ON ASYMPTOTIC CRITICAL VALUES
AND THE RABIER THEOREM

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Abstract. Let $X \subset k^n$ be a smooth affine variety of dimension $n-r$ and let $f = (f_1, \ldots, f_m): X \to k^m$ be a polynomial dominant mapping. It is well-known that the mapping $f$ is a locally trivial fibration outside a small closed set $B(f)$. It can be proved (using a general Fibration Theorem of Rabier) that the set $B(f)$ is contained in the set $K(f)$ of generalized critical values of $f$. In this note we study the Rabier function. We give a few equivalent expressions for this function, in particular we compare this function with the Kuo function and with the (generalized) Gaffney function. As a consequence we give a direct short proof of the fact that $f$ is a locally trivial fibration outside the set $K(f)$ (i.e., that $B(f) \subset K(f)$). This generalizes the previous results of the author for $X = k^r$ (see [2]).

1. Introduction. Let $X$ be a smooth affine variety over $k = \mathbb{R}$ or $k = \mathbb{C}$ of dimension $n-r$ and let $f : X \to k^m$ be a polynomial dominant mapping. It is well-known that the mapping $f$ is a locally trivial fibration outside a bifurcation set $B(f)$, which has a measure 0.

Let us recall that in general the set $B(f)$ is bigger than $K_0(f)$—the set of critical values of $f$. It contains also the set $B_\infty(f)$ of bifurcations points at infinity. Briefly speaking, the set $B_\infty(f)$ consists of points at which $f$ is not a locally trivial fibration at infinity (i.e., outside a compact set). To control the set $B_\infty(f)$ one can use the set of asymptotic critical values at infinity of $f$ (see [6]):

$$K_\infty(f) = \{ y \in k^m : \text{there is a sequence } x_l \to \infty \text{ such that } f(x_l) \to y \text{ and } \|x_l\|\nu(\text{rest}_{T_{x_l}X} df(x_l)) \to 0 \},$$

where we consider the induced Euclidean metric on $X$ and $\nu$ is the function defined by Rabier (see Definition 2.1 below). If $y \notin K_\infty(f)$ we say also that $y$ is Malgrange regular.

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If \( m = 1 \) and \( X = k^n \), then there is a wide literature devoted to different regularity conditions and their comparison (e.g., [8], [9], [10]). It has been proved for instance that the Malgrange regularity is equivalent to another regularity called \( t \)-regularity, by Siersma and Tibar (see [7]). The case \( m > 1 \) and \( X = k^n \) was studied in [1], [2] and [4]. In this paper (and in [3]) we study the case when \( X \) is a smooth affine variety (or even a Stein submanifold of \( \mathbb{C}^m \)) and \( m \leq \dim X \).

Let \( K(f) = K_0(f) \cup K_\infty(f) \) be the set of generalized critical values of \( f \). It can be proved that the set \( K(f) \) is a proper algebraic subset of \( \mathbb{C}^m \)—or proper semi-algebraic in the real case (see [3]). Moreover, we have (e.g., by a general Fibration Theorem of Rabier [6], see also [1]) \( B(f) \subset K(f) \). These two facts together allow us to construct effectively a Zariski open dense subset \( U \subset k^m \) over which the mapping \( f \) is a locally trivial fibration.

In this note we study the Rabier function. As a consequence we give a direct proof of the fact that \( B(f) \subset K(f) \) in the case when \( X \subset k^n \) is a smooth submanifold and \( f : X \to k^m \) is a smooth mapping (moreover, some of these results are used in [3] to study the properties of the set \( K(f) \)).

The fact that \( B(f) \subset K(f) \) follows from a very general Theorem of Rabier (see [6]), but it is so important (e.g., in the study of polynomial mappings) that (as I believe) it is worth to have a simple direct proof of it in a special case of submanifolds of a Euclidean space.

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### 2. On the Rabier function \( \nu \)

Here we give several equivalent expressions for \( \nu \).

