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HARMONIC MAPS AND RIEMANNIAN SUBMERSIONS BETWEEN MANIFOLDS ENDOWED WITH SPECIAL STRUCTURES

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Abstract. It is well known that Riemannian submersions are of interest in physics, owing to their applications in the Yang-Mills theory, Kaluza-Klein theory, supergravity and superstring theories. In this paper we give a survey of harmonic maps and Riemannian submersions between manifolds equipped with certain geometrical structures such as almost Hermitian structures, contact structures, f-structures and quaternionic structures. We also present some new results concerning holomorphic maps and semi-Riemannian submersions between manifolds with metric mixed 3-structures.

1. Introduction. The motivation to study harmonic maps and Riemannian submersions comes from theoretical physics (see e.g. Chapter 8 of [FIP]). Presently, we see

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an increasing interest in harmonic maps between (pseudo-)Riemannian manifolds which are endowed with certain special geometric structure (like almost Hermitian structures [Chi1, Kob], almost metric contact structures [BG, BS, Bur], f-structures [BB, Erd, Fet], quaternionic structures [Sah, SAS, Vil]). In this article we give a survey and some new results concerning harmonic maps and Riemannian submersions between manifolds endowed with remarkable geometric structures. The paper is organized as follows. In Section 2 we recall some definitions and properties of harmonic maps between almost contact manifolds and a generalization of almost contact structures, namely f.pk-structures. In Section 3 we recall the notions of quaternionic manifold, quaternionic submersion and present some properties. In the last two sections of this paper we study holomorphic maps and semi-Riemannian submersions between manifolds endowed with metric mixed 3-structures.

2. Harmonic maps between almost contact manifolds

2.1. Manifolds endowed with almost contact structures. Let M be a differentiable manifold equipped with a triple (φ, ξ, η), where φ is a field of endomorphisms of the tangent spaces, ξ is a vector field and η is a 1-form on M. If

$$\varphi^2 = -Id + \eta \otimes \xi,\ \ \eta(\xi) = 1$$

then we say that (φ, ξ, η) is an almost contact structure on M (see [Bla]). Moreover, if g is a Riemannian metric associated on M, i.e. a metric satisfying, for any sections X and Y in $\Gamma(TM)$,

$$g(\varphi(X), \varphi(Y)) = g(X, Y) - \eta(X)\eta(Y)$$

then we say that (φ, ξ, η, g) is an almost contact metric structure. A manifold equipped with such a structure is called an *almost contact metric manifold*.

If the Nijenhuis tensor N^φ satisfies

$$N^\varphi + 2d\eta \otimes \xi = 0$$

we say that the *almost contact metric structure* (φ, ξ, η, g) is *normal*.

A *contact manifold* is a $(2n + 1)$-dimensional manifold M together with a 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. We say that $(M, \varphi, \xi, \eta, g)$ is a *Sasakian manifold* if it is a normal contact metric manifold such that $\Phi = d\eta$, where Φ is the second fundamental form on M defined for any $X, Y \in \Gamma(TM)$ by

$$\Phi(X, Y) = g(X, \varphi Y). \tag{1}$$

2.2. Holomorphic and harmonic maps. Let $\psi : (M^m, g) \to (N^n, h)$ be a smooth map between two (semi-)Riemannian manifolds. The *norm of* $d\psi$ is given by $\|d\psi\|^2 := Tr_g(\psi^* h)$ and the *energy density* of ψ is a smooth function $e(\psi) : M \to [0, \infty)$ defined by $e(\psi)_x = \frac{1}{2}\|d\psi_x\|^2$ for $x \in M$. We will write ψ_* instead of $d\psi$.

For any compact $\Omega \subseteq M$, the *energy of* ψ over Ω is the integral of its energy density

$$E(\psi; \Omega) = \int_\Omega e(\psi)\vartheta_g,$$

where ϑ_g is the volume measure associated to g. A smooth map $\psi : M \to N$ is said to be a *harmonic map* if $\frac{d}{dt}|_{t=0} E(\psi_t; \Omega) = 0$, for all compact domains $\Omega \subseteq M$ and for all variations $\{\psi_t\}_{t\in(-\epsilon,\epsilon)}$ of ψ supported in Ω, such that $\psi_0 = \psi$. Equivalently, the map ψ is harmonic if the tension field $\tau(\psi)$ of ψ vanishes at each point $x \in M$, where $\tau(\psi)$ is defined as the trace of the second fundamental form α_ψ of ψ, i.e.

