as recently introduced by Connes and Moscovici [CM]: here $[D, a]$ becomes replaced by a twisted commutator $D \circ a - \sigma(a) \circ D$ for an automorphism $\sigma \in \mathrm{Aut}(A)$, and it was shown in [CM] that some core structures of noncommutative geometry generalise to this setting which can be classically motivated by considering the effect that a conformal rescaling of the metric of a spin manifold has on the Dirac operator. We first strip off all analytical considerations and allow A to be any algebra over a field k, H to be any A-module and $D \in \mathrm{End}_k(H)$ be any linear map. Furthermore we consider the essential step in [CM] to be the appearance of a second representation of A on H that is linked to the first one by D, that is, we also replace $\sigma \in \mathrm{Aut}(A)$ by any homomorphism $\sigma : A \to \mathrm{End}_k(H)$. Calling such data in full generality twisted spectral triples one easily proves:

THEOREM 1.1. *Any covariant differential calculus of finite rank over a Hopf algebra with invertible antipode can be realised by means of a twisted spectral triple.*

Turning then to compact quantum groups we show that here the Haar functional allows one to introduce Hilbert space structures in such a way that the elements of the calculi are given by bounded operators:

THEOREM 1.2. *If A is a compact quantum group, then any covariant differential calculus of finite rank over A can be realised by a twisted spectral triple on a Hilbert space H with all elements of the calculus given by bounded operators.*

In the last section we study a specific example for $SU_q(2)$ in which σ is indeed an automorphism as in [CM]. This distinguishes one of Heckenberger's calculi, so it seems an interesting candidate for further study. However, the operator D that we obtain does not have compact resolvent, so an immediate application of the analytic techniques from [CM] is not possible (the same seems to apply to the similarly constructed example from [An]).

2. Covariant differential calculi. In order to fix notations and for the reader's convenience, we survey in this section in some detail Woronowicz's theory of (first-order) covariant differential calculi over Hopf algebras [Wo1, Wo2] (see also [KS] for a detailed account).

Let A be a Hopf algebra with counit ε, coproduct Δ and antipode S.

DEFINITION 2.1. A *covariant A-module* is a (left) A-module and (left) A-comodule M whose action $\triangleright : A \otimes M \to M$ and coaction $\Delta_M : M \to A \otimes M$ are compatible in the sense that $\Delta_M(a \triangleright \omega) = a_1\omega_{-1} \otimes (a_2 \triangleright \omega_0)$ for all $a \in A, \omega \in M$.

Here and in what follows we use Sweedler's notation for coproducts and coactions and write $a_1 \otimes a_2$ and $\omega_{-1} \otimes \omega_0$ for $\Delta(a)$ and $\Delta_M(\omega)$, respectively. Throughout, we work over a field k, and an unadorned $\otimes$ means tensor product over k.

To a covariant A-module M one attaches the k-vector space of invariant elements

$$M_{\mathrm{inv}} := \{\omega \in M \mid \Delta_M(\omega) = 1 \otimes \omega\}. \tag{1}$$

Conversely, $N^A := A \otimes N$ becomes for any vector space N through

$$\Delta_{N^A}(a \otimes n) := a_1 \otimes a_2 \otimes n, \quad a \triangleright (b \otimes n) := ab \otimes n \tag{2}$$

a covariant A-module, and one clearly has $N \simeq (N^A)_{\mathrm{inv}}$ as vector spaces. Finally, there is for any covariant A-module M an isomorphism of covariant A-modules

$$\xi : M \to (M_{\mathrm{inv}})^A, \quad \omega \mapsto \omega_{-2} \otimes S(\omega_{-1}) \triangleright \omega_0, \quad \xi^{-1}(a \otimes \omega) = a \triangleright \omega. \tag{3}$$

Since both $N \mapsto N^A$ and $M \mapsto M_{\mathrm{inv}}$ are functorial, this means:

PROPOSITION 2.2. *The category ${}^A_A\mathbf{Mod}$ of covariant A-modules is equivalent to the category ${}_k\mathbf{Mod}$ of k-vector spaces.*

In particular, a covariant A-module is always free as an A-module, and any vector space basis of M_{inv} is simultaneously a module basis of M.

DEFINITION 2.3. A *covariant A-bimodule* is a covariant A-module and A-bimodule M whose right A-action $\triangleleft : M \otimes A \to M$ is compatible with Δ_M in the sense that $\Delta_M(\omega \triangleleft a) = \omega_{-1}a_1 \otimes (\omega_0 \triangleleft a_2)$ for all $a \in A, \omega \in M$.