Let \( X \cong k^n, Y \cong k^m \) be finite-dimensional vector spaces (over \( k \)). Let us denote by \( \mathcal{L}(X,Y) \) the set of linear mappings from \( X \) to \( Y \) and by \( \Sigma(X,Y) \subset \mathcal{L}(X,Y) \) the set of non-surjective mappings. Let us recall the following ([6]):

**Definition 2.1.** Let \( A \in \mathcal{L}(X,Y) \).

\[
\nu(A) = \inf_{\|\phi\|=1} \|A^*(\phi)\|,
\]

where \( A^* : \mathcal{L}(Y^*,X^*) \) is the adjoint operator and \( \phi \in Y^* \).

In [4] the following characterization of \( \nu \) is given: \( \nu(A) = \text{dist}(A,\Sigma) = \inf_{B \in \Sigma} \|A-B\| \).

Moreover, we have the following useful characterization ([6] and [4]):

**Proposition 2.1.** Let \( A \in \mathcal{L}(X,Y) \). Then

a) \( \nu(A) = \sup \{ r > 0 : B(0,r) \subset A(B(0,1)) \} \), where \( B(0,r) = \{ x \in X : \|x\| \leq r \} \).

b) if \( A \in \text{GL}(X,Y) \) then \( \nu(A) = \|A^{-1}\|^{-1} \).

**Proposition 2.2.** Let \( A = (A_1, \ldots, A_m) \in \mathcal{L}(X,Y) \) and let \( \overline{A}_i = \text{grad} A_i \). Let

\[
\kappa(A) = \min_{1 \leq i \leq m} \text{dist} \left( \overline{A}_i, \left( \overline{A}_j \right)_{j \neq i} \right),
\]

be the Kuo number of \( A \). Then \( \nu(A) \leq \kappa(A) \leq \sqrt{m} \nu(A) \).
We say that \( \nu(A) \) and \( \kappa(A) \) are equivalent and write \( \nu(A) \sim \kappa(A) \). The symbol \( X \sim Y \) means that there are positive constants \( C_1, C_2 \) such that \( C_1 X \leq Y \leq C_2 X \).

**Definition 2.2.** Let \( A \in \mathcal{L}(X,Y) \) and let \( H \subset X \) be a linear subspace. We set
\[
\nu(A,H) = \nu(\text{res}_H A), \quad \kappa(A,H) = \kappa(\text{res}_H A),
\]
where \( \text{res}_H A \) denotes the restriction of \( A \) to \( H \).

From Proposition 2.2 we get immediately:

**Corollary 2.1.** Let \( A \in \mathcal{L}(X,Y) \) and let \( H \subset X \) be a linear subspace. Then
\[
\nu(A,H) \sim \kappa(A,H).
\]

**Proposition 2.3.** Let \( A = (A_1, \ldots, A_m) \in \mathcal{L}(X,Y) \) and let \( H \subset X \) be a linear subspace. Assume that \( H \) is given by a system of linear equations \( B_j = 0, \ j = 1, \ldots, r \). Then
\[
\kappa(A,H) = \min_{1 \leq i \leq m} \text{dist}(\overline{A}_i, (\langle \overline{A}_j \rangle_{j \neq i}; (\overline{B}_j)_{j=1,\ldots,r})),
\]
where \( \overline{A}_i = \text{grad} A_i \) and \( \overline{B}_j = \text{grad} B_j \).

**Proof.** Indeed, every vector \( \overline{A}_i \) can be written as \( a_i + b_i \), where \( a_i \) is orthogonal to the subspace \( B = \langle (\overline{B}_j)_{j=1,\ldots,r} \rangle \) (which means that \( a_i \in H \) and \( b_i \in B \)). Hence
\[
\text{dist}(\overline{A}_i, (\langle \overline{A}_j \rangle_{j \neq i}; (\overline{B}_j)_{j=1,\ldots,r})) = \text{dist}(a_i, (\langle a_j \rangle_{j \neq i}))
\]
and since \( \text{grad}(\text{res}_H A_i) = a_i \), the proof is finished. \( \blacksquare \)

We need also:

