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SPECTRUM GENERATING FUNCTIONS FOR OSCILLATORS IN WIGNER'S QUANTIZATION

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Abstract. The n-dimensional (isotropic and non-isotropic) harmonic oscillator is studied as a Wigner quantum system. In particular, we focus on the energy spectrum of such systems. We show how to solve the compatibility conditions in terms of $\mathfrak{osp}(1|2n)$ generators, and also recall the solution in terms of $\mathfrak{gl}(1|n)$ generators. A method is described for determining a spectrum generating function for an element of the Cartan subalgebra when working with a representation of any Lie (super)algebra. Here, the character of the representation at hand plays a crucial role. This method is then applied to the n-dimensional isotropic harmonic oscillator, yielding explicit formulas for the energy eigenvalues and their multiplicities.

1. Introduction. Wigner quantization is an algebraic alternative to canonical quantization. For a Wigner quantum system described by a Hamiltonian $\hat{H}$, given in terms of (generalized) position operators $\hat{q}_j$ and momentum operators $\hat{p}_j$ $(j = 1, \ldots, n)$, the canonical commutation relations are not required. Instead, the compatibility between the Heisenberg equations and the operator form of Hamilton's equations is required [15, 12, 6, 13]. This algebraic compatibility condition is used as quantization relation. As a consequence, Wigner quantization includes canonical quantization as a special case. But in general, Wigner quantization allows far more solutions than just the canonical one.

Since the compatibility conditions of Wigner quantization are described by algebraic relations in terms of the operators $\hat{q}_j$ and $\hat{p}_j$, one is led to an algebra (in fact a $\star$-

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algebra) and to the (unitary) representations of this algebra. So, the first task in Wigner quantization consists of identifying this algebra (which is, for many examples, a Lie superalgebra). The second task is to study appropriate representations of the algebra, and finally determine the spectrum of physical operators in these representations.

In this contribution, we present this approach for the n-dimensional (isotropic and non-isotropic) harmonic oscillator. We derive the compatibility conditions, and the corresponding algebra. It is shown that the Lie superalgebras $\mathfrak{osp}(1|2n)$ and $\mathfrak{gl}(1|n)$ are realizations of this algebra [10]. For each of these Lie superalgebras, we describe a class of unitary representations characterized by a parameter p. The spectrum of the Hamiltonian $\hat{H}$ is given in these representations, and described by a spectrum generating function. It is interesting to note that for $\mathfrak{osp}(1|2n)$, the case $p = 1$ coincides with the canonical solution; for $p \neq 1$ one obtains "deformations" of the canonical spectrum (but still an infinite spectrum with equidistant energy levels). For $\mathfrak{gl}(1|n)$ one obtains, interestingly, a finite spectrum with equidistant energy levels.

Our approach here is primarily mathematical, but Wigner quantum systems are relevant in a physical context. Wigner quantization belongs to the field of nonstandard quantization, more particularly to the class of models of noncommutative quantum systems (since the operators $\hat{q}_j$ do not commute). There is quite some interest in such models, and more generally in theories with an underlying noncommutative geometry (see e.g. [2] and references therein). This interest is not only purely theoretical, but also inspired by e.g. the prediction of string theory that the geometry of space becomes noncommutative at very small distances [4]. Also q-deformations of canonical commutation relations have drawn further attention to nonstandard commutation relations. In this context, Wigner quantization has the advantage that deformations of commutation relations are not "put in by hand", e.g. by inserting some extra deformation parameters. On the contrary, in Wigner quantization the noncommutativity (or deformation of the canonical commutation relations) simply follows from some other first principles, namely the earlier mentioned compatibility conditions. Furthermore, Wigner quantization often leads to solutions of the quantization conditions in a finite-dimensional Hilbert space. Also here, there seems to be renewed interest in such systems [14]. The solutions discussed in Section 4 offer examples of such finite-dimensional spaces.

We present only a brief overview of the problem and its solutions in this paper; for more details, the reader is referred to [10].

