ALGEBRA, GEOMETRY AND MATHEMATICAL PHYSICS
BANACH CENTER PUBLICATIONS, VOLUME 93
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2011

ON THIRD ORDER SEMIHOLONOMIC PROLONGATION OF A CONNECTION

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Abstract. We recall several different definitions of semiholonomic jet prolongations of a fibered manifold and use them to derive some interesting properties of prolongation of a first order connection to a third order semiholonomic connection.

1. Introduction. The theory of jets has been developed as a unifying tool in the description of many objects within differencial geometry. For example, the frame bundle P^rM of a manifold M with dimension n can be viewed as the space $J^r_0(\mathbb{R}^n, M)$ of jets from $\mathbb{R}^n$ to M with source 0. Consequently, using the notion of jet prolongation of a fibered manifold, one can define even more complicated spaces, e.g. so called r-th order principal prolongation W^rP of a principal bundle $P \to M$ with structure Lie group G as $W^rP = P^rM \times J^rP$ with the appropriate structure Lie group W^r_mG, see [7], [4] for details. Note that principal prolongation of principal bundles plays a fundamental role in gauge theories of mathematical physics, see [3]. For geometric applications see [4]. But not only the spaces can be viewed in the jet setting. By simple identification of horizontal subspaces of the tangent space $T_yY,\ y \in Y$, with the elements $j^1_ys \in J^1_yY,\ s : M \to Y$, such a fundamental object as a connection on the fibered manifold $Y \to M$ can be understood as a mapping $Y \to J^1Y$ to the first jet prolongation of Y. Passing to higher order jet prolongation of a fibered manifold one can define an r-th order general connection Γ on $Y \to M$ as a mapping $\Gamma : Y \to J^rY$ to the r-th order jet prolongation of Y. The role of higher order connections lies in handling higher order geometric structures such as principal prolongations of principal bundles, or e.g. in lifting of geometric objects, see [2] for details. Note that there are three distinguished types of jet prolongations: nonholonomic, semiholonomic and holonomic, $\widetilde{J}^rY,\ \overline{J}^rY,\ J^rY$, satisfying $J^rY \subset \overline{J}^rY \subset \widetilde{J}^rY$,

2010 *Mathematics Subject Classification*: Primary 58A20; Secondary 53C05, 58A05.
Key words and phrases: jet prolongation, connection.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc93-0-22

which will be defined in the first section of this paper. Therefore three appropriate types of higher order general connections are distinguished. We remark that apart from these three main jet types, there are other various significant subspaces of $\widetilde{J}^r Y$, e.g. so called sesqui-holononomic jets, see [4]. We focus on semiholonomic prolongations of a fibered manifold and subsequently on semiholonomic prolongation of connections.

This paper is devoted to semiholonomic prolongation of a general connection $\Gamma : Y \to J^1Y$ to a third order semiholonomic connection $\Gamma_3 : Y \to \overline{J}^3 Y$. We use the fact that for $r = 2$ all natural operators transforming Γ to Γ_2 are described by means of so called Ehresmann prolongation of a connection and the transformations $\overline{J}^2 Y \to \overline{J}^2 Y$, see [9]. By the alternative definition of $\overline{J}^r Y$ we deduce some further characteristics of semiholonomic prolongation of first order connections to third order connections.

2. Jet prolongation of a fibered manifold. In classical theory, the r-th nonholonomic prolongation $\widetilde{J}^r Y$ of Y is defined by the following iteration:

1. $\widetilde{J}^1 Y = J^1 Y$, i.e. $\widetilde{J}^1 Y$ is a space of 1-jets of sections $M \to Y$ over the target space Y.
2. $\widetilde{J}^r Y = J^1(\widetilde{J}^{r-1} Y \to M)$.