If M is a covariant bimodule, then M_{inv} becomes through the adjoint action $\mathrm{ad}(a)\omega := S(a_1) \triangleright \omega \triangleleft a_2$ a right A-module. If N is conversely a right A-module, then the covariant A-module N^A defined above is a covariant bimodule with right action $(a \otimes n) \triangleleft b := ab_1 \otimes nb_2$. In analogy to Proposition 2.2 this gives:

PROPOSITION 2.4. *The category ${}^A_A\mathbf{Mod}_A$ of covariant A-bimodules is equivalent to the category $\mathbf{Mod}_A$ of right A-modules.*

If S is invertible, then an isormorphism similar to (3) shows that any vector space basis of M_{inv} is also a basis of M as right module.

DEFINITION 2.5. An *equivariant derivation* on A with values in $M \in {}^A_A\mathbf{Mod}_A$ is an A-comodule morphism $\mathrm{d} : A \to M$ satisfying $\mathrm{d}(ab) = a \triangleright \mathrm{d}b + (\mathrm{d}a) \triangleleft b$ for $a, b \in A$.

PROPOSITION 2.6. *Let $\mathrm{d} : A \to M$ be an equivariant derivation and consider the right A-module M_{inv} as bimodule with trivial left action $a\omega := \varepsilon(a)\omega$. Then*

$$\mathrm{d}_{\mathrm{inv}} : A \to M_{\mathrm{inv}}, \quad a \mapsto S(a_1) \triangleright \mathrm{d}a_2 \tag{4}$$

is a derivation, any derivation with values in M_{inv} arises in this way, and d *is uniquely determined by* $\mathrm{d}_{\mathrm{inv}}$ *since* $\mathrm{d}a = a_1 \triangleright \mathrm{d}_{\mathrm{inv}}(a_2)$.

This correspondence takes inner derivations to inner ones. If we thus denote by $\mathrm{H}^1_A(A,M)$ the space of equivalence classes of equivariant derivations modulo inner ones and take into account the realisation of Hochschild cohomology of a Hopf algebra described e.g. on p. 197 in [GK], this gives:

Corollary 2.7. *$\mathrm{H}^1_A(A,M) \simeq \mathrm{H}^1(A,M_{\mathrm{inv}}) \simeq \mathrm{Ext}^1_A(k,M_{\mathrm{inv}})$ in ${}_k\mathbf{Mod}$.*

Here in $\mathrm{Ext}^1_A(k,M_{\mathrm{inv}})$, k is considered with trivial A-action and M_{inv} is a left A-module with action $a\omega = \mathrm{ad}(S(a))\omega$.

We will be mainly concerned with the case $\dim_k M_{\mathrm{inv}} < \infty$. Assuming this, fix dual bases $\{\omega^i\}$ and $\{x_i\}$ of M_{inv} and $M^*_{\mathrm{inv}} = \mathrm{Hom}_k(M_{\mathrm{inv}},k)$, respectively, and define linear functionals $X_i, f^j_i \in A^*$ by

$$X_i(a) := x_i(\mathrm{d}_{\mathrm{inv}}(a)), \quad f^j_i(b) := x_i(\mathrm{ad}(b)\omega^j).$$

Recall that the Hopf dual $A^\circ \subset A^*$ consists of those functionals whose kernel contains an ideal of finite codimension.

Proposition 2.8. *One has $X_i, f^j_i \in A^\circ$ with $\varepsilon(X_i)=0$, $\varepsilon(f^i_j) = \delta^i_j$ and*

$$\Delta(X_i) = \sum_j 1 \otimes X_i + X_j \otimes f^j_i, \quad \Delta(f^i_j) = \sum_k f^i_k \otimes f^k_j. \tag{5}$$

Proof. The f^i_j belong as matrix coefficients of a finite-dimensional (anti)representation (namely M_{inv}) to A° and have the desired properties,

$$f^j_i(ab) = x_i(\mathrm{ad}(ab)\omega^j) = x_i\Big(\mathrm{ad}(b)\Big(\sum_k x_k(\mathrm{ad}(a)\omega^j)\omega^k\Big)\Big) = \sum_k f^j_k(a) f^k_i(b), \tag{6}$$

and therefore so do the X_i since

$$X_i(ab) = \varepsilon(a)X_i(b) + x_i(\mathrm{ad}(b)\mathrm{d}_{\mathrm{inv}}(a)) = \varepsilon(a)X_i(b) + \sum_j X_j(a) f^j_i(b). \ \blacksquare \tag{7}$$

