ALGEBRA, GEOMETRY AND MATHEMATICAL PHYSICS
BANACH CENTER PUBLICATIONS, VOLUME 93
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2011

JACOBI OPERATORS OF QUANTUM COUNTERPARTS OF THREE-DIMENSIONAL REAL LIE ALGEBRAS OVER THE HARMONIC OSCILLATOR

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Abstract. Operadic Lax representations for the harmonic oscillator are used to construct the quantum counterparts of three-dimensional real Lie algebras. The Jacobi operators of these quantum algebras are explicitly calculated.

1. Introduction and outline of the paper. In Hamiltonian formalism, a mechanical system is described by the canonical variables q^i, p_i and their time evolution is prescribed by the Hamiltonian equations

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}. \tag{1}$$

By a Lax representation [3] of a mechanical system one means such a pair (L, M) of matrices (linear operators) L, M that the above Hamiltonian system may be represented as the Lax equation

$$\frac{dL}{dt} = ML - LM. \tag{2}$$

Thus, from the algebraic point of view, mechanical systems may be represented by linear operators, i.e by linear maps $V \to V$ of a vector space V. In particular, representation of the physical observables by linear operators is used in quantum mechanics and their time evolution is described by the Heisenberg equations. As a generalization of this one can pose the following question [4]: how can the time evolution of the linear operations (multiplications) $V^{\otimes n} \to V$ be described?

2010 *Mathematics Subject Classification*: Primary 81R05; Secondary 18D50.
Key words and phrases: operad, 3D real Lie algebras, harmonic oscillator, Lax representation, quantization.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc93-0-16

The algebraic operations (multiplications) can be seen as an example of the *operadic* variables [1]. If an operadic system depends on time one can speak about *operadic dynamics* [4]. The latter may be introduced by simple and natural analogy with the Hamiltonian dynamics. In particular, the time evolution of the operadic variables may be given by the operadic Lax equation. In [5, 6, 8], the low-dimensional binary operadic Lax representations for the harmonic oscillator were constructed. In [7] it was shown how the operadic Lax representations are related to the conservation of energy.

In [9], the operadic Lax representations were used to construct the quantum counterparts of the real three-dimensional Lie algebras in Bianchi classification over the harmonic oscillator. In this paper, the Jacobi operators of these quantum algebras are explicitly calculated.

2. Endomorphism operad and Gerstenhaber brackets. Let K be a unital associative commutative ring, V be a unital K-module, and $\mathcal{E}_V^n := \mathcal{E}nd_V^n := \mathrm{Hom}(V^{\otimes n}, V)$ ($n \in \mathbb{N}$). For an *operation* $f \in \mathcal{E}_V^n$, we refer to n as the *degree* of f and often write (when it does not cause confusion) f instead of $\deg f$. For example, $(-1)^f := (-1)^n$, $\mathcal{E}_V^f := \mathcal{E}_V^n$ and $\circ_f := \circ_n$. Also, it is convenient to use the *reduced* degree $|f| := n - 1$. Throughout this paper, we assume that $\otimes := \otimes_K$.

Definition 2.1 (endomorphism operad [1]). For $f \otimes g \in \mathcal{E}_V^f \otimes \mathcal{E}_V^g$ define the *partial compositions*

$$f \circ_i g := (-1)^{i|g|} f \circ (\mathrm{id}_V^{\otimes i} \otimes g \otimes \mathrm{id}_V^{\otimes(|f|-i)}) \in \mathcal{E}_V^{f+|g|}, \quad 0 \le i \le |f|.$$

The sequence $\mathcal{E}_V := \{\mathcal{E}_V^n\}_{n\in\mathbb{N}}$, equipped with the partial compositions $\circ_i$, is called the *endomorphism operad* of V.

Definition 2.2 (total composition [1]). The *total composition* $\circ\colon \mathcal{E}_V^f \otimes \mathcal{E}_V^g \to \mathcal{E}_V^{f+|g|}$ is defined by

$$f \circ g := \sum_{i=0}^{|f|} f \circ_i g \in \mathcal{E}_V^{f+|g|}, \quad |\circ| = 0.$$

The pair $\mathrm{Com}\,\mathcal{E}_V := \{\mathcal{E}_V, \circ\}$ is called the *composition algebra* of $\mathcal{E}_V$.

Definition 2.3 (Gerstenhaber brackets [1]). The *Gerstenhaber brackets* $[\cdot,\cdot]$ are defined in $\mathrm{Com}\,\mathcal{E}_V$ as a graded commutator by

$$[f, g] := f \circ g - (-1)^{|f||g|} g \circ f = -(-1)^{|f||g|}[g, f], \quad |[\cdot,\cdot]| = 0.$$

The *commutator algebra* of $\mathrm{Com}\,\mathcal{E}_V$ is denoted as $\mathrm{Com}^-\mathcal{E}_V := \{\mathcal{E}_V, [\cdot,\cdot]\}$. One can prove (e.g. [1]) that $\mathrm{Com}^-\mathcal{E}_V$ is a *graded Lie algebra*. The Jacobi identity reads

$$(-1)^{|f||h|}[f,[g,h]] + (-1)^{|g||f|}[g,[h,f]] + (-1)^{|h||g|}[h,[f,g]] = 0.$$

3. Operadic Lax pair. Assume that $K := \mathbb{R}$ or $K := \mathbb{C}$ and operations are differentiable. Dynamics in operadic systems (operadic dynamics) may be introduced by

Definition 3.1 (operadic Lax pair [4]). Allow a classical dynamical system to be described by the Hamiltonian system (1). An *operadic Lax pair* is a pair (μ, M) of homogeneous operations $\mu, M \in \mathcal{E}_V$ such that the Hamiltonian system (1) may be represented

as the *operadic Lax equation*

$$\frac{d\mu}{dt} = [M, \mu] := M \circ \mu - (-1)^{|M||\mu|}\mu \circ M.$$

The pair (L, M) is also called an *operadic Lax representations* of/for Hamiltonian system (1).