Clearly, we have an inclusion $J^r Y \subset \widetilde{J}^r Y$ given by $j^r_x \gamma \mapsto j^1_x (j^{r-1}\gamma)$. Further, the r-th semiholonomic prolongation $\overline{J}^r Y \subset \widetilde{J}^r Y$ is defined by the following induction. First, by $\beta_1 = \beta_Y$ we denote the projection $J^1 Y \to Y$ and by $\beta_r = \beta_{\widetilde{J}^{r-1}Y}$ the projection $\widetilde{J}^r Y = J^1 \widetilde{J}^{r-1} Y \to \widetilde{J}^{r-1} Y$, $r = 2, 3, \dots$. If we set $\overline{J}^1 Y = J^1 Y$ and assume we have $\overline{J}^{r-1} Y \subset \widetilde{J}^{r-1} Y$ such that the restriction of the projection $\beta_{r-1} : \widetilde{J}^{r-1} Y \to \widetilde{J}^{r-2} Y$ maps $\overline{J}^{r-1} Y$ into $\overline{J}^{r-2} Y$, we can construct $J^1 \beta_{r-1} : J^1 \overline{J}^{r-1} Y \to J^1 \overline{J}^{r-2} Y$ and define

$$\overline{J}^r Y = \{A \in J^1 \overline{J}^{r-1} Y;\ \beta_r(A) = J^1 \beta_{r-1}(A) \in \overline{J}^{r-1} Y\}.$$

Clearly $J^r Y \subseteq \overline{J}^r Y \subseteq \widetilde{J}^r Y$.. Obviously, $J^r, \overline{J}^r$ and $\widetilde{J}^r$ are bundle functors on the category $\mathcal{FM}_{m,n}$ of fibered manifolds with m-dimensional bases and n-dimensional fibres and locally invertible fiber-preserving mappings.

Alternatively, one can define the r-th order semiholonomic prolongation $\overline{J}^r Y$ by means of natural target projections of nonholonomic jets, see [11]. Note that generally there exist r different projections $\widetilde{J}^r Y \to \widetilde{J}^{r-1} Y$, which in the notation of previous definition of $\overline{J}^r$ are of the form $\beta_r,\ J^1 \beta_{r-1}, \dots, (\underbrace{J^1 \cdots J^1}_{r-1}) \beta_1$. As an example we show the coordinate form of all projections $\widetilde{J}^3 Y \to \widetilde{J}^2 Y$. Local coordinates on $J^1 Y$, $\widetilde{J}^2 Y$ and $\widetilde{J}^3 Y$ are of the form $(x^i, y^p, y^p_i = \frac{\partial y^p}{\partial x^i})$, $(x^i, y^p_{00} = y^p, y^p_{i0} = \frac{\partial y^p}{\partial x^i}, y^p_{0j} = \frac{\partial y^p}{\partial x^j}, y^p_{ij} = \frac{\partial y^p_i}{\partial x^j})$, $(x^i, y^p_{000} = y^p, y^p_{i00} = \frac{\partial y^p}{\partial x^i}, y^p_{0j0} = \frac{\partial y^p}{\partial x^j}, y^p_{ij0} = \frac{\partial y^p_i}{\partial x^j}, y^p_{00k} = \frac{\partial y^p}{\partial x^k}, y^p_{i0k} = \frac{\partial y^p_i}{\partial x^k}, y^p_{0jk} = \frac{\partial y^p_i}{\partial x^k}, y^p_{ijk} = \frac{\partial y^p_{ij}}{\partial x^k})$, respectively, where $i, j, k = 1, \dots, \dim M$, $j = 1, \dots, n$, where n is the fiber dimension of Y. Thus the projections β_1, $J^1 \beta_2$ and $J^1 J^1 \beta_1$ from $\widetilde{J}^3 Y$ onto $\widetilde{J}^2 Y$ are described as

$$\begin{array}{lll}
x^i = x^i, & x^i = x^i, & x^i = x^i, \\
y^p = y^p, & y^p = y^p, & y^p = y^p, \\
y^p_{i0} = y^p_{i00}, & y^p_{i0} = y^p_{i00}, & y^p_{i0} = y^p_{0j0}, \\
y^p_{0j} = y^p_{0j0}, & y^p_{0j} = y^p_{00k}, & y^p_{0j} = y^p_{00k}, \\
y^p_{ij} = y^p_{ij0}, & y^p_{ij} = y^p_{i0k}, & y^p_{ij} = y^p_{0jk},
\end{array}$$

respectively. The problem of projections onto lower order bundle functors including the jet prolongation of fibered manifolds was solved in [1].