In other words, $T_{\mathrm{d}} := \mathrm{span}\,\{X_i\} \oplus k\cdot 1$ is a unital right coideal of A°, and this (almost) determines M and d: the bimodule structure of M is determined by

$$\omega^i \triangleleft a = a_1 S(a_2) \triangleright \omega^i \triangleleft a_3 = a_1 \triangleright \mathrm{ad}(a_2)\omega^i = \sum_j a_1 f^i_j(a_2) \triangleright \omega^j = \sum_j (f^i_j \blacktriangleright a) \triangleright \omega^j, \tag{8}$$

where $X \blacktriangleright a := a_1 X(a_2)$ is the canonical left action of A° on A, and we also have

$$\mathrm{d}a = a_1 \triangleright \mathrm{d}_{\mathrm{inv}}(a_2) = \sum_i a_1 \triangleright x_i(\mathrm{d}_{\mathrm{inv}}(a_2))\omega^i = \sum_i (X_i \blacktriangleright a) \triangleright \omega^i. \tag{9}$$

However, $\mathrm{d}_{\mathrm{inv}}$ is in general not surjective (so that $x_i \mapsto X_i$ is not injective):

Proposition 2.9. *$\mathrm{im}\,\mathrm{d}_{\mathrm{inv}} = \Omega^1_{\mathrm{inv}} \subset M_{\mathrm{inv}}$, where $\Omega^1 := \{\sum_i a_i \mathrm{d}b_i \mid a_i, b_i \in A\} \subset M$.*

Proof. "$\subset$" is clear, and if $\omega = \sum_i a_i \mathrm{d}b_i \in \Omega^1_{\mathrm{inv}}$, then $\omega = \mathrm{d}_{\mathrm{inv}}(\sum_i \varepsilon(a_i) b_i)$. $\blacksquare$

Following Woronowicz we call $\mathrm{d}: A \to \Omega^1$ a (first order) covariant differential calculus over A and $\dim_k \Omega^1_{\mathrm{inv}}$ its dimension. The above considerations now give:

Proposition 2.10. *Finite-dimensional covariant differential calculi over A correspond bijectively to finite-dimensional unital right coideals of A°.*

One calls $T_{\mathrm{d}}^{+} = T_{\mathrm{d}} \cap \ker \varepsilon$ the quantum tangent space of the corresponding calculus. The calculi with trivial class in $\mathrm{Ext}^1_A(k, \Omega^1_{\mathrm{inv}})$ correspond to T_{d}^{+} with basis of the form $X_i = \sum_j f_i^j - 1$, f_i^j being as in (5) a matrix corepresentation.

Since this will be used below, we remark that any differential calculus over A (covariance is irrelevant for this) can be realised as a quotient of a universal one which is $\Omega^1_{\mathrm{univ}} := A \otimes \ker \varepsilon$ with differential $\mathrm{d}_{\mathrm{univ}} a := 1 \otimes (a - \varepsilon(a))$. This implies:

Proposition 2.11. *Two differential calculi* $(\Omega^1_1, \mathrm{d}_1), (\Omega^1_2, \mathrm{d}_2)$ *are isomorphic iff*

$$\sum_r a_r \mathrm{d}_1 b_r = 0 \in \Omega^1_1 \Leftrightarrow \sum_r a_r \mathrm{d}_2 b_r = 0 \in \Omega^1_2 \quad \forall \sum_r a_r \otimes b_r \in A \otimes A. \tag{10}$$

At the end, we briefly consider Hopf $*$-algebras over $k = \mathbb{C}$ (see e.g. [KS], Section 1.2.7 for background on such). Then the compatible structures on the various data we studied above are as follows: on a covariant bimodule M one can consider complex antilinear involutions $* : M \to M$ with $(a \triangleright \omega \triangleleft b)^* = b^* \triangleright \omega^* \triangleleft a^*$ and $\Delta_M(\omega^*) = \omega^*_{-1} \otimes \omega^*_0$. These restrict to M_{inv} and are compatible with the right adjoint action by $(\mathrm{ad}(a)\omega)^* = \mathrm{ad}(\theta(a))\omega^*$, where $\theta = * \circ S$ is the Cartan involution of A (recall that $\theta \circ \theta = \mathrm{id}$). Derivations $\mathrm{d} : A \to M$ can then be required to be $*$-linear, $(\mathrm{d}a)^* = \mathrm{d}(a^*)$, and this happens if and only if $\mathrm{d}_{\mathrm{inv}}$ satisfies $\mathrm{d}_{\mathrm{inv}}(a)^* = \mathrm{ad}(\theta(a_1))\mathrm{d}_{\mathrm{inv}}(a_2^*)$. In terms of quantum tangent spaces, a finite-dimensional covariant differential calculus is $*$ if and only if T_{d} (or equivalently T_{d}^{+}) is invariant under the involution $X^*(a) := \overline{X \circ \theta}$ which turns A° into a Hopf $*$-algebra (see [KS], Proposition 14.6).