2. Wigner quantization of the oscillator system. The Hamiltonian of the n-dimensional harmonic oscillator with mass m is given by

$$\hat{H} = \frac{1}{2m}\sum_{j=1}^{n} \hat{p}_j^2 + \frac{m}{2}\sum_{j=1}^{n} \omega_j^2 \hat{q}_j^2.$$

We shall consider both the non-isotropic case (ω_j's different) and the isotropic case (all ω_j's equal).

In order to treat this as a Wigner quantum system [15, 12], the position and momentum operators are no longer required to satisfy the canonical commutation relations, but one imposes the compatibility of the Heisenberg equations and (the operator form of)

Hamilton's equations. All other axioms of quantum mechanics are retained. In particular, the usual self-adjointness should still hold:

$$\hat{p}_j^\dagger = \hat{p}_j, \qquad \hat{q}_j^\dagger = \hat{q}_j.$$

Under some mild conditions for $\hat{H}$ as a function of the $\hat{q}_j$ and $\hat{p}_j$, the canonical commutation relations imply the compatibility of Heisenberg's and Hamilton's equations. In other words, canonical quantization is a special case of Wigner quantization.

Let us now work out the compatibility conditions for the system under consideration. The operator form of Hamilton's equations reads

$$\dot{\hat{q}}_j = \frac{\partial \hat{H}}{\partial \hat{p}_j} = \frac{1}{m}\hat{p}_j, \qquad \dot{\hat{p}}_j = -\frac{\partial \hat{H}}{\partial \hat{q}_j} = -m\omega_j^2 \hat{q}_j \qquad (j = 1, \ldots, n),$$

whereas the Heisenberg equations read

$$\dot{\hat{q}}_j = \frac{i}{\hbar}[\hat{H}, \hat{q}_j], \qquad \dot{\hat{p}}_j = \frac{i}{\hbar}[\hat{H}, \hat{p}_j] \qquad (j = 1, \ldots, n).$$

So the compatibility conditions are:

$$[\hat{H}, \hat{q}_j] = -i\frac{\hbar}{m}\hat{p}_j, \qquad [\hat{H}, \hat{p}_j] = i\hbar m\omega_j^2 \hat{q}_j \qquad (j = 1, \ldots, n).$$

It is appropriate to introduce linear combinations of the operators $\hat{q}_j$ and $\hat{p}_j$ by

$$a_j^\mp = \sqrt{\frac{m\omega_j}{2\hbar}}\hat{q}_j \pm \frac{i}{\sqrt{2m\hbar\omega_j}}\hat{p}_j \qquad (j = 1, \ldots, n).$$

In terms of these new operators, the expression of the Hamiltonian becomes

$$\hat{H} = \frac{\hbar}{2}\sum_{j=1}^n \omega_j (a_j^+ a_j^- + a_j^- a_j^+) = \frac{\hbar}{2}\sum_{j=1}^n \omega_j \{a_j^+, a_j^-\},$$

and the new form of the compatibility conditions is

$$\sum_{j=1}^n \left[\omega_j \{a_j^+, a_j^-\}, a_k^\pm\right] = \pm 2\omega_k a_k^\pm \qquad (k = 1, \ldots, n). \tag{1}$$

The self-adjointness of $\hat{q}_j$ and $\hat{p}_j$ implies $(a_j^\pm)^\dagger = a_j^\mp$ for $j = 1, \ldots, n$. So we are led to the following definition:

DEFINITION 2.1. Let $\mathcal{A}$ be the $\star$-algebra generated by $2n$ generators $a_j^\pm$ $(j = 1, \ldots, n)$ with $\star$-relations $(a_j^\pm)^\star = a_j^\mp$ and with defining relations (1).

The main questions to address here are: What is the structure of $\mathcal{A}$? Can one give realizations of $\mathcal{A}$ in terms of known algebras? What are the unitary Hilbert space representations of $\mathcal{A}$? Surprisingly, a complete answer to the first question is known only for $n = 1$. For $n > 1$ certain realizations of $\mathcal{A}$ are known, and some (but not all) classes of unitary Hilbert space representations have been studied [7, 10]. We shall discuss two realizations of $\mathcal{A}$ in terms of Lie superalgebras in the following sections, and derive some spectral properties in a class of representations.