REMARK 3.2. Evidently, the degree constraints $|M| = |L| = 0$ give rise to ordinary Lax equation (2) [3]. In this paper we assume that $|M| = 0$.

The Hamiltonian of the harmonic oscillator (HO) is

$$H(q, p) = \frac{1}{2}(p^2 + \omega^2 q^2).$$

Thus, the Hamiltonian system of HO reads

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} = p, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q} = -\omega^2 q. \tag{3}$$

If μ is a linear algebraic operation we can use the above Hamilton equations to obtain

$$\frac{d\mu}{dt} = \frac{\partial \mu}{\partial q}\frac{dq}{dt} + \frac{\partial \mu}{\partial p}\frac{dp}{dt} = p\frac{\partial \mu}{\partial q} - \omega^2 q\frac{\partial \mu}{\partial p} = [M, \mu].$$

Therefore, we get the following linear partial differential equation for $\mu(q, p)$:

$$p\frac{\partial \mu}{\partial q} - \omega^2 q\frac{\partial \mu}{\partial p} = [M, \mu]. \tag{4}$$

By integrating (4) one can get collections of operations called [4] the *operadic* (Lax representations for/of) harmonic oscillator.

4. 3D binary anti-commutative operadic Lax representations for harmonic oscillator.

LEMMA 4.1. *Matrices*

$$L := \begin{pmatrix} p & \omega q & 0 \\ \omega q & -p & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M := \frac{\omega}{2}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

represent a 3D Lax representation for the harmonic oscillator.

DEFINITION 4.2 (quasi-canonical coordinates). For the harmonic oscillator define its *quasi-canonical coordinates* $A_\pm$ by

$$A_+^2 - A_-^2 = 2p, \quad A_+A_- = \omega q. \tag{5}$$

THEOREM 4.3 (see [8]). *Let $C_\nu \in \mathbb{R}$ $(\nu = 1, \ldots, 9)$ be arbitrary real-valued parameters, such that*

$$C_2^2 + C_3^2 + C_5^2 + C_6^2 + C_7^2 + C_8^2 \neq 0. \tag{6}$$

Let M be defined as in Lemma 4.1 and $\mu : V \otimes V \to V$ be an anti-commutative binary

operation in a 3D real vector space V with the structure functions

$$\begin{cases} \mu^1_{11} = \mu^1_{22} = \mu^1_{33} = \mu^2_{11} = \mu^2_{22} = \mu^2_{33} = \mu^3_{11} = \mu^3_{22} = \mu^3_{33} = 0 \\ \mu^1_{23} = -\mu^1_{32} = C_2 p - C_3 \omega q - C_4 \\ \mu^2_{13} = -\mu^2_{31} = C_2 p - C_3 \omega q + C_4 \\ \mu^1_{31} = -\mu^1_{13} = C_2 \omega q + C_3 p - C_1 \\ \mu^2_{23} = -\mu^2_{32} = C_2 \omega q + C_3 p + C_1 \\ \mu^1_{12} = -\mu^1_{21} = C_5 A_+ + C_6 A_- \\ \mu^2_{12} = -\mu^2_{21} = C_5 A_- - C_6 A_+ \\ \mu^3_{13} = -\mu^3_{31} = C_7 A_+ + C_8 A_- \\ \mu^3_{23} = -\mu^3_{32} = C_7 A_- - C_8 A_+ \\ \mu^3_{12} = -\mu^3_{21} = C_9 \end{cases} \tag{7}$$

Then (μ, M) is an operadic Lax pair for the harmonic oscillator.

5. Initial conditions. Now specify the coefficients C_ν in Theorem 4.3 by the initial conditions

$$\mu|_{t=0} = \overset{\circ}{\mu}, \quad p|_{t=0} = p_0, \quad q|_{t=0} = 0.$$

Denoting $E := H|_{t=0}$, the latter together with (5) yield the initial conditions for $A_\pm$:

$$\begin{cases} \left(A_+^2 + A_-^2\right)\big|_{t=0} = 2\sqrt{2E} \\ \left(A_+^2 - A_-^2\right)\big|_{t=0} = 2p_0 \\ A_+ A_-|_{t=0} = 0 \end{cases} \iff \begin{cases} p_0 > 0 \\ A_+^2\big|_{t=0} = 2p_0 \\ A_-|_{t=0} = 0 \end{cases} \vee \begin{cases} p_0 < 0 \\ A_+|_{t=0} = 0 \\ A_-^2\big|_{t=0} = -2p_0 \end{cases}$$

In what follows assume that $p_0 > 0$ and $A_+|_{t=0} = \sqrt{2p_0}$. The other cases can be treated similarly. Note that in this case $p_0 = \sqrt{2E}$. From (7) we get the following linear system:

$$\begin{cases} C_1 = \frac{1}{2}\left(\overset{\circ}{\mu}{}^2_{23} - \overset{\circ}{\mu}{}^1_{31}\right), & C_2 = \frac{1}{2p_0}\left(\overset{\circ}{\mu}{}^2_{13} + \overset{\circ}{\mu}{}^1_{23}\right), & C_3 = \frac{1}{2p_0}\left(\overset{\circ}{\mu}{}^2_{23} + \overset{\circ}{\mu}{}^1_{31}\right) \\ C_4 = \frac{1}{2}\left(\overset{\circ}{\mu}{}^2_{13} - \overset{\circ}{\mu}{}^1_{23}\right), & C_5 = \frac{1}{\sqrt{2p_0}}\overset{\circ}{\mu}{}^1_{12}, & C_6 = -\frac{1}{\sqrt{2p_0}}\overset{\circ}{\mu}{}^2_{12} \\ C_7 = \frac{1}{\sqrt{2p_0}}\overset{\circ}{\mu}{}^3_{13}, & C_8 = -\frac{1}{\sqrt{2p_0}}\overset{\circ}{\mu}{}^3_{23}, & C_9 = \overset{\circ}{\mu}{}^3_{12} \end{cases} \tag{8}$$