**Definition 2.3.** Let \( A \in \mathcal{L}(X,Y) \) (where \( n \geq m + r \)) and let \( H \subset X \) be a linear subspace given by a system of independent linear equations \( B_i = \sum b_{ij} x_j, \ i = 1, \ldots, r \). Let \( a = [a_{ij}] \) be the matrix of \( A \). Let \( c = [c_{kl}] \) be a \( ((m + r) \times n) \) matrix given by the rows \( A_1, \ldots, A_m; B_1, \ldots, B_r \) (we identify \( A_i = \sum a_{ij} x_j \) with the vector \( (a_{i1}, \ldots, a_{in}) \), similarly for \( B_j \)). Let \( M_I \), where \( I = (i_1, \ldots, i_{m+r}) \), denote a \( ((m + r) \times (m + r)) \) minor of \( c \) given by columns indexed by \( I \) and let \( |M_I| \) denote the determinant of \( M_I \). Further, let \( M_J(j) \) denote a \( ((m+r-1) \times (m+r-1)) \) minor given by columns indexed by \( J \) and by deleting the \( j \)-th row, where \( 1 \leq j \leq m \). Then by the generalized Gaffney function of \( A \) with respect to a linear subspace \( H \), we mean the number
\[
g(A,H) = \left( \frac{\sum I |M_I|^2}{\sum J, 1 \leq j \leq m |M_J(j)|^2} \right)^{1/2}.
\]
(If this number is not defined we put \( g(A,H) = 0 \).)

**Remark 2.1.** It is easy to see that \( g(A,H) \) depends on \( A \) and \( H \) only. A particular case of this definition (for \( H = X \)) has been considered by Gaffney—see [1].

**Proposition 2.4.** Let \( A \in \mathcal{L}(X,Y) \) (where \( n \geq m \)) and let \( H \subset X \) be a linear subspace. Then \( g(A,H) \sim \kappa(A,H) \sim \nu(A,H) \).
Proof. By basic properties of the Gram determinant (see e.g., [5]) we have
\[
\text{dist}(\overline{A}_{ij}, \langle (\overline{A}_j)_{j\neq i}; (\overline{B}_j)_{j\in \{1,\ldots,r\}} \rangle) = \frac{G((\overline{A}_j)_{j\in \{1,\ldots,m\}}, (\overline{B}_j)_{j\in \{1,\ldots,r\}})^{1/2}}{G((\overline{A}_j)_{j\neq i}, (\overline{B}_j)_{j\in \{1,\ldots,r\}})^{1/2}} = \frac{(\sum_I |M_I|^2)^{1/2}}{(\sum_J |M_J(i)|^2)^{1/2}}.
\]
Thus \(g(A, H) \leq \kappa(A, H)\). On the other hand there is a number \(i_0\) such that the sum \((\sum_j |M_J(i_0)|^2)^{1/2}\) is maximal. Since
\[
\left(\sum_{j,j} |M_J(j)|^2\right)^{1/2} = \left(\sum_r \left(\sum_j |M_J(r)|^2\right)^{1/2}\right) \leq \sqrt{m} \left(\sum_j |M_J(i_0)|^2\right)^{1/2},
\]
we have
\[
g(A, H) \geq C \frac{(\sum_I |M_I|^2)^{1/2}}{(\sum_J |M_J(i_0)|^2)^{1/2}} = C \text{dist}(\overline{A}_{i_0}, \langle (\overline{A}_j)_{j\neq i_0}; (\overline{B}_j)_{j\in \{1,\ldots,r\}} \rangle) \geq C \kappa(A, H),
\]
where \(C = 1/\sqrt{m}\). ■

Definition 2.4. Let us apply the notation from Definition 2.3. Put
\[
q(A, H) = \max_I \frac{|M_I|}{\max_I, j \subseteq I, j |M_J(j)|},
\]
(where we consider only numbers with \(M_J(j) \neq 0\), if all numbers \(M_J(j)\) are zero, we put \(q(A, H) = 0\)).

Proposition 2.4 can also be formulated in the following way:

Corollary 2.2. We have \(q(A, H) \sim \nu(A, H)\).