$$\tau(\psi)_x = \sum_{i=1}^{m} \epsilon_i \alpha_\psi(e_i, e_i),$$

where $\{e_1, e_2, \ldots, e_m\}$ is a local pseudo-orthonormal frame of T_xM, $x \in M$, with $\epsilon_i = g(e_i, e_i) \in \{\pm 1\}$. The quantity α_ψ is defined by

$$\alpha_\psi(X, Y) = \widetilde{\nabla}_X \psi_* Y - \psi_* \nabla_X Y, \tag{2}$$

for any vector fields X, Y on M, where ∇ is the Levi-Civita connection of M and $\widetilde{\nabla}$ is the pullback of the Levi-Civita connection ∇' of N to the induced vector bundle $f^{-1}(TN)$,

$$\widetilde{\nabla}_X \psi_* Y = \nabla'_{\psi_* X} \psi_* Y.$$

We consider now $\{\psi_{s,t}\}_{s,t\in(-\epsilon,\epsilon)}$ a smooth two-parameter variation of ψ such that $\psi_{0,0} = \psi$ and let $V, W \in \Gamma(\psi^{-1}(TN))$ be the corresponding variational vector fields

$$V = \frac{\partial}{\partial s}(\psi_{s,t})|_{(s,t)=(0,0)},\ W = \frac{\partial}{\partial t}(\psi_{s,t})|_{(s,t)=(0,0)}.$$

The Hessian of a harmonic map ψ is defined by

$$H_\psi(V, W) = \frac{\partial^2}{\partial s \partial t}(E(\psi_{s,t}))|_{(s,t)=(0,0)}.$$

The index of a harmonic map $\psi : (M, g) \to (N, h)$ is defined as the dimension of the largest subspace of $\Gamma(\psi^{-1}(TN))$ on which the Hessian H_ψ is negative definite. A harmonic map ψ is said to be *stable* if the index of ψ is zero and otherwise, is said to be *unstable*. Concerning the stability of the identity map on Sasakian manifolds we have the following result.

THEOREM 2.1 (see [GIP]). *Let $M(\varphi, \xi, \eta, g)$ be a Sasakian compact manifold of constant φ-sectional curvature c, such that $c \leq 1$. If the first eigenvalue of the Laplacian $\triangle_g$ acting on $C^\infty(M, \mathbf{R})$ satisfies*

$$\lambda_1 < c(n + 1) + 3n - 1,$$

then the identity map $1_{|M}$ is a harmonic unstable map.

DEFINITION 2.2 (see [GIP]). A smooth map $\psi : (M^{2m+1}, \varphi, \xi, \eta, g) \to (N^{2n}, h, J)$ from an almost contact manifold to an almost Hermitian manifold is called (φ, J)-*holomorphic* if $\psi_* \circ \varphi = J \circ \psi_*$.

THEOREM 2.3 (see [IP2]). *Let $M(\varphi, \xi, \eta, g)$ be a $(2n + 1)$-dimensional Sasakian compact manifold of constant φ-sectional curvature c and $N(J, h)$ a Kähler manifold. Then any nonconstant (φ, J)-holomorphic map from M to N is an unstable harmonic map if*

$$c > -\frac{3(n-1)}{n+1},\ n \geq 1.$$

DEFINITION 2.4 (see [FIP]). A map $\psi : (M, \varphi, \xi, \eta, g) \to (M', \varphi', \xi', \eta', g')$ between almost contact manifolds is said to be (φ, φ')-*holomorphic* if $\psi_* \circ \varphi = \varphi' \circ \psi_*$.

THEOREM 2.5 (see [IP1]). *Let $M(\varphi, \xi, \eta, g)$ and $N(\varphi', \xi', \eta', g')$ be almost contact manifolds. Then any (φ, φ')-holomorphic map $\psi : M \to N$ is harmonic.*

2.3. A generalization of almost contact structures: *f*.pk-structures. An *f-structure* on a manifold M is a non-vanishing endomorphism of TM, that satisfies $f^3 + f = 0$ and which has constant rank $2n$. We have the splitting of the tangent bundle

$$TM = D \oplus D' = Imf \oplus Kerf$$

into two complementary subbundles. The restriction of f to D determines a complex structure on the subbundle D. The interesting case is when D' is parallelizable; then the structure group is $U(n) \times I_s$, $dimM = 2n + s$ and f is called an f-structure with parallelizable kernel (f.pk-structure). In this case there exists a global frame ξ_i, $1 \leq i \leq s$, for the subbundle D' and 1 - forms η^i, $1 \leq i \leq s$ such that:

$$f^2 = -Id + \eta^i \otimes \xi_i, \ \eta^i(\xi_j) = \delta^i_j.$$

We say that an f-structure is *normal* if

$$N^f + 2d\eta^i \otimes \xi_i = 0.$$

A metric *f*.pk-structure (f, η^i, ξ_i, g), where the Riemannian metric g satisfies

$$g(X, Y) = g(fX, fY) + \eta^i(X)\eta^i(Y)$$

is called a *K-structure* if the corresponding 2-form Φ, defined by $\Phi(X, Y) = g(X, fY)$, for any vector fields X and Y on M, is closed and the normality condition holds.

An *almost C-manifold* is a manifold endowed with a metric f.pk-structure with $d\Phi = 0$ and $d\eta^i = 0$, for any $i \in \{1, \ldots, s\}$. An *almost C-manifold with Kählerian leaves* is an almost C-manifold with any leaf of canonical foliation Kählerian (see also [Ols] for the case of almost cosymplectic manifolds with Kählerian leaves). Concerning (f, J)-holomorphic maps between an almost C-manifold with Kählerian leaves and a Kähler manifold, we have the following.

THEOREM 2.6 (see [IP2]). *Let $M(f, \eta^i, \xi_i, g)$ be an almost C-manifold with Kählerian leaves and $N(J, h)$ a Kähler manifold. Then, any (f, J)-holomorphic map $\psi : M \to N$ is harmonic. Moreover, if M is a compact manifold then ψ is stable.*

3. Harmonic maps and submersions between quaternionic manifolds. If (M, g) and (N, g') are two Riemannian manifolds, then a surjective C^∞-map $\pi : M \to N$ is said to be a *C^∞-submersion* if it has maximal rank at any point of M. Putting $\mathcal{V}_x = Ker\ \pi_{*x}$, for any $x \in M$, we obtain an integrable distribution $\mathcal{V}$; it is called the *vertical* distribution and corresponds to the foliation of M determined by the fibres of π. The complementary distribution $\mathcal{H}$ of $\mathcal{V}$, determined by the Riemannian metric g, is called the *horizontal* distribution. A C^∞-submersion $\pi : M \to N$ between two Riemannian manifolds (M, g) and (N, g') is called a *Riemannian submersion* if, at each point x of M, π_{*x} preserves the length of the horizontal vectors (see [ON1]). We recall that the sections of $\mathcal{V}$, respectively $\mathcal{H}$, are called vertical vector fields, respectively horizontal vector fields.

DEFINITION 3.1 (see [Ish]). An *almost quaternionic structure* on a differentiable manifold M of dimension n is a rank 3-subbundle σ of $End(TM)$ such that a local basis $\{J_1, J_2, J_3\}$

exists on sections of σ satisfying for all $\alpha \in \{1,2,3\}$

$$J_\alpha^2 = -Id,\ J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2}$$

where the indices are taken from $\{1,2,3\}$ modulo 3. Moreover, (M,σ) is said to be an *almost quaternionic manifold.* A Riemannian metric g is said to be *adapted* to an almost quaternionic structure σ on a manifold M if

$$g(J_\alpha X, J_\alpha Y) = g(X,Y), \forall \alpha \in \{1,2,3\}$$

for all vector fields X,Y on M. In this case (M,σ,g) is said to be an *almost quaternionic Hermitian manifold.* Moreover, (M,σ,g) is said to be a *quaternionic Kähler manifold* if the bundle σ is parallel with respect to the Levi-Civita connection ∇ of g.

Definition 3.2 (see [IMV3]). Let (M,σ,g) and (N,σ',g') be two almost quaternionic Hermitian manifolds. A map $f : M \to N$ is called a (σ,σ')-*holomorphic map* at a point x of M if for any $J \in \sigma_x$ exist $J' \in \sigma'_{f(x)}$ such that $f_* \circ J = J' \circ f_*$. Moreover, we say that f is a (σ,σ')-holomorphic map if f is a (σ,σ')-holomorphic map at each point $x \in M$.