6. Bianchi classification of 3D real Lie algebras. We use the Bianchi classification of 3D real Lie algebras [2]. The structure equations of the latter can be presented as follows:

$$[e_1, e_2] = -\alpha e_2 + n^3 e_3, \quad [e_2, e_3] = n^1 e_1, \quad [e_3, e_1] = n^2 e_2 + \alpha e_3.$$

The values of the parameters α, n^1, n^2, n^3 and the corresponding structure constants are presented in Table 1.

7. Dynamical deformations of 3D real Lie algebras. By using the structure constants of the 3D real Lie algebras in the Bianchi classification, Theorem 4.3 and relations (8) one can propose that the time evolution of the 3D real Lie algebras is prescribed [7] as given in Table 2.

Table 1. 3D real Lie algebras in Bianchi classification. Here $a > 0$

Bianchi type	α	(n^1, n^2, n^3)	$\mathring{\mu}^1_{12}$	$\mathring{\mu}^2_{12}$	$\mathring{\mu}^3_{12}$	$\mathring{\mu}^1_{23}$	$\mathring{\mu}^2_{23}$	$\mathring{\mu}^3_{23}$	$\mathring{\mu}^1_{31}$	$\mathring{\mu}^2_{31}$	$\mathring{\mu}^3_{31}$
I	0	$(0,0,0)$	0	0	0	0	0	0	0	0	0
II	0	$(1,0,0)$	0	0	0	1	0	0	0	0	0
VII	0	$(1,1,0)$	0	0	0	1	0	0	0	1	0
VI	0	$(1,-1,0)$	0	0	0	1	0	0	0	-1	0
IX	0	$(1,1,1)$	0	0	1	1	0	0	0	1	0
VIII	0	$(1,1,-1)$	0	0	-1	1	0	0	0	1	0
V	1	$(0,0,0)$	0	-1	0	0	0	0	0	0	1
IV	1	$(0,0,1)$	0	-1	1	0	0	0	0	0	1
VII_a	a	$(0,1,1)$	0	$-a$	1	0	0	0	0	1	a
$\mathrm{III}_{a=1}$	1	$(0,1,-1)$	0	-1	-1	0	0	0	0	1	1
$\mathrm{VI}_{a\neq 1}$	a	$(0,1,-1)$	0	$-a$	-1	0	0	0	0	1	a

Table 2. Time evolution of 3D real Lie algebras. Here $p_0 = \sqrt{2E}$

Dynamical Bianchi type	μ^1_{12}	μ^2_{12}	μ^3_{12}	μ^1_{23}	μ^2_{23}	μ^3_{23}	μ^1_{31}	μ^2_{31}	μ^3_{31}
I^t	0	0	0	0	0	0	0	0	0
II^t	0	0	0	$\frac{p+p_0}{2p_0}$	$\frac{\omega q}{2p_0}$	0	$\frac{\omega q}{2p_0}$	$\frac{p-p_0}{-2p_0}$	0
VII^t	0	0	0	1	0	0	0	1	0
VI^t	0	0	0	$\frac{p}{p_0}$	$\frac{\omega q}{p_0}$	0	$\frac{\omega q}{p_0}$	$-\frac{p}{p_0}$	0
IX^t	0	0	1	1	0	0	0	1	0
VIII^t	0	0	-1	1	0	0	0	1	0
V^t	$\frac{A_-}{\sqrt{2p_0}}$	$\frac{-A_+}{\sqrt{2p_0}}$	0	0	0	$\frac{-A_-}{\sqrt{2p_0}}$	0	0	$\frac{A_+}{\sqrt{2p_0}}$
IV^t	$\frac{A_-}{\sqrt{2p_0}}$	$\frac{-A_+}{\sqrt{2p_0}}$	1	0	0	$\frac{-A_-}{\sqrt{2p_0}}$	0	0	$\frac{A_+}{\sqrt{2p_0}}$
VII^t_a	$\frac{aA_-}{\sqrt{2p_0}}$	$\frac{-aA_+}{\sqrt{2p_0}}$	1	$\frac{p-p_0}{-2p_0}$	$\frac{\omega q}{-2p_0}$	$\frac{-aA_-}{\sqrt{2p_0}}$	$\frac{\omega q}{-2p_0}$	$\frac{p+p_0}{2p_0}$	$\frac{aA_+}{\sqrt{2p_0}}$
$\mathrm{III}^t_{a=1}$	$\frac{A_-}{\sqrt{2p_0}}$	$\frac{-A_+}{\sqrt{2p_0}}$	-1	$\frac{p-p_0}{-2p_0}$	$\frac{\omega q}{-2p_0}$	$\frac{-A_-}{\sqrt{2p_0}}$	$\frac{\omega q}{-2p_0}$	$\frac{p+p_0}{2p_0}$	$\frac{A_+}{\sqrt{2p_0}}$
$\mathrm{VI}^t_{a\neq 1}$	$\frac{aA_-}{\sqrt{2p_0}}$	$\frac{-aA_+}{\sqrt{2p_0}}$	-1	$\frac{p-p_0}{-2p_0}$	$\frac{\omega q}{-2p_0}$	$\frac{-aA_-}{\sqrt{2p_0}}$	$\frac{\omega q}{-2p_0}$	$\frac{p+p_0}{2p_0}$	$\frac{aA_+}{\sqrt{2p_0}}$

