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SYMPLECTIC SINGULARITIES OF ISOTROPIC MAPPINGS

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1. Introduction. A mapping $f : (\mathbf{R}^{n-k}, 0) \to \mathbf{R}^{2n}$ $(0 \leq k \leq n)$ to the symplectic space $\mathbf{R}^{2n}$ with the symplectic form $\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$ is called *isotropic* if $f^*\omega = 0$.

The classification problem of isotropic mappings is one of basic subjects in the symplectic mathematics, in particular, in symplectification of the singularity theory, including the study on Lagrangian varieties and singular curves in the symplectic space. We will give a short survey on the subject and related basic contributions to the classification problem.

One of major motivations for the classification problem comes from classical mechanics and its quantizations, celestial mechanics, field theory, and so on, in the Hamiltonian framework. From the view point of applications, we need to consider also the classification problem in the presence of a system of commuting Hamiltonians.

For example, in the symplectic 4-space $\{(p_1, q_1, p_2, q_2)\}$ consider the energy function $h = \frac{1}{2}(p_1^2 + p_2^2)$, then the problem is to classify curves or isotropic surfaces with singularities by local symplectomorphisms preserving the energy foliation on $\mathbf{R}^4 \setminus h^{-1}(0)$.

In general we consider, in the local framework, coisotropic fibrations $\pi : \mathbf{R}^{2n} \to \mathbf{R}^{n-\ell}$ $(0 \leq \ell \leq n)$. It is defined by a system of Poisson commuting independent $(n-\ell)$-functions. By the classical Jacobi-Liouville theorem, up to local symplectomorphisms,

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we may assume $\pi(p,q) = (q_{\ell+1},\dots,q_n) = \bar{q}$. Then two isotropic map-germs $f,g : (\mathbf{R}^{n-k},0) \to \mathbf{R}^{2n}$ are called *equivalent* if f is transformed to g by a symplectomorphism $\Phi : (\mathbf{R}^{2n}, f(0)) \to (\mathbf{R}^{2n}, g(0))$ preserving π-fibers up to a parametrization, namely, for diffeomorphisms $\sigma : (\mathbf{R}^{n-k},0) \to (\mathbf{R}^{n-k},0)$ and $\varphi : \big(\mathbf{R}^{n-\ell}, \pi(f(0))\big) \to \big(\mathbf{R}^{n-\ell}, \pi(g(0))\big)$, we have $\Phi \circ f = g \circ \sigma$, $\varphi \circ \pi = \pi \circ \Phi$, i.e., the following diagram is commutative:

$$\begin{array}{ccccc} (\mathbf{R}^{n-k},0) & \xrightarrow{f} & \big(\mathbf{R}^{2n}, f(0)\big) & \xrightarrow{\pi} & \big(\mathbf{R}^{n-\ell}, \pi(f(0))\big) \\ \downarrow \sigma & & \downarrow \Phi & & \downarrow \varphi \\ (\mathbf{R}^{n-k},0) & \xrightarrow{g} & \big(\mathbf{R}^{2n}, g(0)\big) & \xrightarrow{\pi} & \big(\mathbf{R}^{n-\ell}, \pi(g(0))\big). \end{array}$$

In the case $\ell = 0$, we call the above equivalence *Lagrange equivalence*. In the case $\ell = n$, we call it *symplectic equivalence*. In the case $\ell = n-1$, we consider a fibration by hypersurfaces. In general, we call the equivalence *flexible equivalence*, or π-*equivalence*, making stress on the fixed coisotropic fibration π.

In the above definition, if we do not impose the condition that Φ is a symplectomorphism but do impose just that Φ is a diffeomorphism, we get the ordinary equivalence of composed mappings (f,π) and (g,π). If $\ell = n$, then it just gives the $\mathcal{A}$-equivalence of mappings f and g.

Apart from generic classification, for Cauchy problem of Hamilton-Jacobi equations, we treat isotropic submanifolds I^{n-1} in a smooth hypersurface $\{H(p,q) = 0\}$ in $\mathbf{R}^{2n}$. Then we parametrize I by $f : \mathbf{R}^{n-1} \to \mathbf{R}^{2n}$, and consider the diagram

$$\mathbf{R}^{n-1} \xrightarrow{f} \mathbf{R}^{2n} \xrightarrow{H} \mathbf{R},$$

with the condition $H \circ f = 0$. Then the problem is reduced to the classification of the reduction $\overline{f} : \mathbf{R}^{n-1} \to \mathbf{R}^{2(n-1)}$, which is a singular isotropic mapping ([12]).

A coisotropic fibration $\pi : (\mathbf{R}^{2n},0) \to (\mathbf{R}^{n-\ell},0)$ induces, via the fiberwise symplectic reduction, the submersion $\bar{\pi} : (\mathbf{R}^{2n},0) \to \big(\mathbf{R}^{2\ell} \times \mathbf{R}^{n-\ell}, (0,0)\big)$ to the family of $\mathbf{R}^{2\ell} = T^*\mathbf{R}^{\ell}$.

Now let $f : (\mathbf{R}^{n-k},0) \to (\mathbf{R}^{2n},0)$ be an isotropic map-germ. If $\pi \circ f$ is a submersion, then, $k \le \ell$ and $\bar{\pi} \circ f : (\mathbf{R}^{n-k},0) \to \big(\mathbf{R}^{2\ell} \times \mathbf{R}^{n-\ell}, (0,0)\big)$ is regarded as an unfolding by isotropic map germs of an isotropic map-germ $(\mathbf{R}^{\ell-k},0) \to (\mathbf{R}^{2\ell},0)$. Then we are led to the classification problem and the unfolding problem of isotropic map-germ $f : (\mathbf{R}^{\ell-k},0) \to (\mathbf{R}^{2\ell},0)$.

In the paper [16] we consider, in particular, the case $\ell = 1$ and $k = 0$. We review shortly the classification result obtained in [16] in Section 2. In particular we have shown in [16] the symplectic codimension of plane curve-germ $f : (\mathbf{R},0) \to (\mathbf{R}^2,0)$ is a diffeomorphism invariant. This is not true for curves $(\mathbf{R},0) \to (\mathbf{R}^{2n},0)$ when $n \ge 2$. However we observe the diffeomorphism invariance of the symplectic codimension of Lagrange variety in Section 6.

Note that the case $\pi \circ f$ is not a submersion and $k = 0$ is treated in [30].

Lagrange singularity theory treats Lagrange equivalence mainly. The objects of the study, then, are the composed mapping

$$(\mathbf{R}^n,0) \xrightarrow{f} (\mathbf{R}^{2n},0) \xrightarrow{\pi} (\mathbf{R}^n,0),$$

consisting of a Lagrange immersion, i.e. an isotropic immersion f with $k = 0$, and a

Lagrange fibration, i.e. a coisotropic fibration π with $\ell = 0$ ([3]). For the classification in the case f is not an immersion, see [4] for example.

Note that a coisotropic fibration $\pi : (\mathbf{R}^{2n}, 0) \to (\mathbf{R}^{n-\ell}, 0)$ can be decomposed into a Lagrange fibration $\Pi : (\mathbf{R}^{2n}, 0) \to (\mathbf{R}^{n}, 0)$ and a fibration $(\mathbf{R}^{n}, 0) \to (\mathbf{R}^{n-\ell}, 0)$. Moreover an isotropic immersion $f : (\mathbf{R}^{n-k}, 0) \to (\mathbf{R}^{2n}, 0)$ can be extended to a Lagrange immersion $F : (\mathbf{R}^{n}, 0) \to (\mathbf{R}^{2n}, 0)$ via an immersion $(\mathbf{R}^{n-k}, 0) \to (\mathbf{R}^{n}, 0)$. Thus we have a composed mapping

$$(\mathbf{R}^{n-k}, 0) \longrightarrow (\mathbf{R}^{n}, 0) \xrightarrow{F} (\mathbf{R}^{2n}, 0) \xrightarrow{\Pi} (\mathbf{R}^{n}, 0) \longrightarrow (\mathbf{R}^{n-\ell}, 0).$$

Though intermediate Lagrange fibrations and Lagrange extensions are not unique, we see this aspect is effective for the classification problem, at least, in the case $\ell = 0$. In Section 4, we review the classification of isotropic immersions under Lagrange equivalence due to Janeczko and Zakalyukin ([20], [29]).

Even if $f : (\mathbf{R}^{n-k}, 0) \to (\mathbf{R}^{2n}, 0)$ is not an immersion, it is effective to consider a Lagrange extension $F : (\mathbf{R}^{n}, 0) \to (\mathbf{R}^{2n}, 0)$ of f. Here we assume $F|_{\mathbf{R}^{n-k}\times 0} = f$ and F may not be an immersion accordingly. Thus we consider the "flag" of isotropic mappings:

$$(\mathbf{R}^{n-k}, 0) \xrightarrow{i} (\mathbf{R}^{n}, 0) \xrightarrow{F} (\mathbf{R}^{2n}, 0).$$

Lastly we consider the case of equivariant isotropic mappings.

2. Symplectic bifurcations of plane curves.

Let $f : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ be a map-germ. We define the codimension (more exactly $\mathcal{A}_e$-codimension) of f by

$$\operatorname{codim}(f) := \dim_{\mathbf{R}} \frac{V_f}{tf(V_1) + wf(V_2)},$$

where $V_f := \{v : (\mathbf{R}, 0) \to T\mathbf{R}^2 \,|\, \pi \circ v = f\}$ is the space of vector field-germs along f, V_1 (resp. V_2) is the space of vector field-germs over $(\mathbf{R}, 0)$ (resp. $(\mathbf{R}^2, 0)$), and $tf : V_1 \to V_f$ (resp. $wf : V_2 \to V_f$) is the homomorphism defined by $tf(\xi) := f_*(\xi)$ (resp. $wf(\eta) := \eta \circ f$). A plane curve f is called $\mathcal{A}$-finite if $\operatorname{codim}(f) < \infty$. Then f has an $\mathcal{A}$-versal unfolding with the parameter dimension $\operatorname{codim}(f)$. If f is analytic, the condition of $\mathcal{A}$-finiteness is equivalent to that the complexification of f has an injective representative.

