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RESTRICTING THE BI-EQUIVARIANT SPECTRAL TRIPLE ON QUANTUM SU(2) TO THE PODLEŚ SPHERES

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Abstract. It is shown that the isospectral bi-equivariant spectral triple on quantum SU(2) and the isospectral equivariant spectral triples on the Podleś spheres are related by restriction. In this approach, the equatorial Podleś sphere is distinguished because only in this case the restricted spectral triple admits an equivariant grading operator together with a real structure (up to infinitesimals of arbitrary high order). The real structure is expressed by the Tomita operator on quantum SU(2) and it is shown that the failure of the real structure to satisfy the commutant property is related to the failure of the universal R-matrix operator to be unitary.

1. Introduction. The search for spectral triples on noncommutative spaces arising in quantum group theory is an active research topic. A typical strategy for finding (equivariant) spectral triples on q-deformed spaces is a case by case study starting with a quantum analogue of the classical spinor bundle and defining the Dirac operator on q-analogues of harmonic spinors (see, e.g., [4–9]). Until now, only few general methods for the construction of spectral triples were found. The most notable examples are the construction of Dirac operators on quantum flag manifolds by Krähmer [14] and the construction of equivariant spectral triples on compact quantum groups by Neshveyev and Tuset [16]. Therefore the question arises whether the latter construction on compact quantum groups can be used to find spectral triples on the associated quantum homogeneous spaces.

We approach this question by studying the relation between the bi-equivariant Dirac operator on quantum SU(2) [8] and spectral triples on the 1-parameter family of Podleś spheres $\mathcal{A}(\mathrm{S}^2_{qc})$, $c \in [0, \infty]$ [6]. This example exhibits already some interesting features. Whereas the standard Podleś sphere $\mathcal{A}(\mathrm{S}^2_{q0})$ is distinguished for being obtained by a

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quotient of quantum groups and admitting a rich non-commutative spin geometry [21], it is the equatorial Podleś sphere $\mathcal{A}(\mathrm{S}^2_{q\infty})$ on the other extreme which distinguishes in the present approach. The restriction of the bi-equivariant Dirac operator on quantum SU(2) to the Podleś spheres $\mathcal{A}(\mathrm{S}^2_{qc})$ does yield a spectral triple for all $c \in [0,\infty]$, but only in the case $c=\infty$ the obtained spectral triple admits an equivariant grading operator.

Having an equivariant even spectral triple on $\mathcal{A}(\mathrm{S}^2_{q\infty})$, one can ask for an equivariant real structure. Again, our aim is to relate the real structure on $\mathcal{A}(\mathrm{S}^2_{q\infty})$ with the one coming from the spectral triple on quantum SU(2). Moreover, and maybe more interestingly, we want to implement the real structure by the Tomita operator on $\mathcal{A}(\mathrm{SU}_q(2))$. It is known that an equivariant real structure for the bi-equivariant spectral triple on quantum SU(2) cannot satisfy the commutant and first order property exactly but does so up to compacts of arbitrary high order [8]. Starting from the Tomita operator on $\mathcal{A}(\mathrm{SU}_q(2))$, we will construct an equivariant operator on the quantum spinor bundle of $\mathcal{A}(\mathrm{S}^2_{q\infty})$ which satisfies the commutant property. This operator is not unitary but its unitary part coincides with restriction of the equivariant real structure on quantum SU(2). The construction uses the R-matrix operator of $\mathcal{U}_q(\mathrm{sl}(2))$ for intertwining tensor product representations. It is argued that the failure of this intertwining operator to be unitary is responsible for the failure of real structure to satisfy the commutant property.

2. Preliminaries

2.1. Algebraic preliminaries. Throughout this paper, q stands for real number such that $0<q<1$, and we set $[x]=[x]_q := \frac{q^x-q^{-x}}{q-q^{-1}}$ for $x\in\mathbb{R}$. All algebras appearing in this paper will be complex and unital. We shall use Sweedlers notation for the coproduct, namely, $\Delta x =: x_{(1)}\otimes x_{(2)}$.

The Hopf $*$-algebra $\mathcal{U}_q(\mathrm{su}(2))$ is generated by e, f, k, k^{-1} with defining relations

$$kk^{-1}=k^{-1}k=1,\quad ek=qke,\quad kf=qfk,\quad fe-ef=(q-q^{-1})^{-1}(k^2-k^{-2}),$$

coproduct $\Delta k = k\otimes k$, $\Delta e = e\otimes k + k^{-1}\otimes e$, $\Delta f = f\otimes k + k^{-1}\otimes f$, counit $\epsilon(k)=1$, $\epsilon(f)=\epsilon(e)=0$, antipode $S(k)=k^{-1}$, $S(f)=-qf$, $S(e)=-q^{-1}e$, and involution $k^*=k$ and $f^*=e$.

The coordinate Hopf $*$-algebra $\mathcal{A}(\mathrm{SU}_q(2))$ of the quantum SU(2) group has two generators a and b satisfying the relations

$$ba=qab,\quad b^*a=qab^*,\quad bb^*=b^*b,\quad a^*a+q^2b^*b=1,\qquad aa^*+bb^*=1.$$

The the counit ε, coproduct Δ and the antipode S are determined by

$$\Delta a = a\otimes a - q\,b\otimes b^*,\quad \Delta b = b\otimes a^* + a\otimes b,\quad \varepsilon(a)=1,\quad \varepsilon(b)=0,$$
$$S(a)=a^*,\quad S(b)=-qb,\quad S(b^*)=-q^{-1}b^*,\quad S(a^*)=a.$$

There is a dual pairing between the Hopf $*$-algebras $\mathcal{U}_q(\mathrm{su}(2))$ and $\mathcal{A}(\mathrm{SU}_q(2))$ given on generators by

$$\langle k^{\pm1},a\rangle = q^{\pm\frac12},\quad \langle k^{\pm1},a^*\rangle = q^{\mp\frac12},\quad \langle f,b\rangle = \langle e,-qb^*\rangle = 1,$$

and zero otherwise. The left action defined by $h\triangleright x := x_{(1)}\langle h, x_{(2)}\rangle$ for $h\in\mathcal{U}_q(\mathrm{su}(2))$ and

$x \in \mathcal{A}(\mathrm{SU}_q(2))$ satisfies

$$h \triangleright (xy) = (h_{(1)} \triangleright x)(h_{(2)} \triangleright y), \quad h \triangleright 1 = \epsilon(h), \quad (h \triangleright x)^* = S(h)^* \triangleright x^*, \tag{1}$$

i.e., $\mathcal{A}(\mathrm{SU}_q(2))$ is a left $\mathcal{U}_q(\mathrm{su}(2))$-module $*$-algebra. Similarly, $x \triangleleft h := \langle h, x_{(1)} \rangle\, x_{(2)}$ defines a right $\mathcal{U}_q(\mathrm{su}(2))$-action on $\mathcal{A}(\mathrm{SU}_q(2))$ such that

$$(xy) \triangleleft h = (x \triangleleft h_{(1)})(y \triangleleft h_{(2)}), \quad 1 \triangleleft h = \epsilon(h), \quad (x \triangleleft h)^* = x^* \triangleleft S(h)^*. \tag{2}$$

We follow [17] and define the Podleś quantum sphere $\mathcal{A}(\mathrm{S}^2_{qc})$, $c \in [0, \infty]$, as the $*$-algebra generated by $A = A^*$ and B with relations

$$BA = q^2 A, \quad B^*B = A - A^2 + c, \quad BB^* = q^2 A - q^4 A^2 + c \quad \text{for } c < \infty,$$
$$BA = q^2 A, \quad B^*B = -A^2 + 1, \quad BB^* = -q^4 A^2 + 1 \quad \text{for } c = \infty.$$

The Podleś quantum sphere $\mathcal{A}(\mathrm{S}^2_{qc})$ can be viewed as a $*$-subalgebra of $\mathcal{A}(\mathrm{SU}_q(2))$ by setting

$$B = c^{1/2} a^{*2} + a^* b - q c^{1/2} b^2, \quad A = c^{1/2} b^* a^* + b b^* + c^{1/2} ab \quad \text{for } c < \infty,$$
$$B = a^{*2} - q b^2, \quad A = b^* a^* + ab \quad \text{for } c = \infty.$$

Then the left $\mathcal{U}_q(\mathrm{su}(2))$-action on $\mathcal{A}(\mathrm{SU}_q(2))$ turns $\mathcal{A}(\mathrm{S}^2_{qc})$ into a left $\mathcal{U}_q(\mathrm{su}(2))$-module $*$-algebra such that the elements

$$x_{-1} := q^{-1}(1+q^2)^{1/2} B, \quad x_0 := 1 - (1+q^2) A, \quad x_1 := -(1+q^2)^{1/2} B^* \quad \text{for } c < \infty,$$
$$x_{-1} := q^{-1}(1+q^2)^{1/2} B, \quad x_0 := -(1+q^2) A, \quad x_1 := -(1+q^2)^{1/2} B^* \quad \text{for } c = \infty$$

transform by a spin 1 representation (see Equation (4)).

2.2. Equivariant representations. Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, V a dense linear subspace, and $\mathcal{A}$ a $*$-algebra. By a $*$-representation of $\mathcal{A}$ on V, we mean a homomorphism $\pi : \mathcal{A} \to \mathrm{End(V)}$ such that $\langle \pi(a) v, w \rangle = \langle v, \pi(a^*) w \rangle$ for all $v, w \in V$ and $a \in \mathcal{A}$.

Now assume that $\mathcal{A}$ is a left $\mathcal{U}$-module $*$-algebra, i.e., there is a left action $\triangleright$ of a Hopf $*$-algebra $\mathcal{U}$ on $\mathcal{A}$ satisfying (1). A $*$-representation π of $\mathcal{A}$ on V is called (left) $\mathcal{U}$-equivariant if there exists a $*$-representation λ of $\mathcal{U}$ on V such that

$$\lambda(h)\, \pi(x) \xi = \pi(h_{(1)} \triangleright x)\, \lambda(h_{(2)}) \xi$$

for all $h \in \mathcal{U}$, $x \in \mathcal{A}$ and $\xi \in V$. We call an operator defined on V equivariant if it commutes on V with $\lambda(h)$ for all $h \in \mathcal{U}$. An antilinear operator T is called equivariant if its domain of definition contains V and if it satisfies on V the relation $T\lambda(h) = \lambda(S(h)^*) T$ for all $h \in \mathcal{U}$. We say that an antiunitary operator is equivariant if it leaves V invariant and if it is the antiunitary part of the polar decomposition of an equivariant antilinear (closed) operator.

