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ON SOME DIRECTIONS IN THE DEVELOPMENT OF JET CALCULUS

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Abstract. Two significant directions in the development of jet calculus are showed. First, jets are generalized to so-called quasijets. Second, jets of foliated and multifoliated manifold morphisms are presented. Although the paper has mainly a survey character, it also includes new results: jets modulo multifoliations are introduced and their relation to (R, S, Q)-jets is demonstrated.

1. Introduction. In this note, we present directions in a generalization of the notion of jet. Thus, roughly speaking, the paper can be read as observations about a development of jet calculus. We start by nonholonomic and semiholonomic jets, which were defined already by Ehresmann; as a more general concept, quasijets were introduced and studied at first by Pradines in [Pra]. So, we recall some facts about nonholonomic jets and quasijets in Section 1. In Section 2, we mention fundamentals from the theory of foliations and show various multifoliated structures. Further, we introduce jet formalism for such multifoliated structures. Here, we have two inspirations: Ikegami's paper [Ike] about jets modulo foliations and the concept of (R, S, Q)-jet, see e.g. [KMS] or [DoK]. We present a generalization and unification (in a way) of both concepts as our new results. Section 3 is devoted to interactions between the above mentioned generalizations, to a Weil algebras approach and a concept of weighted jets. All manifolds and maps are assumed to be of class C^∞.

2. From holonomic jets via nonholonomic jets to quasijets

2.1. Holonomic jets. Jets (holonomic jets) are commonly known as certain equivalence classes of smooth maps between manifolds, which are represented by Taylor polynomials.

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First, we give the definition. Let M and N be two manifolds. Then two maps

$$f\colon M \to N, \quad g\colon M \to N$$

are said to *determine the same r-jet* at $x \in M$ if for every curve

$$\gamma\colon \mathbb{R} \to M \text{ with } \gamma(0) = x$$

the curves

$$f \circ \gamma \text{ and } g \circ \gamma$$

have the r-th order contact at $0 \in \mathbb{R}$. In such a case we write

$$j^r_x f = j^r_x g$$

and an equivalence class of this relation is called an *r-jet* of M into N. The set of all r-jets of M into N is denoted by $J^r(M, N)$. If the *source* of an r-jet is $x \in M$ and the *target* of this jet is $\bar{x} = f(x) \in N$, then

$$\alpha\colon j^r_x f \mapsto x \text{ and } \beta\colon j^r_x f \mapsto \bar{x}$$

are projections of fibered manifolds

$$\alpha\colon J^r(M, N) \to M, \qquad \beta\colon J^r(M, N) \to N.$$

Further, by

$$J^r_x(M, N) \text{ or } J^r(M, N)_{\bar{x}}$$

we denote the set of all r-jets of M into N with the source $x \in M$ or with the target $\bar{x} \in N$, respectively, and we write

$$J^r_x(M, N)_{\bar{x}} = J^r_x(M, N) \cap J^r(M, N)_{\bar{x}}.$$

As r-th order contact of maps is preserved under composition, we define the composition of r-jets as the r-jet of the composed map.

Let $p\colon Y \to M$ be a fibered manifold. The set J^rY of all r-jets of the local sections of Y is called the *r-th jet prolongation of* Y and $J^rY \subset J^r(M, Y)$ is a closed submanifold. (If $Y \to M$ is a vector bundle, then $J^rY \to M$ is also a vector bundle.) Let $q\colon Z \to N$ be another fibered manifold and $f\colon Y \to Z$ a fibered bundle morphism with the property that the base map $f_0\colon M \to N$ is a local diffeomorphism. There is an induced map

$$J^r(f_0, f)(X) := j^r_{\beta(X)} f \circ X \circ j^r_{f_0(\alpha(X))} f_0^{-1}$$

for $X \in J^r(M, Y)$. If we restrict it to local sections, we obtain a map denoted by

$$J^r f\colon J^rY \to J^rZ$$

which is called the *r-th jet prolongation of f*.

It is clear that the trivial choice $Y = M \times N$ in the jet prolongation yields jets of mappings from M to N.