Moreover, in general, we define

$$\text{sp-codim}(f) := \dim_{\mathbf{R}} \frac{V_f}{tf(V_1) + wf(VH_2)},$$

where $VH_2 \subseteq V_2$ means the space of Hamiltonian vector field-germs over the symplectic plane $(\mathbf{R}^2, 0)$. Then clearly

$$\text{sp-codim}(f) \geq \operatorname{codim}(f).$$

In [16] the following is shown:

THEOREM 2.1. *Let $f : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ be an $\mathcal{A}$-finite map-germ.* sp-codim(f) *is an $\mathcal{A}$-invariant (diffeomorphism invariant). In fact we have*

$$\text{sp-codim}(f) = \delta(f) := \dim_{\mathbf{R}} \mathcal{E}_1 / f^* \mathcal{E}_2.$$

Moreover the Milnor number $\mu(f)$ of f is equal to $2\delta(f)$ (cf. [26]). So we have $\text{sp-codim}(f) = \mu(f)/2$.

The classification under $\mathcal{A}$-equivalence of $\mathcal{A}$-simple (0-modal) plane curves is given by Bruce and Gaffney [5]:

$A_{2\ell}$: $t \mapsto (t^2, t^{2\ell+1})$;

$E_{6\ell}$: $t \mapsto (t^3, t^{3\ell+1} \pm t^{3(\ell+p)+2})$, $0 \le p \le \ell - 2$; $t \mapsto (t^3, t^{3\ell+1})$;

$E_{6\ell+2}$: $t \mapsto (t^3, t^{3\ell+2} \pm t^{3(\ell+p)+4})$, $0 \le p \le \ell - 2$; $t \mapsto (t^3, t^{3\ell+2})$;

W_{12}: $t \mapsto (t^4, t^5 \pm t^7)$; $t \mapsto (t^4, t^5)$;

W_{18}: $t \mapsto (t^4, t^7 \pm t^9)$; $t \mapsto (t^4, t^7 \pm t^{13})$; $t \mapsto (t^4, t^7)$;

$W^{\#}_{1,2q-1}$: $t \mapsto (t^4, t^6 + t^{2q+5})$, $q \ge 1$.

Then we have the symplectic classification of them in [16]:

THEOREM 2.2 ([16]).

(1) *Any plane curve germ of type $E_{6\ell}$ ($\ell \ge 2$) is symplectically equivalent to*

$$f_\lambda = \Big(t^3, (\pm 1)^{\ell+1} t^{3\ell+1} + \sum_{j=1}^{\ell-1} \lambda_j t^{3(\ell+j)-1}\Big),$$

for some $\lambda = (\lambda_1, \dots, \lambda_{\ell-1}) \in \mathbf{R}^{\ell-1}$. f_λ and $f_{\lambda'}$ are symplectically equivalent if and only if $\lambda' = (\pm 1)^{\ell-1}\lambda$.

(2) *Any plane curve germ of type $E_{6\ell+2}$ ($\ell \ge 2$) is symplectically equivalent to*

$$f_\lambda = \Big(t^3, (\pm 1)^{\ell} t^{3\ell+2} + \sum_{j=1}^{\ell-1} \lambda_j t^{3(\ell+j)+1}\Big),$$

for some $\lambda = (\lambda_1, \dots, \lambda_{\ell-1}) \in \mathbf{R}^{\ell-1}$. f_λ and $f_{\lambda'}$ are symplectically equivalent if and only if $\lambda' = (\pm 1)^{\ell}\lambda$.

(3) *Any plane curve germ of type W_{12} is symplectically equivalent to*

$$f_\lambda = (t^4, t^5 + \lambda t^7)$$

for some $\lambda \in \mathbf{R}$. Moreover f_λ and $f_{\lambda'}$ are symplectically equivalent if and only if $\lambda' = \lambda$.

(4) *Any plane curve germ of type W_{18} is symplectically equivalent to*

$$f_{\lambda,\mu} = (t^4, t^7 + \lambda t^9 + \mu t^{13})$$

for some $(\lambda, \mu) \in \mathbf{R}^2$. Moreover $f_{\lambda,\mu}$ and $f_{\lambda',\mu'}$ are symplectically equivalent if and only if $(\lambda', \mu') = (\lambda, \mu)$.

(5) *Let $q \ge 1$. Then any plane curve germ of type $W^{\#}_{1,2q-1}$ is symplectically equivalent to*

$$f_{\lambda,\mu} = (t^4, \pm t^6 + \lambda t^{2q+5} + \mu t^{2q+9}),$$

for some $(\lambda, \mu) \in (\mathbf{R} - \{0\}) \times \mathbf{R}$. Moreover $f_{\lambda,\mu}$ and $f_{\lambda',\mu'}$ are symplectically equivalent if and only if $(\lambda', \mu') = \pm(\lambda, \mu)$.

REMARK 2.3. For E_6 and E_8, the symplectomorphism classification and the diffeomorphism classification coincide.

REMARK 2.4. Note that the linear symplectic group $\mathrm{Sp}(\mathbf{R}^2) = \mathrm{SL}(2, \mathbf{R})$ is a normal subgroup of $\mathrm{GL}(2, \mathbf{R})$. If $\mathrm{Symp}(\mathbf{R}^2, 0)$ is a normal subgroup in $\mathrm{Diff}(\mathbf{R}^2, 0)$, then the symplectic classification might be easier. However $\mathrm{Symp}(\mathbf{R}^2, 0)$ is not a normal subgroup in $\mathrm{Diff}(\mathbf{R}^2, 0)$ in fact. For example, define $\tau \in \mathrm{Symp}(\mathbf{R}^2, 0)$ by $\tau(x, y) = (x + y, y)$, and $g \in \mathrm{Diff}(\mathbf{R}^2, 0)$ by $g(x, y) = (\frac{x}{1+x}, y)$. Then $g^{-1}(x, y) = \left(\frac{x}{1-x}, y\right)$ and

$$(g^{-1} \circ \tau \circ g)(x, y) = \left(\frac{x + y(1 + x)}{1 - y(1 + x)}, y\right)$$

is not a symplectomorphism. In fact,

$$(g^{-1} \circ \tau \circ g)^*(dx \wedge dy) = \frac{1}{\big\{1 - y(1 + x)\big\}^2} \, dx \wedge dy \neq dx \wedge dy.$$

The homogeneous space $\mathrm{Diff}(\mathbf{R}^2, 0)/\mathrm{Symp}(\mathbf{R}^2, 0)$ has, via Jacobian, a simple structure (as set):

$$\mathrm{Diff}(\mathbf{R}^2, 0)/\mathrm{Symp}(\mathbf{R}^2, 0) \cong \mathcal{E}_2^{\times},$$

where $\mathcal{E}_2^{\times} = \{h \in \mathcal{E}_2 \mid h(0) \neq 0\}$. This may explain partly the fact observed in Theorem 2.2 that the “symplectic moduli space” has rather simple structure.

In general consider the group $\mathrm{CSymp}(\mathbf{R}^{2n}, 0)$ of conformal symplectomorphisms. Note that $\mathrm{Diff}(\mathbf{R}^2, 0) = \mathrm{CSymp}(\mathbf{R}^2, 0)$. Then

$$\mathrm{CSymp}(\mathbf{R}^{2n}, 0)/\mathrm{Symp}(\mathbf{R}^{2n}, 0) \cong \mathcal{E}_{2n}^{\times}.$$

3. Singular curves in the symplectic space. Arnold [1] gives symplectic classification of singular curves $f : (\mathbf{C}, 0) \to (\mathbf{C}^{2n}, 0)$ with $\mathrm{order}(f) = 2$. This can be applied also to the real case. Also note that any curve-germ $(\mathbf{R}, 0) \to (\mathbf{R}^{2n}, 0)$ is necessarily isotropic.

Now we define the *symplectic codimension* of $f : (\mathbf{R}, 0) \to (\mathbf{R}^{2n}, 0)$ by

$$\text{sp-codim}(f) := \dim_{\mathbf{R}} \frac{VI_f}{tf(V_1) + wf(VH_{2n})},$$

where $VH_{2n} \subseteq V_{2n}$ denotes the space of Hamiltonian vector field-germs over $(\mathbf{R}^{2n}, 0)$. VI_f is the space of infinitesimal isotropic deformations of f.

Then we observe that the symplectic codimension is, in fact, not a diffeomorphism invariant for map-germs $\mathbf{R} \to \mathbf{R}^4$, contrary to the case of symplectic plane curves.

For example, consider map-germs

$$A_{2k,0} : \big(q_1 = t^2, p_1 = t^{2k+1}, q_2 = 0, p_2 = 0\big),$$

and

$$A_{2k,r} : \big(q_1 = t^2, p_1 = t^{2k+1+2r}, q_2 = t^{2k+1}, p_2 = 0\big) \quad (r > 0),$$

from Arnold’s classification [1]. Then, for each $k \geq 1$, all $A_{2k,r}$, $r \geq 0$, are clearly $\mathcal{A}$-equivalent. However we have

$$\text{sp-codim}(A_{2k,0}) = k + 2,$$

and

$$\text{sp-codim}(A_{2k,r}) = k + r + 2.$$

In fact, when $f = A_{2k,0} : (\mathbf{R}, 0) \to (\mathbf{R}^4, 0)$, we can take

$$ {}^t(0, t^{2i-1}, 0, 0) \ (i = 1, 2, \ldots, k), \quad {}^t(0, 0, t, 0), \quad {}^t(0, 0, 0, t) $$

as a basis of the vector space $V_f/\big(tf(V_1) + wf(VH_4)\big)$. For $f = A_{2k,r}$ we need

$$ {}^t(0, t^{2k+2j-1}, 0, 0) \quad (j = 1, 2, \ldots, r) $$

in addition.

Here we do not mention the details of the symplectic classification of curves in $(\mathbf{R}^{2n}, 0)$. However we introduce several natural equivalence groups for a given coisotropic fibration $\pi : (\mathbf{R}^{2n}, 0) \to (\mathbf{R}^{n-\ell}, 0)$.

We denote by π-Symp$(\mathbf{R}^{2n}, 0)$ the group consisting of π-fiber preserving symplectomorphisms $\Phi : (\mathbf{R}^{2n}, 0) \to (\mathbf{R}^{2n}, 0)$. In the similar way, for π-fiber preserving positive-conformally symplectic diffeomorphisms (resp. π-fiber and orientation preserving diffeomorphisms, π-fiber and volume preserving diffeomorphisms), we define the group π-C$^+$Symp$(\mathbf{R}^{2n}, 0)$ (resp. π-Diff$^+(\mathbf{R}^{2n}, 0)$, π-VPDiff$(\mathbf{R}^{2n}, 0)$).