8. Quantum counterparts of 3D real Lie algebras. Let now the harmonic oscillator be quantized, i.e its canonical coordinates satisfy the CCR

$$[\hat{q}, \hat{q}] = 0 = [\hat{p}, \hat{p}], \quad [\hat{p}, \hat{q}] = \hbar/i.$$

Then the classical observables $A_\pm(q,p)$ will be quantized as well and their quantum counterparts are denoted by $\hat{A}_\pm := A_\pm(\hat{q}, \hat{p})$. As a result, the quantum counterparts of the 3D real Lie algebras can be listed as presented in Table 3.

Table 3. Quantum counterparts of 3D real Lie algebras over the harmonic oscillator

Quantum Bianchi type	$\hat{\mu}^1_{12}$	$\hat{\mu}^2_{12}$	$\hat{\mu}^3_{12}$	$\hat{\mu}^1_{23}$	$\hat{\mu}^2_{23}$	$\hat{\mu}^3_{23}$	$\hat{\mu}^1_{31}$	$\hat{\mu}^2_{31}$	$\hat{\mu}^3_{31}$
$\mathrm{I}^{\hbar}$	0	0	0	0	0	0	0	0	0
$\mathrm{II}^{\hbar}$	0	0	0	$\frac{\hat{p}+p_0}{2p_0}$	$\frac{\omega\hat{q}}{2p_0}$	0	$\frac{\omega\hat{q}}{2p_0}$	$\frac{\hat{p}-p_0}{-2p_0}$	0
$\mathrm{VII}^{\hbar}$	0	0	0	1	0	0	0	1	0
$\mathrm{VI}^{\hbar}$	0	0	0	$\frac{\hat{p}}{p_0}$	$\frac{\omega\hat{q}}{p_0}$	0	$\frac{\omega\hat{q}}{p_0}$	$-\frac{\hat{p}}{p_0}$	0
$\mathrm{IX}^{\hbar}$	0	0	1	1	0	0	0	1	0
$\mathrm{VIII}^{\hbar}$	0	0	-1	1	0	0	0	1	0
$\mathrm{V}^{\hbar}$	$\frac{\hat{A}_-}{\sqrt{2p_0}}$	$\frac{-\hat{A}_+}{\sqrt{2p_0}}$	0	0	0	$\frac{-\hat{A}_-}{\sqrt{2p_0}}$	0	0	$\frac{\hat{A}_+}{\sqrt{2p_0}}$
$\mathrm{IV}^{\hbar}$	$\frac{\hat{A}_-}{\sqrt{2p_0}}$	$\frac{-\hat{A}_+}{\sqrt{2p_0}}$	1	0	0	$\frac{-\hat{A}_-}{\sqrt{2p_0}}$	0	0	$\frac{\hat{A}_+}{\sqrt{2p_0}}$
$\mathrm{VII}^{\hbar}_a$	$\frac{a\hat{A}_-}{\sqrt{2p_0}}$	$\frac{-a\hat{A}_+}{\sqrt{2p_0}}$	1	$\frac{\hat{p}-p_0}{-2p_0}$	$\frac{\omega\hat{q}}{-2p_0}$	$\frac{-a\hat{A}_-}{\sqrt{2p_0}}$	$\frac{\omega\hat{q}}{-2p_0}$	$\frac{\hat{p}+p_0}{2p_0}$	$\frac{a\hat{A}_+}{\sqrt{2p_0}}$
$\mathrm{III}^{\hbar}_{a=1}$	$\frac{\hat{A}_-}{\sqrt{2p_0}}$	$\frac{-\hat{A}_+}{\sqrt{2p_0}}$	-1	$\frac{\hat{p}-p_0}{-2p_0}$	$\frac{\omega\hat{q}}{-2p_0}$	$\frac{-\hat{A}_-}{\sqrt{2p_0}}$	$\frac{\omega\hat{q}}{-2p_0}$	$\frac{\hat{p}+p_0}{2p_0}$	$\frac{\hat{A}_+}{\sqrt{2p_0}}$
$\mathrm{VI}^{\hbar}_{a\neq 1}$	$\frac{a\hat{A}_-}{\sqrt{2p_0}}$	$\frac{-a\hat{A}_+}{\sqrt{2p_0}}$	-1	$\frac{\hat{p}-p_0}{-2p_0}$	$\frac{\omega\hat{q}}{-2p_0}$	$\frac{-a\hat{A}_-}{\sqrt{2p_0}}$	$\frac{\omega\hat{q}}{-2p_0}$	$\frac{\hat{p}+p_0}{2p_0}$	$\frac{a\hat{A}_+}{\sqrt{2p_0}}$

One can easily check that $\mathrm{I}^{\hbar}$, $\mathrm{II}^{\hbar}$, $\mathrm{VII}^{\hbar}$, $\mathrm{VI}^{\hbar}$, $\mathrm{IX}^{\hbar}$, $\mathrm{VIII}^{\hbar}$ are Lie algebras. Thus, in what follows, we will only focus on the algebras $\mathrm{V}^{\hbar}$, $\mathrm{IV}^{\hbar}$, $\mathrm{VII}^{\hbar}_a$, $\mathrm{III}^{\hbar}_{a=1}$, $\mathrm{VI}^{\hbar}_{a\neq 1}$, and present the latter more compactly in a separate table.