2.2. Nonholonomic jets. For $r = 1$, the set of nonholonomic 1-jets is

$$\tilde{J}^1(M, N) := J^1(M, N).$$

By induction, let $\alpha\colon \tilde{J}^{r-1}(M,N) \to M$ denote the source projection and $\beta\colon \tilde{J}^{r-1}(M,N) \to N$ the target projection of $(r-1)$-th nonholonomic jets. Then X is said to be a *nonholonomic r-jet* with the source $x \in M$ and the target $\bar{x} \in N$, if there is a local section

$$\sigma\colon M \to \tilde{J}^{r-1}(M,N)$$

such that

$$X = j^1_x\sigma \ \text{ and } \ \beta(\sigma(x)) = \bar{x}.$$

There is a natural embedding $J^r(M,N) \subset \tilde{J}^r(M,N)$.

Every $X \in \tilde{J}^r(M,N)$ induces a map

$$\mu X\colon (\underbrace{T\dots T}_{r\text{-times}} M)_x \to (\underbrace{T\dots T}_{r\text{-times}} N)_{\bar{x}}$$

in the following way. For $r=1$ and $X = j^1_x f$, μX is defined as $T_x f$. By induction, let $X = j^1_x\sigma$ for a local α-section

$$\sigma\colon M \to \tilde{J}^{r-1}(M,N).$$

Then $\sigma(u) \in J^{r-1}_u(M,N)$,

$$\mu(\sigma(u))\colon (\underbrace{T\dots T}_{r-1\text{-times}} M)_u \to (\underbrace{T\dots T}_{r-1\text{-times}} N)_{\beta(\sigma(u))}$$

and we put

$$\mu X = T_x\mu(\sigma(u)).$$

The constructed map

$$\mu X\colon (\underbrace{T\dots T}_{r\text{-times}} M)_x \to (\underbrace{T\dots T}_{r\text{-times}} N)_{\bar{x}}$$

is a vector bundle morphism with respect to all vector bundle structures $\underbrace{T\dots T}_{r\text{-times}} \to \underbrace{T\dots T}_{r-1\text{-times}}$. However, μX is not an entirely general vector bundle morphism of this type.

Let $p\colon Y \to M$ be a fibered manifold. We construct the *r-th jet nonholonomic prolongation of Y* denoted by $\tilde{J}^r Y$ as the set of all nonholonomic r-jets of the local sections of Y. The construction of the *r-th jet nonholonomic prolongation of f* for a fibered bundle morphism $f\colon Y \to Z$ with the property that the base map $f_0\colon M \to N$ is a local diffeomorphism is analogous to the holonomic case.

2.3. Quasijets. We introduce the following notation for projections in the iterated tangent bundle $\underbrace{T\dots T}_{r\text{-times}} M$. For every s, $0 < s \le r$, we denote by

$$\pi^s\colon \underbrace{T\dots T}_{s\text{-times}} M \to M$$

the canonical projection to the base. Further, we denote

$$\pi^s_b := \pi^s_{\underbrace{T\dots T}_{b\text{-times}} M}\colon \tilde{T}^s(\underbrace{T\dots T}_{b\text{-times}} M) \to \underbrace{T\dots T}_{b\text{-times}} M$$

the projection with $\underbrace{T\dots T}_{b\text{-times}} M$ as the base space,

$$_a\pi^s := \underbrace{T\dots T}_{a\text{-times}}\pi^s\colon \underbrace{T\dots T}_{a\text{-times}}(\underbrace{T\dots T}_{s\text{-times}} M) \to \underbrace{T\dots T}_{a\text{-times}} M$$

induced projection originating by the posterior application of the functor $\underbrace{T\dots T}_{a\text{-times}}$,

$$_a\pi^s_b := \underbrace{T\dots T}_{a\text{-times}}\pi^s_{\underbrace{T\dots T}_{b\text{-times}} M}$$

the general case containing both previous cases. If a or b equal zero, we do not write them.