Then we have the square of inclusions:

$$
\begin{array}{ccccc}
 & & \pi\text{-C}^+\mathrm{Symp}(\mathbf{R}^{2n},0) & & \\
 & \nearrow & & \searrow & \\
\pi\text{-Symp}(\mathbf{R}^{2n},0) & & & & \pi\text{-Diff}^+(\mathbf{R}^{2n},0) \\
 & \searrow & & \nearrow & \\
 & & \pi\text{-VPDiff}(\mathbf{R}^{2n},0) & &
\end{array}
$$

Note that, in the case $n = 1$,

$$ \pi\text{-Symp}(\mathbf{R}^2, 0) = \pi\text{-VPDiff}(\mathbf{R}^2, 0), \quad \pi\text{-C}^+\mathrm{Symp}(\mathbf{R}^2, 0) = \pi\text{-Diff}^+(\mathbf{R}^2, 0). $$

4. Isotropic immersions in Lagrange equivalences. Now we consider the case of proper isotropic immersions

$$ i^{n-k} : \mathbf{R}^{n-k} \to (T^*\mathbf{R}^n, \omega), \quad 1 \le k \le n-1, $$

classified with respect to the standard equivalency by symplectomorphisms

$$ \Phi : (T^*\mathbf{R}^n, 0) \to (T^*\mathbf{R}^n, 0), $$

preserving π-fibers up to a parametrization, where the coisotropic fibration $\pi : T^*\mathbf{R}^n \to \mathbf{R}^n$ is a Lagrangian fibration. An each immersed isotropic submanifold-germ $(I^{n-k}, 0) \subset T^*\mathbf{R}^n$ is generated by a smooth generating I-Morse family-germ (cf. [20]) $F : (\mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^k, 0) \to \mathbf{R}$, such that the map-germ

$$ \mathbf{R}^n \times \mathbf{R}^m \ni (q, \lambda) \mapsto \Big(\frac{\partial F}{\partial \lambda}(q, \lambda, 0), \frac{\partial F}{\partial \beta}(q, \lambda, 0)\Big) \in \mathbf{R}^m \times \mathbf{R}^k $$

has exactly rank equal to $m + k$ at 0. Then it is easy to check that

$$ (1) \qquad i^{n-k} : (q, \lambda)|_{\Sigma_F^I} \to \Big(\frac{\partial F}{\partial p}(q, \lambda, 0), q\Big) \in T^*\mathbf{R}^n, $$

where

$$ \Sigma_F^I = \Big\{(q, \lambda) : \frac{\partial F}{\partial \beta}(q, \lambda, 0) = 0 = \frac{\partial F}{\partial \lambda}(q, \lambda, 0)\Big\} \equiv (\mathbf{R}^{n-k}, 0) $$

is an isotropic immersion and $i^{n-k}(\mathbf{R}^{n-k},0) = (I^{n-k},0)$ is an isotropic submanifold-germ. The image $C_I = \pi(I^{n-k})$ was introduced in [20] as a quasicaustic in an aperture diffraction. By this formulation we also see that the isotropic submanifold-germs can be represented as a sub-Lagrangian-germs, i.e., we have a smooth Lagrangian submanifold-germ $\overline{L} \subset T^*(\mathbf{R}^n \times \mathbf{R}^m)$ generated by the function-germ $(q,\lambda) \to f(q,\lambda)$, and the reduction $\pi_C : C \to T^*\mathbf{R}^n$ in $T^*(\mathbf{R}^n \times \mathbf{R}^m)$ defined by the coisotropic space $C = \{(p,q;\mu,\lambda) : \mu = 0\}$. Let $g : (q,\lambda) \to (g_1,\ldots,g_k) \in \mathbf{R}^k$ be a smooth map-germ (coming from the extra k equations of the family F: $g_i(q,\lambda) = \frac{\partial F}{\partial \beta_i}(q,\lambda,0) = 0$). Then I-Morse family-germ implies the transversality of intersection of C with $\overline{L} \cap g^{-1}(0)$ and the image $\pi_C(\bar{L} \cap g^{-1}(0) \cap C)$ is a smooth isotropic germ. All isotropic submanifold-germs can be represented in this way (cf. [20]). To establish the classes of generating I-Morse family-germs which generate Lagrangian equivalent isotropic varieties, to each generating family-germ $(q,\lambda,\beta) \to F(q,\lambda,\beta)$ we prescribe the inactive domain, i.e. the set of function-germs

$$U_F = F + \left\langle \frac{\partial F}{\partial \lambda}, \frac{\partial F}{\partial \beta} \right\rangle^2_{\mathcal{E}_{(q,\lambda,\beta)}} + \langle \beta_1,\ldots,\beta_k \rangle^2_{\mathcal{E}_{(q,\lambda,\beta)}} \ ,$$

which generates as an I-Morse family-germ the same isotropic immersion-germ.

We say that two I-Morse family-germs, $F_1, F_2 : (\mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^k, 0) \to \mathbf{R}$, are R^+-equivalent if there is a diffeomorphism $\Phi : (\mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^k, 0) \to (\mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^k, 0)$ preserving the fibration $\pi_n : (\mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^k, 0) \to (\mathbf{R}^n, 0)$ and such that $\pi_{m+k} \circ \Phi : (\mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^k, 0) \to (\mathbf{R}^m \times \mathbf{R}^k, 0)$ is a q-parametrized family of diffeomorphism-germs on $(\mathbf{R}^m \times \mathbf{R}^k, 0)$ preserving the hypersurface $\{\beta = 0\}$. This group of R^+-equivalences preserves the set U_F. Now we can define the equivalency of I-Morse family-germs representing uniquely the corresponding isotropic immersion-germs. We say that F_1 and F_2 are I-equivalent if there is an element of U_{F_2} which is R^+-equivalent to F_1 and oppositely there is an element of U_{F_1} which is R^+-equivalent to F_2. We say that I-equivalency of I-Morse family-germs is equivalent to the Lagrangian equivalency of the corresponding isotropic immersion-germs.

If we pass to the representing pairs (f,g), then the inactive domains are defined by

$$U_{(f,g)} = f + \left\langle \frac{\partial f}{\partial \lambda}, g \right\rangle^2_{\mathcal{E}_{(q,\lambda)}} ,$$

where $f(q,\lambda) = F(q,\lambda,0)$ and $g_i(q,\lambda) = \frac{\partial F}{\partial \beta}(q,\lambda,0)$. The R^+-equivalence of I-Morse family-germs reduces to the ordinary R^+-equivalence of representing pairs (f,g). Following the methods of [20] and the announced completed classification of [29] we may formulate the following proposition.

Proposition 4.1. *The stable simple isotropic immersion germs, $i^{n-k} : \mathbf{R}^{n-k} \to (T^*\mathbf{R}^n, \omega)$, with the singular quasicaustic C_I are Lagrangian equivalent to the isotropic immersion germs (or their suspensions) generated by the I-Morse family-germs from the following list, or have $k \geq 2$, non-zero 4-jet of $F|_{q=0,\beta=0}$ and non-zero 3-jet of $g_i|_{q=0}$, $i = 1,\ldots,k$.*

1. $$F=\pm\lambda^{s+1}+\sum_{j=1}^{s-1}\lambda^j q_j+\sum_{i=1}^{k}\beta_i\Big(\lambda^l+\sum_{j=0}^{l-1}\lambda^j q_{s+j+(i-1)l}\Big),$$
$$(s+1)/2\le l\le s,\quad s\ge 2,\quad n=kl+s-1;$$

2. $$F=\lambda_1^2\lambda_2\pm\lambda_2^{s-1}+\sum_{j=1}^{s-2}\lambda_2^j q_j+\lambda_1 q_{s-1}+\sum_{i=1}^{k}\beta_i\Big(\lambda_2^l+\sum_{j=0}^{l-1}\lambda_2^j q_{s+j+(i-1)l}\Big),$$
$$s\ge 4,\quad n=kl+s-1;$$

3. $$F=\lambda_1^3\pm\lambda_2^4+q_5\lambda_2^2\lambda_1+q_4\lambda_2\lambda_1+q_3\lambda_2^2+q_2\lambda_1+q_1\lambda_2$$
$$+\sum_{i=1}^{k}\beta_i\big(\lambda_1\lambda_2+\lambda_2^2+\lambda_1 q_{5+3i}+\lambda_2 q_{4+3i}+q_{3+3i}\big),$$
$$n=5+3k;$$

4. $$F=\lambda_1^3\pm\lambda_2^4+q_5\lambda_2^2\lambda_1+q_4\lambda_2\lambda_1+q_3\lambda_2^2+q_2\lambda_1+q_1\lambda_2$$
$$+\sum_{i=1}^{k}\beta_i\big(\lambda_1\lambda_2+\lambda_2^2 q_{5+4i}+\lambda_1 q_{4+4i}+\lambda_2 q_{3+4i}+q_{2+4i}\big),$$
$$n=5+4k;$$

5. $$F=\lambda_1^3\pm\lambda_2^4+q_5\lambda_2^2\lambda_1+q_4\lambda_2\lambda_1+q_3\lambda_2^2+q_2\lambda_1+q_1\lambda_2$$
$$+\sum_{i=1}^{k}\beta_i\big(\lambda_1\lambda_2 q_{5+4i}+\lambda_2^2+\lambda_1 q_{4+4i}+\lambda_2 q_{3+4i}+q_{2+4i}\big),$$
$$n=5+4k;$$

6. $$F=\lambda_1^3\pm\lambda_2^4+q_5\lambda_2^2\lambda_1+q_4\lambda_2\lambda_1+q_3\lambda_2^2+q_2\lambda_1+q_1\lambda_2$$
$$+\sum_{i=1}^{k}\beta_i\big(\lambda_1\lambda_2^2+\lambda_1\lambda_2 q_{5+5i}+\lambda_2^2 q_{4+5i}+\lambda_1 q_{3+5i}+\lambda_2 q_{2+5i}+q_{1+5i}\big),$$
$$n=5+5k;$$

7. $$F=\lambda_1^3\pm\lambda_2^4+q_5\lambda_2^2\lambda_1+q_4\lambda_2\lambda_1+q_3\lambda_2^2+q_2\lambda_1+q_1\lambda_2$$
$$+\sum_{i=1}^{k}\beta_i\big(\lambda_1\lambda_2^2 q_{5+6i}+\lambda_1\lambda_2 q_{4+6i}+\lambda_2^2 q_{3+6i}+\lambda_1 q_{2+6i}+\lambda_2 q_{1+6i}+q_{6i}\big),$$
$$n=5+6k;$$