In the following, by surjection $\pi^r_q : \widetilde{J}^r Y \to \widetilde{J}^q Y$ for $r \geq q > 0$ we understand a composition of any of the above projections. Specifically, π^r_r is the identity on $\widetilde{J}^r Y$. We note that the restriction of these projections to the subspace of semiholonomic jet prolongations will be denoted by the same symbol. For $r \geq q > k > 0$, if we apply the functor $\widetilde{J}^k$, we have also the surjections $\widetilde{J}^k \pi^{r-k}_{q-k} : \widetilde{J}^r Y \to \widetilde{J}^q Y$ and, consequently, the element $X \in \widetilde{J}^r Y$ is semiholonomic if and only if

$$(\widetilde{J}^k \pi^{r-k}_{q-k})(X) = \pi^r_q(X) \text{ for any integers } 1 \leq k \leq q < r. \tag{1}$$

In [11], the author finds this property quite useful when handling semiholonomic connections and their prolongations and also gives a complete proof.

Remark. Local coordinates on $\overline{J}^3 Y \subset \widetilde{J}^3 Y$ are given by

$$y^p_{i00} = y^p_{0i0} = y^p_{00i},\ y^p_{ij0} = y^p_{i0j} = y^p_{0ij}.$$

So the coordinates on $\overline{J}^3 Y$ are

$$\left(x^i,\ y^p,\ y^p_i = \frac{\partial y^p}{\partial x^i},\ y^p_{ij} = \frac{\partial y^p_i}{\partial x^j},\ y^p_{ijk} = \frac{\partial y^p_{ij}}{\partial x^k} \right).$$

Finally, the following functorial definition of semiholonomic prolongation of a fibered manifold can be found in [8]. Assume that the functor $\overline{J}^{r-1}$ comes equipped with the canonical transformation $\overline{J}^{r-1} \to \overline{J}^{r-2}$ given by the restriction of jet target projections. Then there are two canonical transformations $J^1 \overline{J}^{r-1} \to J^1 \overline{J}^{r-2}$ and one can define $\overline{J}^r$ as the equalizer of these two transformations. Then this is equivalent to the definition

$$\overline{J}^r Y = \overline{J}^2 (\overline{J}^{r-2} Y) \cap J^1 (\overline{J}^{r-1} Y).$$

3. Connections and their prolongations. First we recall that a general connection on a fibered manifold $p : Y \to M$ can be defined as a lifting map $\Gamma : Y \times_M TM \to TY$, or, equivalently, as a section $\Gamma : Y \to J^1 Y$ of the first jet prolongation $J^1 Y \to Y$.

Further, let $\widetilde{J}^r Y \to M$ be the r-th nonholonomic jet prolongation of a fibered manifold $p : Y \to M$. In general, an *r-th order nonholonomic connection* on Y is a section $\Gamma : Y \to \widetilde{J}^r Y$. Such a connection is called *semiholonomic* or *holonomic*, if it has values in $\overline{J}^r Y$ or in $J^r Y$, respectively.

The semiholonomity of higher order connections was discussed in [7] for connections in the grupoid form, but some of the considerations can be generalized. Also the following assertion can be found in [7].

Using the property (1) of semiholonomic jet prolongation of a fibered manifold and its notation, we first mention that if $\Gamma : Y \to \widetilde{J}^r Y$ is a general connection on $Y \to M$ of order r, then $\pi^r_q \Gamma := \pi^r_q \circ \Gamma(x),\ x \in M,\ q = 1, \ldots, r-1$, are q-th order connections on $Y \to M$. Now it is easy to see

Proposition 4. *A connection $\Gamma : Y \to \widetilde{J}^r Y$ is semiholonomic if and only if*

$$(\widetilde{J}^k \pi^{r-k}_{q-k})(\Gamma) = \pi^r_q(\Gamma) \textit{ for any integers } 1 \leq k \leq q < r.$$

We now recall some elementary operations on connections. Given two higher order connections $\Gamma : Y \to \widetilde{J}^r Y$ and $\overline{\Gamma} : Y \to \widetilde{J}^s Y$, the product of Γ and $\overline{\Gamma}$ is the $(r+s)$-th order connection $\Gamma * \overline{\Gamma} : Y \to \widetilde{J}^{r+s} Y$ defined by

$$\Gamma * \overline{\Gamma} = \widetilde{J}^s \Gamma \circ \overline{\Gamma}.$$

If we consider two first order connections, the question of semiholonomity of their product was solved in [6], [7] in the following way: for two first order connections Γ and $\overline{\Gamma}$ the product $\Gamma * \overline{\Gamma} : Y \to \widetilde{J}^2 Y$ is semiholonomic if and only if $\Gamma = \overline{\Gamma}$.

