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LIPSCHITZ STRATIFICATIONS AND LIPSCHITZ ISOTOPIES

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Introduction. The motivation of this paper is a question of M. Gromov, communicated by Lev Birbrair. We shall state it after giving, rather informally, a few definitions.

We shall work with the following classes of sets: $\mathfrak{A}_1$ subanalytic, $\mathfrak{A}_2$ semianalytic, $\mathfrak{A}_3$ semialgebraic, $\mathfrak{A}_4$ complex analytic, $\mathfrak{A}_5$ complex algebraic.

Usually we shall not distinguish between sets and their germs at a precised point.

Two subsets $A, B \subset \mathbb{R}^n$ are *Lipschitz equivalent* if there exists a bi-Lipschitz homeomorphism $h : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $h(A) = B$.

Consider now a family of subsets of $\mathbb{R}^n$, i.e. a commuting diagram

$$\begin{array}{ccc} X & \subset & T \times \mathbb{R}^n \\ \downarrow & & \downarrow \pi \\ T & \subset & \mathbb{R}^m \end{array} ,$$

where π is the standard projection $(t, x) \longmapsto t$ and $X, T \in \mathfrak{A}_i$, $i = 1, \dots, 5$. Let $X_t = \pi^{-1}(t) \cap X$ be the fibre over t.

X is *locally Lipschitz trivial* over a subset $T_0 \subset T$ if for every point $t_0 \in T_0$ there exists a neighbourhood $U_0 \subset T_0$ of t_0 and a bi-Lipschitz homeomorphism $h : \pi^{-1}(U_0) \longrightarrow U_0 \times \mathbb{R}^n$ such that the diagram

$$\begin{array}{ccc} \pi^{-1}(U_0) & \xrightarrow{\ h\ } & U_0 \times \mathbb{R}^n \\ & \searrow \quad \swarrow_{pr} & \\ & U_0 & \end{array}$$

commutes, $h : \pi^{-1}(U_0) \cap X \longrightarrow U_0 \times X_{t_0}$ and h is the identity over X_{t_0}.

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Thus h induces a Lipschitz equivalence between X_{t_0} and the fibre X_t over every point t sufficiently close to t_0 in T_0.

Similar definitions can be given for germs at 0 of families of germs at 0 of subsets in each class $\mathfrak{A}_i$.

It is known (see [6] for a review of the results) that *if* $X \longrightarrow T$ *is a family of germs at* 0 *of subsets in any class* $\mathfrak{A}_i$, *then there exists a stratification of* T *with skeletons in the same class such that the family* X *is locally Lipschitz trivial over every stratum.*

Now we pass to curves in the base T.

In this paper a (parametrised subanalytic, abbreviated as s.an.) *curve* in a set $A \subset \mathbb{R}^n$ is a germ at $\mu = 0$ of a subanalytic map

$$p : [0, \varepsilon) \longrightarrow \overline{A}$$

such that $p(\mu) \in A$ for $\mu > 0$.

Let $p(\mu)$, $q(\mu)$ be two curves in the base T; we take an interval $[0, \varepsilon)$ such that both of them are defined on it. We shall say that X is *Lipschitz equivalent* over p and q (or, that p and q are *L-equivalent rel*X) if there exists a mapping $H : (0, \varepsilon) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that:

1° H is continuous,

2° for every $\mu > 0$, $H(\mu, \cdot) : (\mathbb{R}^n, X_{p(\mu)}) = (\mathbb{R}^n, X_{q(\mu)})$ is a bi-Lipschitz homeomorphism and the Lipschitz constant of both $H(\mu, \cdot)$ and its inverse is independent of μ.

Let us restrict ourselves for a moment to semialgebraic sets and semialgebraic curves. A curve $p(\mu)$ in the base T is of *complexity at most* N if its graph can be described (set theoretically) by at most N polynomial equations and inequalities of degree at most N.

