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ON CURVATURE CONSTRUCTIONS OF SYMPLECTIC FORMS

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Abstract. We generalize the result of Lerman [Letters Math. Phys. 15 (1988)] concerning the condition of fatness of the canonical connection in a certain principal fibre bundle. We also describe new classes of symplectically fat bundles: twistor budles over spheres, bundles over quaternionic Kähler homogeneous spaces and locally homogeneous complex manifolds.

1. Introduction. Let there be given a principal fiber bundle $G \to P \to B$ with a connection determined by a horizontal distribution $\mathcal{H}$. Let θ and Ω be the connection form and curvature form of the connection, respectively. The curvature form is a $\mathfrak{g}$-valued 2-form. Denote the pairing between $\mathfrak{g}$ and its dual $\mathfrak{g}^*$ by $\langle , \rangle$. By definition, a vector $u \in \mathfrak{g}^*$ is *fat* if the 2-form

$$(X,Y) \to \langle \Omega(X,Y), u \rangle$$

is non-degenerate for all *horizontal* vector fields X, Y. Note that if a connection admits at least one fat vector, it admits a whole coadjoint orbit of fat vectors. Indeed, since $R_g^*\Omega = Ad\, g^{-1}\Omega$ for any $g \in G$, we have

$$\langle \Omega(X,Y), Adg\, u \rangle = \langle Adg^{-1}\Omega(X,Y), u \rangle = \langle \Omega(R_g^*X, R_g^*Y), u \rangle.$$

The role of fat bundles comes from the following theorem of Sternberg and Weinstein [GLS], [L], [W].

Theorem 1. *Let there be given a principal bundle*

$$G \to P \to B$$

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and a symplectic G-manifold F with hamiltonian G-action and moment map $\mu : F \to \mathfrak{g}^*$. If there exists a connection in the principal bundle such that all vectors in $\mu(F) \subset \mathfrak{g}^*$ are fat, then the total space of the associated bundle

$$F \to P \times_G F \to B$$

admits a fiberwise symplectic structure.

In the sequel, we will call such bundles *symplectically fat*. Clearly, the theorem gives a method to construct fiber bundles with fiberwise symplectic structures. The latter turns out to be important in symplectic geometry. However, such bundles are scarce (compare [C], [GLS]). In fact, known explicit constructions are fiberings of coadjoint orbits over coadjoint orbits [GLS]. There are obstructions to fatness [DR], [W]. The aim of this paper is to describe some new classes of fat bundles. Here are the main points of this note.

1. First, we consider the homogeneous situation, that is, fiber bundles of the form
$$H/V \to K/V \to K/H. \tag{1}$$
We slightly modify a result of [L] showing that for any homogeneous space K/H of semisimple Lie group K with rank K = rank H, the canonical connection in the principal bundle
$$H \to K \to K/H$$
admits fat vectors (Proposition 1).
2. Proposition 1, although simple, yields not only coadjoint orbit hierarchy, but some other interesting symplectically fat fiber bundles: twistor bundles over spheres and homogeneous quaternionic Kähler manifolds (Proposition 2, Proposition 3);
3. The same method yields the structure of a symplectically fat fiber bundle on some locally homogeneous complex manifolds in the sense of Griffiths and Schmid [GrS], (Theorem 3, Corollary 2). A particular case of this was used in [ABCKT] to construct an example of a pseudo-Kähler but non-Kähler manifold.

We complete this introduction with a remark that suggests at least one case, when the fatness condition is easier to check.

LEMMA 1. *Assume that a principal fiber bundle $G \to P \to B$ has a fat vector, say, u. Then the associated bundle*

$$\mathcal{O}(u) \to P \times_G \mathcal{O}(u) \to B$$

is symplectically fat. Here $\mathcal{O}(u)$ denotes the coadjoint orbit of $u \in \mathfrak{g}^*$.

The lemma is straightforward, since we know that $u \in \mathfrak{g}^*$ generates the whole coadjoint orbit of fat vectors, and the moment map for a coadjoint orbit is just an embedding $\mathcal{O}(u) \hookrightarrow \mathfrak{g}^*$ [W].