Considering a connection $\Gamma : Y \to J^1 Y$, one can define an r-th order connection $\Gamma^{(r-1)} : Y \to \widetilde{J}^r Y$ by

$$\Gamma^{(1)} := \Gamma * \Gamma = J^1 \Gamma \circ \Gamma, \qquad \Gamma^{(r-1)} := \Gamma^{(r-2)} * \Gamma = J^1 \Gamma^{(r-2)} \circ \Gamma.$$

The connection $\Gamma^{(r-1)}$ is called the $(r-1)$-st prolongation of Γ in the sense of Ehresmann, shortly $(r-1)$-st Ehresmann prolongation.

The following proposition explains the use of Ehresmann prolongation for the semiholonomic prolongation of a connection:

Proposition 5. *The values of* $\Gamma^{(r-1)}$ *lie in the semiholonomic prolongation* $\overline{J}^r Y$ *and* $\Gamma^{(r-1)}$ *is holonomic if and only if* Γ *is curvature free.*

For the proof see [6], [11].

6. Natural operators transforming first order connections into second order connections. We show that Ehresmann prolongation plays an important role in determining all natural operators transforming first order connections into higher order connections. Let us note that also natural transformations of semiholonomic jet prolongation functor $\overline{J}^r$ are involved. To find the details about this topic we refer to [4],[5], [9]. For our purposes, it is enough to consider $r = 2, 3$. We use the notation of [4], where the map $e : \overline{J}^2 Y \to \overline{J}^2 Y$ is obtained from the natural exchange map $e_\Lambda : J^1 J^1 Y \to J^1 J^1 Y$ as a restriction to the subbundle $\overline{J}^2 Y \subset J^1 J^1 Y$. Note that while e_Λ depends on the linear connection Λ on M, its restriction e is independent of any auxiliary connections. We remark that originally the map e_Λ was introduced by M. Modugno. We also recall that J. Pradines introduced a natural map $\overline{J}^2 Y \to \overline{J}^2 Y$ with the same coordinate expression.

Now we are ready to recall the following assertion, see [9] for the proof.

Proposition 7. *All natural operators transforming first order connection* $\Gamma : Y \to J^1 Y$ *into second order semiholonomic connection* $Y \to \overline{J}^2 Y$ *form a one parameter family*

$$\Gamma \mapsto k \cdot (\Gamma * \Gamma) + (1-k) \cdot e(\Gamma * \Gamma), \qquad k \in \mathbb{R}. \tag{2}$$

In other words, all natural operators from Proposition 7 differ from a constant multiple of Ehresmann prolongation $\Gamma * \Gamma$ by a term, where a natural transformation $\overline{J}^2 \to \overline{J}^2$ is applied to $\Gamma * \Gamma$.

Remark. The bundle $\overline{J}^2 Y \to J^1 Y$ is an affine bundle with the associated vector bundle

$$VY \otimes \overset{2}{\otimes} T^* M = (VY \otimes S^2 T^* M) \oplus (VY \otimes \wedge^2 T^* M),$$

where VY denotes the vertical subbundle of TM. This makes the sum in (2) possible.

Furthermore, by simple rearrangement, (2) can be written as

$$\Gamma \mapsto (\Gamma * \Gamma) + t(\Gamma * \Gamma - e(\Gamma * \Gamma)), \qquad t \in \mathbb{R}. \tag{3}$$

In this form, it is obvious that the term $\delta := \Gamma * \Gamma - e(\Gamma * \Gamma)$ in (3) corresponds to the so–called difference tensor. We recall that the difference tensor $\delta(U)$ of a semiholonomic 2-jet $U \subset \overline{J}^2 Y$ is the map $\delta : \overline{J}^2 Y \to VY \otimes \wedge^2 T^* M$ defined by $\delta(U) := U - e(U)$. The form of the difference tensor can be easily derived from the affine structure of the bundle $\overline{J}^r Y \to \overline{J}^{r-1} Y$, whose associated vector bundle is exactly of the form $VY \otimes \wedge^2 T^* M$.

The result for $r = 3$ is simple. According to [10], the only natural transformation $\overline{J}^3 \to \overline{J}^3$ is the identity and thus no result like that of Proposition 7 can be expected.