Given $\mathcal{U}$ and $\mathcal{A}$ as above, the left crossed product $*$-algebra $\mathcal{A} \rtimes \mathcal{U}$ is defined as the $*$-algebra generated by the two $*$-subalgebras $\mathcal{A}$ and $\mathcal{U}$ with cross commutation relations

$$hx = (h_{(1)} \triangleright x) h_{(2)}, \quad h \in \mathcal{U},\ x \in \mathcal{A}.$$

Thus $\mathcal{U}$-equivariant representations of $\mathcal{A}$ correspond to $*$-representations of $\mathcal{A} \rtimes \mathcal{U}$. As Hilbert space representations of $\mathcal{A}(\mathrm{SU}_q(2)) \rtimes \mathcal{U}_q(\mathrm{su}(2))$ and $\mathcal{A}(\mathrm{S}^2_{qc}) \rtimes \mathcal{U}_q(\mathrm{su}(2))$ have been studied extensively in [18] and [19], we shall mainly consider equivariant representations from this point of view.

Above definitions have their right handed counter parts. For instance, a $*$-representation π of a right $\mathcal{U}$-module $*$-algebra $\mathcal{A}$ (i.e. (2) is satisfied) is called (right) $\mathcal{U}$-equivariant if there exists a $*$-representation ρ of $\mathcal{U}$ on V such that

$$\pi(x)\,\rho(h)\,\xi = \rho(h_{(1)})\,\pi(x \triangleleft h_{(2)}))\,\xi, \quad h \in \mathcal{U},\ \ x \in \mathcal{A},\ \ \xi \in V. \tag{3}$$

Assume that we are given a left and right $\mathcal{U}$-equivariant representation π of $\mathcal{A}$ on V such that $\lambda(h)\rho(g) = \rho(g)\lambda(h)$ for all $h, g \in \mathcal{U}$. Then we say that an operator X on V is bi-equivariant if it commutes with all operators $\lambda(h)$ and $\rho(h)$, $h \in \mathcal{U}$.

The irreducible $*$-representations of $\mathcal{U}_q(\mathrm{su}(2))$ are labeled by non-negative half-integers. For $l \in \frac{1}{2}\mathbb{N}_0$, the corresponding representation σ_l acts on a $2l{+}1$-dimensional Hilbert space V_l with orthonormal basis $\{\, |lm\rangle : m = -l, -l+1, \ldots, l\,\}$ by the formulas

$$\begin{aligned} &\sigma_l(k)\,|lm\rangle = q^m\,|lm\rangle, \quad \sigma_l(f)\,|lm\rangle = \sqrt{[l-m][l+m+1]}\,|l, m+1\rangle, \\ &\sigma_l(e)\,|lm\rangle = \sqrt{[l-m+1][l+m]}\,|l, m-1\rangle. \end{aligned} \tag{4}$$

A $*$-representation of $\mathcal{A}(\mathrm{SU}_q(2)) \rtimes \mathcal{U}_q(\mathrm{su}(2))$ or $\mathcal{A}(\mathrm{S}^2_{qc}) \rtimes \mathcal{U}_q(\mathrm{su}(2))$ is called integrable if its restriction to $\mathcal{U}_q(\mathrm{su}(2))$ is a direct sum of spin l representations σ_l.

Suppose that π is a $*$-representation of $\mathcal{A}(\mathrm{SU}_q(2)) \rtimes \mathcal{U}_q(\mathrm{su}(2))$ (or $\mathcal{A}(\mathrm{S}^2_{qc}) \rtimes \mathcal{U}_q(\mathrm{su}(2))$) on V. Then the tensor product representation $\pi \otimes \sigma_l$ on $V \otimes V_l$ is defined by setting $\pi \otimes \sigma_l(h) := \pi(h_{(1)}) \otimes \sigma_l(h_{(2)})$ for $h \in \mathcal{U}_q(\mathrm{su}(2))$ and $\pi \otimes \sigma_l(x) := \pi(x) \otimes \sigma_l(1)$ for $x \in \mathcal{A}(\mathrm{SU}_q(2))$ (or $x \in \mathcal{A}(\mathrm{S}^2_{qc})$). Straightforward computations show that $\pi \otimes \sigma_l$ yields indeed a $*$-representation of the quoted crossed product $*$-algebras.

2.2.1. *Integrable representations of* $\mathcal{A}(\mathrm{SU}_q(2)) \rtimes \mathcal{U}_q(\mathrm{su}(2))$. Let ψ denote the Haar state of $\mathcal{A}(\mathrm{SU}_q(2))$. From the GNS representation of $\mathcal{A}(\mathrm{SU}_q(2))$ associated to ψ, we derive a unique integrable $*$-representation π_ψ of $\mathcal{A}(\mathrm{SU}_q(2)) \rtimes \mathcal{U}_q(\mathrm{su}(2))$, called the Heisenberg representation [18]. It is obtained as follows. Since ψ is faithful, we can equip $\mathcal{A}(\mathrm{SU}_q(2))$ with the inner product $\langle x, y\rangle := \psi(y^*x)$. The representation is given by the formulas $\pi_\psi(h)x = h \triangleright x$ and $\pi_\psi(y)x = yx$, where $x, y \in \mathcal{A}(\mathrm{SU}_q(2))$ and $h \in \mathcal{U}_q(\mathrm{su}(2))$. Recall that $\mathcal{A}(\mathrm{SU}_q(2))$ has a vector-space basis $\{\, t^l_{mn} : 2l \in \mathbb{N},\ m, n = -l, -l+1, \ldots, l\,\}$ consisting of matrix elements of its finite dimensional irreducible corepresentations [13]. The normalized matrix elements

$$|lmn\rangle := q^n\,[2l+1]^{\frac{1}{2}}\,t^l_{nm}, \quad l \in \tfrac{1}{2}\mathbb{N}_0,\ \ m, n = -l, -l+1, \ldots, l, \tag{5}$$

form an orthonormal basis for $\mathcal{A}(\mathrm{SU}_q(2))$. On

$$V_{ln} := \mathrm{span}\{\, |lmn\rangle : m = -l, -l+1, \ldots, l\,\}, \tag{6}$$

the restriction of π_ψ to $\mathcal{U}_q(\mathrm{su}(2))$ becomes a spin l representation, so π_ψ is integrable.

It follows from [19, Proposition 1.2] that each integrable $*$-representation of the crossed product $*$-algebra $\mathcal{A}(\mathrm{SU}_q(2)) \rtimes \mathcal{U}_q(\mathrm{su}(2))$ is unitarily equivalent to a direct sum of Heisenberg representations. Moreover, an integrable $*$-representation of $\mathcal{A}(\mathrm{SU}_q(2)) \rtimes \mathcal{U}_q(\mathrm{su}(2))$ is irreducible if and only if the vector space of invariant vectors (i.e., vectors belonging to a spin 0 representations) is 1-dimensional. In particular, each irreducible integrable

$*$-representation of $\mathcal{A}(\mathrm{SU}_q(2)) \rtimes \mathcal{U}_q(\mathrm{su}(2))$ is unitarily equivalent to the Heisenberg representation.

Defining

$$\rho_\psi(h)x = x \triangleleft S^{-1}(h), \qquad x \in \mathcal{A}(\mathrm{SU}_q(2)), \tag{7}$$

the left $\mathcal{U}_q(\mathrm{su}(2))$-equivariant representation π_ψ can also be viewed as right $\mathcal{U}_q(\mathrm{su}(2))$-equivariant. One easily shows (see, e.g., [18]) that $V_{lm} = \mathrm{span}\{|lmn\rangle : n = -l, \ldots, l\}$ is an irreducible spin l representation space with highest weight $|lml\rangle$, i.e., $\rho_\psi(k)|lmn\rangle = q^{-n}|lmn\rangle$. Since left and right $\mathcal{U}_q(\mathrm{su}(2))$-actions on $\mathcal{A}(\mathrm{SU}_q(2))$ commute, we have obviously $\pi_\psi(h)\rho_\psi(g) = \rho_\psi(g)\pi_\psi(h)$ for all $g, h \in \mathcal{U}_q(\mathrm{su}(2))$.

2.2.2. *Integrable representations of* $\mathcal{A}(\mathrm{S}^2_{qc}) \rtimes \mathcal{U}_q(\mathrm{su}(2))$*.* The integrable representations of $\mathcal{A}(\mathrm{S}^2_{qc}) \rtimes \mathcal{U}_q(\mathrm{su}(2))$ were completely classified in [19]. It turned out that each integrable representation is a direct sum of irreducible ones. The inequivalent irreducible integrable representation π_j of $\mathcal{A}(\mathrm{S}^2_{qc}) \rtimes \mathcal{U}_q(\mathrm{su}(2))$ are labeled by half-integers $j \in \frac{1}{2}\mathbb{Z}$. Each representation π_j can be realized on an invariant subspace $M_j \subset \mathcal{A}(\mathrm{SU}_q(2))$ by restricting the Heisenberg representation π_ψ of $\mathcal{A}(\mathrm{SU}_q(2)) \rtimes \mathcal{U}_q(\mathrm{su}(2))$ to the $*$-subalgebra $\mathcal{A}(\mathrm{S}^2_{qc}) \rtimes \mathcal{U}_q(\mathrm{su}(2))$. Moreover, $\mathcal{A}(\mathrm{SU}_q(2))$ is the orthogonal direct sum of these invariant subspaces, i.e., $\mathcal{A}(\mathrm{SU}_q(2)) = \oplus_{j \in \frac{1}{2}\mathbb{Z}} M_j$. As a left $\mathcal{A}(\mathrm{S}^2_{qc})$-module, M_j is finitely generated and projective. It is known that M_j can be considered as a line bundle over the quantum sphere S^2_{qc} with winding number $2j$ [1, 11, 15].