Proof. Let \(A\) denote the number of all possible matrices of type \(M_I\) (for all \(I\)) and let \(B\) denote the number of all possible matrices of type \(M_J(j)\) (for all possible \(I, J \subseteq I\) and all \(1 \leq j \leq m\)). Since the norms \(||x|| = \left(\sum |x_i|^2\right)^{1/2}\) and \(||x||' = \sum |x_i|\) are equivalent, we have
\[
g(A, H) \sim \frac{\sum_I |M_I|}{\sum_{I, J \subseteq I, j |M_J(j)|}.}
\]
On the other hand
\[
(1/B) \frac{\max_I |M_I|}{\max_I, j \subseteq I, j |M_J(j)|} \leq \frac{\sum_I |M_I|}{\sum_{I, J \subseteq I, j |M_J(j)|}} \leq A \frac{\max_I |M_I|}{\max_I, j \subseteq I, j |M_J(j)|}
\]
and consequently \(g(A, H) \sim q(A, H)\). Now we finish the proof by Proposition 2.4. ■

At the end of this section we introduce another important function (the notation is as in Definition 2.3):

Definition 2.5. We define the function
\[
g'(A, H) = \max_I \left\{\min_{J \subseteq I, 1 \leq j \leq m} \frac{|M_I|}{|M_J(j)|}\right\},
\]
(where we consider only numbers with \(M_J(j) \neq 0\), if all numbers \(M_J(j)\) are zero, we put \(g'(A, H) = 0\)).
Proposition 2.5. We have $g'(A, H) \sim g(A, H)$.

Proof. First we prove that there is a constant $C > 0$ such that $g'(A, H) \leq C g(A, H)$. Let us fix an index $I = (i_1, \ldots, i_{m+r})$ such that $|M_I| \neq 0$ and consider the numbers $|M_I|/|M_J(s)|$, where $J \subset I$ and $1 \leq s \leq m$. For simplicity we can assume that $I = (1, \ldots, m+r)$. Let the subspace $H$ be given by a system of independent linear equations $B_i = \sum b_{ij} x_j$, $i = 1, \ldots, r$, and let $a = [a_{ij}]$ be the matrix of $A$.

Consider the system of linear equations:

\[
\begin{align*}
\sum_{j=1}^n a_{1j} x_j &= y_1, \\
& \quad \ldots \quad \ldots \\
\sum_{j=1}^n a_{mj} x_j &= y_m, \\
\sum_{j=1}^n b_{1j} x_j &= 0, \\
& \quad \ldots \quad \ldots \\
\sum_{j=1}^n b_{rj} x_j &= 0, \\
x_{m+r+1} &= 0, \\
& \quad \ldots \quad \ldots \\
x_n &= 0.
\end{align*}
\]

We can solve this system using the Cramer rules. Let $M_{ki} := M_J(i)$ for $J = I \setminus \{k\}$. We have

\[
\begin{align*}
x_1 &= \sum_{k=1}^m (-1)^{1+k} y_k M_{1k}/M_I, \\
& \quad \ldots \quad \ldots \\
x_{m+r} &= \sum_{k=1}^m (-1)^{m+r+k} y_k M_{(m+r)k}/M_I, \\
x_{m+r+1} &= 0, \\
& \quad \ldots \quad \ldots \\
x_n &= 0.
\end{align*}
\]

In particular we have $||x|| \leq \left( \max |M_J(i)|/|M_I| \right) ||y||$. Consequently we see that the image of a unit ball in the subspace $H' = \{ x \in H : x_{m+r+1} = 0, \ldots, x_n = 0 \}$ by the mapping $A$ contains a ball of radius $\min_{J \subset I, 1 \leq j \leq m} |M_I|/|M_J(j)|$. Now by Proposition 2.1a), we see that $\min_{J \subset I, 1 \leq j \leq m} |M_I|/|M_J(j)| \leq \nu(A, H') \leq \nu(A, H)$. Finally we get

\[
\nu(A, H) \geq \max_I \left\{ \min_{J \subset I, 1 \leq j \leq m} \frac{|M_I|}{|M_J(j)|} \right\} = g'(A, H).
\]

In particular there is a constant $C$ such that $C g(A, H) \geq g'(A, H)$.