Definition 3.3 (see [IMV2]). Let (M,σ,g) and (N,σ',g') be two almost quaternionic Hermitian manifolds. A Riemannian submersion $\pi : M \to N$ which is a (σ,σ')-holomorphic map is called a *quaternionic submersion.* Moreover, if (M,σ,g) is a quaternionic Kähler manifold, then we say that π is a *quaternionic Kähler submersion.*

Theorem 3.4 (see [IMV3]). *Let (M,σ,g) and (N,σ',g') be two quaternionic Kähler manifolds. If $f : M \to N$ is a (σ,σ')-holomorphic map such that, for any local section $J \in \Gamma(\sigma)$ and corresponding $J' \in \Gamma(\sigma')$ one has $(\nabla'_{f_* X} J') \circ f_* = f_* \circ (\nabla_X J)$, for any local vector field X on M, then f is a harmonic map.*

Corollary 3.5 (see [IMV2]). *Any quaternionic Kähler submersion is a harmonic map.*

Theorem 3.6 (Stability of (σ,σ')-holomorphic maps, [IMV3]). *Let (M^{4m},σ,g) and (N^{4n},σ',g') be two quaternionic Kähler manifolds such that M is compact, N has non positive scalar curvature and, at any point $p \in M$, there exists a basis $\{J_1,J_2,J_3\}$ of σ_p such that one of J_1, J_2 or J_3 is parallel. If $f : M \to N$ is a (σ,σ')-holomorphic map such that, for any local section $J \in \Gamma(\sigma)$ and corresponding $J' \in \Gamma(\sigma')$ one has $(\nabla'_{f_* X} J') \circ f_* = f_* \circ (\nabla_X J)$, for any local vector field X on M, then f is stable.*

Corollary 3.7 (see [IMV2]). *If $\pi : (M,\sigma,g) \to (N,\sigma',g')$ is a quaternionic Kähler submersion, then the fibres are totally geodesic quaternionic Kähler submanifolds.*

Example 3.8. Let (M,σ,g) be an almost quaternionic hermitian manifold and TM be the tangent bundle, endowed with the metric:

$$G(A,B) = g(KA,KB) + g(\pi_* A, \pi_* B),\ \forall A,B \in T(TM),$$

where π is the natural projection of TM onto M and K is the connection map (see [Dom]).

We remark that if $X \in \Gamma(TM)$, then there exists exactly one vector field on TM called the "horizontal lift" (resp. "vertical lift") of X such that for all $U \in TM$,

$$\pi_* X_U^h = X_{\pi(U)},\ \pi_* X_U^v = 0_{\pi(U)},\ K X_U^h = 0_{\pi(U)},\ K X_U^v = X_{\pi(U)}.$$

We define three tensor fields J'_1, J'_2, J'_3 on TM by the equalities

$$J'_\alpha X^h = (J_\alpha X)^h,\ J'_\alpha X^v = (J_\alpha X)^v,\ \forall \alpha \in \{1,2,3\},$$

where $\{J_1, J_2, J_3\}$ is a canonical local basis of σ.

If we consider now the vector bundle σ' over TM generated by $\{J'_1, J'_2, J'_3\}$, then we have that (TM, σ', G) is an almost quaternionic hermitian manifold and the natural projection $\pi : TM \to M$ is a quaternionic submersion (see [IMV2]).

We note that the results from this section were recently extended to the class of manifolds endowed with quaternionic structures of second kind (paraquaternionic structures) and compatible metrics in [Cal]. The counterpart in odd dimension of a paraquaternionic structure, called mixed 3-structure, was introduced in [IMV1]. This concept, which arises in a natural way on lightlike hypersurfaces in paraquaternionic manifolds, has been refined in [CP], where the authors have introduced positive and negative metric mixed 3-structures. Next we study holomorphic maps from manifolds endowed with such kind of structures.

4. Holomorphic maps between manifolds endowed with mixed 3-structures and compatible metrics. Let M be a smooth manifold equipped with a triple (φ, ξ, η), where ϕ is a field of endomorphisms of the tangent spaces, ξ is a vector field and η is a 1-form on M. If we have:

$$\varphi^2 = Id - \eta \otimes \xi, \quad \eta(\xi) = 1$$

then we say that (φ, ξ, η) is an almost paracontact structure on M (cf. [Sat]).