For the convenience of the reader, we recall from [19] the explicit description of the irreducible representations π_j, $j \in \frac{1}{2}\mathbb{Z}$. The Hilbert space is the orthogonal direct sum $\bigoplus_{l=|j|,|j|+1,\ldots} V^l$, where V^l is a spin l-representation space with an orthonormal basis of weight vectors $\{v^l_{k,j} : k = -l, -l+1, \ldots, l\}$. The generators e, f, k of $\mathcal{U}_q(\mathrm{su}(2))$ act on V^l by (4). The actions of the generators x_1, x_0, x_{-1} of $\mathcal{A}(\mathrm{S}^2_{qc})$ are determined by

$$\begin{aligned}\pi_j(x_1)v^l_{k,j} = {} & q^{-l+k}[l{+}k{+}1]^{1/2}[l{+}k{+}2]^{1/2}[2l{+}1]^{-1/2}[2l{+}2]^{-1/2}\alpha_j(l)v^{l+1}_{k+1,j} \\ & - q^{k+2}[l{-}k]^{1/2}[l{+}k{+}1]^{1/2}[2]^{1/2}[2l]^{-1}\beta_j(l)v^l_{k+1,j} \\ & - q^{l+k+1}[l{-}k{-}1]^{1/2}[l{-}k]^{1/2}[2l{-}1]^{-1/2}[2l]^{-1/2}\alpha_j(l{-}1)v^{l-1}_{k+1,j},\end{aligned} \tag{8}$$

$$\begin{aligned}\pi_j(x_0)v^l_{k,j} = {} & q^k[l{-}k{+}1]^{1/2}[l{+}k{+}1]^{1/2}[2]^{1/2}[2l{+}1]^{-1/2}[2l{+}2]^{-1/2}\alpha_j(l)v^{l+1}_{k,j} \\ & + \big(1 - q^{l+k+1}[l{-}k][2][2l]^{-1}\big)\beta_j(l)v^l_{k,j} \\ & + q^k[l{-}k]^{1/2}[l{+}k]^{1/2}[2]^{1/2}[2l{-}1]^{-1/2}[2l]^{-1/2}\alpha_j(l{-}1)v^{l-1}_{k,j}\end{aligned} \tag{9}$$

and $\pi_j(x_{-1}) = -q^{-1}\pi_j(x_{-1})^*$. For $c < \infty$, the real numbers $\beta_j(l)$ and $\alpha_j(l)$ are defined by

$$\begin{aligned}\beta_j(l) &= [2l{+}2]^{-1}\big([2|j|](q^{-2}\lambda_\pm - \lambda_\mp) + (1 - q^{-2})[|j|]\,[|j|{+}1] - (1 - q^{-2})[l][l{+}1]\big), \\ \alpha_j(l) &= [2]^{-1/2}[2l{+}3]^{-1/2}[2l{+}2]^{1/2}\big(1 + [2]^2c - (1 - q^2)\beta_j(l) - q^2(\beta_j(l))^2\big)^{1/2},\end{aligned}$$

where $\lambda_\pm = 1/2 \pm (c + 1/4)^{1/2}$. For $c = \infty$, $\beta_j(l)$ and $\alpha_j(l)$ are given by

$$\beta_j(l) = \mathrm{sign}(j)\, q^{-1}[2l{+}2]^{-1}[2]\,[2|j|], \;\; \alpha_j(l) = [2]^{-1/2}[2l{+}3]^{-1/2}[2l{+}2]^{1/2}\big([2]^2 - q^2(\beta_j(l))^2\big)^{1/2}.$$

In the case $l = k = j = 0$, Equation (9) becomes $\pi_0(x_0)v^0_{0,0} = \alpha_0(0)\, v^1_{0,0}$.

In the present paper, we are particularly interested in the representation π_0 acting on the trivial line bundle $M_0 \cong \mathcal{A}(\mathrm{S}^2_{qc})$. This representation can also be obtained from the GNS representation associated to Haar state $\tilde{\psi}$ on $\mathcal{A}(\mathrm{S}^2_{qc})$. By the uniqueness of the Haar state, one can take $\tilde{\psi}$ to be the restriction of ψ on $\mathcal{A}(\mathrm{SU}_q(2))$ to $\mathcal{A}(\mathrm{S}^2_{qc})$. Analogously to the Heisenberg representation from the previous subsection, we have $\langle x, y\rangle := \tilde{\psi}(y^*x)$, $\pi_0(y)x = yx$, and $\pi_0(h)x = h \triangleright x$, where $x, y \in \mathcal{A}(\mathrm{S}^2_{qc})$ and $h \in \mathcal{U}_q(\mathrm{su}(2))$.

2.3. Spectral triples. By a (compact) spectral triple $(\mathcal{A}, \mathcal{H}, D)$, we mean a $*$-algebra $\mathcal{A}$, a bounded $*$-representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, and a self-adjoint operator D on $\mathcal{H}$ such that [2]

(*i*) $(D - \zeta)^{-1}$ is a compact operator for all $\zeta \in \mathbb{C} \setminus \mathbb{R}$,
(*ii*) the commutators $[D, \pi(a)]$ are bounded for all $a \in \mathcal{A}$.

If there exists an $n \in \mathbb{N}_0$ such that the asymptotic behavior of the eigenvalues $0 \le \mu_1 \le \mu_2 \le \dots$ of $|D|^{-n}$ is given by $\mu_k = \mathrm{O}(k^{-1})$ as $k \to \infty$, then the spectral triple is said to be n^+-summable.

Let $\mathcal{U}$ be a Hopf $*$-algebra, $\mathcal{A}$ a left and/or right $\mathcal{U}$-module and π a $\mathcal{U}$-equivariant representation on $\mathcal{H}$. The spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is called left or right $\mathcal{U}$-equivariant if D is a left or right equivariant operator. We call it bi-equivariant if D is left and right $\mathcal{U}$-equivariant.

An (equivariant) spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is called even if there exists an (equivariant) grading operator γ on $\mathcal{H}$ such that $\gamma^* = \gamma$, $\gamma^2 = 1$, $\gamma D = -D\gamma$, and $\gamma\pi(a) = \pi(a)\gamma$ for all $a \in \mathcal{A}$.

In the seminal paper [3], a real structure J on a spectral triple was defined by an antiunitary operator J on $\mathcal{H}$ satisfying

$$[\pi(x), J\pi(y)J^{-1}] = 0, \qquad [[D, \pi(x)], J\pi(y)J^{-1}] = 0, \qquad x, y \in \mathcal{A}, \tag{10}$$

$J^2 = \pm 1$, $JD = \pm DJ$ and, for even spectral triples, $J\gamma = \pm\gamma J$. The real structure is called equivariant, if J is equivariant in the sense of Section 2.2.

It was noted in [8] that, by requiring equivariance of J, it is not possible to satisfy (10). However, the problem was overcome in [8] by requiring that (10) holds up to an operator ideal contained in the ideal of infinitesimals of arbitrary high order. Here, a compact operator A is called an infinitesimal of arbitrary high order if its singular values $s_n(A)$ satisfy $\lim_{n\to\infty} n^p s_n(A) = 0$ for all $p > 0$.

3. Equivariant spectral triples

3.1. The equivariant Dirac operator on $\mathcal{A}(\mathrm{SU}_q(2))$. In this section, we summarize the results from [8] concerning the equivariant isospectral Dirac operator on $\mathcal{A}(\mathrm{SU}_q(2))$. Starting point of the construction is the Heisenberg representation π_ψ of the left crossed product $*$-algebra $\mathcal{A}(\mathrm{SU}_q(2)) \rtimes \mathcal{U}_q(\mathrm{su}(2))$ on $V := \mathcal{A}(\mathrm{SU}_q(2))$. The spin representation π is given by the tensor product representation $\pi := \pi_\psi \otimes \sigma_{\frac{1}{2}}$ acting on $W := V \otimes V_{\frac{1}{2}}$. The Hilbert space completion of W will be denoted by $\mathcal{H}$. Setting $\rho := \rho_\psi \otimes \mathrm{id}$, the left $\mathcal{U}_q(\mathrm{su}(2))$-equivariant representation π becomes also right $\mathcal{U}_q(\mathrm{su}(2))$-equivariant and we have $\pi(h)\rho(g) = \rho(g)\pi(h)$ for all $h, g \in \mathcal{U}_q(\mathrm{su}(2))$.

Recall that the set of vectors defined in (5) forms an orthonormal basis for $\mathcal{A}(\mathrm{SU}_q(2))$. Set

$$H_l := \mathrm{span}\{\, |lmn\rangle : m,n=-l,-l+1,\dots,l\,\}. \tag{11}$$

Then, by (6), $H_l = \oplus_{n=-l}^{l} V_{ln}$ is the $2l+1$-fold orthogonal sum of irreducible spin l representation spaces. As before, let V_l, $l \in \frac{1}{2}\mathbb{N}_0$, denote the irreducible spin l representation space. From the Clebsch-Gordan decomposition, it is known that

$$V_l \otimes V_{\frac{1}{2}} = V_{l-\frac{1}{2}} \oplus V_{l+\frac{1}{2}}, \quad l=\tfrac{1}{2},1,\dots, \qquad V_0 \otimes V_{\frac{1}{2}} = V_{\frac{1}{2}}. \tag{12}$$

Hence we can write

$$H_l \otimes V_{\frac{1}{2}} = W^{\uparrow}_{l-\frac{1}{2}} \oplus W^{\downarrow}_{l+\frac{1}{2}}, \quad l=\tfrac{1}{2},1,\dots, \qquad H_0 \otimes V_{\frac{1}{2}} = W^{\downarrow}_{\frac{1}{2}}, \tag{13}$$

where $W^{\uparrow}_{l-\frac{1}{2}}$ and $W^{\downarrow}_{l+\frac{1}{2}}$ are the linear spaces of vectors from $H_l \otimes V_{\frac{1}{2}}$ belonging to spin $l-\frac{1}{2}$ and spin $l+\frac{1}{2}$ representations, respectively. Since $V = \oplus_{l\in\frac{1}{2}\mathbb{N}_0} H_l$, it follows that the representation space $W = V \otimes V_{\frac{1}{2}}$ decomposes into

$$W = \bigoplus_{l\in\frac{1}{2}\mathbb{N}_0} W^{\uparrow}_l \oplus \bigoplus_{l\in\frac{1}{2}\mathbb{N}} W^{\downarrow}_l. \tag{14}$$

By (11)–(13), we have $\dim W^{\uparrow}_l = (2l+1)(2l+2)$ and $\dim W^{\downarrow}_l = 2l(2l+1)$.