2. Fatness of canonical connections in principal bundles $K \to K/H$. In [L] Lerman expressed the fatness condition, in terms of Lie algebra data, for the canonical connection in the principal bundle

$$H \to K \to K/H$$

provided that K is compact and semisimple, and K/H is a coadjoint orbit. In fact, a weaker restriction that H is compact and has maximal rank in K is sufficient. For that reason, we present a somewhat modified and generalized proof of Lerman's result. We don't assume that K is compact, but we do assume that H is compact, and K is semisimple. Let us start with several known facts from the Lie theory and introduce notation. We denote by $\mathfrak{k}$ the Lie algebra of a Lie group K. Symbol $\mathfrak{k}^c$ denotes the complexification. Let $\mathfrak{t}$ be a maximal abelian subalgebra in $\mathfrak{k}$. Then $\mathfrak{t}^c$ will be a Cartan subalgebra in $\mathfrak{k}^c$. We denote by $\Delta = \Delta(\mathfrak{k}^c, \mathfrak{t}^c)$ the root system of $\mathfrak{k}^c$ with respect to $\mathfrak{t}^c$. Also, if $\mathfrak{h}^c$ is a maximal rank subalgebra of $\mathfrak{k}^c$, we choose $\mathfrak{t}^c$ in a way to get $\mathfrak{t}^c \subset \mathfrak{h}^c \subset \mathfrak{k}^c$. Under these choices the root system for $(\mathfrak{h}^c, \mathfrak{t}^c)$ is a subsystem of Δ. Denote this subsystem as $\Delta(\mathfrak{h})$. Consider the orthogonal complement $\mathfrak{m}^c$ to $\mathfrak{h}^c$ with respect to the Killing form (note that the decomposition $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$ complexifies to $\mathfrak{k}^c = \mathfrak{h}^c \oplus \mathfrak{m}^c$). Thus, we have root decompositions:

$$\mathfrak{k}^c = \mathfrak{t}^c + \sum_{\alpha \in \Delta} \mathfrak{k}^\alpha, \ \mathfrak{h}^c = \mathfrak{t}^c + \sum_{\alpha \in \Delta(\mathfrak{h})} \mathfrak{k}^\alpha, \quad \mathfrak{m}^c = \sum_{\alpha \in \Delta \setminus \Delta(\mathfrak{h})} \mathfrak{k}^\alpha.$$

Since, by assumption, K is semisimple, the Killing form B is non-degenerate, hence we can identify Lie algebra $\mathfrak{k}$ with its dual $\mathfrak{k}^*$ via B. Denote this identification by $u \to X_u$ $(u \in \mathfrak{k}^*)$. Let $C \subset \mathfrak{t}^c$ be a closed Weyl chamber and let C_α denote its wall determined by the root α.

Proposition 1. *Let K be a semisimple Lie group, and H its compact subgroup of maximal rank. Then, for the canonical connection in the principal bundle*

$$H \to K \to K/H$$

a vector $u \in \mathfrak{h}^*$ is fat if and only if X_u does not belong to the set

$$Ad(H)(\cup_{\alpha \in \Delta \setminus \Delta(\mathfrak{h})} C_\alpha \cap \mathfrak{t}).$$

Proof. We know from the general theory of invariant connections that the curvature form of the canonical connection in the given principal bundle has the form

$$\Omega(X^*, Y^*) = -\frac{1}{2}[X, Y]_{\mathfrak{h}}, \qquad X, Y \in \mathfrak{m}.$$