We can now state Gromov's question: *is the set of L-equivalence classes rel* X *of curves of complexity at most* N *finite?*

The answer to this question is affirmative.

This answer is an immediate corollary to two propositions which we shall now state; they hold in the subanalytic and semialgebraic categories and constitute the main results of the paper.

To state the first proposition we fix some notation. Let us write $\mathbb{R}^m = \mathbb{R}^m_t$, the ambient space of the base T, and $\mathbb{R}^n = \mathbb{R}^n_x$, the ambient space of the fibres; by t or x we shall denote points of $\mathbb{R}^m_t$ or $\mathbb{R}^n_x$.

A *stratification* $\mathcal{Z} = \{Z^j\}$ of some Euclidean space $\mathbb{R}^N$ with skeletons Z^j in any of the classes $\mathfrak{A}_i$ is a sequence of sets

$$\mathbb{R}^N \supset Z^{N-1} \supset Z^{N-2} \supset \dots$$

such that all $Z^j \in \mathfrak{A}_i$ and every

$$\mathring{Z}^j = Z^j \setminus Z^{j-1}$$

is either empty or smooth j-dimensional; Z^j are skeletons of $\mathcal{Z}$ and $\mathring{Z}^j$ strata (thus strata are not assumed to be connected).

A stratification $\mathcal{Z}$ is *compatible* with a set if this set is a union of some connected components of strata.

A vector field v defined on a subset of $\mathbb{R}^N$ is *tangent* to $\mathcal{Z}$ (or *compatible* with $\mathcal{Z}$) if for every $x \in \mathring{Z}^j$, $v(x) \in T_x\mathring{Z}^j$, provided that $v(x)$ is defined. More generally, if v depends on some parameters μ, then v is tangent (compatible) to $\mathcal{Z}$ if for every $x \in \mathring{Z}^j$, $v(\mu, x) \in T_x\mathring{Z}^j$, provided that $v(\mu, x)$ is defined.

The flow of a vector field v will be denoted by χ^v_λ; λ is "time".

Let us now return to the family $X \longrightarrow T$. To be slightly more general, suppose we are given finitely many subsets $X_s \subset X$, also considered as families over T, with fibres $X_{s,t}$. Assume that X, X_s, $T \in \mathfrak{A}_i$, $i = 1, 3, 4, 5$.

Let $B^n_x \subset \mathbb{R}^n_x$ be the closed unit ball. Assume that X, X_s are subsets of B^n_x.

Proposition 1. *There exists a stratification $\mathfrak{T} = \{T^j\}$ of $\mathbb{R}^m_t$, compatible with T, with skeletons in $\mathfrak{A}_i$, with the following property: for every Lipschitz vector field v on $\mathbb{R}^m_t$, tangent to $\mathfrak{T}$, and every stratum $\mathring{T}^j$, there exists a function*

$$H_\lambda : \mathring{T}^j \times \mathbb{R}^n_x \longrightarrow \mathbb{R}^n_x, \qquad \lambda \in [0, 1],$$

such that:

1° *$H_\lambda(t, x)$ is continuous with respect to all variables,*

2° *for every λ and $t \in \mathring{T}^j$*

$$H_\lambda(t, \cdot) : \mathbb{R}^n_x \longrightarrow \mathbb{R}^n_x$$

is a bi-Lipschitz homeomorphism, and the Lipschitz constants of $H_\lambda(t, \cdot)$ and its inverse are independent of λ, t,

3° *for every λ, t*

$$H_\lambda(t, \cdot) : X_t \longrightarrow X_{\chi^v_\lambda(t)}$$

and, more generally,

$$H_\lambda(t, \cdot) : X_{s,t} \longrightarrow X_{s,\chi^v_\lambda(t)}.$$

Remarks.