3. Solutions in terms of $\mathfrak{osp}(1|2n)$. The Lie superalgebra $\mathfrak{osp}(1|2n)$ can be defined as an algebra with $2n$ generators $b_j^\pm$ subject to certain triple relations in terms of commutators and anti-commutators:

THEOREM 3.1 (Ganchev, Palev [3]). *The Lie superalgebra generated by* $2n$ *odd generators* $b_j^\pm$ $(j = 1, \ldots, n)$ *subject to the relations*

$$[\{b_j^\xi, b_k^\eta\}, b_l^\epsilon] = (\epsilon - \xi)\delta_{jl} b_k^\eta + (\epsilon - \eta)\delta_{kl} b_j^\xi,$$

where $j, k, l \in \{1, \ldots, n\}$ *and* $\eta, \epsilon, \xi \in \{+, -\}$ *(to be interpreted as* $+1$ *and* -1 *in the above algebraic relation) is the orthosymplectic Lie superalgebra* $\mathfrak{osp}(1|2n)$.

Now it is easy to verify that

$$a_j^- = \sigma_j b_j^-, \quad a_j^+ = \sigma_j^* b_j^+, \quad \text{with} \quad |\sigma_j| = 1$$

satisfy indeed (1), and that the $\star$-relations are equivalent with $(b_j^\pm)^\dagger = b_j^\mp$. Thus $\mathfrak{osp}(1|2n)$ is a *realization* of $\mathcal{A}$. Consequently, any unitary representation of $\mathfrak{osp}(1|2n)$ is also a $\star$-representation for $\mathcal{A}$.

Note that, in this realization, the Hamiltonian takes the following form:

$$\hat{H} = \frac{\hbar}{2}\sum_{j=1}^n \omega_j \{a_j^-, a_j^+\} = \frac{\hbar}{2}\sum_{j=1}^n \omega_j \{b_j^-, b_j^+\} = \hbar \sum_{j=1}^n \omega_j h_j, \tag{2}$$

where the $h_j = \{b_j^-, b_j^+\}/2$ $(j = 1, \ldots, n)$ span the Cartan subalgebra of $\mathfrak{osp}(1|2n)$.

The unitary representations of $\mathfrak{osp}(1|2n)$ for which the relations $(b_j^\pm)^\dagger = b_j^\mp$ hold are infinite-dimensional. The structure of all these representations is not known, but one interesting class of unitary irreducible representations (unirrep) is known. These are the *paraboson Fock spaces* $V(p)$, labeled by a parameter p [9]. We shall consider this class of representations here.

THEOREM 3.2 (Lievens, Stoilova, Van der Jeugt [9]). *The* $\mathfrak{osp}(1|2n)$ *representation* $V(p)$ *with lowest weight* $(\frac{p}{2}, \ldots, \frac{p}{2})$ *is a unirrep if and only if* $p \in \{1, 2, \ldots, n-1\}$ *or* $p > n-1$. *For* $p > n-1$, *the character of this representation is given by*

$$\operatorname{char} V(p) = \frac{(x_1 \cdots x_n)^{p/2}}{\prod_i (1 - x_i) \prod_{j<k} (1 - x_j x_k)} = (x_1 \cdots x_n)^{p/2} \sum_\lambda s_\lambda(x_1, \ldots, x_n),$$

where the sum is over all partitions λ *and* s_λ *is the common Schur symmetric function in the variables* $x_1, \ldots, x_n$. *For* $p \in \{1, 2, \ldots, n-1\}$, *the character of* $V(p)$ *is*

$$\operatorname{char} V(p) = (x_1 \cdots x_n)^{p/2} \sum_{\lambda,\ \ell(\lambda) \le p} s_\lambda(x_1, \ldots, x_n)$$

where $\ell(\lambda)$ *is the* length *of the partition* λ. *So in this case, the sum is over all partitions* λ *of length at most* p.

For partitions, symmetric functions, Schur functions etc. we follow the standard notation of Macdonald [11].