8. Semiholonomic connections of order three. In this section we will show that among operators transforming first order connections into third order semiholonomic connections, Ehresmann prolongation plays a significant role. Direct computations of the coordinate form of all natural operators in question becomes technically complicated. Thus we use the property (1) of semiholonomic jet prolongation of fibered manifolds to derive some interesting properties of Ehresmann prolongation. Let us note that the following proposition is strongly motivated by a result of J. Virsik, [11], where the author handles the connections and their prolongations in the groupoid form and thus also the prolongations of connections differ from the notion used in this paper. This is why we present a modified proof and refer to [11] for the original version.

Proposition 9. *Let Σ be an arbitrary r-th order connection and denote $\Gamma := \pi_1^r \Sigma$ the first order underlying connection. If the connection $\Sigma * \Gamma$ is semiholonomic, then $\Sigma = \Gamma^{(r-1)}$ and thus Σ is also semiholonomic.*

Proof. First note that $\pi_1^r \Sigma = \pi_1^{r+1}(\Sigma * \Gamma) = \Gamma$. Now consider the connection $\pi_2^{r+1}(\Sigma * \Gamma)$. If $\Sigma * \Gamma$ is semiholonomic, according to Proposition 4 and the definition of Ehresmann prolongation, $\pi_2^{r+1}(\Sigma * \Gamma) = J^1 \pi_1^r (J^1 \Sigma \circ \Gamma)$. Thus for any $u \in Y$ we have an element $J^1 \pi_1^r (J^1 \Sigma(\Gamma(u)) \in \overline{J}^2 Y$. But J^1 is a functor on the category of fibered manifolds and fiber respecting mappings and thus

$$J^1 \pi_1^r (J^1 \Sigma(\Gamma(u))) = J^1(\pi_1^r \Sigma(\Gamma(u))) = J^1 \Gamma(\Gamma(u)) = (\Gamma * \Gamma)(u)$$

for any $u \in Y$, i.e. $\pi_2^{r+1}(\Sigma * \Gamma) = \Gamma * \Gamma$. Now if we consider connections $\pi_q^{r+1}(\Sigma * \Gamma)$, $q = 3, ..., r$, we proceed recurrently with respect to q and therefore prove that for $q = r$

$$\Sigma = \pi_r^r \Sigma = \Gamma * \Gamma * ... * \Gamma = \Gamma^{(r-1)}. \ \blacksquare$$

Remark. If a connection Σ is of the form $\Gamma^{(r-1)}$, then the prolongation $\Sigma * \Gamma = \Gamma^{(r-1)} * \Gamma = \Gamma^{(r)}$ is also semiholonomic according to Proposition 5 and thus, trivially, the converse to Proposition 9 is valid.

Let us now consider a second order connection in the form (3) and denote the element corresponding to the difference tensor by δ, i.e. we consider a connection $\Sigma := \Gamma * \Gamma + t\delta$, where $\Gamma : Y \to J^1 Y$ is an arbitrary first order general connection. Then we have

Proposition 10. *The prolongation $\Sigma * \Gamma$ of $\Sigma = \Gamma * \Gamma + t\delta$, $t \in \mathbb{R}$, $t \neq 0$, is semiholonomic if and only if $\delta = 0$.*

Proof. Clearly, for $t = 0$ the connection $\Sigma * \Gamma$ is semiholonomic, too. Let us consider $t \neq 0$. If $\delta = 0$, then $\Sigma * \Gamma = \Gamma * \Gamma * \Gamma$ is semiholonomic according to Proposition 5. On the other hand, if $\Sigma * \Gamma$ is semiholonomic, then $\Sigma = \Gamma * \Gamma$ according to Proposition 9 due to the obvious fact that $\pi^2_1 \Sigma = \Gamma$. ∎

Conclusions. Ehresmann prolongation plays an important role in semiholonomic prolongation of connections. All possible prolongations of a connection into a second order semiholonomic connection can be described by means of Ehresmann prolongation and natural transformations of the functor $\overline{J}^2$ of second order jet prolongation of a fibered manifold. In order three, the only natural transformation of $\overline{J}^3$ is the identity and thus no similar result can be expected. But if we transform any second order semiholonomic connection of the form (3) to a third order connection by means of prolongation, only exactly third Ehresmann prolongation returns a semiholonomic connection. This leads to the conjecture that Ehresmann prolongation is the only natural operator transforming first order connections into third order semiholonomic connections, but an exact proof is still missing.

Acknowledgments. The author was supported by the Czech Science Foundation (GAČR, No. 201/08/P230).

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Received January 30, 2010; Revised January 21, 2011