On the other hand, there exists $I_0$ such that the minor $M_{I_0}$ has a maximal norm.
Since
\[ g(A, H) = \frac{\left( \sum_l |M_l|^2 \right)^{1/2}}{\left( \sum_j |M_j(j)|^2 \right)^{1/2}} \leq \left( \frac{n}{m + r} \right)^{1/2} \frac{|M_0|}{\left( \sum_j |M_j(j)|^2 \right)^{1/2}} \]
\[ \leq \left( \frac{n}{m + r} \right)^{1/2} \min_{j \leq i_0, 1 \leq j \leq m} \frac{|M_0|}{|M_j(j)|} \leq \left( \frac{n}{m + r} \right)^{1/2} g'(A, H), \]
we deduce that there is a constant \( C' > 0 \) such that \( g(A, H) \leq C'g'(A, H) \).

**Corollary 2.3.** We have \( g'(A, H) \sim \nu(A, H) \).

**3. Main result.** In this section we give a short direct proof of the fact \( B(f) \subset K(f) \) for a smooth mapping \( f : X \to k^m \), where \( X \) is a smooth submanifold of \( k^m \). Let us recall the following basic definition:

**Definition 3.1.** Let \( k = \mathbb{C} \) or \( k = \mathbb{R} \) and let \( X \) be a smooth submanifold of \( k^n \). Let \( f : X \to k^m \) be a \( k \)-smooth mapping. Then we define the set of generalized critical values \( K(f) = K_0(f) \cup K_\infty(f) \), where \( K_0(f) \) is the set of critical values of \( f \) and

\[ K_\infty(f) = \{ y \in k^m : \text{there is a sequence } x_l \to \infty \text{ such that } f(x_l) \to y, \text{ and } ||x_l||\nu(df(x_l), T_{x_l}X) \to 0 \} \]
is the set of critical values at infinity.

**Remark 3.1.** Note that by virtue of results of Section 2, in place of the function \( \nu \) above we can put also \( \kappa, g, q \) or \( g' \).

We have the following simple observation (see [2], [6]):

**Proposition 3.1.** Let \( k = \mathbb{C} \) or \( k = \mathbb{R} \) and let \( X \) be a smooth affine variety over \( k \). Let \( f : X \to k^m \) be a \( k \)-smooth mapping. Then the set \( K(f) = K_0(f) \cup K_\infty(f) \) is closed.

We need also the following lemma (see [2]):

**Lemma 3.1.** Let \( U \subset k^n \) be an open set and \( V : U \to k^n \) be a smooth mapping. Let \( y \in U \) and let

\[ x'(t) = V(x), \text{ with } x(0) = y, \]
be a differential equation. Let \( x(y, t), t \in [0, t_0) \), be a solution of this equation. Assume that for \( ||x(y, t)|| \) large enough, we have \( ||V(x(y, t))|| < M||x(y, t)|| \). Then this trajectory is bounded. In particular this trajectory either is defined for every \( t > 0 \) or intersects the boundary \( \partial U \) of \( U \).

Now we give a short direct proof of the fact that \( B(f) \subset K(f) \), which is a particular version of a very general result of Rabier [6] (see also [1]).

**Theorem 3.1.** Let \( k = \mathbb{C} \) or \( k = \mathbb{R} \) and let \( X \subset k^n \) be a smooth submanifold (i.e., \( X \) is smooth for \( k = \mathbb{R} \) or Stein for \( k = \mathbb{C} \)). Let \( f : X \to k^m \) be a \( k \)-smooth mapping (i.e., \( f \) is smooth for \( k = \mathbb{R} \) or holomorphic for \( k = \mathbb{C} \)). Then

\[ B(f) \subset K(f) = K_0(f) \cup K_\infty(f), \]
i.e., the mapping \( f \) is a locally trivial fibration outside the set \( K(f) \).
**Proof.** It is well-known that we can assume that \( f \) can be extended to a \( k \)-smooth mapping \( \overline{f} \) on the whole \( k^{n} \) (in real case it is an easy exercise, in complex it follows from the theory of Stein manifolds).