1° It is pleasant to consider the map

$$(\lambda, x) \longmapsto \big(\chi^v_\lambda(t), H_\lambda(t, x)\big)$$

as a lifting of the isotopy $\lambda \longmapsto \chi^v_\lambda(t)$ of the point t; this lifting is thus bi-Lipschitz and preserves fibres of X and X_s's.

2° The Lipschitz constant of $H_\lambda(t, \cdot)$ and its inverse depend only on X, X_s, T, $\mathfrak{T}$ and v.

3° In the sequel we shall need a slight generalisation of Proposition 1 to the case of Lipschitz families of vector fields which depend continuously on one parameter μ (of course one could treat in the same way the case of more parameters).

Definition. *A Lipschitz family* v_μ of vector fields is a function $v_\mu(x)$, continuous with respect to all variables, Lipschitz with respect to x, with a Lipschitz constant independent of μ.

Proposition 1′. *In the notation of Proposition* 1, *there exists a stratification* $\mathfrak{T}$ *of* $\mathbb{R}^m_t$, *compatible with* T, *with the following property*: *for every Lipschitz family* v_μ *of vector fields on* $\mathbb{R}^m_t$, $\mu \in (0,\varepsilon)$, *tangent to* $\mathfrak{T}$, *and every stratum* $\mathring{T}^j$, *there exists a function*

$$H_{\mu,\lambda} : \mathring{T}^j \times \mathbb{R}^n_x \longrightarrow \mathbb{R}^n_x, \qquad \lambda \in [0,1], \quad \mu \in (0,\varepsilon),$$

which depends continuously on all variables μ, λ, t, x *and has all the properties of Proposition* 1; *in particular the Lipschitz constant of* $H_{\mu,\lambda}(t,\cdot)$ *and its inverse are independent of* μ, λ, t.

Since the sets X_s present no difficulty, we shall simply omit them in the sequel.

The second problem that we shall study deals with the following situation. Suppose we have two curves $p = p(\mu)$ and $q = q(\mu)$ in a stratum of some stratification of a space $\mathbb{R}^n$. We want to know when one of these curves, say p, can be "pushed" to the other one by the flow of a Lipschitz family v_μ of vector fields tangent to this stratification, i.e.

$$q(\mu) = \chi_1^{v_\mu}\big(p(\mu)\big) \quad \text{for all } \mu > 0.$$

There is an obvious obstacle: orders of distances of $p(\mu)$ and $q(\mu)$ to skeletons of this stratification must be the same.

Let us precise this point.

If $p : [0,\varepsilon) \longrightarrow \mathbb{R}^n$ is a curve and $A \subset \mathbb{R}^n$ a set in any class $\mathfrak{A}_i$, then, by Puiseux,

$$\mathrm{dist}(p(\mu), A) = c\mu^\gamma + o(\mu^\gamma)$$

for some $c > 0$, $\gamma \in \mathbb{Q} \cup \{\infty\}$, $\gamma \geq 0$. The exponent γ is the *order of the distance* from $p(\mu)$ to A.

Now let $q(\mu)$ be another (s.an.) curve in $\mathbb{R}^n$ related to $p(\mu)$ by the formula

$$q(\mu) = \chi_1^{v_\mu}(p(\mu)),$$

where v_μ is a Lipschitz family of vector fields which preserve A, i.e. for all μ and λ

$$\chi_\lambda^{v_\mu}(A) \subset A.$$

Then, as we shall see in detail in Section 1.5, the distances of $p(\mu)$ and $q(\mu)$ to A are of the same order.

In particular, if v_μ is tangent to a stratification with skeletons in $\mathfrak{A}_i$, then the distances of $p(\mu)$ and $q(\mu)$ to every skeleton are of the same order.