Let β, γ, a, b be real-valued parameters from Table 4 and let $\mathcal{A}^{\hbar}$ denote an entry from the first column of Table 3. Algebras $\mathrm{V}^{\hbar}$, $\mathrm{IV}^{\hbar}$, $\mathrm{VII}^{\hbar}_a$, $\mathrm{III}^{\hbar}_{a=1}$, $\mathrm{VI}^{\hbar}_{a\neq 1}$ from Table 3 can be presented as Table 5.

Table 4. Values of β, γ, a, b for quantum algebras $\mathcal{A}^{\hbar}$. Here $a > 0$

$\mathcal{A}^{\hbar}$	β	γ	a	b
$\mathrm{V}^{\hbar}$	0	0	1	0
$\mathrm{IV}^{\hbar}$	0	0	1	1
$\mathrm{VII}^{\hbar}_a$	1	1	a	1
$\mathrm{III}^{\hbar}_{a=1}$	1	1	1	1
$\mathrm{VI}^{\hbar}_{a\neq 1}$	1	1	$a \neq 1$	-1

Table 5. $\mathcal{A}^{\hbar}$

Quantum Bianchi type	$\hat{\mu}^1_{12}$	$\hat{\mu}^2_{12}$	$\hat{\mu}^3_{12}$	$\hat{\mu}^1_{23}$	$\hat{\mu}^2_{23}$	$\hat{\mu}^3_{23}$	$\hat{\mu}^1_{31}$	$\hat{\mu}^2_{31}$	$\hat{\mu}^3_{31}$
$\mathcal{A}^{\hbar}$	$\frac{a\hat{A}_-}{\sqrt{2p_0}}$	$\frac{-a\hat{A}_+}{\sqrt{2p_0}}$	b	$\frac{-\gamma(\hat{p}-p_0)}{2p_0}$	$\frac{-\beta\omega\hat{q}}{2p_0}$	$\frac{-a\hat{A}_-}{\sqrt{2p_0}}$	$\frac{-\beta\omega\hat{q}}{2p_0}$	$\frac{\gamma(\hat{p}+p_0)}{2p_0}$	$\frac{a\hat{A}_+}{\sqrt{2p_0}}$

Let $\mathcal{A}_{HO}$ denote the state space of the quantum harmonic oscillator and $\{e_1, e_2, \ldots\}$ be its basis. By using Table 5 we define the structure equations in $\mathcal{A}_{HO}$ by

$$[e_i, e_j]_\hbar := \hat{\mu}_{ij}^s e_s,$$

where the structure operators $\hat{\mu}_{ij}^s$ for $i, j, s \le 3$ are defined by Table 5 and $\hat{\mu}_{ij}^s := 0$ for $i, j, s > 3$. For $x, y \in \mathcal{A}_{HO}$, their quantum multiplication is defined by

$$[x, y]_\hbar := \hat{\mu}_{jk}^i x^j y^k e_i = \hat{\mu}_{jk}^1 x^j y^k e_1 + \hat{\mu}_{jk}^2 x^j y^k e_2 + \hat{\mu}_{jk}^3 x^j y^k e_3,$$

where we omitted the trivial terms, because $\hat{\mu}_{jk}^i = 0$ for $i > 3$.

9. Jacobi operators. For $x, y, z \in \mathcal{A}_{HO}$, their quantum Jacobi operator is defined by

$$\begin{aligned}\hat{J}_\hbar(x; y; z) &:= [x, [y, z]_\hbar]_\hbar + [y, [z, x]_\hbar]_\hbar + [z, [x, y]_\hbar]_\hbar \\ &= \hat{J}_\hbar^1(x; y; z)e_1 + \hat{J}_\hbar^2(x; y; z)e_2 + \hat{J}_\hbar^3(x; y; z)e_3,\end{aligned}$$

where we again omitted the trivial terms, because $\hat{J}_\hbar^i = 0$ for $i > 3$. In [9] the quantum Jacobi operators were presented for all real three-dimensional Lie algebras. In this paper, we present a calculation of the Jacobi operators. Denote

$$(x, y, z) := \begin{vmatrix} x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \\ z^1 & z^2 & z^3 \end{vmatrix}, \quad \xi_\pm := \beta\omega\hat{q}\hat{A}_\mp \pm \gamma(\hat{p} \mp p_0)\hat{A}_\pm.$$

Then we have

THEOREM 9.1. *The Jacobi operator components of $\mathcal{A}^\hbar$ read*

$$\begin{aligned}\hat{J}_\hbar^1(x; y; z) &= -\frac{a(x, y, z)}{\sqrt{2p_0^3}}\hat{\xi}_+, \\ \hat{J}_\hbar^2(x; y; z) &= -\frac{a(x, y, z)}{\sqrt{2p_0^3}}\hat{\xi}_-, \\ \hat{J}_\hbar^3(x; y; z) &= \frac{a^2(x, y, z)}{p_0}[\hat{A}_+, \hat{A}_-].\end{aligned}$$