Now consider the self-adjoint operator D on $\mathcal{H}$ determined by

$$Dw^{\uparrow}_l = (l+\tfrac{1}{2})w^{\uparrow}_l, \ \ w^{\uparrow}_l \in W^{\uparrow}_l, \qquad Dw^{\downarrow}_l = -(l+\tfrac{1}{2})w^{\downarrow}_l, \ \ w^{\downarrow}_l \in W^{\downarrow}_l. \tag{15}$$

It was proved in [8] that $(\mathcal{A}(\mathrm{SU}_q(2)), \mathcal{H}, D)$ is a bi-equivariant spectral triple. The eigenvalues of D are $l+\frac{1}{2}$ with multiplicities $(2l+1)(2l+2)$ and $-(l+\frac{1}{2})$ with multiplicities $2l(2l+1)$, where $l \in \frac{1}{2}\mathbb{N}_0$ and $l \in \frac{1}{2}\mathbb{N}$, respectively. Using the results from [12], one easily checks that the eigenvalues and multiplicities of $2D-\frac{1}{2}$ coincide with those of a classical Dirac operator on $\mathbb{S}^3 \approx \mathrm{SU}(2)$ equipped with a $\mathrm{SU}(2)\times\mathrm{SU}(2)$-invariant metric (set $\lambda=-1$ in [12, Proposition 3.2]). So we have an isospectral deformation of a SU(2)-bi-invariant classical spectral triple.

3.2. The equivariant Dirac operator on $\mathcal{A}(\mathrm{S}^2_{qc})$. Our aim is to show that restricting the Dirac operator on $\mathcal{A}(\mathrm{SU}_q(2))$ to the quantum spinor bundle $\mathcal{A}(\mathrm{S}^2_{qc}) \otimes V_{\frac{1}{2}} \subset \mathcal{A}(\mathrm{SU}_q(2)) \otimes V_{\frac{1}{2}}$ yields a spectral triple on $\mathcal{A}(\mathrm{S}^2_{qc})$.

To begin, recall from Section 2.2.2 that π_0 is the $*$-representation of $\mathcal{A}(\mathrm{S}^2_{qc}) \rtimes \mathcal{U}_q(\mathrm{su}(2))$ obtained by restricting the Heisenberg representation of $\mathcal{A}(\mathrm{SU}_q(2)) \rtimes \mathcal{U}_q(\mathrm{su}(2))$ to its subalgebra $\mathcal{A}(\mathrm{S}^2_{qc}) \rtimes \mathcal{U}_q(\mathrm{su}(2))$ and to the subspace $M_0 = \mathcal{A}(\mathrm{S}^2_{qc})$ of $V = \mathcal{A}(\mathrm{SU}_q(2))$. Along the lines of the previous subsection, we take the tensor product representation $\tilde{\pi} := \pi_0 \otimes \sigma_{\frac{1}{2}}$ on $\tilde{W} := M_0 \otimes V_{\frac{1}{2}}$ as spin representation. Furthermore, the Hilbert space completion of $\tilde{W}$, say $\tilde{\mathcal{H}}$, will be considered as Hilbert space of spinors.

Let $\tilde{V}_l := \mathrm{span}\{\, v^l_{m,0} : m=-l,\dots,l\,\}$. In Section 2.2.2, we saw that $M_0 = \oplus_{l\in\mathbb{N}_0} \tilde{V}_l$ is an orthogonal sum of irreducible spin l representation spaces. The Clebsch-Gordan decomposition yields

$$\tilde{V}_l \otimes V_{\frac{1}{2}} = \tilde{W}^{\uparrow}_{l-\frac{1}{2}} \oplus \tilde{W}^{\downarrow}_{l+\frac{1}{2}}, \quad l=1,2,\dots, \qquad \tilde{V}_0 \otimes V_{\frac{1}{2}} = \tilde{W}^{\downarrow}_{\frac{1}{2}}, \tag{16}$$

where the restriction of $\pi_0 \otimes \sigma_{\frac{1}{2}}$ to $\tilde{W}_l^{\uparrow}$ or $\tilde{W}_l^{\downarrow}$ is an irreducible spin l representation of $\mathcal{U}_q(\mathrm{su}(2))$. Clearly,

$$\tilde{W} = \bigoplus_{l\in\mathbb{N}_0} \tilde{W}^{\uparrow}_{l+\frac{1}{2}} \oplus \tilde{W}^{\downarrow}_{l+\frac{1}{2}}. \tag{17}$$

As $M_0 = \oplus_{l\in\mathbb{N}_0} \tilde{V}_l \subset V = \oplus_{l\in\frac{1}{2}\mathbb{N}_0} H_l$ and $\tilde{V}_l$ is a spin l representation space, we have $\tilde{V}_l \subset H_l$. Comparing (13) and (16) shows that $\tilde{W}_l^{\uparrow} \subset W_l^{\uparrow}$ and $\tilde{W}_l^{\downarrow} \subset W_l^{\downarrow}$, where $l = \frac{1}{2}, \frac{3}{2}, \ldots$. The operator D from Subsection 3.1 acts on each $W_l^{\uparrow}$ and $W_l^{\downarrow}$ as a multiple of the identity. In particular, D leaves the subspaces $\tilde{W}_l^{\uparrow}$ and $\tilde{W}_l^{\downarrow}$ invariant. Let $\tilde{D}$ denote (the closure of) the restriction of D to $\tilde{W}$. By (15),

$$\tilde{D}\tilde{w}_l^{\uparrow} = (l+\tfrac{1}{2})\tilde{w}_l^{\uparrow},\ \ \tilde{w}_l^{\uparrow} \in \tilde{W}_l^{\uparrow}, \qquad \tilde{D}\tilde{w}_l^{\downarrow} = -(l+\tfrac{1}{2})\tilde{w}_l^{\downarrow},\ \ \tilde{w}_l^{\downarrow} \in \tilde{W}_l^{\downarrow}, \qquad l = \tfrac{1}{2}, \tfrac{3}{2}, \ldots. \tag{18}$$

Since the spin representation $\pi_0 \otimes \sigma_{\frac{1}{2}}$ is obtained by restricting $\pi_\psi \otimes \sigma_{\frac{1}{2}}$ to the subalgebra $\mathcal{A}(\mathrm{S}^2_{qc}) \rtimes \mathcal{U}_q(\mathrm{su}(2))$ of $\mathcal{A}(\mathrm{SU}_q(2)) \rtimes \mathcal{U}_q(\mathrm{su}(2))$ and to the subspace $\tilde{W} \subset W$, we can now apply verbatim the results from [8]. Therefore, $[\tilde{D}, \pi_0 \otimes \sigma_{\frac{1}{2}}(x)]$ is bounded for all $x \in \mathcal{A}(\mathrm{S}^2_{qc})$ and $\tilde{D}$ is left $\mathcal{U}_q(\mathrm{su}(2))$-equivariant because the same is true for $[D, \pi_\psi \otimes \sigma_{\frac{1}{2}}(x)]$ and D.

Comparing the eigenvalues and the corresponding multiplicities with those of the Dirac operator on the Riemannian 2-sphere with the standard metric (see, e.g., [10]), one sees that $(\mathcal{A}(\mathrm{S}^2_{qc}), \tilde{\mathcal{H}}, \tilde{D})$ is an isospectral deformation of the classical spectral triple. From the asymptotic behavior of the eigenvalues, one readily concludes that it is 2^+-summable.

Summarizing, we arrive at the following proposition.

Proposition 3.1. *Restricting the spectral triple $(\mathcal{A}(\mathrm{SU}_q(2)), \mathcal{H}, D)$ to $\mathcal{A}(\mathrm{S}^2_{qc}) \otimes V_{\frac{1}{2}} \subset \mathcal{A}(\mathrm{SU}_q(2)) \otimes V_{\frac{1}{2}}$ (considered as subspaces of $\mathcal{H}$) gives rise to a left $\mathcal{U}_q(\mathrm{su}(2))$-equivariant, 2^+-summable spectral triple $(\mathcal{A}(\mathrm{S}^2_{qc}), \tilde{\mathcal{H}}, \tilde{D})$, where $\tilde{\mathcal{H}}$ denotes the closure of $\mathcal{A}(\mathrm{S}^2_{qc}) \otimes V_{\frac{1}{2}}$ in $\mathcal{H}$. It is an isospectral deformation of the classical spectral triple on the Riemannian 2-sphere with the standard metric.*

Remark. The fact that $(\mathcal{A}(\mathrm{S}^2_{qc}), \tilde{\mathcal{H}}, \tilde{D})$ defines a left $\mathcal{U}_q(\mathrm{su}(2))$-equivariant spectral triple on the Podleś spheres has been proved in [7] for $c = \infty$ and in [6] for all $c \in [0, \infty]$ by direct computations.

3.3. Equivariant grading operator on $(\mathcal{A}(\mathrm{S}^2_{q\infty}), \tilde{\mathcal{H}}, \tilde{D})$.

As $(\mathcal{A}(\mathrm{S}^2_{qc}), \tilde{\mathcal{H}}, \tilde{D})$ is an isospectral deformation and 2^+-summable, its classical dimension is 2. For this reason and in analogy with the classical picture, we are interested in obtaining an even spectral triple. The next proposition shows that an equivariant grading operator exists only for the spectral triple $(\mathcal{A}(\mathrm{S}^2_{q\infty}), \tilde{\mathcal{H}}, \tilde{D})$. The "only if" part of the proposition is an interesting result since it seems to contradict [6], where equivariant even spectral triples for all Podleś spheres were constructed. In fact, one of the main purposes of this paper is to point out that the construction of spectral triples by restriction may be possible but extra care has to be taken when trying to satisfy additional structures, for instance, when passing from odd to even ones. The reason behind the seeming contradiction between [6] and Proposition 3.2 will be explained after the proof of the proposition. Roughly speaking, it arises because the spectral triples from [6] and Proposition 3.1 are unitarily equivalent but, for $c<\infty$, the unitary operators implementing the equivalence are not compatible with the (unique) equivariant grading operator.