Note that, in general, the character of a representation R is an expression (in this case a series) in the variables $x_1, \dots, x_n$ of the form

$$\operatorname{char} R = \sum_{r_1,\dots,r_n} d_{r_1,\dots,r_n} x_1^{r_1} \cdots x_n^{r_n},$$

where $(r_1, \dots, r_n)$ is a weight of the representation and $d_{r_1,\dots,r_n}$ is the dimension of the corresponding weight space (in other words, the *multiplicity* of the weight). Since the action of the Hamiltonian (2) on a vector v of weight $(r_1, \dots, r_n)$ is given by $\hat{H}\, v = (\sum \hbar\omega_j r_j)\, v$, it follows that the spectrum generating function for $\hat{H}$ in such a representation R is given by

$$\operatorname{spec} \hat{H} = \sum_{r_1,\dots,r_n} d_{r_1,\dots,r_n} t^{\alpha_1 r_1} \cdots t^{\alpha_n r_n}.$$

In other words, to get the spectrum generating function, one should make the formal substitution $x_j \to t^{\hbar\omega_j}$ in the character of the representation.

Let us perform this procedure for the unirreps under consideration. This gives (in the case $p \in \{1, 2, \dots, n-1\}$; the other case is similar)

$$\operatorname{spec} \hat{H} = t^{\hbar p(\omega_1+\cdots+\omega_n)/2} \sum_{\lambda,\ \ell(\lambda)\le p} s_\lambda(t^{\hbar\omega_1}, \dots, t^{\hbar\omega_n}).$$

This expression simplifies a lot in the *isotropic case*, with

$$\omega_1 = \cdots = \omega_n = \omega.$$

It should be emphasized that in the remainder of this section we are dealing with this isotropic case. Putting $z = t^{\hbar\omega}$, all Schur functions appearing in the spectrum generating function are of the form $s_\lambda(z, \dots, z)$, and for these there is a well known expression [11]:

$$s_\lambda(z, \dots, z) = z^{|\lambda|} s_\lambda(1, \dots, 1) = z^{|\lambda|} \binom{n}{\lambda'},$$

this last symbol being the generalized binomial coefficient [11, I.3, example 4]. Thus one obtains:

$$\begin{aligned}\operatorname{spec} \hat{H} &= z^{np/2} \sum_{\lambda,\ \ell(\lambda)\le p} s_\lambda(z, \dots, z)\\ &= \sum_{k\ge 0}\ \sum_{\lambda,\ |\lambda|=k,\ \ell(\lambda)\le p} s_\lambda(1, \dots, 1) t^{\hbar\omega(np/2+k)}.\end{aligned}$$

So we have [10]

Theorem 3.3. *In the representations $V(p)$ with $p \in \{1, 2, \dots, n-1\}$ described in Theorem 3.2, the spectrum of the Hamiltonian consists of equidistant energy levels*

$$E_k^{(p)} = \hbar\omega(np/2 + k), \quad k = 0, 1, 2, 3, \dots$$

with spacing $\hbar\omega$ and with multiplicities (degeneracies) given by

$$\mu(E_k^{(p)}) \equiv \sum_{\lambda,\ |\lambda|=k,\ \ell(\lambda)\le p} s_\lambda(1, \dots, 1) = \sum_{\lambda,\ |\lambda|=k,\ \ell(\lambda)\le p} \binom{n}{\lambda'}. \tag{3}$$

In the representations $V(p)$ with $p > n-1$, the expressions are the same, but the condition $\ell(\lambda) \le p$ should be dropped in (3).

The multiplicities can be computed more explicitly, see [10] for more details. For example, for $p = 1$:

$$\mu(E_k^{(p=1)}) = \binom{n}{(k)'} = (-1)^k \binom{-n}{(k)} = (-1)^k \binom{-n}{k} = \binom{n+k-1}{k}.$$

Since $p = 1$ corresponds to the canonical case, we obtain here the well known energy multiplicities. Next, for $p = 2$ one finds

$$\mu(E_{2k}^{(p=2)}) = \binom{n+k-1}{k}^2, \qquad \mu(E_{2k+1}^{(p=2)}) = \binom{n+k-1}{k}\binom{n+k}{k+1}.$$

In the generic case, for $p > n-1$, one has

$$\mu(E_k^{(p>n-1)}) = \sum_{l=0}^{k} (-1)^l \binom{\binom{n+1}{2} + k - l - 1}{k-l} \binom{\binom{n}{2} + l - 1}{l}.$$

Let us summarize these results in a table, for $n = 3$ (the 3-dimensional harmonic oscillator), where we give for each of the relevant p-values the spectrum generating function (GF) as a function of $z = t^{\hbar\omega}$, the energy levels E_k ($k = 0, 1, 2, \ldots$), and the corresponding multiplicities (Table 1). More explicitly, let us also present the values of the multiplicities for the first few energy levels in a table (Table 2).