Proof. As an example, calculate $\hat{J}_\hbar^1(x; y; z)$. First find the products $[x, y]_\hbar$, $[y, z]_\hbar$ and $[z, x]_\hbar$ in $\mathcal{A}^\hbar$. Denote $\Delta := (x, y, z)$ and let Δ^{ij} be the cofactor (signed minor) of the element of Δ in the i-th row and j-th column. Calculate

$$\begin{aligned}[x, y]_\hbar &= [x, y]_\hbar^i e_i = \hat{\mu}_{jk}^i x^j y^k e_i \\ &= \left(\hat{\mu}_{12}^1\Delta^{33} - \hat{\mu}_{13}^1\Delta^{32} + \hat{\mu}_{23}^1\Delta^{31}\right)e_1 + \left(\hat{\mu}_{12}^2\Delta^{33} - \hat{\mu}_{13}^2\Delta^{32} + \hat{\mu}_{23}^2\Delta^{31}\right)e_2 \\ &\quad + \left(b\Delta^{33} - \hat{\mu}_{13}^3\Delta^{32} + \hat{\mu}_{23}^3\Delta^{31}\right)e_3 \\ &= \left(\frac{a\hat{A}_-}{\sqrt{2p_0}}\Delta^{33} - \beta\frac{\omega\hat{q}}{2p_0}\Delta^{32} - \gamma\frac{\hat{p} - p_0}{2p_0}\Delta^{31}\right)e_1 \\ &\quad + \left(\frac{-a\hat{A}_+}{\sqrt{2p_0}}\Delta^{33} + \gamma\frac{\hat{p} + p_0}{2p_0}\Delta^{32} - \beta\frac{\omega\hat{q}}{2p_0}\Delta^{31}\right)e_2 \\ &\quad + \left(b\Delta^{33} + \frac{a\hat{A}_+}{\sqrt{2p_0}}\Delta^{32} - \frac{a\hat{A}_-}{\sqrt{2p_0}}\Delta^{31}\right)e_3.\end{aligned}$$

In the same way, we can see that

$$\begin{aligned}
[y,z]_\hbar &= [y,z]^i_\hbar e_i = \hat{\mu}^i_{jk} y^j z^k e_i \\
&= \left(\frac{a\hat{A}_-}{\sqrt{2p_0}} \Delta^{13} - \beta \frac{\omega \hat{q}}{2p_0} \Delta^{12} - \gamma \frac{\hat{p}-p_0}{2p_0} \Delta^{11} \right) e_1 \\
&+ \left(\frac{-a\hat{A}_+}{\sqrt{2p_0}} \Delta^{13} + \gamma \frac{\hat{p}+p_0}{2p_0} \Delta^{12} - \beta \frac{\omega \hat{q}}{2p_0} \Delta^{11} \right) e_2 \\
&+ \left(b\Delta^{13} + \frac{a\hat{A}_+}{\sqrt{2p_0}} \Delta^{12} - \frac{a\hat{A}_-}{\sqrt{2p_0}} \Delta^{11} \right) e_3
\end{aligned}$$

and also

$$\begin{aligned}
[z,x]_\hbar &= [z,x]^i_\hbar e_i = \hat{\mu}^i_{jk} z^j x^j e_i \\
&= \left(\frac{a\hat{A}_-}{\sqrt{2p_0}} \Delta^{23} - \beta \frac{\omega \hat{q}}{2p_0} \Delta^{22} - \gamma \frac{\hat{p}-p_0}{2p_0} \Delta^{21} \right) e_1 \\
&+ \left(\frac{-a\hat{A}_+}{\sqrt{2p_0}} \Delta^{23} + \gamma \frac{\hat{p}+p_0}{2p_0} \Delta^{22} - \beta \frac{\omega \hat{q}}{2p_0} \Delta^{21} \right) e_2 \\
&+ \left(b\Delta^{23} + \frac{a\hat{A}_+}{\sqrt{2p_0}} \Delta^{22} - \frac{a\hat{A}_-}{\sqrt{2p_0}} \Delta^{21} \right) e_3.
\end{aligned}$$

Now calculate the first component of the Jacobi operator:

$$\begin{aligned}
\hat{J}^1_\hbar(x;y;z) &= [x,[y,z]_\hbar]^1_\hbar + [y,[z,x]_\hbar]^1_\hbar + [z,[x,y]_\hbar]^1_\hbar \\
&= \hat{\mu}^1_{jk} x^j [y,z]^k_\hbar + \hat{\mu}^1_{jk} y^j [z,x]^k_\hbar + \hat{\mu}^1_{jk} z^j [x,y]^k_\hbar \\
&= \hat{\mu}^1_{12} \left(x^1 [y,z]^2_\hbar - x^2 [y,z]^1_\hbar \right) + \hat{\mu}^1_{13} \left(x^1 [y,z]^3_\hbar - x^3 [y,z]^1_\hbar \right) \\
&+ \hat{\mu}^1_{23} \left(x^2 [y,z]^3_\hbar - x^3 [y,z]^2_\hbar \right) + \hat{\mu}^1_{12} \left(y^1 [z,x]^2_\hbar - y^2 [z,x]^1_\hbar \right) \\
&+ \hat{\mu}^1_{13} \left(y^1 [z,x]^3_\hbar - y^3 [z,x]^1_\hbar \right) + \hat{\mu}^1_{23} \left(y^2 [z,x]^3_\hbar - y^3 [z,x]^2_\hbar \right) \\
&+ \hat{\mu}^1_{12} \left(z^1 [x,y]^2_\hbar - z^2 [x,y]^1_\hbar \right) + \hat{\mu}^1_{13} \left(z^1 [x,y]^3_\hbar - z^3 [x,y]^1_\hbar \right) \\
&+ \hat{\mu}^1_{23} \left(z^2 [x,y]^3_\hbar - z^3 [x,y]^2_\hbar \right) \\
&= \frac{a\hat{A}_-}{\sqrt{2p_0}} \left\{ x^1 \left(\frac{-a\hat{A}_+}{\sqrt{2p_0}} \Delta^{13} + \gamma \frac{\hat{p}+p_0}{2p_0} \Delta^{12} - \beta \frac{\omega \hat{q}}{2p_0} \Delta^{11} \right) \right. \\
&\qquad \left. - x^2 \left(\frac{a\hat{A}_-}{\sqrt{2p_0}} \Delta^{13} - \beta \frac{\omega \hat{q}}{2p_0} \Delta^{12} - \gamma \frac{\hat{p}-p_0}{2p_0} \Delta^{11} \right) \right\} \\
&+ \beta \frac{\omega \hat{q}}{2p_0} \left\{ x^1 \left(b\Delta^{13} + \frac{a\hat{A}_+}{\sqrt{2p_0}} \Delta^{12} - \frac{a\hat{A}_-}{\sqrt{2p_0}} \Delta^{11} \right) \right. \\
&\qquad \left. - x^3 \left(\frac{a\hat{A}_-}{\sqrt{2p_0}} \Delta^{13} - \beta \frac{\omega \hat{q}}{2p_0} \Delta^{12} - \gamma \frac{\hat{p}-p_0}{2p_0} \Delta^{11} \right) \right\}
\end{aligned}$$