PROPOSITION 3.2. *The spectral triple* $(\mathcal{A}(\mathrm{S}^2_{qc}),\tilde{\mathcal{H}},\tilde{D})$ *from Proposition 3.1 admits an equivariant grading operator of an even spectral triple if and only if* $c=\infty$.

Proof. The assumptions on γ imply that it commutes with all elements from the crossed product algebra $\mathcal{A}(\mathrm{S}^2_{qc})\rtimes\mathcal{U}_q(\mathrm{su}(2))$. In [19, Proposition 4.4], it has been shown that the tensor product representation $\pi_0\otimes\sigma_{\frac12}$ on $M_0\otimes V_{\frac12}$ decomposes into the direct sum of the irreducible representations $\pi_{-\frac12}$ and $\pi_{\frac12}$ on $M_{-\frac12}$ and $M_{\frac12}$, respectively. From $\gamma^*=\gamma$ and $\gamma^2=1$, it follows that γ has eigenvalues ±1. Since the irreducible representations $\pi_{-\frac12}$ and $\pi_{\frac12}$ are nonequivalent and integrable, we conclude that γ acts on $M_{-\frac12}$ and $M_{\frac12}$ by $\pm\,\mathrm{id}$, with opposite sign on each space. In the notation of Section 2.2.2, we can assume without loss of generality that $\tilde{W}=M_{-\frac12}\oplus M_{\frac12}$ and

$$\gamma v^l_{m,-\frac12}=-v^l_{m,-\frac12},\qquad \gamma v^l_{m,\frac12}=v^l_{m,\frac12}. \tag{19}$$

Next, the relation $\gamma\tilde{D}=-\tilde{D}\gamma$ forces γ to map $\tilde{W}^{\downarrow}_l$ into $\tilde{W}^{\uparrow}_l$ and $\tilde{W}^{\uparrow}_l$ into $\tilde{W}^{\downarrow}_l$. In addition, the equivariance of γ implies that we can choose a basis $\{|lm{\downarrow}\rangle : m=-l,\dots,l\}$ for $\tilde{W}^{\uparrow}_l$ and a basis $\{|lm{\uparrow}\rangle : m=-l,\dots,l\}$ for $\tilde{W}^{\uparrow}_l$ such that the action of $\mathcal{U}_q(\mathrm{su}(2))$ on these vectors is given by (4) and

$$\gamma\,|lm{\downarrow}\rangle=|lm{\uparrow}\rangle,\qquad \gamma\,|lm{\uparrow}\rangle=|lm{\downarrow}\rangle. \tag{20}$$

Assume now that γ is an operator on $\tilde{W}$ satisfying Equations (19) and (20). Using $\mathrm{span}\{v^{\frac12}_{\frac12,-\frac12},v^{\frac12}_{\frac12,\frac12}\}=\mathrm{span}\{|\frac12,\frac12,{\downarrow}\rangle,|\frac12,\frac12,{\uparrow}\rangle\}$ and applying (19) and (20), we can write

$$|\tfrac12,\tfrac12,{\downarrow}\rangle=s\,v^{\frac12}_{\frac12,\frac12}+t\,v^{\frac12}_{\frac12,-\frac12},\qquad |\tfrac12,\tfrac12,{\uparrow}\rangle=s\,v^{\frac12}_{\frac12,\frac12}-t\,v^{\frac12}_{\frac12,-\frac12},$$

where $s,t\in\mathbb{C}$ such that $|s|^2+|t|^2=1$. Moreover, $\langle\frac12,\frac12,{\downarrow}\,|\,\frac12,\frac12,{\uparrow}\rangle=0$ implies $|s|^2=|t|^2=\frac12$. In the notation of (4), let $V_{\frac12}=\mathrm{span}\{|\frac12,-\frac12\rangle,|\frac12,\frac12\rangle\}$. From (16), it follows that $|\frac12,\frac12,{\downarrow}\rangle=\exp(\mathrm{i}\omega)\,v^0_{0,0}\otimes|\frac12,\frac12\rangle$, where $\omega\in[0,2\pi)$. Applying the formulas from Section 2.2.2, we compute

$$0=\langle v^0_{0,0},\pi_0(x_0)v^0_{0,0}\rangle=\langle\tfrac12,\tfrac12,{\downarrow}|\tilde{\pi}(x_0)|\tfrac12,\tfrac12,{\downarrow}\rangle=\tfrac12\big(\beta_{\frac12}(\tfrac12)+\beta_{-\frac12}(\tfrac12)\big). \tag{21}$$

For $c<\infty$, we obtain a contradiction since $\beta_{\frac12}(\frac12)+\beta_{-\frac12}(\frac12)=[3]^{-1}(q^{-2}-1)\neq0$. Therefore a grading operator satisfying Equations (19) and (20) can only exist in the case of the equatorial Podleś sphere $\mathcal{A}(\mathrm{S}^2_{q\infty})$.

Let $c=\infty$. Our aim is to find orthonormal vectors $|lm{\downarrow}\rangle,|lm{\uparrow}\rangle\in\mathrm{span}\{v^l_{m,-\frac12},v^l_{m,\frac12}\}$ such that γ is given by (20). To begin, consider

$$\begin{aligned}
|ll{\downarrow}\rangle &:= v^{l-\frac12}_{l-\frac12,0}\otimes|\tfrac12,\tfrac12\rangle,\\
|ll{\uparrow}\rangle &:= [2l+2]^{-\frac12}\big(-q^{\frac12}[2l+1]^{\frac12}\,v^{l+\frac12}_{l+\frac12,0}\otimes|\tfrac12,-\tfrac12\rangle+q^{-l-\frac12}\,v^{l+\frac12}_{l-\frac12,0}\otimes|\tfrac12,\tfrac12\rangle\big).
\end{aligned} \tag{22}$$

Since $|ll{\downarrow}\rangle$ and $|ll{\uparrow}\rangle$ are highest weight vectors of weight q^l, it follows that both belong to $\mathrm{span}\{v^l_{l,-\frac12},v^l_{l,\frac12}\}$. Moreover, by (16), $|ll{\downarrow}\rangle\in\tilde{W}^{\downarrow}_l$ and $|ll{\uparrow}\rangle\in\tilde{W}^{\uparrow}_l$. We claim that, for some $\omega_l,\ \phi_l \in [0, 2\pi)$,

$$\tfrac{1}{\sqrt{2}}(|ll\!\downarrow\rangle + |ll\!\uparrow\rangle) = \exp(\mathrm{i}\omega_l)\, v^l_{l,\frac12}, \qquad \tfrac{1}{\sqrt{2}}(|ll\!\downarrow\rangle - |ll\!\uparrow\rangle) = \exp(\mathrm{i}\phi_l)\, v^l_{l,-\frac12}. \tag{23}$$

Before justifying the claim, we observe that, for $w = x\, v^l_{l,\frac12} + y\, v^l_{l,-\frac12}$ with $x, y \in \mathbb{C}$ and $|x|^2 + |y|^2 = 1$, we have $w = \exp(\mathrm{i}\omega_l)\, v^l_{l,\pm\frac12}$ if and only if $\langle w, \pi(x_0) w\rangle = \beta_{\pm\frac12}(l)$. This is apparent from the equality $\langle w, \pi(x_0)w\rangle = |x|^2\beta_{\frac12}(l) + |y|^2\beta_{-\frac12}(l)$ since $\beta_{\frac12}(l) > 0$ and $\beta_{-\frac12}(l) < 0$. Applying the formulas from Subsection 2.2.2 (note that $\beta_0(l) = 0$ for all $l \in \mathbb{N}_0$) gives

$$\begin{aligned}
&\tfrac12(\langle ll\!\downarrow\!| \pm \langle ll\!\uparrow\!|)\,\tilde\pi(x_0)\,(|ll\!\downarrow\rangle \pm |ll\!\uparrow\rangle)\\
&\qquad = \tfrac12[2l+2]^{-\frac12} q^{-l-\frac12}\big(\pm\langle v^{l+\frac12}_{l-\frac12,0}, \pi_0(x_0) v^{l-\frac12}_{l-\frac12,0}\rangle \pm \langle v^{l-\frac12}_{l-\frac12,0}, \pi_0(x_0) v^{l+\frac12}_{l-\frac12,0}\rangle\big)\\
&\qquad = \pm q^{-1}[2l+2]^{-1}[2] = \beta_{\pm\frac12}(l)
\end{aligned}$$

from which the claim follows.

With e denoting one of the generators of $\mathcal{U}_q(\mathrm{su}(2))$, set

$$|lm\!\downarrow\rangle := ||\tilde\pi(e)^{l-m}\,|ll\!\downarrow\rangle||^{-1}\,\tilde\pi(e)^{l-m}\,|ll\!\downarrow\rangle, \qquad |lm\!\uparrow\rangle := ||\tilde\pi(e)^{l-m}\,|ll\!\uparrow\rangle||^{-1}\,\tilde\pi(e)^{l-m}\,|ll\!\uparrow\rangle. \tag{24}$$

Then (23) implies that

$$\tfrac{1}{\sqrt{2}}(|lm\!\downarrow\rangle + |lm\!\uparrow\rangle) = \exp(\mathrm{i}\omega_l)\, v^l_{m,\frac12}, \qquad \tfrac{1}{\sqrt{2}}(|lm\!\downarrow\rangle - |lm\!\uparrow\rangle) = \exp(\mathrm{i}\phi_l)\, v^l_{m,-\frac12}, \tag{25}$$

and the operator γ given by Equation (20) satisfies (19). Clearly, this operator meets all the requirements on an equivariant grading operator. ∎

Let us now explain why $(\mathcal{A}(\mathrm{S}^2_{qc}), \tilde{\mathcal{H}}, \tilde D)$ does not admit an equivariant grading operator for $c < \infty$ although equivariant even spectral triples with the same spectral properties were constructed in [6] for all c. The Dirac operators from [6] are unitarily equivalent to $\tilde D$, and it follows from [6, Equation (5.1)] that the unitary equivalence is determined by unitary operators

$$U_l\ :\ \mathrm{span}\{|ll\!\downarrow\rangle, |ll\!\uparrow\rangle\} \longrightarrow \mathrm{span}\{v^l_{l,-\frac12}, v^l_{l,\frac12}\}, \qquad l = \tfrac12, \tfrac32, \ldots .$$

Now Equation (23) tells us that the existence of an equivariant grading operator requires that the unitary transformations between $\mathrm{span}\{|ll\!\downarrow\rangle, |ll\!\uparrow\rangle\}$ and $\mathrm{span}\{v^l_{l,-\frac12}, v^l_{l,\frac12}\}$ are given by matrices of the type

$$\tfrac{1}{\sqrt{2}}\begin{pmatrix} \exp(\mathrm{i}\omega_l) & \exp(\mathrm{i}\omega_l)\\ \exp(\mathrm{i}\phi_l) & -\exp(\mathrm{i}\phi_l)\end{pmatrix}, \qquad \omega_l, \phi_l \in [0, 2\pi),$$

but the contradiction obtained below Equation (21) proves that, for $c < \infty$, the matrix corresponding to the unitary operator $U_{\frac12}$ does not have the above form.