Let $x \in M$, $\bar{x} \in N$. A map

$$\phi\colon (\underbrace{T\dots T}_{r\text{-times}} M)_x \to (\underbrace{T\dots T}_{r\text{-times}} N)_{\bar{x}}$$

is said to be a *quasijet* of order r with the source x and the target $\bar{x}$, if it is a vector bundle morphism with respect to all vector bundle structures

$$_a\pi^1_b\colon (\underbrace{T\dots T}_{r\text{-times}} M)_x \to (\underbrace{T\dots T}_{r-1\text{-times}} M)_x$$

and

$$_a\pi^1_b\colon (\underbrace{T\dots T}_{r\text{-times}} N)_{\bar{x}} \to (\underbrace{T\dots T}_{r-1\text{-times}} N)_{\bar{x}},$$

$a+b=r-1$. The set of all such quasijets is denoted by $QJ^r_x(M,N)_{\bar{x}}$ and $QJ^r(M,N)$ means the set of all quasijets from M to N.

There is a bundle structure $QJ^r(M,N) \to M \times N$ and, analogously to J^r, the set QJ^rY of all r-jets of the local sections of a fibered manifold $Y \to M$ Y is called the *r-th quasijet prolongation of* Y. We compose quasijets as maps. Further, let $q\colon Z \to N$ be another fibered manifold and $f\colon Y \to Z$ a fibered bundle morphism with the property that the base map $f_0\colon M \to N$ is a local diffeomorphism. There is an induced map

$$QJ^r(f_0,f)(X) = j^r_{\beta(X)} f \circ X \circ j^r_{f_0(\alpha(X))} f_0^{-1}$$

for $X \in QJ^r(M,Y)$. The composition denotes the composition of quasijets, where the holonomic jets $j^r_{\beta(X)} f$, $j^r_{f_0(\alpha(X))} f_0^{-1}$ are considered as quasijets by the use of the map μ from Section 2. If we confine ourselves to local sections, we obtain a map denoted by $QJ^rf\colon QJ^rY \to QJ^rZ$ which is called the *r-th quasijet prolongation of* f.

3. Jets preserving foliated structures

3.1. Fundamentals from the theory of foliations. Let M be a m-dimensional smooth manifold, $m = p+q$, $m \in \mathbb{N}$, $p,q \in \mathbb{N} \cup \{0\}$, $(x,y) = (x^1,\dots,x^p,y^1,\dots,y^q) \in \mathbb{R}^p \times \mathbb{R}^q = \mathbb{R}^m$. For constants $\bar{c} \in \mathbb{R}^p$, $c \in \mathbb{R}^q$, we consider spaces $\mathbb{R}^q_{\bar{c}} = \{(x,y) \in \mathbb{R}^m; x^1 = \bar{c}^1,\dots,x^p = \bar{c}^p\}$ and $\mathbb{R}^p_c = \{(x,y) \in \mathbb{R}^m; y^1 = c^1,\dots,y^q = c^q\}$. Intersections of $\mathbb{R}^q_{\bar{c}}$ and $\mathbb{R}^p_c$ with open sets (balls) with respect to the standard topology are denoted by $P^q_{\bar{c}}$ and P^p_c and called $(\bar{c},q)$-*coplaques* and (p,c)-*plaques* in $\mathbb{R}^m$. Suppose that $\mathcal{F} = \{L_t\}_{t\in J}$ is a partition of M into connected subsets, $M = \bigcup_{t\in J} L_t$, $L_t \cap L_s = \emptyset$ for $t \neq s$. Further, we consider a *foliated atlas* on M, i.e., a collection $\{U_i, \varphi_i\}_{i\in I}$, $\varphi_i = \alpha_i \times \beta_i$, $\alpha_i : U_i \to \mathbb{R}^p$, $\beta_i : U_i \to \mathbb{R}^q$, of charts satisfying

(i) $\{U_i\}_{i\in I}$ is a cover of M by open sets,
(ii) each connected component of $L_t \cap U_i$ (for all $i \in I$, $t \in J$) is mapped by φ_i onto a (p, c)-plaque in $\mathbb{R}^m$, i.e., for $u \in U_i$

$$\begin{aligned} x^a &= \alpha_i^a(u), && a = 1, \dots, p, \\ y^b &= \beta_i^b(u) = c^b, && b = 1, \dots, q, \end{aligned}$$

(iii) transition functions $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$ on $U_i \cap U_j$, $\varphi_{ij} = \alpha_{ij} \times \beta_{ij}$, send (p, c)-plaques onto (p, c)-plaques, i.e.

$$\begin{aligned} x^a &= \alpha_{ij}^a(x, y), && a = 1, \dots, p, \\ y^b &= \beta_{ij}^b(y), && b = 1, \dots, q. \end{aligned}$$

Then $\mathcal{F}$ is called a *foliation* of M of *dimension* p and *codimension* q, L_t, $t \in J$ *leaves* of $\mathcal{F}$ and M a *foliated manifold* written briefly $(M, \mathcal{F})$. Trivial cases arise for $p = 0$, $q = m$ (leaves = points) and for $p = m$, $q = 0$ (the unique leaf = M).