$$
\begin{aligned}
&- \gamma\frac{\hat{p}-p_0}{2p_0}\left\{x^2\left(b\Delta^{13}+\frac{a\hat{A}_+}{\sqrt{2p_0}}\Delta^{12}-\frac{a\hat{A}_-}{\sqrt{2p_0}}\Delta^{11}\right)\right.\\
&\qquad\left.- x^3\left(\frac{-a\hat{A}_+}{\sqrt{2p_0}}\Delta^{13}+\gamma\frac{\hat{p}+p_0}{2p_0}\Delta^{12}-\beta\frac{\omega\hat{q}}{2p_0}\Delta^{11}\right)\right\}\\
&+\frac{a\hat{A}_-}{\sqrt{2p_0}}\left\{y^1\left(\frac{-a\hat{A}_+}{\sqrt{2p_0}}\Delta^{23}+\gamma\frac{\hat{p}+p_0}{2p_0}\Delta^{22}-\beta\frac{\omega\hat{q}}{2p_0}\Delta^{21}\right)\right.\\
&\qquad\left.- y^2\left(\frac{a\hat{A}_-}{\sqrt{2p_0}}\Delta^{23}-\beta\frac{\omega\hat{q}}{2p_0}\Delta^{22}-\gamma\frac{\hat{p}-p_0}{2p_0}\Delta^{21}\right)\right\}\\
&+\beta\frac{\omega\hat{q}}{2p_0}\left\{y^1\left(b\Delta^{23}+\frac{a\hat{A}_+}{\sqrt{2p_0}}\Delta^{22}-\frac{a\hat{A}_-}{\sqrt{2p_0}}\Delta^{21}\right)\right.\\
&\qquad\left.- y^3\left(\frac{a\hat{A}_-}{\sqrt{2p_0}}\Delta^{23}-\beta\frac{\omega\hat{q}}{2p_0}\Delta^{22}-\gamma\frac{\hat{p}-p_0}{2p_0}\Delta^{21}\right)\right\}\\
&- \gamma\frac{\hat{p}-p_0}{2p_0}\left\{y^2\left(b\Delta^{23}+\frac{a\hat{A}_+}{\sqrt{2p_0}}\Delta^{22}-\frac{a\hat{A}_-}{\sqrt{2p_0}}\Delta^{21}\right)\right.\\
&\qquad\left.- y^3\left(\frac{-a\hat{A}_+}{\sqrt{2p_0}}\Delta^{23}+\gamma\frac{\hat{p}+p_0}{2p_0}\Delta^{22}-\beta\frac{\omega\hat{q}}{2p_0}\Delta^{21}\right)\right\}\\
&+\frac{a\hat{A}_-}{\sqrt{2p_0}}\left\{z^1\left(\frac{-a\hat{A}_+}{\sqrt{2p_0}}\Delta^{33}+\gamma\frac{\hat{p}+p_0}{2p_0}\Delta^{32}-\beta\frac{\omega\hat{q}}{2p_0}\Delta^{31}\right)\right.\\
&\qquad\left.- z^2\left(\frac{a\hat{A}_-}{\sqrt{2p_0}}\Delta^{33}-\beta\frac{\omega\hat{q}}{2p_0}\Delta^{32}-\gamma\frac{\hat{p}-p_0}{2p_0}\Delta^{31}\right)\right\}\\
&+\beta\frac{\omega\hat{q}}{2p_0}\left\{z^1\left(b\Delta^{33}+\frac{a\hat{A}_+}{\sqrt{2p_0}}\Delta^{32}-\frac{a\hat{A}_-}{\sqrt{2p_0}}\Delta^{31}\right)\right.\\
&\qquad\left.- z^3\left(\frac{a\hat{A}_-}{\sqrt{2p_0}}\Delta^{33}-\beta\frac{\omega\hat{q}}{2p_0}\Delta^{32}-\gamma\frac{\hat{p}-p_0}{2p_0}\Delta^{31}\right)\right\}\\
&- \gamma\frac{\hat{p}-p_0}{2p_0}\left\{z^2\left(b\Delta^{33}+\frac{a\hat{A}_+}{\sqrt{2p_0}}\Delta^{32}-\frac{a\hat{A}_-}{\sqrt{2p_0}}\Delta^{31}\right)\right.\\
&\qquad\left.- z^3\left(\frac{-a\hat{A}_+}{\sqrt{2p_0}}\Delta^{33}+\gamma\frac{\hat{p}+p_0}{2p_0}\Delta^{32}-\beta\frac{\omega\hat{q}}{2p_0}\Delta^{31}\right)\right\}.
\end{aligned}
$$