Definition 4.1 ([CP]). A *mixed 3-structure* on a smooth manifold M is a triple of structures $(\varphi_\alpha, \xi_\alpha, \eta_\alpha)$, $\alpha \in \{1,2,3\}$, which are almost paracontact structures for $\alpha = 1, 2$ and almost contact structure for $\alpha = 3$, satisfying the following conditions:

$$\begin{gathered} \eta_\alpha(\xi_\beta) = 0, \\ \varphi_\alpha(\xi_\beta) = \tau_\beta \xi_\gamma, \quad \varphi_\beta(\xi_\alpha) = -\tau_\alpha \xi_\gamma, \\ \eta_\alpha \circ \varphi_\beta = -\eta_\beta \circ \varphi_\alpha = \tau_\gamma \eta_\gamma \end{gathered} \tag{3}$$

$$\varphi_\alpha \varphi_\beta - \tau_\alpha \eta_\beta \otimes \xi_\alpha = -\varphi_\beta \varphi_\alpha + \tau_\beta \eta_\alpha \otimes \xi_\beta = \tau_\gamma \varphi_\gamma\,, \tag{4}$$

where (α, β, γ) is an even permutation of (1,2,3) and $\tau_1 = \tau_2 = -\tau_3 = -1$.

Moreover, if a manifold M with a mixed 3-structure $(\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}$ admits a semi-Riemannian metric g such that

$$g(\varphi_\alpha X, \varphi_\alpha Y) = \tau_\alpha[g(X,Y) - \varepsilon_\alpha \eta_\alpha(X)\eta_\alpha(Y)], \tag{5}$$

for all $X, Y \in \Gamma(TM)$ and $\alpha \in \{1,2,3\}$, where $\varepsilon_\alpha = g(\xi_\alpha, \xi_\alpha) = \pm 1$, then we say that M has a *metric mixed 3-structure* and g is called a *compatible* metric.

Remark 4.2. If $(M, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ is a manifold with a metric mixed 3-structure then from (3) and (5) we can easily obtain

$$\eta_\alpha(X) = \varepsilon_\alpha g(X, \xi_\alpha),\ g(\varphi_\alpha X, Y) = -g(X, \varphi_\alpha Y) \tag{6}$$

and

$$g(\xi_1, \xi_1) = g(\xi_2, \xi_2) = -g(\xi_3, \xi_3).$$

Hence the vector fields ξ_1 and ξ_2 are both either space-like or time-like and these force the causal character of the third vector field ξ_3. We may therefore distinguish between positive and negative metric mixed 3-structures, according as ξ_1 and ξ_2 are both space-like, or both time-like vector fields. Because at each point of M, there always exists a pseudo-orthonormal frame field given by $\{(E_i, \varphi_1 E_i, \varphi_2 E_i, \varphi_3 E_i)_{i=\overline{1,n}}, \xi_1, \xi_2, \xi_3\}$ we conclude that the dimension of the manifold is $4n+3$ and the signature of $\overline{g}$ is $(2n+1, 2n+2)$ if the metric mixed 3-structure is positive (i.e. $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = 1$), or the signature of g is $(2n+2, 2n+1)$ if the metric mixed 3-structure is negative (i.e. $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = -1$).

Definition 4.3 ([CP]). Let $(M, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ be a manifold with a metric mixed 3-structure.

(i) If $(\varphi_1, \xi_1, \eta_1, g)$, $(\varphi_2, \xi_2, \eta_2, g)$ are para-cosymplectic structures and $(\varphi_3, \xi_3, \eta_3, g)$ is a cosymplectic structure, i.e. the Levi-Civita connection ∇ of g satisfies

$$\nabla \varphi_\alpha = 0 \tag{7}$$

for all $\alpha \in \{1, 2, 3\}$, then $((\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, \overline{g})$ is said to be a mixed 3-cosymplectic structure on M.

(ii) If $(\varphi_1, \xi_1, \eta_1, g)$, $(\varphi_2, \xi_2, \eta_2, g)$ are para-Sasakian structures and $(\varphi_3, \xi_3, \eta_3, g)$ is a Sasakian structure, i.e.

$$(\nabla_X \varphi_\alpha)Y = \tau_\alpha[g(X, Y)\xi_\alpha - \epsilon_\alpha \eta_\alpha(Y)X] \tag{8}$$

for all $X, Y \in \Gamma(TM)$ and $\alpha \in \{1, 2, 3\}$, then $((\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ is said to be a mixed 3-Sasakian structure on M.

Remark that from (7) it follows that

$$\nabla \xi_\alpha = 0,\ \nabla \eta_\alpha = 0 \tag{9}$$

and from (8) we obtain

$$\nabla_X \xi_\alpha = -\varepsilon_\alpha \varphi_\alpha X, \tag{10}$$

for all $\alpha \in \{1, 2, 3\}$ and $X \in \Gamma(TM)$.

We also note that the main property of a manifold endowed with a mixed 3-Sasakian structure is given by the following theorem (see [CP]).