Table 1. Spectrum generating functions (GF) for the 3-dimensional harmonic oscillator; energy levels, and corresponding multiplicities (for different p-values)

p	GF	levels	multiplicities
1	$\dfrac{z^{1/2}}{(1-z)^3}$	$\hbar\omega(\frac{3}{2}+k)$	$\mu(E_k^{(1)}) = \binom{k+2}{2}$
2	$\dfrac{z^3(1+z+z^2)}{(1-z^2)^3(1-z)^2}$	$\hbar\omega(3+k)$	$\mu(E_{2k}^{(2)}) = \binom{k+2}{2}^2$, $\mu(E_{2k+1}^{(2)}) = \binom{k+2}{2}\binom{k+3}{2}$
> 2	$\dfrac{z^{3p/2}}{(1-z^2)^3(1-z)^3}$	$\hbar\omega(\frac{3p}{2}+k)$	$\mu(E_{2k}) = \frac{4k+5}{5}\binom{k+4}{4}$, $\mu(E_{2k+1}) = \frac{4k+15}{5}\binom{k+4}{4}$

Table 2. Explicit multiplicities of some energy levels for the 3-dimensional harmonic oscillator

	$\mu(E_0^{(p)})$	$\mu(E_1^{(p)})$	$\mu(E_2^{(p)})$	$\mu(E_3^{(p)})$	$\mu(E_4^{(p)})$	$\mu(E_5^{(p)})$	$\mu(E_6^{(p)})$
$p = 1$	1	3	6	10	15	21	28
$p = 2$	1	3	9	18	36	60	100
$p > 2$	1	3	9	19	39	69	119

The conclusion of this $\mathfrak{osp}(1|2n)$ type of solution for the Wigner quantization is clear: the quantization involves an extra parameter p. When $p = 1$, the canonical quantization is obtained. For other values of p, the spectrum itself is similar to that of canonical quantization. The main difference is the lowest energy level (which is p-dependent), and the multiplicities of the energy levels (which are also p-dependent).

4. Solutions in terms of $\mathfrak{gl}(1|n)$. A second algebraic solution of the compatibility conditions is obtained in terms of the general linear Lie superalgebra $\mathfrak{gl}(1|n)$ (for $n > 1$).

In the defining representation, $\mathfrak{gl}(1|n)$ has a basis consisting of $(n+1)\times(n+1)$ matrices e_{ij} $(i,j=0,1,\ldots,n)$ with $(e_{ij})_{kl}=\delta_{ik}\delta_{jl}$. The odd basis elements (those of degree 1) of $\mathfrak{gl}(1|n)$ are e_{0j} and e_{j0} $(j=1,\ldots,n)$; all other basis elements are even (of degree 0). The Lie superalgebra bracket (which can be a commutator or anti-commutator for basis elements) is determined by

$$[\![e_{ij},e_{kl}]\!]=\delta_{jk}e_{il}-(-1)^{\mathsf{deg}(e_{ij})\,\mathsf{deg}(e_{kl})}\delta_{il}e_{kj}\qquad(i,j,k,l=0,1,\ldots,n).$$

Now one can verify that

$$a_j^-=\sqrt{2|\beta_j|/\omega_j}\,e_{j0},\qquad a_j^+=\mathrm{sign}(\beta_j)\sqrt{2|\beta_j|/\omega_j}\,e_{0j},$$

with coefficients

$$\beta_j=-\omega_j+(\sum_{k=1}^n\omega_k)/(n-1)$$

also satisfy (1), and that the $\star$-relations are equivalent to $(e_{0j})^\dagger=\mathrm{sign}(\beta_j)e_{j0}$. In other words, $\mathfrak{gl}(1|n)$ is a second realization of $\mathcal{A}$, and all unitary representations of $\mathfrak{gl}(1|n)$ are $\star$-representations of $\mathcal{A}$. The choice of unitary representations depends on the choice of the $\star$-relations, i.e. on the choice of the signs of the β_j. It is appropriate to choose all β_j positive, because then a class of unitary representations is actually known: implicitly, this means we are working with the compact form $\mathfrak{u}(1|n)$ of $\mathfrak{gl}(1|n)$. So from now on let us assume that $\beta_j>0$ for all j.