3.4. The real structure on $(\mathcal{A}(\mathrm{SU}_q(2)), \mathcal{H}, D)$**.** In this section, we give a brief summary of the results of [8] on the real structure. Set

$$C_{jm} := q^{-(j+m)/2}\,[j-m]^{1/2}\,[2j]^{-1/2}, \qquad S_{jm} := q^{(j-m)/2}\,[j+m]^{-1/2}\,[2j]^{-1/2}. \tag{26}$$

With $|lmn\rangle$ defined in (5) and $\{|\frac{1}{2},-\frac{1}{2}\rangle, |\frac{1}{2},\frac{1}{2}\rangle\}$ being a orthonormal basis of $V_{\frac{1}{2}}$, let

$$|j\,m\nu{\downarrow}\rangle := C_{jm}\,|j-\tfrac{1}{2},m+\tfrac{1}{2},\nu\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle + S_{jm}\,|j-\tfrac{1}{2},m-\tfrac{1}{2},\nu\rangle\otimes|\tfrac{1}{2},+\tfrac{1}{2}\rangle, \tag{27}$$

$$|j\,m\mu{\uparrow}\rangle := -S_{j+1,m}\,|j+\tfrac{1}{2},m+\tfrac{1}{2},\mu\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle + C_{j+1,m}\,|j+\tfrac{1}{2},m-\tfrac{1}{2},\mu\rangle\otimes|\tfrac{1}{2},+\tfrac{1}{2}\rangle. \tag{28}$$

According to the Clebsch-Gordan decomposition of the tensor product representation $\sigma_l\otimes\sigma_{\frac{1}{2}}$ on $V_l\otimes V_{\frac{1}{2}}$, we have

$$W_j^{\downarrow} = \mathrm{span}\{\,|j\,m\nu{\downarrow}\rangle \,:\, m=-j,\dots,j,\;\; \nu=-j+\tfrac{1}{2},\dots,j-\tfrac{1}{2}\,\},\quad j=\tfrac{1}{2},1,\dots, \tag{29}$$

$$W_j^{\uparrow} = \mathrm{span}\{\,|j\,m\mu{\uparrow}\rangle \,:\, m=-j,\dots,j,\;\; \mu=-j-\tfrac{1}{2},\dots,j+\tfrac{1}{2}\,\},\quad j=0,\tfrac{1}{2},\dots, \tag{30}$$

and the sets on the right hand side are orthonormal bases. The set of all vectors $|j\,m\nu{\downarrow}\rangle$ and $|j\,m\mu{\uparrow}\rangle$ forms an orthonormal basis for the Hilbert space of spinors $\mathcal{H}$. Define an antiunitary operator J on $\mathcal{H}$ by

$$J\,|jmn{\uparrow}\rangle = \mathrm{i}^{2(2j+m+n)}\,|j,-m,-n,\uparrow\rangle,\quad J\,|jmn{\downarrow}\rangle = \mathrm{i}^{2(2j-m-n)}\,|j,-m,-n,\downarrow\rangle. \tag{31}$$

The proof of the following facts can be found in [8]:

Let $(\mathcal{A}(\mathrm{SU}_q(2)),\mathcal{H},D)$ be the spectral triple from Subsection 3.1. The antiunitary operator J defined above satisfies $J^2=-1$, $JD=DJ$ and

$$J\pi(h)J^{-1} = \pi(kS(h)^*k^{-1}),\qquad h\in\mathcal{U}_q(\mathrm{su}(2)). \tag{32}$$

The commutators $[\pi(x),J\pi(y)J^{-1}]$ and $[[D,\pi(x)],J\pi(y)J^{-1}]$ are infinitesimals of arbitrary high order for all $x,y\in\mathcal{A}(\mathrm{SU}_q(2))$, so J satisfies in this sense the condition of a real structure on the spectral triple $(\mathcal{A}(\mathrm{SU}_q(2)),\mathcal{H},D)$. Moreover, J is equivariant since we can consider it, for instance, as the antiunitary part of (the closure of) the equivariant antilinear operator $J\pi(k)$.

The assembly $(\mathcal{A}(\mathrm{SU}_q(2)),\mathcal{H},D,J)$ is viewed as an equivariant real spectral triple on $\mathcal{A}(\mathrm{SU}_q(2))$.

3.5. Implementation of the real structure by the Tomita operator on $\mathcal{A}(\mathrm{SU}_q(2))$. For a GNS-representation of a von Neumann algebra, the modular conjugation from the Tomita-Takesaki theory [20] can be used to introduce a reality operator [3]. The objective of this section is to relate the real structure J on the Hilbert space of spinors to the modular conjugation associated with the GNS-representation π_ψ of $\mathcal{A}(\mathrm{SU}_q(2))$.

To begin, define an antilinear operator T_ψ on $V=\mathcal{A}(\mathrm{SU}_q(2))$ by

$$T_\psi(x)=x^*,\qquad x\in\mathcal{A}(\mathrm{SU}_q(2)).$$

Obviously, $T_\psi^2=1$. Recall that the inner product on $\mathcal{A}(\mathrm{SU}_q(2))$ is given by $\langle x,y\rangle=\psi(y^*x)$ and that the Haar state ψ has the property (see, e.g., [13])

$$\psi(xy)=\psi((k^{-2}\triangleright y\triangleleft k^{-2})x),\quad x,y\in\mathcal{A}(\mathrm{SU}_q(2)).$$

Using this relation together with Equations (1), (2) and (7), we compute

$$\langle y,T_\psi x\rangle=\psi(xy)=\psi((k^2\triangleright y^*\triangleleft k^2)^*x)=\langle x,\pi_\psi(k^2)\rho_\psi(k^{-2})T_\psi(y)\rangle,\quad x,y\in\mathcal{A}(\mathrm{SU}_q(2)).$$

Hence T_ψ^* acts on $\mathcal{A}(\mathrm{SU}_q(2))$ by $\pi_\psi(k^2)\rho_\psi(k^{-2})T_\psi$ and T_ψ is closable. In the Tomita-Takesaki theory, the closure of T_ψ is referred to as Tomita operator. By a slight abuse of notation, we denote in the sequel a closable operator and its closure by the same symbol.

Let $T_\psi = J_\psi |T_\psi|$ be the polar decomposition of the Tomita operator. The antiunitary operator J_ψ is known as modular conjugation. Since $T_\psi^* T_\psi\lceil_{\mathcal{A}(\mathrm{SU}_q(2))} = \pi_\psi(k^2)\,\rho_\psi(k^{-2})$, we have

$$J_\psi\, x = T_\psi\, \pi_\psi(k^{-1})\, \rho_\psi(k)\, x = \pi_\psi(k)\, \rho_\psi(k^{-1})\, T_\psi\, x, \qquad x \in \mathcal{A}(\mathrm{SU}_q(2)).$$

Equation (1) implies that $T_\psi\, \pi_\psi(h) = \pi_\psi(S(h)^*) T_\psi$ for $h \in \mathcal{U}_q(\mathrm{su}(2))$. Therefore

$$J_\psi \pi_\psi(h) J_\psi^{-1} = \pi_\psi(kS(h)^* k^{-1}), \tag{33}$$

exactly as in Equation (32). Note that $J_\psi^2 = 1$.

Our next aim is to define an antilinear "Tomita" operator T on the tensor product $W = \mathcal{A}(\mathrm{SU}_q(2)) \otimes V_{\frac{1}{2}}$ satisfying $T\,\pi(h) = \pi(S(h)^*)T$ for $h \in \mathcal{U}_q(\mathrm{su}(2))$. To begin, we look for an antilinear operator $T_{\frac{1}{2}}$ on $V_{\frac{1}{2}}$ such that $T_{\frac{1}{2}}\, \sigma_{\frac{1}{2}}(h) = \sigma_{\frac{1}{2}}(S(h)^*) T_{\frac{1}{2}}$. A convenient choice is given by

$$T_{\frac{1}{2}} |\tfrac{1}{2}, \tfrac{1}{2}\rangle = \mathrm{i} q^{1/2} |\tfrac{1}{2}, -\tfrac{1}{2}\rangle, \qquad T_{\frac{1}{2}} |\tfrac{1}{2}, -\tfrac{1}{2}\rangle = -\mathrm{i} q^{-1/2} |\tfrac{1}{2}, \tfrac{1}{2}\rangle. \tag{34}$$

Then $J_{\frac{1}{2}} := \sigma_{\frac{1}{2}}(k) T_{\frac{1}{2}} = T_{\frac{1}{2}} \sigma_{\frac{1}{2}}(k^{-1})$ is an antiunitary operator and $J_{\frac{1}{2}}^2 = -1$.