Here X^*, Y^* denote the horizontal lifts of the vector fields on K/H determined by X, Y. Hence the fatness condition is expressed as the non-degeneracy of the form

$$(X, Y) \to B(X_u, [X, Y]_{\mathfrak{h}}).$$

Here and throughout the paper, the subscript $\mathfrak{h}$ denotes the projection onto the $\mathfrak{h}$-summand with respect to a vector space decomposition with summand $\mathfrak{h}$. Recall that here the pairing is given by the Killing form. Assume first that $X_u \in \mathfrak{t}$. Then, $B(X_u, \mathfrak{m}) = 0$. Hence

$$B(X_u, [X, Y]_{\mathfrak{h}}) = B(X_u, [X, Y]) = B([X_u, X], Y)$$

(the latter is a property of the Killing form). But B is non-degenerate, as also is its restriction on $\mathfrak{h}$ (since $\mathfrak{h}$ is compact). Thus the latter form is non-degenerate if and only if $[X_u, X] \neq 0$. This can be written as $\operatorname{Ker} ad\, X_u \cap \mathfrak{m} = 0$, or, equivalently, as

$$\mathfrak{z}_{\mathfrak{k}}(X_u) \cap \mathfrak{m} = 0. \tag{2}$$

Clearly, the latter can be equivalently complexified. However, since, by assumption, $X_u \in \mathfrak{t}^c \subset \mathfrak{h}^c$, the equality $[X_u, X] = 0$ shows that either $X \in \mathfrak{t}^c$, or $\alpha(X_u) = 0$ for some root $\alpha \in \Delta$. Taking into consideration the root decompositions above, we see that (2) is equivalent to the fact that X_u cannot centralize any element in $\mathfrak{m}^c$, and thus $\alpha(X_u) \neq 0$ if and only if $\alpha \in \Delta \setminus \Delta(\mathfrak{h})$. Thus, if $X_u \in \mathfrak{t}^c$, it is fat if and only if it does not belong to $\mathfrak{t} \cap C_\alpha$, for some $\alpha \in \Delta \setminus \Delta(\mathfrak{h})$. The general case follows, since $\mathfrak{h} = \bigcup_{h \in H} Ad\, h(\mathfrak{t})$ for compact H (all Cartan subalgebras of $\mathfrak{h}$ are conjugate, and each point of $\mathfrak{h}$ is touched by a Cartan subalgebra [K]). Here we again identify adjoint and coadjoint actions. ∎

COROLLARY 1. *Let K be a semisimple Lie group of non-compact type, which is a real form of a semisimple complex Lie group K^C. Assume that $V = K \cap P$ is the compact intersection of a parabolic subgroup $P \subset K^C$ with K. Let H be a maximal compact subgroup in K containing V. Then the conclusions of Proposition 1 hold.*

Proof. It is sufficient to show that $\operatorname{rank} K = \operatorname{rank} H$. Let $\mathfrak{k}^c$ be a complexification of $\mathfrak{k}$, and let $\mathfrak{c} \supset \mathfrak{h}$ be a maximal compact subalgebra in $\mathfrak{k}^c$ (it is a compact real form of $\mathfrak{k}^c$). It is shown in [GrS] that $\operatorname{rank} \mathfrak{c} = \operatorname{rank} \mathfrak{h}$. But this means that

$$\operatorname{rank} \mathfrak{k} = \operatorname{rank} \mathfrak{k}^c = \operatorname{rank} \mathfrak{c} = \operatorname{rank} \mathfrak{h}$$

and $\operatorname{rank} K = \operatorname{rank} H$. ∎

This modification of Lerman's proof enables us to solve the problem of fatness completely in the case of canonical connection and fibers which are coadjoint orbits.

THEOREM 2. *The canonical connection in the bundle*

$$H \to K \to K/H$$

determined by a semisimple Lie group K and a compact subgroup H admits fat vectors if rank $K =$ rank H. *If K is compact, the converse is also true.*