Definition. A subset $A \subset \mathring{Z}^j$ is *Lipschitz homogeneous with respect to* $\mathfrak{Z} = \{Z^j\}$ (abbreviated as *LHrel* $\mathfrak{Z}$) if there exists an N with the following property: for every pair $p(\mu), q(\mu)$ of curves in A having the same orders of distances to all skeletons Z^k, $k < j$, there exists a sequence of curves in A:

$$p = p_1, p_2, \ldots, p_N = q$$

and $N-1$ families $v_{1,\mu}, \ldots, v_{N-1,\mu}$ of Lipschitz vector fields on $\mathbb{R}^n$, tangent to $\mathfrak{Z}$, such that for all $i = 1, \ldots, N-1$

$$p_{i+1}(\mu) = \chi_1^{v_{i,\mu}}(p_i(\mu)).$$

Remark. We do not require $v_{i,\mu}$'s to preserve A.

Our second result is the following proposition.

Proposition 2. *Given any finite number of sets in* $\mathbb{R}^n$ *in any class* $\mathfrak{A}_i$, $i = 1, 3$, *there exists a stratification* $\mathfrak{Z} = \{Z^j\}$ *of* $\mathbb{R}^n$, *compatible with all of these sets, with skeletons* Z^j *in* $\mathfrak{A}_i$, *such that every stratum* $\mathring{Z}^j$ *is a finite union, not necessarily disjoint, of sets in* $\mathfrak{A}_i$ *which are LHrel* $\mathfrak{Z}$:

$$\mathring{Z}^j = \bigcup A^j_\beta, \quad A^j_\beta \text{ are } LHrel\, \mathfrak{Z}.$$

Corollary 1. *Let* $X \longrightarrow T$ *be a family as in Proposition* $1'$, $X, T \in \mathfrak{A}_i$, $i = 1, 3$. *Then there exists a stratification* $\mathfrak{T}$ *of* $\mathbb{R}^m_t$ *having both properties of Propositions* $1'$ *and* 2.

In fact, take any stratification of $\mathbb{R}^m_t$ satisfying the conclusion of Proposition $1'$; by Proposition 2 we can refine it to get also the conclusion of Proposition 2.

Another Lipschitz homogeneity property of subanalytic sets will be given in Proposition 4; it will be used in the proof of Proposition 2.

We shall now show how the above corollary yields an answer to Gromov's question.

Let $X, T \in \mathfrak{A}_3$. Take a stratification $\mathfrak{T} = \{T^j\}$ of $\mathbb{R}^m_t$ as in the corollary and decompose every $\mathring{T}^j$

$$\mathring{T}^j = \bigcup A^j_\beta, \quad A^j_\beta \text{ are } LHrel\, \mathfrak{T}.$$

The space $\mathfrak{F}_N$ of all curves in T of complexity not greater than N is the union of the spaces $\mathfrak{F}_{Nj\beta}$ of curves of complexity not greater than N in A^j_β. The bound of complexity implies that there are only finitely many rationals which are orders of distances of curves in $\mathfrak{F}_N$ to skeletons of $\mathfrak{T}$. In other words, if

$$\widetilde{\gamma} = (\gamma(0), \gamma(1), \dots, \gamma(m))$$

is any sequence of rationals and $\mathfrak{F}_{Nj\beta\widetilde{\gamma}} \subset \mathfrak{F}_{Nj\beta}$ the space of all curves in A^j_β having $\gamma(k)$ (for every $k < j$) as the order of distance to T^k, then, for only finitely many $\widetilde{\gamma}$, $\mathfrak{F}_{Nj\beta\widetilde{\gamma}} \neq \varnothing$ and

$$\mathfrak{F}_{Nj\beta} = \bigcup \mathfrak{F}_{Nj\beta\widetilde{\gamma}}.$$

It is enough to prove that any two $p, q \in \mathfrak{F}_{Nj\beta\widetilde{\gamma}}$ are *L-equivalent rel* X.