Theorem 4.4. *Any $(4n+3)$-dimensional manifold endowed with a mixed 3-Sasakian structure is an Einstein space with Einstein constant $\lambda = (4n+2)\varepsilon$, with $\varepsilon = \mp 1$, according as the metric mixed 3-structure is positive or negative, respectively.*

Definition 4.5. Let $(M, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ and $(N, (\varphi'_\alpha, \xi'_\alpha, \eta'_\alpha)_{\alpha=\overline{1,3}}, g')$ be two manifolds endowed with metric mixed 3-structures. We say that a smooth map $f : M \to N$ is *holomorphic* if the equation

$$f_* \circ \varphi_\alpha = \varphi'_\alpha \circ f_* \tag{11}$$

holds for all $\alpha \in \{1, 2, 3\}$.

Example 4.6. If $(M, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ is a manifold endowed with a metric mixed 3-structure and M' is an invariant submanifold of M (i.e. a non-degenerate submanifold of M such that $\varphi_\alpha(T_pM') \subset T_pM'$, for all $p \in M'$ and $\alpha = 1, 2, 3$), tangent to the structure vector fields, then the restriction of $((\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ to M' is a metric mixed 3-structure and the inclusion map $i : M' \to M$ is holomorphic.

Now we are able to state the following.

THEOREM 4.7. *Let* $(M, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ *and* $(N, (\varphi'_\alpha, \xi'_\alpha, \eta'_\alpha)_{\alpha=\overline{1,3}}, g')$ *be two mixed 3-cosymplectic or mixed 3-Sasakian manifolds. If* $f : M \to N$ *is a holomorphic map, then* f *is a harmonic map.*

Proof. Let $\{(E_i, \varphi_1 E_i, \varphi_2 E_i, \varphi_3 E_i)_{i=\overline{1,n}}, \xi_1, \xi_2, \xi_3\}$ be a local pseudo-orthonormal basis of vector fields tangent to M, with $\epsilon_i = g(E_i, E_i) \in \{\pm 1\}$. Then we have from (5) and (6) that

$$g(\varphi_j E_i, \varphi_j E_i) = \tau_j \epsilon_i, \ j \in \{1, 2, 3\}$$

and we deduce that the tension field of f is given by

$$\tau(f) = \sum_{i=1}^{n} \epsilon_i [\alpha_f(E_i, E_i) + \sum_{j=1}^{3} \tau_j \alpha_f(\varphi_j E_i, \varphi_j E_i)] + \sum_{j=1}^{3} \varepsilon_j \alpha_f(\xi_j, \xi_j). \tag{12}$$

We remark now that in both cases (mixed 3-cosymplectic and mixed 3-Sasakian), we obtain from (9) or (10) that

$$\nabla_{\xi_j} \xi_j = 0, \ \nabla'_{\xi'_j} \xi'_j = 0,$$

since $\varphi_j \xi_j = 0$, for all $j \in \{1, 2, 3\}$ (see e.g. [Bla]). Taking into account now that there exists a positive real number r such that $f_* \xi_j = r \xi'_j$ (see [IP1]), we deduce

$$\alpha_f(\xi_j, \xi_j) = 0. \tag{13}$$

Using now (7) or (8), according as the manifold is mixed 3-cosymplectic or mixed 3-Sasakian, we can easily obtain, for all $j \in \{1, 2, 3\}$ and $i \in \{1, \ldots, n\}$,

$$\nabla_{E_i} E_i = -\tau_j \varphi_j \nabla_{E_i} \varphi_j E_i \tag{14}$$

and

$$\nabla_{\varphi_j E_i} \varphi_j E_i = \varphi_j \nabla_{\varphi_j E_i} E_i. \tag{15}$$

From (14) and (15) we derive

$$\nabla_{E_i} E_i + \tau_j \nabla_{\varphi_j E_i} \varphi_j E_i = \tau_j \varphi_j [\varphi_j E_i, E_i]. \tag{16}$$

Similarly, since f is holomorphic, we obtain

$$\widetilde{\nabla}_{E_i} f_* E_i + \tau_j \widetilde{\nabla}_{\varphi_j E_i} f_* \varphi_j E_i = \tau_j \varphi'_j [f_* \varphi_j E_i, f_* E_i]. \tag{17}$$

From (2), (16) and (17), taking account of (11), we derive

$$\alpha_f(E_i, E_i) + \tau_j \alpha_f(\varphi_j E_i, \varphi_j E_i) = 0, \tag{18}$$

for all $j \in \{1, 2, 3\}$.