3. Realisations by twisted spectral triples. Let H be a k-vector space, $A \subset \mathrm{End}_k(H)$ be a unital associative k-algebra and $\sigma : A \to \mathrm{End}_k(H)$ be a second representation of A on H. Then $\mathrm{End}_k(H)$ becomes an A-bimodule via

$$a \triangleright \omega \triangleleft b := \sigma(a) \circ \omega \circ b, \quad a, b \in A, \omega \in \mathrm{End}_k(H), \tag{11}$$

and any $D \in \mathrm{End}_k(H)$ defines a derivation

$$\mathrm{d} : A \to \mathrm{End}_k(H), \quad a \mapsto \mathrm{d}a := D \circ a - \sigma(a) \circ D, \tag{12}$$

so

$$\Omega^1_{(A,H,\sigma,D)} := \mathrm{span}_k\{a \triangleright \mathrm{d}b \in \mathrm{End}_k(H) \mid a, b \in A\} \tag{13}$$

becomes a differential calculus over A. We say in this case that (A, H, σ, D) defines a realisation of this calculus.

For $\sigma = \mathrm{id}$ the above is the algebraic structure underlying A. Connes' spectral triples, and the idea to introduce the twist σ appeared recently in [CM] (for the case $\sigma \in \mathrm{Aut}(A)$). Therein it was shown that some core ideas of noncommutative geometry can be generalised to such "twisted" spectral triples, taking into account as well analytic aspects. In the present paper we focus mainly on algebraic questions and speak for simplicity of (A, H, σ, D) in full generality of a twisted spectral triple over A. Our main aim is to remark that the following fact holds:

Theorem 3.1. *Any finite-dimensional covariant differential calculus over a Hopf algebra with invertible antipode can be realised by means of a twisted spectral triple.*

Proof. Let (Ω^1, d) be d-dimensional and $\{\omega^i\}$, $\{X_i\}$ and f^i_j be as in the previous section. We first define

$$s^j_i := S^{-1}(f^j_i), \quad \partial_i := \sum_{j=1}^{d} s^j_i X_j \in A^\circ. \tag{14}$$

Then one has

$$\Delta(\partial_i) = \partial_i \otimes 1 + s^j_i \otimes \partial_j, \quad \Delta(s^j_i) = \sum_k s^k_i \otimes s^j_k. \tag{15}$$

The vector space H will be constructed now as a free module,

$$H := A \otimes V, \tag{16}$$

with V a vector space, and the second representation σ and D will be given by

$$\sigma(a) = \sum_{ij} (s^i_j \blacktriangleright a) \otimes E^j_i, \quad D = \sum_k \partial_k \otimes \gamma^k, \tag{17}$$

where E^j_i, γ^k are suitable k-linear maps on V. The actions of $(s^i_j \blacktriangleright a) \in A$ and of $\partial_k \in A^\circ$ on A are given by left multiplication and by $\blacktriangleright$, respectively. One then has

$$\sigma(a)\sigma(b) = \sum_{ijmn} (s^i_j \blacktriangleright a)(s^m_n \blacktriangleright b) \otimes E^j_i \circ E^n_m, \tag{18}$$

so σ will be a representation provided that

$$E^j_i \circ E^n_m = \delta^n_i E^j_m, \tag{19}$$

since $X \blacktriangleright (ab) = (X_1 \blacktriangleright a)(X_2 \blacktriangleright b)$. For example, we can choose

$$V = \Omega^1_{\mathrm{inv}}, \quad E^j_i : \omega^k \mapsto \delta^k_i \omega^j. \tag{20}$$