8. $$F=\lambda_1^3+\lambda_1\lambda_2^3+q_6\lambda_2^4+q_5\lambda_2^3+q_4\lambda_2\lambda_1+q_3\lambda_2^2+q_2\lambda_1+q_1\lambda_2$$
$$+\sum_{i=1}^{k}\beta_i\big(\lambda_1\lambda_2+\lambda_2^2 q_{6+4i}+\lambda_1 q_{5+4i}+\lambda_2 q_{4+4i}+q_{3+4i}\big),$$
$$n=6+4k;$$

9. $$F=\lambda_1^3+\lambda_1\lambda_2^3+q_6\lambda_2^4+q_5\lambda_2^3+q_4\lambda_2\lambda_1+q_3\lambda_2^2+q_2\lambda_1+q_1\lambda_2$$
$$+\sum_{i=1}^{k}\beta_i\big(\lambda_2^3+\lambda_1\lambda_2 q_{6+5i}+\lambda_2^2 q_{5+5i}+\lambda_1 q_{4+5i}+\lambda_2 q_{3+5i}+q_{2+5i}\big),$$
$$n=6+5k;$$

10. $$F = \lambda_1^3 + \lambda_1\lambda_2^3 + q_6\lambda_2^4 + q_5\lambda_2^3 + q_4\lambda_2\lambda_1 + q_3\lambda_2^2 + q_2\lambda_1 + q_1\lambda_2 + \sum_{i=1}^{k} \beta_i\big(\lambda_2^4 + \lambda_2^3 q_{6+6i} + \lambda_1\lambda_2 q_{5+6i} + \lambda_2^2 q_{4+6i} + \lambda_1 q_{3+6i} + \lambda_2 q_{2+6i} + q_{1+6i}\big),$$
$n = 6 + 6k$;

11. $$F = \lambda_1^3 + \lambda_1\lambda_2^3 + q_6\lambda_2^4 + q_5\lambda_2^3 + q_4\lambda_2\lambda_1 + q_3\lambda_2^2 + q_2\lambda_1 + q_1\lambda_2 + \sum_{i=1}^{k} \beta_i\big(\lambda_2^4 q_{6+7i} + \lambda_2^3 q_{5+7i} + \lambda_1\lambda_2 q_{4+7i} + \lambda_2^2 q_{3+7i} + \lambda_1 q_{2+7i} + \lambda_2 q_{1+7i} + q_{7i}\big),$$
$n = 6 + 7k$;

12. $$F = \lambda_1^3 + \lambda_2^5 + q_7\lambda_2^3\lambda_1 + q_6\lambda_2^2\lambda_1 + q_5\lambda_2^3 + q_4\lambda_1\lambda_2 + q_3\lambda_2^2 + q_2\lambda_1 + q_1\lambda_2 + \sum_{i=1}^{k} \beta_i\big(\lambda_1\lambda_2 + \lambda_2^2 q_{7+4i} + \lambda_1 q_{6+4i} + \lambda_2 q_{5+4i} + q_{4+4i}\big),$$
$n = 7 + 4k$;

13. $$F = \lambda_1^3 + \lambda_2^5 + q_7\lambda_2^3\lambda_1 + q_6\lambda_2^2\lambda_1 + q_5\lambda_2^3 + q_4\lambda_1\lambda_2 + q_3\lambda_2^2 + q_2\lambda_1 + q_1\lambda_2 + \sum_{i=1}^{k} \beta_i\big(\lambda_2^3 + \lambda_1\lambda_2 q_{7+5i} + \lambda_2^2 q_{6+5i} + \lambda_1 q_{5+5i} + \lambda_2 q_{4+5i} + q_{3+5i}\big),$$
$n = 7 + 5k$;

14. $$F = \lambda_1^3 + \lambda_2^5 + q_7\lambda_2^3\lambda_1 + q_6\lambda_2^2\lambda_1 + q_5\lambda_2^3 + q_4\lambda_1\lambda_2 + q_3\lambda_2^2 + q_2\lambda_1 + q_1\lambda_2 + \sum_{i=1}^{k} \beta_i\big(\lambda_1\lambda_2^2 + \lambda_2^3 q_{7+6i} + \lambda_1\lambda_2 q_{6+6i} + \lambda_2^2 q_{5+6i} + \lambda_1 q_{4+6i} + \lambda_2 q_{3+6i} + q_{2+6i}\big),$$
$n = 7 + 6k$;

15. $$F = \lambda_1^3 + \lambda_2^5 + q_7\lambda_2^3\lambda_1 + q_6\lambda_2^2\lambda_1 + q_5\lambda_2^3 + q_4\lambda_1\lambda_2 + q_3\lambda_2^2 + q_2\lambda_1 + q_1\lambda_2 + \sum_{i=1}^{k} \beta_i\big(\lambda_1\lambda_2^3 + \lambda_1\lambda_2^2 q_{7+7i} + \lambda_2^3 q_{6+7i} + \lambda_1\lambda_2 q_{5+7i} + \lambda_2^2 q_{4+7i} + \lambda_1 q_{3+7i} + \lambda_2 q_{2+7i} + q_{1+7i}\big),$$
$n = 7 + 7k$;

16. $$F = \lambda_1^3 + \lambda_2^5 + q_7\lambda_2^3\lambda_1 + q_6\lambda_2^2\lambda_1 + q_5\lambda_2^3 + q_4\lambda_1\lambda_2 + q_3\lambda_2^2 + q_2\lambda_1 + q_1\lambda_2 + \sum_{i=1}^{k} \beta_i\big(\lambda_1\lambda_2^3 q_{7+8i} + \lambda_1\lambda_2^2 q_{6+8i} + \lambda_2^3 q_{5+8i} + \lambda_1\lambda_2 q_{4+8i} + \lambda_2^2 q_{3+8i} + \lambda_1 q_{2+8i} + \lambda_2 q_{1+8i} + q_{8i}\big),$$
$n = 7 + 8k$.

Note that there exists another natural class of immersions in a symplectic space: coisotropic immersions. Though we treat isotropic mappings in this paper mainly, we recall here the basic construction of coisotropic immersions briefly.

Let $c^{n+k} : (\mathbf{R}^{n+k}, 0) \to (T^*\mathbf{R}^n, \omega)$ be a coisotropic immersion-germ ($1 \le k \le n-1$), classified with respect to the standard Lagrangian equivalence. Coisotropic submanifold-germs are generated by the corresponding generating families (cf. [19]) called C-Morse families. To each coisotropic immersed submanifold-germ $(C, 0) \subset (T^*\mathbf{R}^n, \omega)$ there exists a C-Morse family-germ $F : (\mathbf{R}^n \times \mathbf{R}^k \times \mathbf{R}^m, 0) \to \mathbf{R}$, such that the smooth map-germ

$$\mathbf{R}^n \times \mathbf{R}^k \times \mathbf{R}^m \ni (q, \alpha, \lambda) \mapsto \Big(\frac{\partial F}{\partial \alpha}(q, \alpha, \lambda), \alpha\Big)$$

is regular on the stationary set

$$\Sigma_F^C = \Big\{(q, \alpha, \lambda) : \frac{\partial F}{\partial \lambda}(q, \alpha, \lambda) = 0\Big\}$$

and

$$\Sigma_F^C \ni (q, \alpha, \lambda) \mapsto \Big(\frac{\partial F}{\partial q}(q, \alpha, \lambda), q\Big)$$

is a coisotropic immersion-germ.

We leave the classification of the coisotropic singularities to the forthcoming paper.

5. Singular isotropic mappings. In this section, we first observe that any family of curves produces an isotropic mapping of corank not greater than 1.

Let $\Psi : \big(\mathbf{R} \times \mathbf{R}^{n-k-1}, (0,0)\big) \to T^*\mathbf{R}^{k+1}$ be a family of curves:

$$\Psi(t, \lambda) = \psi_\lambda(t) = \big(a_1(t, \lambda), \ldots, a_{k+1}(t, \lambda), b_1(t, \lambda), \ldots, b_{k+1}(t, \lambda)\big).$$

Set

$$e(t, \lambda) := \int_0^t \Big(a_1 \frac{\partial b_1}{\partial t} + \ldots + a_{k+1} \frac{\partial b_{k+1}}{\partial t}\Big) dt.$$

Then we define $\widetilde{\Psi} : \mathbf{R} \times \mathbf{R}^{n-k-1} \to T^*\mathbf{R}^n$ by

$$(p \circ \widetilde{\Psi})(t, \lambda) = a(t, \lambda), \quad (q \circ \widetilde{\Psi})(t, \lambda) = b(t, \lambda), \quad (q' \circ \widetilde{\Psi})(t, \lambda) = \lambda,$$

and

$$(p' \circ \widetilde{\Psi})(t, \lambda) = \big(c_1(t, \lambda), \ldots, c_{n-k-1}(t, \lambda)\big),$$

where, for $1 \le j \le n-k-1$,

$$c_j(t, \lambda) = \int_0^t \Big(\sum_{i=1}^{k+1} \frac{\partial a_i}{\partial \lambda_j} \frac{\partial b_i}{\partial t} - \frac{\partial a_i}{\partial t} \frac{\partial b_i}{\partial \lambda_j}\Big) dt.$$

In fact, the p'-components are given by the condition

$$a_1\, db_1 + \ldots + a_{k+1}\, db_{k+1} + c_1\, d\lambda_1 + \ldots + c_{n-k-1}\, d\lambda_{n-k-1} = de,$$

namely by

$$a_1 \frac{\partial b_1}{\partial \lambda_j} + \ldots + a_{k+1} \frac{\partial b_{k+1}}{\partial \lambda_j} + c_j = \frac{\partial e}{\partial \lambda_j}\,.$$

Then $\widetilde{\Psi}$ is an isotropic mapping uniquely determined from Ψ up to symplectomorphisms. We call $\widetilde{\Psi}$ the *isotropic lifting* of Ψ (cf. [16]).