Now open the parentheses and rearrange the terms. Then we have

$$
\begin{aligned}
\hat{J}^1_\hbar(x;y;z) = &- \frac{a\hat{A}_-}{\sqrt{2p_0}}\frac{a\hat{A}_+}{\sqrt{2p_0}}\underbrace{\left(x^1\Delta^{13}+y^1\Delta^{23}+z^1\Delta^{33}\right)}_{0}\\
&+\frac{a\hat{A}_-}{\sqrt{2p_0}}\gamma\frac{\hat{p}+p_0}{2p_0}\underbrace{\left(x^1\Delta^{12}+y^1\Delta^{22}+z^1\Delta^{32}\right)}_{0}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{a\hat{A}_-}{\sqrt{2p_0}}\beta\frac{\omega\hat{q}}{2p_0}\underbrace{\left(x^1\Delta^{11}+y^1\Delta^{21}+z^1\Delta^{31}\right)}_{(x,y,z)}\\
&- \frac{a\hat{A}_-}{\sqrt{2p_0}}\frac{a\hat{A}_-}{\sqrt{2p_0}}\underbrace{\left(x^2\Delta^{13}+y^2\Delta^{23}+z^2\Delta^{33}\right)}_{0}\\
&+ \frac{a\hat{A}_-}{\sqrt{2p_0}}\beta\frac{\omega\hat{q}}{2p_0}\underbrace{\left(x^2\Delta^{12}+y^2\Delta^{22}+z^2\Delta^{32}\right)}_{(x,y,z)}\\
&+ \frac{a\hat{A}_-}{\sqrt{2p_0}}\gamma\frac{\hat{p}-p_0}{2p_0}\underbrace{\left(x^2\Delta^{11}+y^2\Delta^{21}+z^2\Delta^{31}\right)}_{0}\\
&+ \beta\frac{\omega\hat{q}}{2p_0}b\underbrace{\left(x^1\Delta^{13}+y^1\Delta^{23}+z^1\Delta^{33}\right)}_{0}\\
&+ \beta\frac{\omega\hat{q}}{2p_0}\frac{a\hat{A}_+}{\sqrt{2p_0}}\underbrace{\left(x^1\Delta^{12}+y^1\Delta^{22}+z^1\Delta^{32}\right)}_{0}\\
&- \beta\frac{\omega\hat{q}}{2p_0}\frac{a\hat{A}_-}{\sqrt{2p_0}}\underbrace{\left(x^1\Delta^{11}+y^1\Delta^{21}+z^1\Delta^{31}\right)}_{(x,y,z)}\\
&- \beta\frac{\omega\hat{q}}{2p_0}\frac{a\hat{A}_-}{\sqrt{2p_0}}\underbrace{\left(x^3\Delta^{13}+y^3\Delta^{23}+z^3\Delta^{33}\right)}_{(x,y,z)}\\
&+ \beta\frac{\omega\hat{q}}{2p_0}\beta\frac{\omega\hat{q}}{2p_0}\underbrace{\left(x^3\Delta^{12}+y^3\Delta^{22}+z^3\Delta^{32}\right)}_{0}\\
&+ \beta\frac{\omega\hat{q}}{2p_0}\gamma\frac{\hat{p}-p_0}{2p_0}\underbrace{\left(x^3\Delta^{11}+y^3\Delta^{21}+z^3\Delta^{31}\right)}_{0}\\
&- \gamma\frac{\hat{p}-p_0}{2p_0}b\underbrace{\left(x^2\Delta^{13}+y^2\Delta^{23}+z^2\Delta^{33}\right)}_{0}\\
&- \gamma\frac{\hat{p}-p_0}{2p_0}\frac{a\hat{A}_+}{\sqrt{2p_0}}\underbrace{\left(x^2\Delta^{12}+y^2\Delta^{22}+z^2\Delta^{32}\right)}_{(x,y,z)}\\
&+ \gamma\frac{\hat{p}-p_0}{2p_0}\frac{a\hat{A}_-}{\sqrt{2p_0}}\underbrace{\left(x^2\Delta^{11}+y^2\Delta^{21}+z^2\Delta^{31}\right)}_{0}\\
&- \gamma\frac{\hat{p}-p_0}{2p_0}\frac{a\hat{A}_+}{\sqrt{2p_0}}\underbrace{\left(x^3\Delta^{13}+y^3\Delta^{23}+z^3\Delta^{33}\right)}_{(x,y,z)}\\
&+ \gamma\frac{\hat{p}-p_0}{2p_0}\gamma\frac{\hat{p}+p_0}{2p_0}\underbrace{\left(x^3\Delta^{12}+y^3\Delta^{22}+z^3\Delta^{32}\right)}_{0}
\end{aligned}
$$

$$-\gamma\frac{\hat{p}-p_0}{2p_0}\beta\frac{\omega\hat{q}}{2p_0}\underbrace{\left(x^3\Delta^{11}+y^3\Delta^{21}+z^3\Delta^{31}\right)}_{0}$$

$$=-\frac{a(x,y,z)}{\sqrt{2p_0^3}}(\beta\omega\hat{q}\hat{A}_-+\gamma(\hat{p}-p_0)\hat{A}_+).$$

The remaining operators $\hat{J}^2_\hbar(x;y;z)$ and $\hat{J}^3_\hbar(x;y;z)$ can be calculated in the same way. ∎

REMARK 9.2. By the direct calculations one can see that the Jacobi operators of $\mathrm{II}^\hbar$ and $\mathrm{VI}^\hbar$ turn out to be zero.

Acknowledgement. The research was in part supported by the Estonian Science Foundation, Grant ETF-6912.

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Received January 25, 2010; Revised February 2, 2011