Furthermore, we have

$$\begin{aligned} D \circ a - \sigma(a) \circ D &= \sum_k (\partial_k \blacktriangleright a) \otimes \gamma^k \\ &= \sum_k \sigma(X_k \blacktriangleright a) \circ (1 \otimes \gamma^k) \\ &= \sum_k (X_k \blacktriangleright a) \triangleright (1 \otimes \gamma^k), \end{aligned} \tag{21}$$

provided that

$$E^j_i \circ \gamma^k = \delta^k_i \gamma^j. \tag{22}$$

It now follows from (21), (9) and Proposition 2.11 that the abstract calculus (Ω^1, d) we started with is in this case isomorphic to the one defined by the twisted spectral triple (A, H, σ, D) as long as the γ^k are linearly independent over k. For example, (22) follows from (19) provided that

$$\gamma^k = \sum_n E^k_n \circ \hat{\gamma}^n, \tag{23}$$

with arbitrary $\hat{\gamma}^n \in \mathrm{End}_k(V)$. Thus we can choose in any case the simple ansatz (20) with $\hat{\gamma}^n = 1$ for all n. ∎

The twisted spectral triple from the proof is not really interesting and does not take into account spinorial effects at all. It lives on $H = \Omega^1$ itself with a somehow artificial commutator realisation of d that becomes possible as a consequence of (9). On the other hand, we will see below that for concrete examples of Hopf algebras there are plenty of possibilites to define more interesting triples realising a given calculus. In particular, if $s^i_j = \delta^i_j s$ for some group-like element $s \in A^\circ$, then the condition on the E^i_j reduces to

$$\sum_{ij} E^i_i \circ E^j_j = \sum_k E^k_k, \tag{24}$$

so one can choose e.g. $E^i_j = \frac{1}{d}\delta^i_j$. We can simply choose V arbitrary and $E^i_j = 1$ in order to obtain a representation σ on H.

4. Compact quantum groups. A major aim of noncommutative geometry is to obtain in a Hilbert space framework index formulas for the operator D of a spectral triple. Thus one usually considers spectral triples only for $*$-algebras A of bounded operators on a Hilbert space which is a completion of H from the preceding section, and one assumes that the differentials da, $a \in A$ are also given by bounded operators, although D itself is as an analogue of a first-order differential operator typically unbounded.

If we want to bring our algebraic considerations into contact with these ideas, then A should resemble from now on an algebra of smooth functions on a compact Lie group. Woronowicz also developed a suitable framework for this:

Definition 4.1. A *compact quantum group* is a Hopf $*$-algebra A equipped with a functional $h : A \to \mathbb{C}$ satsifying $h(a_1)a_2 = h(a)$ and $h(a^*a) > 0$ for all $a \in A \setminus \{0\}$.

Thus the ground field becomes restricted to $k = \mathbb{C}$, and A is a $*$-algebra whose involution $a \mapsto a^*$ is compatible with Δ. The functional h is called the Haar state of A since it is classically given by integration of a function with respect to Haar measure. It is unique up to normalisation which we assume to be $h(1) = 1$, and it also satisfies $a_1 h(a_2) = h(a)$ for all $a \in A$ (for proofs and more details see e.g. [KS]).

Left multiplication in A defines a $*$-representation of A by bounded operators on the pre-Hilbert space A with Hermitian form given by (see [KS], p. 420-421)

$$(a, b) := h(a^*b), \quad a, b \in A. \tag{25}$$

Since $X(1) = \varepsilon(X)$ for all $X \in A^\circ$, the defining property of h implies $h(X \blacktriangleright a) = \varepsilon(X)h(a)$ for all $a \in A$. Furthermore, the $*$-structure of A induces one on A° given by $X^*(a) := \overline{X(S(a)^*)}$. Then $(X \blacktriangleright a)^* = S(X)^* \blacktriangleright a^*$. Finally, one has in any Hopf $*$-algebra $S^{-1} = * \circ S \circ *$ ([KS], Section 1.2.7). Thus

$$\begin{aligned}(a, X \blacktriangleright b) &= h((\varepsilon(X_1) \blacktriangleright a^*)(X_2 \blacktriangleright b)) = h(((X_2 S^{-1}(X_1)) \blacktriangleright a^*)(X_3 \blacktriangleright b)) \\ &= h(X_2 \blacktriangleright ((S^{-1}(X_1) \blacktriangleright a^*)b)) = h((S^{-1}(X) \blacktriangleright a^*)b) \\ &= h((X^* \blacktriangleright a)^*b) = (X^* \blacktriangleright a, b).\end{aligned} \tag{26}$$

What we want to point out is that the realisation of covariant differential calculi from Theorem 3.1 satisfies at least the minimal requirment that one can have:

Theorem 4.2. *If A is a compact quantum group, then any finite-dimensional covariant differential calculus over A can be realised by a twisted spectral triple on a Hilbert space H with all elements of the calculus given by bounded operators.*