By the same construction as above, we see that any isotropic unfolding $\Psi : (\mathbf{R}^m \times \mathbf{R}^s, 0) \to (\mathbf{R}^{2n} \times \mathbf{R}^s, 0)$ of corank not greater than 1, lifts to an isotropic mapping $\widetilde{\Psi} : (\mathbf{R}^{m+s}, 0) \to (\mathbf{R}^{2(n+s)}, 0)$. In fact, since Ψ is an unfolding by isotropic germs, setting

$$\Psi(x, \lambda) = \big(a_1(x, \lambda), \ldots, a_n(x, \lambda), b_1(x, \lambda), \ldots, b_n(x, \lambda), \lambda\big),$$

we see that there exists a function $e(x,\lambda)$ satisfying

$$\sum_{i=1}^{n} a_i \frac{\partial b_i}{\partial x_j} = \frac{\partial e}{\partial x_j},$$

for each j $(1 \le j \le m)$. We set

$$c_k(x,\lambda) := \frac{\partial e}{\partial \lambda_k} - \sum_{i=1}^{n} a_i \frac{\partial b_i}{\partial \lambda_k},$$

for each k $(1 \le k \le s)$. Then we have the isotropic lifting of Ψ by setting

$$\widetilde{\Psi}(x,\lambda) = \big(a(x,\lambda), b(x,\lambda), \lambda, c(x,\lambda)\big).$$

Let $\pi : (T^*\mathbf{R}^n, 0) = (\mathbf{R}^{2n}, 0) \to (\mathbf{R}^{n-\ell}, 0)$ be a coisotropic fibration. We suppose that $\pi(p,q) = (q_{\ell+1}, \ldots, q_n) =: \bar{q}$. Let r be an integer with $0 \le r \le n - \ell$. Then set

$$q = (q', \bar{q}), \quad q' = (q_1, \ldots, q_\ell), \quad q'' = (q_{\ell+1}, \ldots, q_{n-r}), \quad q''' = (q_{n-r+1}, \ldots, q_n),$$

so that $\bar{q} = (q'', q''') = (q_{\ell+1}, \ldots, q_n)$. Similarly we decompose $p = (p', \bar{p}) = (p', p'', p''')$. We define $\pi_0 : (\mathbf{R}^{2(n-r)}, 0) \to (\mathbf{R}^{n-\ell-r}, 0)$ by $\pi_0(p', p'', q', q'') = q''$.

Then, as a converse of the lifting construction, we have a kind of "rank theorem":

PROPOSITION 5.1. *Let $f : (\mathbf{R}^{n-k}, 0) \to (\mathbf{R}^{2n}, 0)$ be an isotropic map-germ. Assume $k \le \ell$ and* $\operatorname{rank}(\pi \circ f) = r\,(\le n - \ell)$. *Then f is π-equivalent to an isotropic unfolding of $f_0 : (\mathbf{R}^{n-k-r}, 0) \to (\mathbf{R}^{2(n-r)}, 0)$ with the coisotropic fibration $\pi_0 : (\mathbf{R}^{2(n-r)}, 0) \to (\mathbf{R}^{n-\ell-r}, 0)$.*

Next we remark that the construction of isotropic jet spaces discussed in the case $k = 0$ in [13] works also for the general case, and the isotropic transversality theorem holds as well.

Let X be an $(n-k)$-dimensional manifold, M a symplectic $2n$-dimensional manifold. We set

$$J_I^r(X, M) := \big\{ j^r f(x) \,|\, f : (X, x) \to M \text{ is isotropic} \big\} \subseteq J^r(X, M),$$

and

$$R^r(X, M) := J_I^r(X, M) \setminus \overline{\Sigma^2}(X, M) \subset J^r(X, M).$$

Here $\overline{\Sigma^2}(X, M)$ denotes the set of isotropic jets with corank not smaller than 2.

We set

$$J_I^r(n-k, 2n) := \{ j^f(0) \,|\, f : (\mathbf{R}^{n-k}, 0) \to \mathbf{R}^{2n} \text{ is isotropic} \} \subseteq J^r(n-k, 2n),$$

and

$$R^r(n-k, 2n) := \{ j^f(0) \,|\, f : (\mathbf{R}^{n-k}, 0) \to \mathbf{R}^{2n} \text{ is isotropic of corank at most } 1 \}.$$

PROPOSITION 5.2. *$R^r(X, M)$ is a submanifold of $J^r(X, M)$ of codimension*

$$(n-k) \cdot \binom{n-k+r}{r} - \binom{n-k+r+1}{r+1} + 1.$$

Proof. Set $m = n - k$. Since the assertion is a local one, it is sufficient to show it in the case $X = \mathbf{R}^m$ and $M = \mathbf{R}^{2n}$. Moreover, since

$$R^r(\mathbf{R}^m, \mathbf{R}^{2n}) = R^r(m, 2n) \times \mathbf{R}^m \times \mathbf{R}^{2n},$$

under the trivialization $J^r(\mathbf{R}^m, \mathbf{R}^{2n}) = J^r(m, 2n) \times \mathbf{R}^m \times \mathbf{R}^{2n}$, it is sufficient to show the assertion for $R^r(m, 2n) \subseteq J^r(m, 2n)$. Set $V^r = J^r(m, 2n)$ and denote by Λ^r the space of $(r-1)$-jets of closed 2-forms on $(\mathbf{R}^m, 0)$. Define $\rho : V^r \to \Lambda^r$ by $\rho(j^r\phi(0)) = j^{r-1}(\phi^*\omega)(0)$.

It is sufficient to show that $\rho : V^r \to \Lambda^r$ is a submersion along $(\Sigma^0 \cup \Sigma^1)(m, 2n)$, since $\rho^{-1}(0) = J^r_I(m, 2n)$. Set $z = j^r\phi(0)$ and

$$\phi = (P, Q) = (P_1, \ldots, P_n, Q_1, \ldots, Q_n).$$

We set $\phi_t = (P + t\widetilde{P}, Q + t\widetilde{Q})$. Then

$$\frac{d}{dt}\phi_t^*\omega|_{t=0} = \sum_{i=1}^n d\widetilde{P}_i \wedge dQ_i - \sum_{i=1}^n d\widetilde{Q_i} \wedge dP_i = d\Big(\sum_{i=1}^n \widetilde{P}_i\, dQ_i - \widetilde{Q_i}\, dP_i\Big).$$

Then, for any 1-form E on $\mathbf{R}^m$, we find $\widetilde{P}_i, \widetilde{Q_i}$ and e in $\mathcal{E}_m$ with

$$\sum_{i=1}^n \widetilde{P}_i\, dQ_i - \widetilde{Q_i}\, dP_i = E + de.$$

Set $Q = \big(x', u_1(x', x_m), \ldots, u_{n-m+1}(x', x_m)\big)$, where $x' = (x_1, \ldots, x_{m-1})$. Then, setting $E = \sum_{i=1}^m E_i\, dx_i$, $\widetilde{Q} = 0$, $\widetilde{P}_m = 0, \ldots, \widetilde{P}_n = 0$, we obtain

$$\sum_{i=1}^{m-1} \Big(\widetilde{P}_i - E_i - \frac{\partial e}{\partial x_i}\Big) dx_i - \Big(E_m + \frac{\partial e}{\partial x_m}\Big) dx_m = 0.$$

Then we can find e and then consequently find $\widetilde{P}_i$, $i = 1, \ldots, m-1$. Thus we see that ρ_* is surjective. ■

For an $(n-k)$-manifold X and for a symplectic $2n$-manifold M, we denote by $C^\infty_I(X, M)^1$ the set of C^∞ isotropic mappings $f : X \to M$ of corank not greater than 1 everywhere on X, endowed with Whitney C^∞ topology.

In [13], we prove "isotropic transversality theorem" in the case $k = 0$. The same proof works also for $k > 0$:

Proposition 5.3. *Let X be an $(n-k)$-manifold, M a symplectic $2n$-manifold, r a non-negative integer and U a locally finite family of submanifolds of $R^r(X, M)$. Then the subspace*

$$T_U := \{f \in C^\infty_I(X, M)^1 \,|\, j^r f \text{ is transverse to all elements of } U\}$$

is dense in $C^\infty_I(X, M)^1$.

Next we give a remark on "isotropic Thom-Boardman singularity". We define, for a sufficiently large r,

$$\Sigma^1_I(n-k, 2n) := \Sigma^1(n-k, 2n) \cap J^r_I(n-k, 2n),$$

and

$$\Sigma^{1_s}_I(n-k, 2n) := \Sigma^{1_s}(n-k, 2n) \cap J^r_I(n-k, 2n).$$

Here 1_s means the s time iteration of 1, and $s \le r$. Moreover $\Sigma^1_{2,I}(n-k, 2n; \ell)$ denotes the subset of $J^r_I(n-k, 2n)$ consisting of isotropic r-jets $j^r f(0)$ with $\operatorname{corank}_0(\pi \circ f) = 2$.

Proposition 5.4. $\operatorname{codim} \Sigma^1_I(n-k, 2n) = 2(k+1)$. *In general* $\operatorname{codim} \Sigma^{1_s}_I(n-k, 2n) = 2(k+1)s$. *Suppose* $0 \le \ell \le k$. *Then* $\operatorname{codim} \Sigma^1_{2,I}(n-k, 2n; \ell) = 3k - \ell + 4$.

By the isotropic transversality theorem (Proposition 5.3), we see that there exists an isotropic mapping $\mathbf{R}^7 \to T^*\mathbf{R}^8 = \mathbf{R}^{16}$ intersecting $\Sigma_2^1(7,16;0)$ transversely at the origin. Also we see that there exists an isotropic mapping $\mathbf{R}^6 \to T^*\mathbf{R}^7 = \mathbf{R}^{14}$ intersecting $\Sigma_2^1(6,14;1)$ transversely at the origin.

Lastly, in this section, we prepare several infinitesimal notions.

For the infinitesimal study on isotropic deformations, we need to consider the canonical lifting $\widetilde{\omega}$ on $T\mathbf{R}^{2n}$ of the symplectic form ω on $\mathbf{R}^{2n}$. In fact we set

$$\widetilde{\omega} = \sum_{i=1}^{n} (d\varphi_i \wedge dq_i + dp_i \wedge d\kappa_i),$$

where (p, q, φ, κ) is the coordinate system on $T\mathbf{R}^{2n} = \mathbf{R}^{4n}$ defined by $\varphi_i(v) = v(dp_i)$ and $\kappa_i(v) = v(dq_i)$ for $v \in T\mathbf{R}^{2n}$. Then $\widetilde{\omega} = d\theta^{\#}$, where

$$\theta^{\#} = \sum_{i=1}^{n} (\varphi_i \, dq_i - \kappa_i \, dp_i).$$

Let $f : (\mathbf{R}^{n-k}, 0) \to \mathbf{R}^{2n}$ be an isotropic map-germ. Then we set, as the space of infinitesimal isotropic deformations of f, at least formally,

$$VI_f := \left\{ v : (\mathbf{R}^{n-k}, 0) \to T\mathbf{R}^{2n} \,\middle|\, \Pi \circ v = f, \ v^*\widetilde{\omega} = 0 \right\},$$

where Π is the canonical projection $\Pi : T\mathbf{R}^{2n} \to \mathbf{R}^{2n}$.