Let $\mathcal{F}$, $\mathcal{F}'$ be two foliations of M with dimensions p and p'. Then $\mathcal{F}'$ is called a *subfoliation* of $\mathcal{F}$ and $\mathcal{F}$ is called a *superfoliation* of $\mathcal{F}'$, denoted by $\mathcal{F}' \preceq \mathcal{F}$, if the following conditions hold:

(i) $0 \le p' \le p \le m$,
(ii) for any leaf L' of $\mathcal{F}'$, there exists a leaf L of $\mathcal{F}$ such that $L' \subseteq L$, and the restriction of $\mathcal{F}'$ on a leaf L of $\mathcal{F}$ is a foliation of dimension $p - p'$ of L.

The relation $\preceq$ is an order in the set of foliations of M.

Fibered manifolds are canonically foliated, their fibers can be viewed as leaves. On the other hand, there exist manifolds which are foliated but not fibered.

3.2. Transversality of maps, transversality of foliations and induced multifoliations. Let Δ be an integer greater than 1. Let us consider manifolds H_δ, $\delta = 1, \dots, \Delta$, and M. Let $f_\delta : H_\delta \to M$, $\delta = 1, \dots, \Delta$, be (smooth) maps.

We take an arbitrary non-empty subset $E \subseteq \{1, \dots, \Delta\}$ and denote by $\mathrm{Im} f_E$ the intersection of all images of f_ϵ, $\epsilon \in E$.

For $u_E \in \mathrm{Im} f_E$ and every $\epsilon \in E$, let $(Tf_\epsilon)_{u_E}$ denote the image of the tangent map to f_ϵ in u_E; tangent vectors belonging to $(Tf_\epsilon)_{u_E}$ generate a vector subspace of $T_{u_E}M$; we denote it by $\langle (Tf_\epsilon)_{u_E} \rangle$. Further, we denote by $\langle \bigcup_E (Tf_\epsilon)_{u_E} \rangle$ the vector space generated by the union of vectors in all $(Tf_\epsilon)_{u_E}$, $\epsilon \in E$, and by $\langle \bigcap_E (Tf_\epsilon)_{u_E} \rangle$ the vector space generated by vectors belonging to the intersection of all $(Tf_\epsilon)_{u_E}$, $\epsilon \in E$.

For simplicity, we consider only maps for which the vector spaces above have constant dimensions for all $u_E \in \mathrm{Im} f_E$.

Now, it is evident that for every chosen $\epsilon_0 \in E$

$$0 \le \dim \left\langle \bigcap_E (Tf_\epsilon)_{u_E} \right\rangle \le \dim \langle (Tf_{\epsilon_0})_{u_E} \rangle \le \dim \left\langle \bigcup_E (Tf_\epsilon)_{u_E} \right\rangle \le m,$$

or, in the codimension language,

$$m \geq \operatorname{codim}\Big\langle \bigcap_E (Tf_\epsilon)_{u_E} \Big\rangle \geq \operatorname{codim}\langle (Tf_{\epsilon_0})_{u_E}\rangle \geq \operatorname{codim}\Big\langle \bigcup_E (Tf_\epsilon)_{u_E} \Big\rangle \geq 0.$$

Maps $f_\delta \colon H_\delta \to M$, $\delta = 1, \dots, \Delta$, are said to be

- $\cap$-*transversal*, if
$$\operatorname{codim}\Big\langle \bigcap_E (Tf_\epsilon)_{u_E} \Big\rangle = \sum_E \operatorname{codim}\langle (Tf_\epsilon)_{u_E}\rangle$$
for all $E \subseteq \{1, \dots, \Delta\}$;
- $\cup$-*transversal*, if
$$\sum_E \dim\langle (Tf_\epsilon)_{u_E}\rangle = \dim\Big\langle \bigcup_E (Tf_\epsilon)_{u_E} \Big\rangle$$
for all $E \subseteq \{1, \dots, \Delta\}$.