Applying repeatedly (18) and making use of (4) and (6), we obtain

$$\begin{aligned} \alpha_f(E_i, E_i) &= -\alpha_f(\varphi_3 E_i, \varphi_3 E_i) = -\alpha_f(\varphi_2 \varphi_3 E_i, \varphi_2 \varphi_3 E_i) \\ &= -\alpha_f(-\varphi_1 E_i - \eta_3(E_i)\xi_2, -\varphi_1 E_i - \eta_3(E_i)\xi_2) \\ &= -\alpha_f(\varphi_1 E_i, \varphi_1 E_i) = -\alpha_f(E_i, E_i) \end{aligned}$$

and so we conclude that

$$\alpha_f(E_i, E_i) = 0. \tag{19}$$

From (18) and (19) we obtain

$$\alpha_f(\varphi_j E_i, \varphi_j E_i) = 0,\ j \in \{1, 2, 3\}. \tag{20}$$

Using now (13), (19) and (20) in (12) we derive $\tau(f) = 0$ and the conclusion follows. ∎

5. Semi-Riemannian submersions from manifolds endowed with metric mixed 3-structures. An almost para-hypercomplex structure on a smooth manifold M is a triple $H = (J_\alpha)_{\alpha=\overline{1,3}}$, where J_1, J_2 are almost product structures on M and J_3 is an almost complex structure on M, satisfying

$$J_\alpha J_\beta = -J_\beta J_\alpha = \tau_\gamma J_\gamma,$$

for every even permutation (α, β, γ) of (1,2,3), where $\tau_1 = \tau_2 = -\tau_3 = -1$.

A semi-Riemannian metric g on (M, H) is said to be compatible or adapted to the almost para-hypercomplex structure $H = (J_\alpha)_{\alpha=\overline{1,3}}$ if it satisfies

$$g(J_\alpha X, J_\alpha Y) = \tau_\alpha g(X, Y)$$

for all vector fields X,Y on M. Moreover, the triple (M, H, g) is said to be an almost para-hyperhermitian manifold. If $\{J_1, J_2, J_3\}$ are parallel with respect to the Levi-Civita connection of g, then the manifold is called para-hyper-Kähler.

Let (M, g) and (M', g') be two connected semi-Riemannian manifold of index s $(0 \leq s \leq dimM)$ and s' $(0 \leq s' \leq dimM')$ respectively, with $s' \leq s$. The concept of semi-Riemannian submersion was introduced by O'Neill (see [ON2]) as a smooth map $\pi : M \to M'$ which is onto and satisfies the following conditions:

(*i*) $\pi_*|_p$ is onto for all $p \in M$;
(*ii*) The fibres $\pi^{-1}(p')$, $p' \in M'$, are semi-Riemannian submanifolds of M;
(*iii*) π_* preserves scalar products of vectors normal to fibres.

A semi-Riemannian submersion $\pi : M \to M'$ determines, as in the Riemannian case (see [ON1]), two (1,2) tensor fields T and A on M, by the formulas

$$T(E, F) = T_E F = h\nabla_{vE} vF + v\nabla_{vE} hF$$

and respectively

$$A(E, F) = A_E F = v\nabla_{hE} hF + h\nabla_{hE} vF$$

for any $E, F \in \Gamma(TM)$, where v and h are the vertical and horizontal projection. We remark that for $U, V \in \Gamma(\mathcal{V})$, $T_U V$ coincides with the second fundamental form of the immersion of the fibre submanifolds.

A horizontal vector field X on M is said to be *basic* if X is π-related to a vector field X' on M'. It is clear that every vector field X' on M' has a unique horizontal lift X to M and X is basic.

REMARK 5.1. If $\pi : M \to M'$ is a semi-Riemannian submersion and X, Y are basic vector fields on M, π-related to X' and Y' on M', then (see [ON2]):

(*i*) $h[X, Y]$ is a basic vector field and $\pi_* h[X, Y] = [X', Y'] \circ \pi$;

(*ii*) $h(\nabla_X Y)$ is a basic vector field π-related to $\nabla'_{X'}Y'$, where ∇ and ∇' are the Levi-Civita connections on M and M';
(*iii*) $[E,U] \in \Gamma(\mathcal{V}), \forall U \in \Gamma(\mathcal{V})$ and $\forall E \in \Gamma(TM)$.