For a $v \in VI_f$, we have $0 = v^*\widetilde{\omega} = d(v^*\theta^{\#})$. Then there exists a function $e \in \mathcal{E}_{n-k}$, which is called a *generating function*, satisfying $de = v^*\theta^{\#}$, up to a constant. We set

$$R_f := \{ h \in \mathcal{E}_{n-k} \mid dh \in \mathcal{E}_{n-k} \, df \}.$$

Then any generating function of a vector field in VI_f belongs to R_f. Thus we define the linear mapping $e : VI_f \to R_f/\mathbf{R}$ by taking a generating function up to constant.

We denote by V_{n-k} (resp. VH_{2n}) the space of vector field-germs over $(\mathbf{R}^{n-k}, 0)$ (resp. the space of Hamiltonian vector field-germs over $(\mathbf{R}^{2n}, 0)$). Then we define the basic operations $tf : V_{n-k} \to VI_f$ and $wf : VH_{2n} \to VI_f$ induced by f in the infinitesimal way: $tf(\xi) := f_*(\xi)$ and $wf(\eta) := \eta \circ f$, where $\xi \in V_{n-k}, \eta \in VH_{2n}$ and $f_* : T\mathbf{R}^{n-k} \to T\mathbf{R}^{2n}$ being the differential of f.

Note that $H \circ f$ is a generating function of $wf(X_H)$, where X_H is the Hamiltonian vector field with Hamiltonian $H \in \mathcal{E}_{2n}$. Moreover we have $(tf(\xi))^*(\theta^{\#}) = 0$, and therefore $e(tf(\xi)) = 0$, for any $\xi \in V_{n-k}$.

We define $VL_\pi \subseteq VH_{2n}$, the subspace consisting of Hamiltonian vector fields which project to vector fields over $(\mathbf{R}^{n-\ell}, 0)$, via the differential $\pi_* : T\mathbf{R}^{2n} \to T\mathbf{R}^{n-\ell}$ of the given coisotropic fibration $\pi : \mathbf{R}^{2n} \to \mathbf{R}^{n-\ell}$. Then we have the following basic result:

PROPOSITION 5.5. *A Hamiltonian vector field $\eta \in VH_{2n}$ projects to a vector field over $(\mathbf{R}^{n-\ell}, 0)$ via $\pi_* : T\mathbf{R}^{2n} \to T\mathbf{R}^{n-\ell}$ if and only if η has a Hamiltonian function H which is inhomogeneous linear with respect to π:*

$$H(p, q) = a_0(p', q) + a_{\ell+1}(\bar{q})p_{\ell+1} + \ldots + a_n(\bar{q})p_n,$$

where $p' = (p_1, \ldots, p_\ell)$, $\bar{q} = \pi(p, q) = (q_{\ell+1}, \ldots, q_n)$, and $a_{\ell+1}, \ldots, a_n$ are smooth functions.

Proof. Let $H = H(p, q)$ be a Hamiltonian of η; $-dH = i_\eta\omega$. Then

$$\eta = -\frac{\partial H}{\partial q'}\frac{\partial}{\partial p'} - \frac{\partial H}{\partial \bar{q}}\frac{\partial}{\partial \bar{p}} + \frac{\partial H}{\partial p'}\frac{\partial}{\partial q'} + \frac{\partial H}{\partial \bar{p}}\frac{\partial}{\partial \bar{q}}.$$

The condition is that $\frac{\partial H}{\partial \bar{p}}$ depends only on $\bar{q}$. Set $\frac{\partial H}{\partial p_i} = a_i(\bar{q})$, $\ell + 1 \le i \le n$. Then we have the required result. ∎

We denote by $\mathcal{H}_\pi$ the set of inhomogeneous linear functions with respect to π.

By taking generating functions, we have defined the linear mapping $e : VI_f \to R_f/\mathbf{R}$.

Lemma 5.6 (Basic exact sequence). *The linear mapping $e : VI_f \to R_f/\mathbf{R}$ induces the following exact sequence*:

$$0 \to \frac{VI'_f}{tf(V_{n-k}) + wf(VL'_{f,\pi})} \to \frac{VI_f}{tf(V_{n-k}) + wf(VL_\pi)} \to \frac{R_f}{f^*(\mathcal{H}_\pi)} \to 0,$$

where

$$VI'_f := \{v \in VI_f \mid v^*\theta^\# = 0\},$$

and

$$VL'_{f,\pi} := \{X_H \in VL_\pi \mid H \circ f = 0\}.$$

Now we define the *symplectic codimension* of an isotropic map-germ $f : (\mathbf{R}^{n-k}, 0) \to (\mathbf{R}^{2n}, 0)$ with respect to a coisotropic fibration $\pi : (\mathbf{R}^{2n}, 0) \to (\mathbf{R}^{n-\ell}, 0)$, from the infinitesimal aspect, by

$$\pi\text{-sp-codim}(f) = \dim_{\mathbf{R}} \frac{VI_f}{tf(V_{n-k}) + wf(VL_\pi)}.$$

Simply f is called π-*stable* if π-sp-codim$(f) = 0$, namely if $VI_f = tf(V_{n-k}) + wf(VL_\pi)$. This stability condition coincides with the stability for isotropic deformations under π-equivalence, at least if corank$(f) \le 1$.

In the case $\ell = 0$, namely in the case that π is a Lagrange fibration, f is called *Lagrange stable* if f is π-stable. Lagrange stable immersions are classified by means of their generating families ([3]). Lagrange stable mappings of corank not greater than 1 are classified partly in [4].

If $\ell = n$, then we omit π:

$$\text{sp-codim}(f) = \dim_{\mathbf{R}} \frac{VI_f}{tf(V_{n-k}) + wf(VH_{2n})}.$$

f is called *symplectically stable* if sp-codim$(f) = 0$. Symplectically stable isotropic map-germs are classified under the symplectic equivalence completely in [14].

6. Symplectic codimension of Lagrange varieties. We understand the reason of the fact proved in [16] that the symplectic codimension of a plane curve on the symplectic plane is a diffeomorphism invariant (Theorem 2.1) is simply that a plane curve is a Lagrange variety. Now we show that the symplectic codimension of an isotropic map-germ $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^{2n}, 0)$, satisfying a mild condition, is a diffeomorphism invariant (i.e. $\mathcal{A}$-invariant).

We call a map-germ $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ $(n < p)$ a C^∞-*normalization* if f is $\mathcal{A}$-equivalent to an analytic normalization $(\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ to the image ([9], [27]). Note that if a plane curve $f : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ is $\mathcal{A}$-finite then f is a C^∞-normalization.

LEMMA 6.1. *Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$ $(n < p)$ be a C^∞-normalization, and η a C^∞ vector field over $(\mathbf{R}^p, 0)$. If, for any regular point $x \in (\mathbf{R}^n, 0)$, the vector $\eta(f(x)) \in T_{f(x)}\mathbf{R}^p$ belongs to $f_*(T_x\mathbf{R}^n)$, then there exists a C^∞ vector field ξ over $(\mathbf{R}^n, 0)$ such that $f_*(\xi) = \eta \circ f$.*

Proof. Let φ_t denote the flow generated by η. Then $\varphi_t \circ f$ has the same image as f. Therefore, by Corollary 2.5 of [9], there exists a unique diffeomorphism-germ ψ_t satisfying $f \circ \psi_t = \varphi_t \circ f$, for each t. Note that, since $n < p$, f is a critical normalization in the sense of [9]. Then ψ_t $(t \in \mathbf{R})$ defines a local flow on $(\mathbf{R}^n, 0)$. The differentiability of ψ_t for t can be obtained by applying Corollary 2.5 of [9] to the trivial unfolding $F := f \times \mathrm{id}_{\mathbf{R}} : (\mathbf{R}^{n+1}, 0) \to (\mathbf{R}^{p+1}, 0)$, which is also a C^∞, therefore critical, normalization. Now we set $\xi := \frac{d\psi_t}{dt}|_{t=0}$. Then, from the equality $f \circ \psi_t = \varphi_t \circ f$, we have $f_*(\xi) = \eta \circ f$. ■

Recall that we have defined the symplectic codimension of an isotropic map-germ $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^{2n}, 0)$ $(k = 0, \ell = n)$, by

$$\text{sp-codim}(f) = \dim_{\mathbf{R}} \frac{VI_f}{tf(V_n) + wf(VH_{2n})}.$$

Then we have the following:

THEOREM 6.2. *Let $f, g : (\mathbf{R}^n, 0) \to (\mathbf{R}^{2n}, 0)$ be isotropic C^∞-normalizations. If f and g are $\mathcal{A}$-equivalent, then* $\text{sp-codim}(f) = \text{sp-codim}(g)$.

Proof. First we remark that if f and g are $\mathcal{R}$-equivalent, then

$$\text{sp-codim}(f) = \text{sp-codim}(g).$$

So we may assume $g = \Phi \circ f$, Φ being a diffeomorphism on $(\mathbf{R}^{2n}, 0)$.

Consider the basic exact sequence for f:

$$0 \longrightarrow \frac{VI'_f}{tf(V_n) + wf(VH'_{f,2n})} \longrightarrow \frac{VI_f}{tf(V_n) + wf(VH_{2n})} \longrightarrow \frac{R_f}{f^*\mathcal{E}_{2n}} \to 0,$$

where

$$VH'_{f,2n} := \{X_H \in VH_{2n} \,|\, H \circ f = 0\}.$$

Then by Lemma 6.1, $wf(VH'_{f,2n}) \subseteq tf(V_n)$. Thus we have the following exact sequence:

$$0 \longrightarrow \frac{VI'_f}{tf(V_n)} \longrightarrow \frac{VI_f}{tf(V_n) + wf(VH_{2n})} \longrightarrow \frac{R_f}{f^*\mathcal{E}_{2n}} \to 0.$$

Also for g, we have the corresponding exact sequence.