Proof. Use the algebraic realisation as in the previous section and complete $H = A \otimes V$ to a Hilbert space using $(\cdot,\cdot)$ on A and any Hermitian product on V. Then the elements of the calculus act by bounded operators since they are given by finite matrices with entries in A. ■

5. Example: Quantum $SU(2)$ **[Wo1].** Let $A := \mathbb{C}_q[SU(2)]$, $q \in (0,1)$, be the Hopf $*$-algebra with generators a, b, c, d, relations

$$\begin{aligned} &ab = qba, \quad ac = qca, \quad bd = qdb,\\ &cd = qdc, \quad bc = cb, \quad ad - da = (q - q^{-1})bc,\\ &ad - qbc = 1, \quad a^* = d, \quad b^* = -qc \end{aligned} \tag{27}$$

and the usual Hopf algebra structure, see e.g. [KS]. This is a compact quantum group and has a Peter-Weyl-type vector space basis

$$\{t^l_{mn} \in A \,|\, l \in \mathbb{N}/2, m, n = -l, \dots, l\} \tag{28}$$

consisting of the matrix coefficients of the finite-dimensional irreducible corepresentations, with the Haar functional h given by projection onto $t^0_{00} = 1$.

The Hopf dual A° contains the Hopf $*$-subalgebra $U := U_q(\mathfrak{su}(2))$ with generators $K^{\pm 1} = (K^{\pm 1})^*, E, F = E^*$ having relations

$$KEK^{-1} = qE, \quad KFK^{-1} = q^{-1}F, \quad EF - FE = \frac{K^2 - K^{-2}}{q - q^{-1}} \tag{29}$$

and coproduct

$$\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}, \quad \Delta(X) = X \otimes K + K^{-1} \otimes X, \quad X = E, F. \tag{30}$$

The nonvanishing pairings of these generators with those of A are

$$\langle K^{\pm 1}, a\rangle = \langle K^{\mp 1}, d\rangle = q^{\mp 1/2}, \quad \langle E, c\rangle = \langle F, b\rangle = 1. \tag{31}$$

In [He] Heckenberger classified the 3-dimensional covariant differential calculi over $A = \mathbb{C}_q[SU(2)]$. Requiring that the universal higher order calculi share some natural properties with the classical de Rham complex of $SU(2)$ he obtained a list of 7 nonisomorphic calculi which have essentially all the same algebraic properties.

The aim of this section is to realise one of them, namely No. 10 in Heckenberger's final list [He], pp. 234-235, by a twisted spectral triple. The functionals X_i of this calculus are given by

$$X_1 = X_1^* := \frac{2}{q - q^{-1}}(1 - K^2), \quad X_2 := q^{-1/2}FK, \quad X_3 := X_2^* = q^{-1/2}KE, \tag{32}$$

and the appearing corepresentation of U is simply

$$f^i_j = \delta_{ij}K^2. \tag{33}$$

In other words, one has

$$\Delta(X_i) = 1 \otimes X_i + X_i \otimes K^2, \quad i = 1, 2, 3, \tag{34}$$

and for all $a \in A, \omega \in \Omega^1_{\mathrm{inv}}$

$$\omega \triangleleft a = \sigma^{-1}(a) \triangleright \omega, \quad \sigma(a) := K^{-2} \blacktriangleright a. \tag{35}$$

To obtain this calculus as in Section 3 we can use as V the classical spinor space $\mathbb{C}^4$, that is, we set $H := A^2$ whose elements will be written as column vectors $\psi = \left(\begin{smallmatrix}\psi_+\\ \psi_-\end{smallmatrix}\right)$, $\psi_\pm \in A$. We fix quantum gamma-matrices $\gamma^i \in M_2(\mathbb{C})$ as

$$\gamma^1 := \gamma^0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^2 := \lambda\gamma^0 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \gamma^3 := \bar{\lambda}\gamma^0 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{36}$$

with

$$\lambda \in \mathbb{C} \setminus \{0\}, \quad \gamma^0 := \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix} \tag{37}$$

and then introduce $\partial_i \in U = U_q(\mathfrak{su}(2)), i = 1, 2, 3$, and D by

$$\partial_i := K^{-2} X_i, \quad D := \sum_{i=1}^{3} \gamma^i \partial_i, \tag{38}$$

where X_i are as in (32). Then by the reasoning in Section 3 the calculus associated to (A, H, D) is isomorphic to Heckenberger's abstract one. Note that this is not the calculus studied in [FP] which is in fact not compatible with the $*$-structure of $SU_q(2)$ but rather with that of $SL_q(2, \mathbb{R})$ defined for $q \in \mathbb{C}, |q| = 1$. Note further that this calculus is also different from Woronowicz's original 3D-calculus.