Proof. By Proposition 1, if rank $K =$ rank H, there exist fat vectors (for example, those which lie in the interior of the Weyl chamber). Assume that rank $K >$ rank H. Assume that there exists a fat vector $u \in \mathfrak{h}^*$. Then the whole orbit $H/V \cong \mathcal{O}(u) \subset \mathfrak{h}^*$ is fat. One can check that the moment map of $\mathcal{O}$ is just an embedding $\mathcal{O} \hookrightarrow \mathfrak{h}^*$. Hence, the associated bundle

$$\mathcal{O} \to K \times_H \mathcal{O} \to K/H$$

admits a symplectic structure. But the total space of the latter bundle is

$$K \times_H \mathcal{O}(u) \cong K \times_H (H/V) \cong K/V.$$

Hence, K/V admits a symplectic structure, say, ω. Let $[\omega] \in H^2(K/V, \mathbb{R})$ be the cohomology class of ω. It is well known and easy to see that $[\omega]$ always contains also a K-invariant representative, say $\tilde{\omega}$. Since $[\omega]^m \neq 0$, for $m = \dim(K/V)$, we have that necessarily $\tilde{\omega}^m$ is a nowhere vanishing m-form. Hence, $\tilde{\omega}$ is non-degenerate. Finally, we have a K-invariant symplectic form on K/V. By the classical Borel's theorem, $V = Z_K(S)$, where S is some torus in K. In particular, one must have rank $K =$ rank V, a contradiction with rank $V =$ rank $H <$ rank K (again, for the same reason, since coadjoint orbit $\mathcal{O}(u) \cong H/V$ is symplectic). ∎

Now our aim is to describe (semisimple) symplectic fiber bundles of the form

$$H/V \to K/V \to K/H$$

with symplectic fiber and total space. Note that K/H itself need not be symplectic. This yields more symplectic fiber bundles than could be obtained by Thurston's theorem [McD], since we don't require K/H to be symplectic. Note that any compact simply connected homogeneous symplectic manifold is symplectomorphic to a coadjoint orbit with its standard Kostant-Kirillov-Souriau symplectic form [CGR]. However, our results are applicable only to coadjoint orbits $\mathcal{O}(u)$ of u determined by Proposition 1.

REMARK 1. Note that one cannot extend the class of symplectic fat bundles by the following construction: take any symplectic K-manifold (F, K, ω_F) and require that $\mu(F) = \mathcal{O}(u)$ for some fat u (for example, u from the interior of the Weyl chamber). The result in [Bi] shows that only homogeneous spaces can have coadjoint orbits as images of the moment map.

Here is an explicit example of the situation described in Theorem 2.

EXAMPLE 1. For manifolds of the form $F = H/T$, where T is a maximal torus, the fiber bundle

$$H/T \to K/T \to K/H$$

is always symplectically fat.

Proof. In [GS], it is shown that the image of the moment map $\mu : H/T \to \mathfrak{h}^*$ is of the form $\mathcal{O}(u)$, where u belongs to the *interior* of the Weyl chamber (see the proof of Proposition 2.2 there). By Theorem 2, the given bundle has to be symplectically fat. ■

3. Twistor bundles and locally homogeneous complex manifolds as fat bundles. We see that fatness can be fully described for associated bundles over homogeneous spaces K/H of equal rank pairs (K, H), with compact H. If H is a centralizer of a torus, K/H is symplectic, and we get a hierarchy of coadjoint orbits [L]. However, if K/H is *not* symplectic, then there are some interesting examples as well. In the sequel, for brevity, we call the coadjoint orbit $\mathcal{O}(u)$ "admissible", if u does not belong to walls, which are "forbidden" by Proposition 1, that is, are hyperplanes determined by roots $\alpha \in \Delta \setminus \Delta(\mathfrak{h})$.

3.1. Twistors over spheres. There are various notions of twistor spaces (compare, for example, [Be] and [OR]). Here we think of it simply as an associated bundle of an $SO(2n)$-principal bundle over a $2n$-manifold M with fiber $SO(2n)/U(n)$. This is the case, for example, when the twistor space $\mathcal{T}(M)$ consists of orthogonal orientation preserving complex structures on tangent spaces to a $2n$-dimensional Riemannian manifold M:

$$\mathcal{T}(M) = \{J_x : T_xM \to T_xM,\ J_x^2 = -\,\mathrm{id}\ \text{ for all}\, x \in M\}.$$

It is known that in some cases $\mathcal{T}(M)$ admits a symplectic structure [AGI], [R] (e.g. $M = S^{2n}$, or M is a particular quaternionic Kähler manifold). Our approach suggests that there may be a unified explanation of these results since we show that twistor bundles over spheres are symplectically fat.