Now we have $R_g = R_f$, $g^*\mathcal{E}_{2n} = f^*\mathcal{E}_{2n}$, and $\Phi_*(tf(V_n)) = tg(V_n)$. Moreover, in general, we have

$$VI'_f \subseteq \{v : (\mathbf{R}^n, 0) \to T\mathbf{R}^{2n} \,|\, \pi \circ v = f,\ v(x) \in f_*(T_x\mathbf{R}^n) \text{ for all } x \in \mathrm{Reg}(f)\}.$$

In fact, suppose $v \in VI'_f$. Then, for all $\xi \in T_x\mathbf{R}^n$, we have $0 = \langle \widetilde{v}^*\theta, \xi \rangle = \langle \theta(\widetilde{v}(x)), \widetilde{v}_*\xi \rangle = \langle \widetilde{v}(x), \pi_*\widetilde{v}_*\xi \rangle = \langle \widetilde{v}, f_*\xi \rangle = \omega(v(x), f_*\xi)$. So, for any $x \in \mathrm{Reg}(f)$, $f_*(T_x\mathbf{R}^n)$ is Lagrangian, and therefore $v(x) \in f_*(T_x\mathbf{R}^n)$.

Since f is a C^∞-normalization, we see $\mathrm{Reg}(f)$ is dense. Therefore we have

$$VI'_f = \{v : (\mathbf{R}^n, 0) \to T\mathbf{R}^{2n} \,|\, \pi \circ v = f,\ v(x) \in f_*(T_x\mathbf{R}^n),\ \text{for all } x \in \mathrm{Reg}(f)\},$$

Hence $\Phi_*(VI'_f) = VI'_g$. Thus

$$\begin{aligned}\text{sp-codim}(g) &= \dim_{\mathbf{R}} \frac{R_g}{g^*\mathcal{E}_{2n}} + \dim_{\mathbf{R}} \frac{VI'_g}{tg(V_n) + wg(VH'_{2n})} \\ &= \dim_{\mathbf{R}} \frac{R_f}{f^*\mathcal{E}_{2n}} + \dim_{\mathbf{R}} \frac{VI'_f}{tf(V_n) + wf(VH'_{2n})} = \text{sp-codim}(f). \quad \blacksquare\end{aligned}$$

Moreover we define a variant of symplectic codimension, called the *reduced symplectic codimension* of f, by

$$\widetilde{\text{sp}}\text{-codim}(f) = \dim_{\mathbf{R}} \frac{VI_f}{tf(V_n) + wf(V_{2n}) \cap VI_f}.$$

Then we have:

THEOREM 6.3. $\widetilde{\text{sp}}\text{-codim}(f)$ *is a diffeomorphism invariant.*

Proof. Set

$$G_f := \{h \in \mathcal{E}_n \,|\, dh \in \langle df \rangle_{f^*\mathcal{E}_{2n}}\}.$$

We have the following exact sequence:

$$0 \longrightarrow \frac{VI'_f}{tf(V_n)} \longrightarrow \frac{VI_f}{tf(V_n) + wf(V_{2n}) \cap VI_f} \longrightarrow \frac{R_f}{G_f} \to 0.$$

Then we see easily that $\widetilde{\text{sp}}\text{-codim}(f)$ is an $\mathcal{A}$-invariant. $\blacksquare$

We set

$$\text{sd}(f) := \text{sp-codim}(f) - \widetilde{\text{sp}}\text{-codim}(f).$$

Then we have:

COROLLARY 6.4. $\text{sd}(f)$ *is a diffeomorphism invariant. Moreover we see that*

$$\text{sd}(f) = \dim_{\mathbf{R}} \frac{tf(V_n) + wf(V_{2n}) \cap VI_f}{tf(V_n) + wf(VH_{2n})} = \dim_{\mathbf{R}} \frac{G_f}{f^*\mathcal{E}_{2n}}.$$

REMARK 6.5. We can also treat the case of multi-germs: Let $S \subset \mathbf{R}^n$ be a finite subset. A map-germ $f : (\mathbf{R}^n, S) \to (\mathbf{R}^{2n}, 0)$, $f(S) = \{0\}$, is called *isotropic* if $f^*\omega = 0$. Then if f is a C^∞-normalization, then we have the exact sequence

$$0 \longrightarrow \frac{VI'_f}{tf(V_S)} \longrightarrow \frac{VI_f}{tf(V_S) + wf(VH_{2n})} \longrightarrow \frac{R_f}{f^*\mathcal{E}_{2n}} \to 0,$$

where VI_f, VI'_f, V_S and R_f are defined similarly to the case $S = \{0\}$. For example,

$$R_f := \{e \in \mathcal{E}_S \,|\, de \in \mathcal{E}_S\, df\}.$$

Moreover we see that

$$VI'_f = \{v \in V_f \,|\, v(x) \in f_*(T_x\mathbf{R}^n) \text{ for all } x \in \mathrm{Reg}(f)\}.$$

Thus we conclude that

$$\text{sp-codim}(f) := \dim_{\mathbf{R}} \frac{VI_f}{tf(V_S) + wf(VH_{2n})}$$

is a diffeomorphism invariant ($\mathcal{A}$-invariant).

7. Lagrange extensions of isotropic mappings. Any isotropic immersion-germ can be extended to a Lagrange immersion. Then we consider singular Lagrange extensions of singular isotropic mappings.

First we observe, at least, any generic isotropic map-germ of corank at most 1 has a Lagrange extension of corank not greater than 1. We recall the open Whitney umbrellas $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^{2n}, 0)$ ([14]). For an integer m with $0 \leq 2m \leq n$, we set

$$u(x_1, x_2, \ldots, x_n) := \frac{x_n^{m+1}}{(m+1)!} + x_1 \frac{x_n^{m-1}}{(m-1)!} + \ldots + x_{m-1}x_n,$$

and

$$v(x_1, x_2, \ldots, x_n) := x_m \frac{x_n^m}{m!} + \ldots + x_{2m-1}x_n.$$

Then the pair (u, v) can be regarded as a family of plane curves $\Psi : (\mathbf{R}^{n-1} \times \mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ with parameters $x_1, \ldots, x_{n-1}$. We remark that just $x_1, \ldots, x_{2m-1}$ appear explicitly in u, v, and that $2m - 1 \leq n - 1$. Thus, we get the isotropic lifting

$$f_{n,m} := \widetilde{\Psi} : (\mathbf{R}^n, 0) = (\mathbf{R}^{n-1} \times \mathbf{R}, 0) \to (\mathbf{R}^{2n}, 0).$$

See Section 5. The explicit form of $\widetilde{\Psi}$ is given in [14]. We call an isotropic map-germ the *open Whitney umbrella of type* m if it is symplectically equivalent to $f_{n,m}$. General open Whitney umbrellas are introduced also in [20].

Now we have:

PROPOSITION 7.1. *Any generic isotropic mapping* $f : (\mathbf{R}^{n-k}, 0) \to (\mathbf{R}^{2n}, 0)$ *of corank not exceeding* 1 *is symplectically equivalent to an open Whitney umbrella* $F : (\mathbf{R}^n, 0) \to (\mathbf{R}^{2n}, 0)$ *composed with an embedding* $i : (\mathbf{R}^{n-k}, 0) \to (\mathbf{R}^n, 0)$.

Proof. There exists a Lagrange fibration $\pi : (\mathbf{R}^{2n}, 0) \to (\mathbf{R}^n, 0)$ such that $\pi \circ f : (\mathbf{R}^{n-k}, 0) \to (\mathbf{R}^n, 0)$ is of corank at most 1. Any perturbation of $\pi \circ f$ lifts to an isotropic perturbation of the original f. Thus we may assume $\pi \circ f$ is a generic map-germ of corank not greater than 1. Then we see $\pi \circ f$ is $\mathcal{A}$-equivalent to

$$\begin{aligned}
y_1 &= \frac{t^{m+1}}{(m+1)!} + \lambda_1 \frac{t^{m-1}}{(m-1)!} + \ldots + \lambda_{m-1}t, \\
y_2 &= \lambda_m \frac{t^m}{m!} + \lambda_{m+1} \frac{t^{m-1}}{(m-1)!} + \ldots + \lambda_{2m-1}t, \\
\cdots\cdots &\quad \cdots\cdots \\
y_{k+1} &= \lambda_{(k+1)m} \frac{t^m}{m!} + \lambda_{(k+1)m+1} \frac{t^{m-1}}{(m-1)!} + \ldots + \lambda_{(k+2)m-1}t, \\
y_{k+2} &= \lambda_1, \\
\cdots\cdots &\quad \cdots\cdots \\
y_{k+(k+2)m} &= \lambda_{(k+2)m-1}, \\
\cdots\cdots &\quad \cdots\cdots \\
y_n &= \lambda_{n-k-1}.
\end{aligned}$$

with $(k+2)m \leq n-k$. Then we define the embedding $i : (\mathbf{R}^{n-k}, 0) \to (\mathbf{R}^n, 0)$, $i(\lambda, t) =$

$(x_1, \ldots, x_{n-1}, x_n)$ by

$$\begin{aligned}
x_1 &= \lambda_1, \\
\cdots\cdots & \quad \cdots\cdots \\
x_{m-1} &= \lambda_{m-1}, \\
x_m &= \lambda_m, \\
\cdots\cdots & \quad \cdots\cdots \\
x_{2m-1} &= \lambda_{2m-1}, \\
x_{2m} &= \lambda_{2m} \frac{t^m}{m!} + \lambda_{2m+1} \frac{t^{m-1}}{(m-1)!} + \ldots + \lambda_{3m-1} t, \\
\cdots\cdots & \quad \cdots\cdots \\
x_{n-1} &= \lambda_{n-k-1}, \\
x_n &= t.
\end{aligned}$$

Then $\pi \circ f = \Psi \circ i$, and we see that f is symplectically equivalent to $\widetilde{\Psi} \circ i$. ∎

EXAMPLE 7.2. Let us consider the open Whitney umbrella $F = f_{2,1} : (\mathbf{R}^2, 0) \to (\mathbf{R}^4, 0)$ defined by

$$F(u, v) = (q_1, q_2, p_1, p_2) = \left(u^2, v, uv, \frac{2}{3} u^3\right),$$

and its restrictions. Consider an embedding $i : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ with $u = t$, $v = \varphi(t)$. Then $f := F \circ i : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ is given by $f(t) = (t^2, \varphi(t), t\varphi(t), \frac{2}{3} t^3)$. Then f is an immersion if and only if $\mathrm{ord}(\varphi) = 1$. f is of type $A_{2,0}$ if and only if $\mathrm{ord}(\varphi) = 2$. f is of type $A_{2,r}$ for some $r \geq 1$ if and only if $\mathrm{ord}(\varphi) \geq 3$.