The explicit form of the gamma-matrices was chosen in particular to have an essentially self-adjoint operator D when turning H into a pre-Hilbert space using

$$(\!(\phi, \psi)\!) := (\phi_+, \psi_+) + (\phi_-, \psi_-), \quad \phi, \psi \in H. \tag{39}$$

In fact, the γ^i need only to be linearly independent for the differential calculus to be the one we want to represent, and D is symmetric on H, $(\!(D\phi, \psi)\!) = (\!(\phi, D\psi)\!)$ for all $\phi, \psi \in H$, provided that

$$(\gamma^1)^* = \gamma^1, \quad (\gamma^2)^* = q^{-2}\gamma^3. \tag{40}$$

If $H_l \subset H$ consists of $\psi \in H$ whose components are linear combinations of the $t^l_{mn} \in A$ from (28) with fixed l, then $H = \bigoplus_l H_l$, $H_l \perp H_{l'}$, $\dim(H_l) < \infty$, and $DH_l \subset H_l$. Thus D is even essentially self-adjoint.

The covariance of the triple (A, H, D) can be implemented in form of a right action of $U = U_q(\mathfrak{su}(2))$ on H that commutes with D. We point out that in order to have compatibility of the $*$-structure of U with the one arising on $\mathrm{End}_\mathbb{C}(H)$ from $(\!(\cdot,\cdot)\!)$ one has to define the right U-action in a nonstandard way: there is a canonical right action of A° on A given by

$$a \blacktriangleleft X := a_2 X_(a_1), \quad a \in A, X \in A^\circ. \tag{41}$$

However, for this action a computation as in (26) gives

$$\begin{aligned}(a, b \blacktriangleleft X) &= h((a^* \blacktriangleleft \varepsilon(X_1))(b \blacktriangleleft X_2)) = h((a^* \blacktriangleleft (S(X_1)X_2))(X_3 \blacktriangleright b)) \\ &= h(((a^* \blacktriangleleft S(X_1))b) \blacktriangleleft X_2) = h((a^* \blacktriangleleft S(X))b) \\ &= h((a \blacktriangleleft S^2(X)^*)^* b) = (a \blacktriangleleft S^2(X)^*, b).\end{aligned} \tag{42}$$

Since $S^2(X) = K^2 X K^{-2}$ and $K = K^*$, the twisted right action

$$\pi(X)a := a \blacktriangleleft K^{-1} X K \tag{43}$$

thus defines a right $*$-action of U on A and hence on H,

$$(\pi(X)a, b) = (a, \pi(X^*)b) \quad \Rightarrow \quad ((\pi(X)\phi, \psi)) = ((\phi, \pi(X^*)\psi)). \tag{44}$$

Since $\blacktriangleright, \blacktriangleleft$ turn A and hence H into a U-bimodule, and since the action of D is based on $\blacktriangleright$, we have

$$[D, \pi(X)] = 0 \in \mathrm{End}_{\mathbb{C}}(H) \quad \forall X \in U. \tag{45}$$

Since the ∂_i commute according to

$$\begin{aligned} &q\partial_2\partial_1 - q^{-1}\partial_1\partial_2 = -2\partial_2, \\ &q^{-1}\partial_3\partial_1 - q\partial_1\partial_3 = 2\partial_3, \\ &q^{-1}\partial_3\partial_2 - q\partial_2\partial_3 = -\partial_1 - \frac{q-q^{-1}}{4}\partial_1^2, \end{aligned} \tag{46}$$

the square of D is given for general γ^i by

$$\begin{aligned} D^2 = &\left((\gamma^1)^2 - \frac{q-q^{-1}}{4}q\gamma^3\gamma^2\right)\partial_1^2 + (\gamma^2)^2\partial_2^2 + (\gamma^3)^2\partial_3^2 \\ &+ q^{-1}(q\gamma^1\gamma^2 + q^{-1}\gamma^2\gamma^1)\partial_1\partial_2 + q(q^{-1}\gamma^1\gamma^3 + q\gamma^3\gamma^1)\partial_1\partial_3 \\ &+ (q^{-1}\gamma^2\gamma^3 + q\gamma^3\gamma^2)q\partial_2\partial_3 - q\gamma^3\gamma^2\partial_1 - 2q^{-1}\gamma^2\gamma^1\partial_2 + 2q\gamma^3\gamma^1\partial_3. \end{aligned}$$