Let $K/H = SO(2n+1)/SO(2n) = S^{2n}$. We see that

$$\operatorname{rank} SO(2n+1) = \operatorname{rank} SO(2n) = n.$$

Hence, the even dimensional sphere satisfies the assumptions of Theorem 2. Note that $SO(2n)/U(n)$ is known to be symplectic. Thus, it is a coadjoint orbit. Moreover, it is symplectomorphic to a coadjoint orbit $\mathcal{O}(u)$ which satisfies the assumptions of Proposition 1. This can be checked as follows. On the Lie algebra level, we have the embedding $\mathfrak{u}(n) \to \mathfrak{so}(2n)$ of the form

$$A + iB \to \begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

where $A^T = -A$, $B^T = B$. Consider the complexifications of both algebras, and their root systems. Let $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C}), \mathfrak{h} = \mathfrak{u}^c$. We calculate Δ and $\Delta(\mathfrak{h})$ *with respect to the Cartan subalgebra* given by the matrices of the form

$$\begin{pmatrix} 0 & \operatorname{diag}(x_1, ..., x_n) \\ -\operatorname{diag}(x_1, ..., x_n) & 0 \end{pmatrix},$$

$x_i \in \mathbb{C}$. In this case Δ consists of linear functionals of the form

$$\alpha_{ij} = \varepsilon_i - \varepsilon_j \ (i \neq j), \quad i, j = 1, ..., n,$$
$$\beta_{ij} = \varepsilon_i + \varepsilon_j, -\beta_{ij}, (i < j), i, j = 1, ..., n,$$

where

$$\varepsilon_i(\operatorname{diag}(x_1, ..., x_n, -x_1, ..., -x_n)) = x_i - x_j.$$

In the same way, $\Delta(\mathfrak{h})$ consists of the functionals α_{ij} only. To do this, one modifies standard formulas from [OV] which are derived for the Cartan subalgebra consisting of the matrices

$$\operatorname{diag}(x_1, ..., x_n, -x_1, ..., -x_n)$$

in the (equivalent) representation of $\mathfrak{so}(2n, \mathbb{C})$ by matrices of the form

$$\begin{pmatrix} X & Y \\ Z & -X^T \end{pmatrix}$$

with $Z^T = -Z$, $Y^T = -Y$, $X \in \mathfrak{gl}(n)$.

It follows that the element

$$X_u = \begin{pmatrix} 0 & xE_n \\ -xE_n & 0 \end{pmatrix}$$

does not belong to the forbidden wall, since $\beta_{ij}(X_u) \neq 0$. One can check that in this representation $\mathfrak{u}(n)$ is a centralizer of X_u, and, therefore, $SO(2n)/U(n)$ is symplectomorphic to $\mathcal{O}(u)$, where u is dual to X_u. Hence, by Theorem 2, we get the following result.

PROPOSITION 2. *The twistor bundle*

$$SO(2n)/U(n) \to SO(2n+1)/U(2n) \to S^{2n}$$

over the even-dimensional sphere is a fat symplectic bundle.

3.2. Bundles over quaternionic Kähler homogeneous spaces. We refer to [Be] for basic facts on quaternionic Kähler manifolods. Recall that a Riemannian $4n$-manifold is called quaternionic Kähler if the holonomy group of the Riemannian metric is contained in $Sp(n)Sp(1)$. This class of manifolds is important in constructing Kähler-Einstein metrics. For example, by Salamon's theorem [Be], twistor bundles over compact quaternionic Kähler manifolds of positive scalar curvature admit Kähler-Einstein metrics.