In the present realization, the Hamiltonian can be rewritten as

$$\hat{H}=\hbar\Big(\beta e_{00}+\sum_{j=1}^n\beta_je_{jj}\Big)\qquad\Big(\beta=\sum_{j=1}^n\beta_j\Big),$$

i.e. $\hat{H}$ is again an element of the Cartan subalgebra of $\mathfrak{gl}(1|n)$, for which the elements e_{jj} $(j=0,1,\ldots,n)$ form a basis.

The unitary representations of $\mathfrak{u}(1|n)$ are finite-dimensional, and they have been classified [5, 8]. They consist of 3 classes: covariant tensor representations, contravariant tensor representations, and certain typical representations (with a condition for the highest weight). Let us consider here the class of covariant tensor representations; these are characterized by a partition $\lambda=(\lambda_1,\lambda_2,\ldots)$ with $\lambda_2\le n$. The character of this representation is known, and given by a so-called supersymmetric Schur function $s_\lambda(x_0|x)=s_\lambda(x_0|x_1,x_2,\ldots,x_n)$ [1]. Following the technique of the previous section, it is now a matter of making the substitution $x_0\to t^{\hbar\beta}$ and $x_j\to t^{\hbar\beta_j}$ in this supersymmetric Schur function in order to find the spectrum generating function. Once more, these Schur functions simplify in the isotropic case, where $\omega_1=\cdots=\omega_n=\omega$; then

$$\beta_k=\frac{\omega}{n-1},\qquad\beta=\frac{n\omega}{n-1}.$$

In other words, one finds the spectrum generating function by means of the substitution

$$x_0\to t^{\hbar\beta}=t^{\frac{\hbar\omega n}{n-1}}=:z^n,\qquad x_k\to t^{\hbar\beta_k}=t^{\frac{\hbar\omega}{n-1}}=:z\qquad(k=1,\ldots,n).$$

As an example, consider the spectrum generating function in the case that the covariant representation is *typical* (this means here $\lambda_1 \geq n$). This gives:

$$\begin{aligned}\operatorname{spec}\hat{H} &= s_\lambda(z^n|z,z,\ldots,z) \qquad (\text{where } z = e^{\hbar\omega/(n-1)})\\ &= t^{\hbar\omega(|\lambda|/(n-1)+\lambda_1-n)}(1+t^{\hbar\omega})^n\binom{n}{(\lambda_2,\lambda_3,\ldots)}\end{aligned}$$

where the last symbol is again a generalized binomial [11, I.3, ex. 4]. The important point to notice here is that this results once again in equidistant energy levels. However, the spectrum is *finite* (after all, we are dealing with finite-dimensional unirreps). An expression for the ground level and the highest level is [10]:

$$E_0^{(\lambda)} = \hbar\omega\left(\frac{|\lambda|}{n-1}+\lambda_1-n\right), \qquad E_n^{(\lambda)} = \hbar\omega\left(\frac{|\lambda|}{n-1}+\lambda_1\right).$$

The multiplicity of the k-th energy level is given by

$$\mu(E_k^{(\lambda)}) = \binom{n}{k}\binom{n}{(\lambda_2,\lambda_3,\ldots)}.$$

We have presented here only the case of typical covariant representations; for more information on the atypical covariant representations, see [10].

The conclusion of the $\mathfrak{gl}(1|n)$ type of solution for the Wigner quantization is: this quantization involves extra partition parameters $\lambda_1, \lambda_2, \ldots$. The canonical quantization does not belong to this class. The energy spectrum is equidistant (with distances $\hbar\omega$) but *finite*; the lowest and highest energy levels – and their multiplicities – depend on the representation parameters λ.

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