With our particular ansatz for the γ^i we have

$$\begin{aligned} &(\gamma^0)^2 = (\gamma^1)^2 = [2]_q\gamma^0 - 1, \quad (\gamma^2)^2 = (\gamma^3)^2 = 0, \\ &q^{-1}\gamma^2\gamma^1 = -q\gamma^1\gamma^2 = -\gamma^2, \quad q\gamma^3\gamma^1 = -q^{-1}\gamma^1\gamma^3 = \gamma^3, \\ &q\gamma^3\gamma^2 = -q^{-1}\gamma^2\gamma^3 + |\lambda|^2\gamma^0 = \frac{|\lambda|^2}{q-q^{-1}}(q\gamma^0 - 1) = \frac{|\lambda|^2}{[2]_q}(1 - q\gamma^1) \end{aligned} \tag{47}$$

with $[n]_q := \frac{q^n - q^{-n}}{q-q^{-1}}$. Therefore we get

$$\begin{aligned} D^2 - 2D &= \left((\gamma^1)^2 - \frac{q-q^{-1}}{4}q\gamma^3\gamma^2\right)\partial_1^2 - (2\gamma^1 + q\gamma^3\gamma^2)\partial_1 + |\lambda|^2\gamma^0 q\partial_2\partial_3 \\ &= \left([2]_q\gamma^0 - 1 - \frac{|\lambda|^2}{4}(q\gamma^0 - 1)\right)\partial_1^2 + |\lambda|^2\gamma^0 q\partial_2\partial_3 \\ &\quad + \left(2\left(\frac{[2]_q}{q-q^{-1}}\gamma^0 - \frac{2}{q-q^{-1}}\right) - \frac{|\lambda|^2}{q-q^{-1}}(q\gamma^0 - 1)\right)\partial_1. \end{aligned} \tag{48}$$

This in turn suggests to take $\lambda = 2$ since then the above reduces to

$$\ldots = \gamma^0(q^{-1}\gamma^0\partial_1^2 - 2\partial_1 + 4q\partial_2\partial_3) = 4\gamma^0 K^{-2}\left(C - \frac{[2]_q}{(q-q^{-1})^2}\right), \tag{49}$$

where

$$C = EF + \frac{q^{-1}K^2 + qK^{-2}}{(q-q^{-1})^2} = FE + \frac{qK^2 + q^{-1}K^{-2}}{(q-q^{-1})^2} \tag{50}$$

is the quantum Casimir operator of U which is related to the ∂_i by

$$\frac{q^{-1}}{4}\partial_1^2 + \mu\partial_1 + q\partial_2\partial_3 = K^{-2}\left(C + 2\frac{(q-q^{-1})\mu - q^{-1}}{(q-q^{-1})^2}\right) - \frac{1+2\mu}{q-q^{-1}}. \tag{51}$$

Both C and K^{-2} act diagonally on the Peter-Weyl basis (28), so $(D-1)^2$ acts diagonally on the basis of H given by

$$\psi^{l+}_{mn} := \begin{pmatrix} t^l_{mn} \\ 0 \end{pmatrix}, \quad \psi^{l-}_{mn} := \begin{pmatrix} 0 \\ t^l_{mn} \end{pmatrix}, \quad l \in \mathbb{N}/2, m, n = -l, \dots, l. \tag{52}$$

Inserting the explicit formulas for the action of C and K from [KS], pp. 61-62 (note that there K^2 is written as K and that the formulas for C is slightly incorrect),

$$K \blacktriangleright t^l_{mn} = q^n t^l_{mn}, \quad C \blacktriangleright t^l_{mn} = \frac{q^{2l+1} + q^{-(2l+1)}}{(q-q^{-1})^2} t^l_{mn} \tag{53}$$

one obtains the corresponding eigenvalues

$$\lambda^{l\pm}_{mn} = 1 + \frac{4q^{\mp 1}}{(q-q^{-1})^2} q^{-2n}((q^{2l+1} + q^{-2l-1}) - [2]_q) \tag{54}$$

which in the classical limit $q \to 1$ become

$$\lim_{q\to 1} \lambda^{l\pm}_{mn} = (2l+1)^2. \tag{55}$$

On the other hand, we have for $n = -l$

$$\lim_{l\to\infty} \lambda^{l\pm}_{m-l} = 1 + q^{\pm 1} \frac{4q^{-1}}{(q-q^{-1})^2} \tag{56}$$

so that the resolvent of D is not compact. The analytic theory from [CM] is therefore not directly applicable to this spectral triple. In a word, we think the mystery in the interplay between spectral triples and covariant differential calculi on quantum groups will live on.

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