Since the antipode is a coalgebra *anti*-homomorphism, i.e., $\Delta S(h) = S(h_{(2)}) \otimes S(h_{(1)})$, we combine $T_\psi \otimes T_{\frac{1}{2}}$ with the flip operator on tensor products and set

$$T_0 := \tau \circ (T_\psi \otimes T_{\frac{1}{2}}) \,:\, \mathcal{A}(\mathrm{SU}_q(2)) \otimes V_{\frac{1}{2}} \to V_{\frac{1}{2}} \otimes \mathcal{A}(\mathrm{SU}_q(2)),$$

where τ is defined by $\tau(x \otimes y) = y \otimes x$. By construction, the antilinear operator T_0 satisfies

$$T_0\, \pi(h) = (\sigma_{\frac{1}{2}} \otimes \pi_\psi)(S(h)^*)\, T_0, \qquad h \in \mathcal{U}_q(\mathrm{su}(2)).$$

To obtain a mapping from W into itself, we compose T_0 with an operator intertwining the tensor product representations $\sigma_{\frac{1}{2}} \otimes \sigma_l$ and $\sigma_l \otimes \sigma_{\frac{1}{2}}$. Such an operator is provided by the universal R-matrix of $\mathcal{U}_q(\mathrm{su}(2))$ (see, e.g., [13]). For a tensor product representation with $\sigma_{\frac{1}{2}}$ as left tensor factor, it can be expressed by

$$R = (\sigma_{\frac{1}{2}} \otimes \pi_\psi)\big(qfe \otimes k + qef \otimes k^{-1} + (q - q^{-1}) q^{1/2} f \otimes e\big). \tag{35}$$

Let $\hat{R} := \tau \circ R$. It follows from the properties of the R-matrix (or can be checked directly) that

$$\pi(h) \circ \hat{R} = \hat{R} \circ (\sigma_{\frac{1}{2}} \otimes \pi_\psi)(h), \qquad h \in \mathcal{U}_q(\mathrm{su}(2)).$$

Therefore the antilinear operator

$$T := \hat{R} \circ T_0$$

fulfills $T\pi(h) = \pi(S(h)^*)T$ for $h \in \mathcal{U}_q(\mathrm{su}(2))$ as required.

To describe the action of T on W, we need at first explicit formulas for the action of T_ψ on $\mathcal{A}(\mathrm{SU}_q(2))$. On writing the matrix element t^l_{mn} in (5) in terms of the generators of $\mathcal{A}(\mathrm{SU}_q(2))$ (see, e.g., [8] or [13]), one easily sees that

$$T_\psi\, |lmn\rangle = (-1)^{2l+m+n} q^{m+n}\, |l, -m, -n\rangle. \tag{36}$$

From (26)–(28), (35) and (36), we obtain after a direct calculation

$$\begin{aligned} T\, |l\, m\, \nu\!\downarrow\rangle &= \mathrm{i}^{2(2l-m-\nu)}\, q^{l+m+\nu+\frac{1}{2}}\, |l, -m, -\nu, \downarrow\rangle, \\ T\, |l\, m\, \mu\!\uparrow\rangle &= \mathrm{i}^{2(2l+m+\mu)}\, q^{-l+m+\mu-\frac{1}{2}}\, |l, -m, -\mu, \uparrow\rangle, \end{aligned} \tag{37}$$

where we used also the fact that $2(m+\nu)-1$ is an even integer. Equation (37) implies that T maps $W_l^{\uparrow}$ and $W_l^{\downarrow}$ into themselves. As a consequence, $TD = DT$.

Remarkably, we even have $[\pi(x), T\pi(y)T^{-1}] = 0$ for all $x, y \in \mathcal{A}(\mathrm{SU}_q(2))$. To see this, one uses $(\Delta\otimes\mathrm{id})\hat{R} = (\sum_i \mathrm{id}\otimes h_i\otimes g_i)(\sum_j h_j\otimes\mathrm{id}\otimes g_j)$, where $\hat{R} = \sum_i h_i\otimes g_i$, which can be deduced from general properties of R-matrices [13]. Then a straightforward computation shows that $T\pi(y)T^{-1}(w\otimes v) = \sum_i w(h_i \triangleright y^*)\otimes\sigma_{\frac{1}{2}}(g_i)v$ for all $w\otimes v \in \mathcal{A}(\mathrm{SU}_q(2))\otimes V_{\frac{1}{2}}$. Since $T\pi(y)T^{-1}$ acts by right multiplication on the first tensor factor, and $\pi(x)$ by left multiplication, it is clear that $\pi(x)$ and $T\pi(y)T^{-1}$ commute.

Observe that

$$\begin{aligned} T^*\,|l\,m\,\nu{\downarrow}\rangle &= \mathrm{i}^{2(2l+m+\nu)}\, q^{l-m-\nu+\frac{1}{2}}\,|l,-m,-\nu,{\downarrow}\rangle,\\ T^*\,|l\,m\,\mu{\uparrow}\rangle &= \mathrm{i}^{2(2l-m-\mu)}\, q^{-l-m-\mu-\frac{1}{2}}\,|l,-m,-\mu,{\uparrow}\rangle. \end{aligned} \tag{38}$$

In particular, T^* is densely defined, and T is closable. By the convention made above, its closure will again be denoted by T. In analogy with the Tomita operator T_ψ, define an antilinear operator J by the unique polar decomposition $T = J|T|$. Comparing Equations (37) and (38) with Equation (31) shows that this J actually coincides with that from Section 3.4. Moreover, $|T|$ is given on W by

$$|T|\,w = \pi(k)\,\rho(k^{-1})\,q^{-D}\,w, \qquad w\in W, \tag{39}$$

where $\rho(h) := \rho_\psi(h)\otimes\mathrm{id}$ for $h\in\mathcal{U}_q(\mathrm{su}(2))$. Thus we arrive at the following Proposition.

Proposition 3.3. *The antilinear operator J from Equation* (31) *can be expressed by*

$$J\,w = T\,\pi(k^{-1})\,\rho(k)\,q^D\,w = \pi(k)\,\rho(k^{-1})\,q^D\,T\,w, \qquad w\in W.$$

Proposition 3.3 yields another proof of the invariance relation (32). Since q^D and $\rho(k)$ commute with $\pi(h)$ and since $T\pi(h) = \pi(S(h)^*)T$ for all $h\in\mathcal{U}_q(\mathrm{su}(2))$, J and $\pi(h)$ satisfy the same commutation relation as J_ψ and $\pi_\psi(h)$ in Equation (33).

We showed above that $[\pi(x), T\pi(y)T^{-1}] = 0$ for all $x, y\in\mathcal{A}(\mathrm{SU}_q(2))$. The operator $J_0 := T\pi(k^{-1})\rho(k)$ still satisfies $[\pi(x), J_0\pi(y)J_0^{-1}] = 0$ for all $x, y\in\mathcal{A}(\mathrm{SU}_q(2))$ since $\pi(k^{-1})\rho(k)\pi(y)\rho(k^{-1})\pi(k) = \pi(k^{-1}\triangleright y\triangleleft k^{-1})$ by the equivariance of $\pi = \pi_\psi\otimes\mathrm{id}$. However, it was argued in [8] that J does not have this property. This is due to the operator $q^D = |\hat{R}^*|^{-1}$ ensuring the (anti)unitarity of J. To verify $|\hat{R}^*| = q^{-D}$, observe that

$$(T_\psi\otimes T_{\frac{1}{2}})^*w = -(\pi_\psi(k^2)\rho_\psi(k^{-2})T_\psi\otimes\sigma_{\frac{1}{2}}(k^2)T_{\frac{1}{2}})w = -\pi(k^2)\rho(k^{-2})(T_\psi\otimes T_{\frac{1}{2}})w \quad w\in W.$$

Moreover, $\hat{R}(T_\psi\otimes T_{\frac{1}{2}}) = (T_\psi\otimes T_{\frac{1}{2}})\hat{R}^*$ and $\hat{R}^*(T_\psi\otimes T_{\frac{1}{2}}) = (T_\psi\otimes T_{\frac{1}{2}})\hat{R}$ since $S\otimes S(\hat{R}) = \hat{R}$. Hence

$$T^*\,T\,w = (T_\psi\otimes T_{\frac{1}{2}})^*\,(T_\psi\otimes T_{\frac{1}{2}})\,\hat{R}\,\hat{R}^*\,w = \pi(k^2)\rho(k^{-2})|\hat{R}^*|^2 w, \quad w\in W.$$

Comparing this equation with (39) gives $|\hat{R}^*| = q^{-D}$ since $\pi(k)$ and $\rho(k)$ are invertible on W.

3.6. Equivariant real even spectral triple for $\mathcal{A}(\mathrm{S}^2_{q\infty})$. For an 2^+-summable even spectral triple with grading operator γ, the requirements on a real structure J include the commutation relation $J\gamma = -\gamma J$. By Proposition 3.2, only $(\mathcal{A}(\mathrm{S}^2_{q\infty}), \tilde{\mathcal{H}}, \tilde{D})$ admits a grading operator of an even spectral triple. For this reason, we restrict the following discussion to the equatorial Podleś sphere $\mathcal{A}(\mathrm{S}^2_{q\infty})$ although most of the results remain valid in the general case.

We proceed as in Section 3.5 and define an antilinear operator $\tilde{T}_\psi$ on $M_0 = \mathcal{A}(\mathrm{S}^2_{q\infty})$ by

$$\tilde{T}_\psi(x) = x^*, \qquad x \in \mathcal{A}(\mathrm{S}^2_{q\infty}).$$

By Equation (1), since $\pi_0(h)x = h \triangleright x$, we have $\tilde{T}_\psi \pi_0(h) = \pi_0(S(h)^*)\tilde{T}_\psi$ for all $h \in \mathcal{U}_q(\mathrm{su}(2))$. From [19, Lemma 6.3], it follows that the Haar state $\tilde{\psi}$ on $\mathcal{A}(\mathrm{S}^2_{q\infty})$ satisfies

$$\tilde{\psi}(xy) = \tilde{\psi}((k^{-2} \triangleright y)x), \qquad x, y \in \mathcal{A}(\mathrm{S}^2_{q\infty}).$$

Analogously to Section 3.5, $\tilde{T}_\psi^* \lceil_{M_0} = \pi_0(k^2)\tilde{T}_\psi$ and $\tilde{T}_\psi$ is closable (with closure denoted again by $\tilde{T}_\psi$). Moreover, $|\tilde{T}_\psi|\lceil_{M_0} = \pi_0(k)$, and the antiunitary operator $\tilde{J}_\psi$ from the polar decomposition $\tilde{T}_\psi = \tilde{J}_\psi |\tilde{T}_\psi|$ is given on M_0 by

$$\tilde{J}_\psi\, x = \tilde{T}_\psi\, \pi_0(k^{-1})\, x = \pi_0(k)\, \tilde{T}_\psi\, x, \qquad x \in M_0.$$