The definition implies that f_δ can be $\cap$-transversal only for

$$\sum_{\delta=1}^{\Delta} \operatorname{codim}\langle (Tf_\delta)_{u_{\{1,\dots,\Delta\}}}\rangle \leq m$$

and, analogously, f_δ can be $\cup$-transversal only for

$$\sum_{\delta=1}^{\Delta} \dim\langle (Tf_\delta)_{u_{\{1,\dots,\Delta\}}}\rangle \leq m.$$

It is easy to show that

$$\sum_{\delta=1}^{\Delta} \operatorname{codim}\langle (Tf_\delta)_{u_{\{1,\dots,\Delta\}}}\rangle \leq m \quad \text{and} \quad \sum_{\delta=1}^{\Delta} \dim\langle (Tf_\delta)_{u_{\{1,\dots,\Delta\}}}\rangle \leq m$$

occur simultaneously only for the case $\Delta = 2$ and $\operatorname{codim}\langle (Tf_1)_{u_{\{1,2\}}}\rangle + \operatorname{codim}\langle (Tf_2)_{u_{\{1,2\}}}\rangle = \dim\langle (Tf_1)_{u_{\{1,2\}}}\rangle + \dim\langle (Tf_2)_{u_{\{1,2\}}}\rangle = m$. In this special case, the concepts of $\cap$-transversality and $\cup$-transversality are identical. (Sometimes, exactly this is understood as transversality: cf. e.g. [Mat], Definition 7.5, where a decomposition of a tangent space into two complementary subspaces is claimed.)

Let us consider $\cap$-transversal maps $f_\delta \colon H_\delta \to M$, $\delta = 1, \dots, \Delta$ in the following situation: H_δ are subsets (submanifolds) of M and $f_\delta \colon H_\delta \to M$ are their inclusion maps (immersions). Then H_δ are called $\cap$-*transversal*, too. Moreover, if we have Δ foliations F_δ of M, we take in every $u \in M$ their leaves: if they are $\cap$-*transversal* for each choice of u, we say that the foliations F_δ of M are $\cap$-*transversal*.

The concept of $\cup$-*transversal* foliations F_δ of M is quite analogous.

A collection $\mathbf{F} = \{\mathcal{F}_\delta\}_{\delta=1}^{\Delta}$ of foliations of M ($\dim M = m$) with dimensions p_δ and codimensions q_δ is called a $\cap$-*multifoliation* ($\cup$-*multifoliation*), if the foliations $\mathcal{F}_\delta$ are $\cap$-transversal ($\cup$-transversal). In particular, the $\cap$-multifoliation ($\cup$-multifoliation) is called a *total* $\cap$-*multifoliation* (*total* $\cup$-*multifoliation*) if $\Delta = m$.

It is clear that $q_1 = \ldots = q_\Delta = 1$ for a total $\cap$-multifoliation and $p_1 = \ldots = p_\Delta = 1$ for a total $\cup$-multifoliation.

3.3. Multifoliations according to Kodaira and Spencer. Kodaira and Spencer [KoS] introduced the following concept of a multifoliation: let $(P,\geq)$ be a partially ordered set with Δ elements and let us consider a surjective map

$$p\colon \{1,\dots,m\} \to P.$$

(Thus, $m \geq \Delta$.) We set

$$a^i_j = 0 \ \text{ for } \ p(j) \ngeq p(i)$$

and denote by

$$\mathrm{GL}(m,\mathbb{R};P,p)$$

the subgroup of $\mathrm{GL}(m,\mathbb{R})$ of linear transformations $\mathbb{R}^m \to \mathbb{R}^m$ given by $A = (a^i_j)$. Further, we denote by $\Gamma(P,p)$ the pseudogroup of all local diffeomorphisms

$$g\colon U \to V, \ \ U,V \subseteq \mathbb{R}^m,$$

such that

$$\mathrm{d}g_x \in \mathrm{GL}(m,\mathbb{R};P,p) \ \text{ for all } \ x \in U.$$

A maximal atlas compatible with $\Gamma(P,p)$ is called a (P,p)-*multifoliated structure* and M endowed with a (P,p)-multifoliated structure is called a (P,p)-*multifoliated manifold* or a manifold with a (P,p)-multifoliation.