First assume that \( X \) is a global complete intersection, i.e. \( X = \{ b_1 = 0, \ldots, b_r = 0 \} \) and \( \text{rank}\{ d_x b_1, \ldots, d_x b_r \} = r \) for every \( x \in X \).

Let \( a \notin K(f) \). Without loss of generality we can assume that \( a = 0 \). We have \( a \notin K_0(f) \) and \( a \notin K_\infty(f) \). This implies that there are \( R > 0, \epsilon > 0, \eta > 0 \), such that for every \( x \in X \) with \( ||x|| \geq R \) and \( ||f(x)|| < \eta \), we have

\[
\max\left\{ \min_{J \subset I, 1 \leq j \leq m} ||x|| \frac{|M_I|}{|M_J(j)|} \right\} > \epsilon.
\]

Moreover, there is \( \omega > 0 \) such that for every \( x \in X \) with \( ||x|| \leq R \) and \( ||f(x)|| < \eta \), we have \( \max_I |M_I(x)| \geq \omega \).

Let \( U = \{ y \in k^{m} : ||y|| < \eta \} \) and let \( \Gamma = f^{-1}(0) \). We show that \( f^{-1}(U) \cong \Gamma \times U \) and \( f \) is a projection \( \Gamma \times U \ni (\gamma, u) \mapsto u \in U \). Indeed, let us define a set

\[
U_I = \left\{ x \in f^{-1}(U) : \text{if } ||x|| \geq R \text{ then } \min_{J \subset I, 1 \leq j \leq m} ||x|| \frac{|M_I|}{|M_J(j)|} \geq \epsilon, \right. \\
\left. \text{if } ||x|| \leq R \text{ then } |M_I(x)| \geq \omega \right\}.
\]

Further, let

\[
V_I = \left\{ x \in f^{-1}(U) : \text{if } ||x|| \geq R \text{ then } \min_{J \subset I, 1 \leq j \leq m} ||x|| \frac{|M_I|}{|M_J(j)|} \leq \epsilon/2, \right. \\
\left. \text{if } ||x|| \leq R \text{ then } |M_I(x)| \leq \omega/2 \right\}.
\]

The sets \( V_I \) and \( U_I \) are disjoint. Consequently there is a \( C^\infty \) function \( \delta_I : k^{n} \to [0, 1] \), which is equal to 1 on \( U_I \) and to 0 on \( V_I \). It is easy to see that the sets \( H_I = \{ x : \delta_I(x) > 0 \} \) cover the set \( f^{-1}(U) \). Now take \( \delta = \sum_I \delta_I \) and let \( \Delta_I = \delta_I / \delta \).

Take \( y = (y_1, \ldots, y_n) \in U \). Take the index \( I = (1, \ldots, m + r) \) and consider a (formal) system of differential equations:

\[
\sum_{j=1}^{n} \frac{\partial f_j}{\partial x_j}(x(t))x_j(t)' = y_1,
\]

\[
\ldots \ldots
d
\]

\[
\sum_{j=1}^{n} \frac{\partial f_m}{\partial x_j}(x(t))x_j(t)' = y_m,
\]

\[
\sum_{j=1}^{n} \frac{\partial b_1}{\partial x_j}(x(t))x_j(t)' = 0,
\]

\[
\ldots \ldots
d
\]

\[
\sum_{j=1}^{n} \frac{\partial b_r}{\partial x_j}(x(t))x_j(t)' = 0,
\]

\[
x_{m+r+1}(t)' = 0,
\]

\[
\ldots \ldots
d
\]

\[
x_n(t)' = 0.
\]
We can solve this system using the Cramer rules (at least in $U_I$). Let $M_{ki} := M_f(i)$ for $J = I \setminus \{k\}$. We have

$$x_1(t)' = \sum_{k=1}^{m} (-1)^{1+k} y_k M_{1k}/M_I,$$

$$\ldots \ldots$$

$$x_{m+r}(t)' = \sum_{k=1}^{m} (-1)^{m+r+k} y_k M_{(m+r)k}/M_I,$$

$$x_{m+r+1}(t)' = 0,$$

$$\ldots \ldots$$

$$x_n(t)' = 0.$$

We can write this system shortly as

$$x(t)' = V_I(y, x(t)).$$

By the Cramer rules, we have $df(V_I(y, x)) = y$ and $db(V_I(y, x)) = 0$. In an analogous way we can define $V_I$ for an arbitrary index $I = (i_1, \ldots, i_m)$.