Since the entries of the R-matrix in (35) are elements from $\mathcal{U}_q(\mathrm{su}(2))$, the restriction of $\hat{R}$ (again denoted by $\hat{R}$) to $\tilde{W} = M_0 \otimes V_{\frac{1}{2}}$ leaves $\tilde{W}$ invariant. Thus, with $T_{\frac{1}{2}}$ from the previous subsection,

$$\tilde{T} := \hat{R}\,(\tilde{T}_\psi \otimes T_{\frac{1}{2}}). \tag{40}$$

defines an antilinear operator on $\tilde{W}$. By construction, $\tilde{T}\tilde{\pi}(h) = \tilde{\pi}(S(h)^*)\tilde{T}$ for all h in $\mathcal{U}_q(\mathrm{su}(2))$. Its adjoint $\tilde{T}^*$ acts on $\tilde{W}$ by

$$\tilde{T}^* w = -\tilde{\pi}(k^2)\,(\tilde{T}_\psi \otimes T_{\frac{1}{2}})\,\hat{R}^*\, w, \qquad w \in \tilde{W}.$$

Recall that $|\hat{R}^*| = q^{-D}$, $\hat{R}(\tilde{T}_\psi \otimes T_{\frac{1}{2}}) = (\tilde{T}_\psi \otimes T_{\frac{1}{2}})\hat{R}^*$ and $\hat{R}^*(\tilde{T}_\psi \otimes T_{\frac{1}{2}}) = (\tilde{T}_\psi \otimes T_{\frac{1}{2}})\hat{R}$. Hence

$$\tilde{T}^*\tilde{T}\, w = \tilde{\pi}(k^2)\,\hat{R}\,\hat{R}^*\, w = \tilde{\pi}(k^2)\, q^{-2D}\, w, \quad w \in \tilde{W}.$$

Clearly, $\tilde{T}^*$ is densely defined and, therefore, $\tilde{T}$ is closable. Denoting its closure again by $\tilde{T}$, we can write $|\tilde{T}|\lceil_{\tilde{W}} = \tilde{\pi}(k)\, q^{-\tilde{D}}$ since $D\lceil_{\tilde{\mathcal{H}}} = \tilde{D}$.

Now we define an antiunitary operator $\tilde{J}$ by the polar decomposition $\tilde{T} = \tilde{J}|\tilde{T}|$. From the preceding, it follows that

$$\tilde{J}w = \tilde{T}\,\tilde{\pi}(k^{-1})\, q^{\tilde{D}}\, w = \hat{R}\,(\tilde{T}_\psi \otimes T_{\frac{1}{2}})\,\tilde{\pi}(k^{-1})\, q^{\tilde{D}}\, w, \qquad w \in \tilde{W}. \tag{41}$$

Our next aim is to give explicit formulas for the action of $\tilde{J}$. Let $|lm{\downarrow}\rangle$ and $|lm{\uparrow}\rangle$ denote the vectors defined by Equations (22) and (24). The set of all these vectors forms an orthonormal basis for $\tilde{\mathcal{H}}$. Inserting (22) into (24), one easily verifies that

$$|l\,m{\downarrow}\rangle := C_{lm}\, v^{l-\frac{1}{2}}_{m+\frac{1}{2},0} \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle + S_{lm}\, v^{l-\frac{1}{2}}_{m-\frac{1}{2},0} \otimes |\tfrac{1}{2}, +\tfrac{1}{2}\rangle, \tag{42}$$

$$|l\,m{\uparrow}\rangle := -S_{l+1,m}\, v^{l+\frac{1}{2}}_{m+\frac{1}{2},0} \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle + C_{l+1,m}\, v^{j+\frac{1}{2}}_{m-\frac{1}{2},0} \otimes |\tfrac{1}{2}, +\tfrac{1}{2}\rangle \tag{43}$$

with C_{lm} and S_{lm} given by (26).

To determine $\tilde{T}_\psi$, we use the identification $M_0 = \mathcal{A}(\mathrm{S}^2_{q\infty})$. Then $v^0_{0,0} = 1$ and, thus,

$$v^{l+1}_{l+1,0} = (\Pi^l_{k=0}\alpha_0(k))^{-1}\pi_0(x_1)^l v^0_{0,0} = (\Pi^l_{k=0}\alpha_0(k))^{-1} x_1^l.$$

Similarly, $v^{l+1}_{-l-1,0} = (\Pi^l_{k=0}\alpha_0(k))^{-1}x^l_{-1}$. This gives $\tilde{T}_\psi v^l_{l,0} = (-q)^l v^l_{-l,0}$ since $x_1 = -qx^*_{-1}$. Computing both sides of $\tilde{T}_\psi\,\tilde{\pi}(e)^k\, v^l_{l,0} = (-q)^{-k}\,\tilde{\pi}(f)^k\,\tilde{T}_\psi\, v^l_{l,0} = (-q)^{l-k}\,\tilde{\pi}(f)^k\, v^l_{-l,0}$, we finally get

$$\tilde{T}_\psi\, v^l_{m,0} = (-q)^m\, v^l_{-m,0}, \qquad l \in \mathbb{N}_0,\ \ m = -l, \dots, l.$$

Using these formulas, the action of $\tilde{T} = \hat{R}(\tilde{T}_\psi \otimes T_{\frac{1}{2}})$ on $|lm{\downarrow}\rangle$ and $|lm{\uparrow}\rangle$ can be computed directly. Analogously to Equation (37), we find

$$\tilde{T}\,|l\,m{\downarrow}\rangle = \mathrm{i}^{2m}\, q^{l+m+\frac{1}{2}}\,|l,-m,\downarrow\rangle, \quad \tilde{T}\,|l\,m{\uparrow}\rangle = -\mathrm{i}^{2m}\, q^{-l+m-\frac{1}{2}}\,|l,-m,\uparrow\rangle.$$

Consequently, by (41),

$$\tilde{J}\,|l\,m{\downarrow}\rangle = \mathrm{i}^{2m}\,|l,-m,\downarrow\rangle, \qquad \tilde{J}\,|l\,m{\uparrow}\rangle = -\mathrm{i}^{2m}\,|l,-m,\uparrow\rangle.$$

Therefore, by (25) (up to unitary equivalence),

$$\tilde{J}\, v^l_{m,\pm\frac{1}{2}} = \mathrm{i}^{2m}\, v^l_{-m,\mp\frac{1}{2}},$$

where $l = \frac{1}{2}, \frac{3}{2}, \dots$ and $m = -l, \dots, l$.

The last equation shows that $\tilde{J}$ coincides with the real structure defined in [7]. The results in [7] (or [6]) tell us that $[\tilde{\pi}(a), \tilde{J}\tilde{\pi}(b)\tilde{J}^{-1}]$ and $[[D, \tilde{\pi}(a)], \tilde{J}\tilde{\pi}(b)\tilde{J}^{-1}]$ are infinitesimals of arbitrary high order for all $a, b \in \mathcal{A}(\mathrm{S}^2_{q\infty})$.

Finally let us discuss how $\tilde{T}$ and $\tilde{J}$ are related to T and J from Section 3.5. Since $\tilde{T}_\psi = T_\psi\lceil_{\mathcal{A}(\mathrm{S}^2_{q\infty})}$, it follows from the definitions that $T\lceil_{\tilde{W}} = \tilde{T}$. In particular, as shown above, $[\pi(x), \tilde{T}\pi(y)\tilde{T}^{-1}] = 0$ for all $x, y \in \mathcal{A}(\mathrm{S}^2_{q\infty})$. On the other hand, we do not have $J\lceil_{\tilde{W}} = \tilde{J}$. This is due to the fact that the adjoint $T^*_\psi(x) = k^2 \triangleright x^* \triangleleft k^2$ does not map $\mathcal{A}(\mathrm{S}^2_{q\infty})$ into itself. In general, $y \triangleleft k^2 \notin \mathcal{A}(\mathrm{S}^2_{q\infty})$ for $y \in \mathcal{A}(\mathrm{S}^2_{q\infty})$.

Summarizing our conclusions, we can now state the main theorem of this paper.

THEOREM 3.4. *Let $(\mathcal{A}(\mathrm{SU}_q(2)), \mathcal{H}, D)$ denote the spectral triple described in Section 3.1. The embedding $\mathcal{A}(\mathrm{S}^2_{qc}) \otimes V_{\frac{1}{2}} \subset \mathcal{A}(\mathrm{SU}_q(2)) \otimes V_{\frac{1}{2}}$ gives rise to an equivariant real even spectral triple $(\mathcal{A}(\mathrm{S}^2_{qc}), \tilde{\mathcal{H}}, \tilde{D}, \tilde{J}, \gamma)$ if and only if $c = \infty$. The equivariant representation $\tilde{\pi}$ on $\mathcal{A}(\mathrm{S}^2_{q\infty}) \otimes V_{\frac{1}{2}}$ is given by restricting the $*$-representation of $\mathcal{A}(\mathrm{SU}_q(2)) \rtimes \mathcal{U}_q(\mathrm{su}(2))$ on $\mathcal{A}(\mathrm{SU}_q(2)) \otimes V_{\frac{1}{2}}$ to a $*$-representation of $\mathcal{A}(\mathrm{S}^2_{q\infty}) \rtimes \mathcal{U}_q(\mathrm{su}(2))$ on $\mathcal{A}(\mathrm{S}^2_{q\infty}) \otimes V_{\frac{1}{2}}$. The Dirac operator $\tilde{D}$ is the closure of the restriction of D to the invariant subspace $\mathcal{A}(\mathrm{S}^2_{q\infty}) \otimes V_{\frac{1}{2}}$. The decomposition of $\mathcal{A}(\mathrm{S}^2_{q\infty}) \otimes V_{\frac{1}{2}}$ into eigenspaces corresponding to the eigenvalues ± 1 of γ coincides with the decomposition into subspaces corresponding to irreducible $*$-representations of $\mathcal{A}(\mathrm{S}^2_{q\infty}) \rtimes \mathcal{U}_q(\mathrm{su}(2))$. The real structure $\tilde{J}$ is the antiunitary part of the equivariant (closed) Tomita operator defined in Equation (40). The commutators $[\tilde{\pi}(a), \tilde{J}\tilde{\pi}(b)\tilde{J}^{-1}]$ and $[[\tilde{D}, \tilde{\pi}(a)], \tilde{J}\tilde{\pi}(b)\tilde{J}^{-1}]$ are infinitesimals of arbitrary high order for all $a, b \in \mathcal{A}(\mathrm{S}^2_{q\infty})$.*

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