DEFINITION 5.2. Let $(M, (\varphi_\alpha, \xi_\alpha, \eta_\alpha)_{\alpha=\overline{1,3}}, g)$ and $(N, (\varphi'_\alpha, \xi'_\alpha, \eta'_\alpha)_{\alpha=\overline{1,3}}, g')$ be two manifolds endowed with metric mixed 3-structures. A semi-Riemannian submersion $\pi : M \to N$ is said to be a *mixed 3-submersion* if it is a holomorphic map and the structure vector field ξ_α on M is a basic vector field π-related to the structure vector field ξ'_α on N, for all $\alpha \in \{1,2,3\}$.

Using the same techniques as in [Wat] (see also [Chi2, IMV2, TM]), we can prove the following.

THEOREM 5.3. *Let $\pi : M \to N$ be a mixed 3-submersion. Then:*

(*i*) *The vertical and horizontal distributions induced by π are invariant under each φ_α, $\alpha \in \{1,2,3\}$.*
(*ii*) *The fibres of the submersion are almost para-hyperhermitian manifolds.*
(*iii*) *If M is a mixed 3-cosymplectic manifold, then the base space N is also a mixed 3-cosymplectic manifold. Moreover, the fibres are totaly geodesic para-hyper-Kähler submanifolds.*
(*iv*) *If M is a mixed 3-Sasakian manifold, then π is a semi-Riemannian covering map.*

Proof. (i) Let $V \in \Gamma(\mathcal{V})$. Then, we have $\pi_* \varphi_\alpha V = \varphi'_\alpha \pi_* V = 0$, and so we conclude that $\varphi_\alpha(\mathcal{V}) \subset \mathcal{V}$. On another hand, for any $X \in \Gamma(\mathcal{H})$ and $V \in \Gamma(\mathcal{V})$, we derive from (6) that $g(\varphi_\alpha X, V) = -g(X, \varphi_\alpha V) = 0$ and thus we obtain $\varphi_\alpha(\mathcal{H}) \subset \mathcal{H}$.

(ii) If we denote by J_α the restriction of φ_α to $\mathcal{V}$, then for any vertical vector field V we have

$$J_\alpha^2 V = \varphi_\alpha^2 V = \tau_\alpha[-V + \eta_\alpha(V)\xi_\alpha] = -\tau_\alpha V,$$

and from (4) we obtain

$$J_\alpha J_\beta V = -J_\beta J_\alpha V = \tau_\gamma J_\gamma V,$$

since ξ_α is horizontal. On the other hand, from (5) we deduce that the restriction of g to any fibre F is compatible with $\{J_\alpha\}_{\alpha=\overline{1,3}}$ defined above and so we conclude that $(F, \{J_\alpha\}_{\alpha=\overline{1,3}}, g_{|F})$ is an almost para-hyperhermitian manifold.

(iii) For any basic vector fields X, Y on M, π-related with X' and Y' on N, we deduce from (7) that

$$\pi_*(\nabla_X \varphi_\alpha Y) - \pi_* \varphi_\alpha \nabla_X Y = 0, \ \alpha \in \{1,2,3\}.$$

Since $\varphi_\alpha Y$ is a basic vector field π-related with $\varphi'_\alpha Y'$, using (11) and Remark 5.1 we obtain

$$\nabla'_{X'} \varphi'_\alpha Y' - \varphi'_\alpha \nabla'_{X'} Y' = 0, \ \alpha \in \{1,2,3\}$$

and thus N is a mixed 3-cosymplectic manifold.

Using the Gauss's formula and (7), by identifying the tangential and normal components to a fibre F, we obtain, for any vector fields U, V tangent to F,

$$(\nabla_U J_\alpha) V = 0 \tag{21}$$

and

$$T_U J_\alpha V = \varphi_\alpha(T_U V). \tag{22}$$

From (21) it follows that $(F, \{J_\alpha\}_{\alpha=\overline{1,3}}, g_{|F})$ is a para-hyper-Kähler manifold and applying repeatedly (22) we obtain $T = 0$. So F is a totaly geodesic submanifold.

(iv) Because $d\eta(V, \varphi_\alpha V) = 0$, for any vector field V tangent to a fibre F and $\Phi_\alpha = d\eta_\alpha$, it follows from (1) that $g(V, V) = 0$. Therefore, since $g_{|F}$ is non-degenerate, we obtain that the fibres are discrete and the conclusion follows.

The proof is now complete. ■

Corollary 5.4. *Any mixed 3-submersion from a mixed 3-cosymplectic manifold is a harmonic map.*

Proof. The statement is obvious since it is well known that a semi-Riemannian submersion is a harmonic map if and only if each fibre is a minimal submanifold (see e.g. [FIP]). ■

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