Now consider a vector field $V(y, x) = \sum_I \Delta_I V_I(y, x)$ in a domain $\overline{f}^{-1}(U)$. By the construction, we have $||V(x)|| \leq 2m\eta/\epsilon||x||$ for $||x|| \geq R$ and $x \in X$. Let us consider the differential equation

$$(2) \quad x(t)' = V(y, x(t)), \quad x(0) = \gamma,$$

where $\gamma \in \Gamma$. Let us note that

$$df(V(y, x)) = df\left(\sum_I \Delta_I V_I(y, x)\right) = \sum_I df(\Delta_I V_I(y, x)) = \sum_I \Delta_I y.$$}

Similarly $db(V(x, y)) = 0$. Consequently, if $x(t, y, \gamma)$ is a solution of system (2), then the trajectory is contained in $X$ and $yt = \overline{f}(x(t), y, \gamma) = f(x(t), y, \gamma)$. Since $y \in U$, we see that the trajectory $x(t, y, \gamma)$, $t \in [0, t_0]$ does not cross the border $\partial f^{-1}(U)$ for every $0 \leq t_0 \leq 1 + \delta$, for some $\delta > 0$. Consequently by Lemma 3.1 the trajectory $x(t, y, \gamma)$ is defined on the whole $[0, 1]$ and is contained in $X$. Since $f(x(t, y, \gamma)) = yt$, the phase flow $x(t, y, \gamma)$, $t \in [0, 1]$, transforms $f^{-1}(0) = \Gamma$ into $f^{-1}(y)$ (in fact, by the symmetry, it transforms $\Gamma$ onto $f^{-1}(y)$). Let

$$\Phi : \Gamma \times U \ni (\gamma, y) \mapsto x(1, y, \gamma) \in f^{-1}(U).$$

It is easy to see that $\Phi$ is a diffeomorphism. Thus $0 \notin B(f)$.

In the general case we can choose a locally finite cover $\{U_i\}$ of $k^n$ such that in each $U_i$ the manifold $X \cap X_i$ is a complete intersection. Now we can construct vector fields $V_i$ on $U_i$ (construction is as above) and then glue them to one field $V$ by a partition of unity subordinate to the cover $\{U_i\}$. The rest of the proof is the same as above. ■

At the end of this note we give two simple examples.

**Example 3.1.** Let us consider a Stein curve $\Gamma = \{(x, y) \in \mathbb{C}^2 : \exp(xy) = 2\}$. Let us consider the projection $f : \Gamma \ni (x, y) \mapsto y \in \mathbb{C}$. Using the generalized Gaffney function
we see that
\[ K_0(f) = f\{(x, y) \in \Gamma : y \exp(xy) = 0\} = \emptyset \]
and
\[ K_\infty(f) = \{\lim f(x_n, y_n) = y_n; \text{ where } (|x_n| + |y_n|) \to \infty \text{ and } |y_n| \to 0\} = \{0\}. \]
Hence finally \( K(f) = \{0\} \) and indeed we can check directly that in this case \( B(f) = K(f) = \{0\} \) (in fact \( f \) is a topological covering outside 0). Note that the mapping \( f \) has no usual critical values.

**Example 3.2.** Let us consider a smooth mapping
\[ f : \mathbb{C}^3 \ni (x, y, z) \mapsto (x \exp(z), y \exp(z)) \in \mathbb{C}^2. \]
Using the function \( g \) we can easily compute that \( K(f) = \{0\} \). But the function \( f \) is a global fibration of \( \mathbb{C}^3 \) — in fact it gives a fibration
\[ \mathbb{C}^2 \times \mathbb{C} \ni ((x, y), z) \mapsto (x \exp(-z), y \exp(-z), z) \in \mathbb{C}^3. \]
Thus in general \( B(f) \neq K(f) \).

**References**