3.4. Jets modulo multifoliations. G. Ikegami [Ike] has defined jets modulo foliations. (However, we refer also to approach of Doupovec, Kolář and Mikulski, [DKM].) We generalize Ikegami's concept by the following definition. (In this section, by a multifoliation we mean either a $\cap$-multifoliation or a $\cup$-multifoliation or a (P,p)-multifoliation.)

Definition 3.1. Let H, M be two manifolds, $f,g\colon H \to M$ maps satisfying $f(h) = g(h) = u \in M$ and let $\mathbf{F} = \{\mathcal{F}_\delta\}_{\delta=1}^{\Delta}$ be a multifoliation of M. Then f is said to have the $(r_1,\dots,r_\Delta)$-*multiorder contact modulo* $\mathbf{F}$ with g at u, if for every Δ-tuple of charts $\left\{U^\delta \ni u, \varphi^\delta\right\}_{1\leq\delta\leq\Delta}$ the maps

$$\alpha^\delta \circ f\colon U^\delta \to \mathbb{R}^{p_\delta} \quad \text{and} \quad \alpha^\delta \circ g\colon U^\delta \to \mathbb{R}^{p_\delta}$$

belong to the same (classical) r_δ-jet at u. (It means that for every curve $\gamma\colon\mathbb{R} \to H$ with $\gamma(0) = h$, the curves $\alpha^\delta \circ f \circ \gamma$ and $\alpha^\delta \circ g \circ \gamma$ have r_δ-order contact at zero.) As the relation of having the $(r_1,\dots,r_\Delta)$-multiorder contact modulo $\mathbf{F}$ is evidently an equivalence relation, we denote the class of maps having the $(r_1,\dots,r_\Delta)$-multiorder contact modulo $\mathbf{F}$ with f at u by

$$j^{r_1,\dots,r_\Delta}_h f \bmod F$$

and call it an $(r_1,\dots,r_\Delta)$-*jet modulo the multifoliation* $\mathbf{F}$ with the *source* $h \in H$ and the *target* $u = f(h) \in M$.

We denote by $J^{r_1,\dots,r_\Delta}_h(H,M;\mathbf{F})_u$ the set of all $(r_1,\dots,r_\Delta)$-jets modulo the multifoliation $\mathbf{F}$ with the source h and the target u. Further, we denote

$$J^{r_1,\dots,r_\Delta}_h(H,M;\mathbf{F}) = \bigcup_{u\in M} J^{r_1,\dots,r_\Delta}_h(H,M;\mathbf{F})_u,$$

$$J^{r_1,\dots,r_\Delta}(H,M;\mathbf{F})_u = \bigcup_{h\in H} J^{r_1,\dots,r_\Delta}_h(H,M;\mathbf{F})_u$$

and

$$J^{r_1,\dots,r_\Delta}(H,M;\mathbf{F}) = \bigcup_{u\in M}\bigcup_{h\in H} J_h^{r_1,\dots,r_\Delta}(H,M;\mathbf{F})_u.$$

Theorem 3.2. *For manifolds H and M and a multifoliation $\mathbf{F}$ of M, the spaces*

$$J_h^{r_1,\dots,r_\Delta}(H,M;\mathbf{F})_u,\quad J_h^{r_1,\dots,r_\Delta}(H,M;\mathbf{F}),\quad J^{r_1,\dots,r_\Delta}(H,M;\mathbf{F})_u,\quad J^{r_1,\dots,r_\Delta}(H,M;\mathbf{F})$$

have a smooth manifold structure. Further, there are bundle projections

$$J^{r_1,\dots,r_\Delta}(H,M;\mathbf{F}) \to H \quad \textit{and} \quad J^{r_1,\dots,r_\Delta}(H,M;\mathbf{F}) \to M$$