REMARK 7.3. In the case of analytic curves, we have a stronger result: Let $f : (\mathbf{R}, 0) \to \mathbf{R}^4$ be a real analytic curve-germ. Then there exists a real analytic isotropic map-germ $F : (\mathbf{R}^2, 0) \to T^*\mathbf{R}^2$ of corank at most 1 such that $F|_{\mathbf{R} \times \{0\}} = f$.

8. Symmetry on symplectic singularities. In [21], an equivariant classification of isotropic immersions is given. Here we study the stability and versality problems of equivariant singular isotropic mappings.

Let G be a compact Lie group. We suppose a representation $\rho : G \to \mathrm{Diff}(\mathbf{R}^{n-k}, 0)$ (resp. $\rho' : G \to \mathrm{Symp}(\mathbf{R}^{2n}, 0)$, $\rho'' : G \to \mathrm{Diff}(\mathbf{R}^{n-\ell}, 0)$) is given. We call a map-germ $f : (\mathbf{R}^{n-k}, 0) \to (\mathbf{R}^{2n}, 0)$ (resp. $\pi : (\mathbf{R}^{2n}, 0) \to (\mathbf{R}^{n-\ell}, 0)$) *$G$-equivariant* if, for any $g \in G$, $f \circ \rho(g) = \rho'(g) \circ f$ (resp. $\pi \circ \rho'(g) = \rho''(g) \circ \pi$).

We can formulate G-equivariant stability of G-equivariant isotropic map-germs of corank not exceeding 1, for a fixed coisotropic submersion $\pi : (\mathbf{R}^{2n}, 0) \to (\mathbf{R}^{n-\ell}, 0)$, and introduce corresponding infinitesimal condition

$$VI_f^G = tf(V_{n-k}^G) + wf(VL_\pi^G),$$

where VI_f^G (resp. V_{n-k}^G, VL_π^G) is the set of G-equivariant isotropic vector fields along f (resp. G-invariant vector fields over $(\mathbf{R}^{n-k}, 0)$, G-invariant Hamiltonian vector fields over $(\mathbf{R}^{2n}, 0)$ covering a vector field over $(\mathbf{R}^{n-\ell}, 0)$ via π). We endow $T\mathbf{R}^{2n}$ with the naturally lifted G-action from the G-action on $(\mathbf{R}^{2n},0)$. We call a G-equivariant isotropic map-germ f *G-π-stable* if

$$VI_f^G = tf(V_{n-k}^G) + wf(VL_\pi^G).$$

If $\ell = 0$ (resp. if $\ell = n$), then we call it also *G-Lagrange stable* (resp. G-symplectically stable).

By using the infinitesimal criterion for the stability, we obtain:

PROPOSITION 8.1. *Let $f : (\mathbf{R}^{n-k},0) \to (\mathbf{R}^{2n},0)$ be an isotropic mapping of corank at most* 1. *If f is G-equivariant and π-stable, then f is G-π-stable.*

Proof. We will show that the condition $VI_f = tf(V_{n-k}) + wf(VL_\pi)$ implies $VI_f^G = tf(V_{n-k}^G) + wf(VL_\pi^G)$. Let $v \in VI_f$ (resp. $\xi \in V_{n-k}$, $\eta \in VL_\pi$). For $g \in G$, we set $g_*v := \rho'(g)_* \circ v \circ \rho(g)^{-1}$ (resp. $g_*\xi := \rho(g)_* \circ \xi \circ \rho(g)^{-1}$, $g_*\eta := \rho'(g)_* \circ \eta \circ \rho'(g)^{-1}$). Then $v \in VI_f^G$ (resp. $\xi \in V_{n-k}^G$, $\eta \in VL_\pi^G$) if and only if $g_*v = v$ (resp. $g_*\xi = \xi$, $g_*\eta = \eta$) for any $g \in G$.

Now let $v \in VI_f^G$. From the condition $VI_f = tf(V_{n-k}) + wf(VL_\pi)$, there exist $\xi \in V_{n-k}$, $\eta \in VL_\pi$ satisfying $v = tf(\xi) + wf(\eta)$. Since f is G-equivariant, we have

$$v = g_*v = g_*(tf(\xi) + wf(\eta)) = tf(g_*\xi) + wf(g_*\eta).$$

By means of the invariant integral over G, we set

$$\overline{\xi} := \int_G g_*\xi, \quad \overline{\eta} := \int_G g_*\eta.$$

Then we see that $\overline{\xi} \in V_{n-k}^G$, $\overline{\eta} \in VL_\pi^G$, and that $v = tf(\overline{\xi}) + wf(\overline{\eta})$. Therefore $VI_f^G = tf(V_{n-k}^G) + wf(VL_\pi^G)$. ∎

EXAMPLE 8.2. Let us consider again the open Whitney umbrella $f : (\mathbf{R}^2,0) \to (\mathbf{R}^4,0)$,

$$f(u,v) = (q_1,q_2,p_1,p_2) = \Big(u^2, v, uv, \frac{2}{3}u^3\Big).$$

Let $G = \mathbf{Z}/2\mathbf{Z} = \langle 1, \sigma \rangle$. G acts on $(\mathbf{R}^2,0)$ (resp. on $(\mathbf{R}^4,0)$ symplectically, on $(\mathbf{R}^2,0)$) by $\sigma(u,v) = (-u,-v)$ (resp. $\sigma(q_1,q_2,p_1,p_2) = (q_1,-q_2,p_1,-p_2)$, $\sigma(q_1,q_2) = (q_1,-q_2)$). Then f is G-equivariant. Moreover f is Lagrange stable ([10], [11]). Therefore f is G-Lagrange stable.

We have G-equivariant versality theorem and that the infinitesimal G-stability is equivalent to the G-stability in the case of corank not greater than 1 (cf. [8], [15]).

EXAMPLE 8.3. Let $G = \mathbf{Z}/2\mathbf{Z} = \langle 1, \sigma \rangle$. G acts on $(\mathbf{R},0)$ by $\sigma(t) = -t$ and acts on $(\mathbf{R}^2,0)$ symplectically by $\sigma(x_1,x_2) = (-x_1,-x_2)$. Let $f : (\mathbf{R},0) \to (\mathbf{R}^2,0)$ be defined by $f(t) = (t^3,t^5)$. Then f is G-equivariant. We see $F : (\mathbf{R}\times\mathbf{R},0) \to (\mathbf{R}^2\times\mathbf{R},0)$ defined by $F(t,\lambda) = (t^3+\lambda t, t^5+\lambda t, \lambda)$ is a G-versal isotropic unfolding of f.

Moreover we can define the *G-π-symplectic codimension* of a G-equivariant isotropic map-germ $f : (\mathbf{R}^{n-k},0) \to (\mathbf{R}^{2n},0)$ for a fixed G-equivariant coisotropic fibration $\pi : (\mathbf{R}^{2n},0) \to (\mathbf{R}^{n-\ell},0)$ by

$$G\text{-}\pi\text{-sp-codim}(f) := \dim_{\mathbf{R}} \frac{VI_f^G}{tf(V_{n-k}^G) + wf(VL_\pi^G)},$$

and, if $\ell = 0$, we call it simply *G-symplectic codimension* of f:

$$G\text{-sp-codim}(f) := \dim_{\mathbf{R}} \frac{VI_f^G}{tf(V_{n-k}^G) + wf(VH_{2n}^G)}\,.$$

Then we have

PROPOSITION 8.4. *Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^{2n}, 0)$ be a G-equivariant isotropic map-germ. Suppose f is a C^∞-normalization. Then G-sp-codim(f) is a G-diffeomorphism invariant. Namely, if G-equivariant isotropic map-germs $f, f' : (\mathbf{R}^n, 0) \to (\mathbf{R}^{2n}, 0)$ are $\mathcal{A}$-equivalent by G-equivariant diffeomorphisms: the diagram*

$$\begin{array}{ccc} (\mathbf{R}^n, 0) & \xrightarrow{f} & (\mathbf{R}^{2n}, 0) \\ \downarrow \sigma & & \downarrow \Phi \\ (\mathbf{R}^n, 0) & \xrightarrow{f'} & (\mathbf{R}^{2n}, 0), \end{array}$$

commutes for G-equivariant diffeomorphism-germs σ and Φ, then

$$G\text{-sp-codim}(f) = G\text{-sp-codim}(f').$$

Proof. We have the exact sequence

$$0 \longrightarrow \frac{VI_f'^G}{tf(V_n^G)} \longrightarrow \frac{VI_f^G}{tf(V_n^G) + wf(VH_{2n}^G)} \longrightarrow \frac{R_f^G}{f^*\mathcal{E}_{2n}^G} \to 0,$$

where $VH_{2n}^G, R_f^G, \mathcal{E}_{2n}^G$ are corresponding G-objects to the case that G is trivial. Actually, VH_{2n}^G is the space of G-equivariant Hamiltonian vector fields over $(\mathbf{R}^{2n}, 0)$, $\mathcal{E}_{2n}^G$ is the space of G-invariant functions on $(\mathbf{R}^{2n}, 0)$, and R_f^G the space of functions e on $(\mathbf{R}^n, 0)$ such that de is a functional linear combination of $d(q_1 \circ f), \dots, d(q_n \circ f), d(p_1 \circ f), \dots, d(p_n \circ f)$ with G-invariant functions on $(\mathbf{R}^n, 0)$ as coefficients.

Moreover we have

$$VI_f'^G = \{v \in V_f^G \mid v(x) \in f_*(T_x\mathbf{R}^n) \text{ for all } x \in \mathrm{Reg}(f)\}.$$

Then the invariance of G-sp-codim(f) under G-equivariant diffeomorphisms follows from that all $VI_f'^G, tf(V_n^G), R_f^G, f^*\mathcal{E}_{2n}^G$ are defined by means of invariant notions under G-diffeomorphisms. ∎

The details will be given in forthcoming papers.

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