PROPOSITION 3. *Any homogeneous quaternionic Kähler symmetric Riemannian manifold K/H of non-zero Ricci curvature is a base of a fat symplectic fiber bundle for any coadjoint orbit $H/V \cong \mathcal{O}(u)$, with admissible u.*

Proof. The classification of compact quaternionic Kähler Riemannian symmetric manifolds of non-zero Ricci curvature is known (see [Be]). All of them are irreducible, simply connected and are divided into two classes according to the sign of the Ricci curvature. If the Ricci curvature is positive, these manifolds are compact and are contained in the following list:

$$Sp(n+1)/Sp(n)Sp(1), SU(n+2)/S(U(n)U(2)),$$

$$SO(n+4)/SO(n)SO(4), G_2/SO(4),$$

$$F_4/Sp(3)/Sp(1), E_6/SU(6)Sp(1),$$

$$E_7/Spin(12)Sp(1), E_8/E_7Sp(1).$$

If the Ricci curvature is negative, they are non-compact, and are dual to symmetric spaces from the list. Clearly, all these homogeneous spaces are formed by equal rank pairs. ∎

3.3. Locally homogeneous complex manifolds. Assume that we are given a Lie group G which is semisimple of *non-compact type* and which is a real form of a complex semisimple Lie group G^c. Let P be a parabolic subgroup in G^c, and let K be a maximal compact subgroup in G. Choose a cocompact lattice $\Gamma \subset G$. Assume that $P \cap K = V$ is compact. Then we get a fiber bundle

$$K/V \to \Gamma\backslash G/V \to \Gamma\backslash G/K$$

over locally symmetric Riemannian space $\Gamma\backslash G/K$ with structure group K. Following Griffiths-Schmid [GrS], we will call $\Gamma\backslash G/V$ *locally-homogeneous complex manifold.* The latter is indeed complex, since $G/V \subset G^c/P$ is an open subvariety of a complex projective variety G^c/P, and therefore, inherits the G-invariant complex structure.

THEOREM 3. *A locally homogeneous complex manifold $\Gamma\backslash G/V$ fibers over locally symmetric Riemannian homogeneous space $\Gamma\backslash G/K$ with fiber K/V. If K/V is symplectomorphic to a coadjoint orbit $\mathcal{O}(u)$ with admissible u, the corresponding bundle is fat. In particular, for such V, G/V is symplectic.*

Proof. First, V is a centralizer of a torus in K. This follows, since it is shown in [GrS] that K/V is Kähler, and Borel's theorem applies. We use Corollary 1 and get rank $G =$ rank K. We get that the fiber bundle

$$K/V \to G/V \to G/K$$

is fat, applying Theorem 2, and the assumption that $K/V \cong \mathcal{O}(u)$ for admissible u to the pair (G, K). Now, if Γ is a lattice in G, we have a commutative diagram of principal bundles

$$\begin{array}{ccccc} K & \longrightarrow & G & \longrightarrow & G/K \\ \downarrow & & \downarrow & & \downarrow \\ K & \longrightarrow & \Gamma\backslash G & \longrightarrow & \Gamma\backslash G/K \end{array}$$

where the second and the third arrows are coverings. Hence, fatness condition, which depends on tangent spaces only, is satisfied for the second row in the diagram, as required, and the proof follows. ∎

Corollary 2. *The following twistor bundle over locally symmetric homogeneous space is fat:*

$$SO(2n)/U(n) \to \Gamma\backslash SO(2n,p)/(U(n) \times SO(p)) \to$$

$$\Gamma\backslash SO(2n,p)/(SO(2n) \times SO(p)), n > 1, p > 2.$$

Proof. The proof follows from Theorem 2 and the proof of Proposition 2. The latter is used to show that the fiber is an admissible coadjoint orbit. ∎

Remark 2. The following question was posed in [W]: are there symplectic fat bundles whose total spaces do not carry Kähler structures? Corollary 2 yields a positive answer, since

$$\mathcal{T} = \Gamma\backslash SO(2n,p)/(SO(2n) \times SO(p))$$

is non-Kähler. The latter follows, since Γ is the fundamental group of $\mathcal{T}$, and is a non-Kähler group [ABCKT].

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