as well as canonical bundle projections

$$J^{r_1,\dots,r_\Delta}(H,M;\mathbf{F}) \to J^{\tilde{r}_1,\dots,\tilde{r}_\Delta}(H,M;\mathbf{F})$$

by restricting the multiorder, i.e. for $0 \le \tilde{r}_1 \le r_1, \dots, 0 \le \tilde{r}_\Delta \le r_\Delta$. Moreover

$$J^{0,\dots,0}(H,M;\mathbf{F}) = H \times M.$$

Now, we show that (R,S,Q)-jets are included in the concept of the $(r_1,\dots,r_\Delta)$-jet modulo the multifoliation $\mathbf{F}$. We recall that two morphisms of fibered manifolds determine the same (R,S,Q)*-jet* ($R \le S$, $R \le Q$) at a point y if they have the same R-jet at y, their restrictions to the fiber through y have the same S-jet at y, and their base maps have the same Q-jet at the base point of y.

Let $Y \to M$ be a fibered manifold, $\dim M = q$, $\dim Y = p+q$. The fibered manifold structure of $Y \to M$ determines the foliation $\mathcal{F}_1$ with p-dimensional leaves (leaves = fibers). Fiber bundles do not have global sections in general. However, if $Y \to M$ allow global sections (e.g. for a vector bundle with smooth sections including zero section, such as constant smooth sections), we have also a foliation $\mathcal{F}_2$ with q-dimensional leaves (leaves = sections); of course, such $\mathcal{F}_2$ is not determined uniquely. But in the case we have obtained a (non-unique) multifoliation which is simultaneously $\cup$-multifoliation and $\cap$-multifoliations. Our construction implies:

Theorem 3.3. *Let $\mathbf{F}$ be a multifoliation given by the fibration as stated above. Then there is a representation of every (R,S,Q)-jet as a (S,Q)-jet modulo $\mathbf{F}$.*

4. Final remarks

Remark 4.1 (On a unification). Tomáš has generalized the concept of (R,S,Q)-jet to a concept of the nonholonomic (R,S,Q)-jet in [To1]. He derived a composition of such jets as well as some properties of a corresponding bundle functor. That is the only attempt to unify the above mentioned generalizations of jets up to now.

Remark 4.2 (Weil algebras approach). Product preserving bundle functors on the category of smooth manifolds and smooth maps are in bijection with Weil algebras and the natural transformations are in bijection with the Weil algebra homomorphisms. The main example is the Weil algebra of the functor of k-dimensional r-th order velocities $J_0(\mathbb{R}^k, M)$. For nonholonomic velocities or even quasivelocities, Weil algebras are described, see e.g. [To2]. Further, it is known that product preserving bundle functors on the category of fibered manifolds are in bijection with algebra homomorphisms acting between two Weil algebras. These facts were recently also generalized. For (P, p)-multifoliated manifolds, Shurygin has defined in [Shu] an *inductive system of Weil algebra homomorphisms over* P as a collection $\mu = (A_\alpha, \mu^\beta_\alpha, P)$ consisting of Weil algebras A_α, $\alpha \in P$ and Weil algebra homomorphisms $\mu^\beta_\alpha \colon A_\beta \to A_\alpha$, $\beta \leq \alpha$ and has proved that product preserving bundle functors on (P, p)-multifoliated manifolds are uniquely determined by inductive systems of Weil algebra homomorphisms.

REMARK 4.3 (Weighted jets). The concept of weighted jet bundles of sections of vector bundles over filtered manifolds was introduced by Morimoto ([Mor]) in order to study differential equations on filtered manifolds. His weighted jet formalism is used in some problems connected with parabolic geometries (they represent special cases of Cartan's "espaces généralisés" which are geometric structures that have homogeneous spaces G/P, where G is a Lie group and P a subgroup, as their models).

Kunzinger and Popovych ([KuP]) identify differential consequences of a system of differential equations with a system of algebraic equations in the jet space. For certain purposes, e.g. for different potential and pseudo-potential frames, they find it useful to introduce the notion of weight of differential variables instead of the order. Namely, for each variable in the infinite-order jet space there are weights of dependent variables (coordinates in the target space) specified by the definition, with respect to preservation of usual rules for derivations of higher order jet coordinates.

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