Let $p = p_1, \dots, p_N = q$ be curves in A^j_β such that

$$p_{i+1}(\mu) = \chi_1^{v_{i,\mu}}(p_i(\mu)).$$

By Proposition $1'$ to every $v_{i,\mu}$ there corresponds a function

$$H^i_{\mu,\lambda} : \mathring{T}^j \times \mathbb{R}^n_x \longrightarrow \mathbb{R}^n_x, \qquad \lambda \in [0,1],\ \mu \in (0, \varepsilon).$$

If

$$H^{(i)}_\mu = H^{(i)}_{\mu,1}\big(p_i(\mu), \cdot\big) : \mathbb{R}^n_x \longrightarrow \mathbb{R}^n_x,$$

then the composition of the $H^{(i)}_\mu$:

$$H_\mu = H^{(N-1)}_\mu \circ \dots \circ H^{(1)}_\mu : \mathbb{R}^n_x \longrightarrow \mathbb{R}^n_x$$

establishes L-equivalence of p and q *rel* X.

REMARK. As was pointed out by K. Kurdyka and A. Parusiński, the answer to Gromov's question can be obtained directly from Proposition 1.

In fact, the space C of all (germs of) curves in T of complexity at most N has a natural structure of a finite-dimensional semi-algebraic set. For every $p \in C$ choose $\varepsilon = \varepsilon(p)$ such that $p(\mu)$ is defined on $[0, \varepsilon(p)]$ and $\varepsilon(p)$ is a semi-algebraic function. Let

$$\widetilde{T} = \big\{(p, \mu) : p \in C,\ \mu \in [0, \varepsilon(p)]\big\}$$

and let $\widetilde{T} \to T$ be defined by

$$(p, \mu) \to p(\mu).$$

It is enough to apply Proposition 1 to the family over $\widetilde{T}$ induced from X by this map.

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1. Preliminaries

1.1. *The symbols* $\lesssim, \simeq$. We write, for non-negative functions,
$\varphi \lesssim \psi \Leftrightarrow \varphi \le C\psi$ for some constant C,
$\varphi \simeq \psi \Leftrightarrow \varphi \lesssim \psi$ and $\psi \lesssim \varphi$.
If φ, ψ depend also on parameters, we ask C to be independent of them.

1.2. *Boundary of a set.* $\partial A = \overline{A} \setminus A$.

1.3. *Distance to a set.* It is denoted by $d(x, A) = d_A(x)$; the Hausdorff distance of two (non-empty) sets A, B is $d(A, B)$, i.e. $d(A, B) = \inf\{d(a, B) : a \in A\}$; distance d_{Z^i} to a skeleton Z^i in a stratification will be abbreviated as d_i.

1.4. *Kirszbraun's theorem.* (See [1].) We need only a weak version of it. *If $f : A \longrightarrow \mathbb{R}$ is Lipschitz with a constant C, $A \subset \mathbb{R}^n$, then the formula*

$$F(x) = \sup_{a \in A}\big(f(a) - C|x - a|\big), \quad x \in \mathbb{R}^n$$

gives an extension of f being Lipschitz with the same constant C.

It follows that if f depends continuously on some parameters μ, i.e. $f(x, \mu)$, $x \in A$, is continuous as a function of all variables, and is Lipschitz with respect to x with a Lipschitz constant independent of μ, then f has a Lipschitz extension $F(x, \mu)$, $x \in \mathbb{R}^n$, with a Lipschitz constant independent of μ.

We shall write usually a vector field in the form

$$v = \sum v_i\, \partial/\partial x_i$$

and identify it with the sequence $(v_1, \ldots, v_n)$ of its components; so v can be identified with a mapping with values in $\mathbb{R}^n$.

Applying Kirszbraun's theorem to every component of v we get the following observation:

Let v be a Lipschitz vector field defined on a subset $A \subset \mathbb{R}^n$, which depends continuously on some parameters μ, with a Lipschitz constant C (as on p. 181); then *there exists a Lipschitz vector field V on $\mathbb{R}^n$, which depends continuously on μ, with the Lipschitz constant $C\sqrt{n}$* (of course we use the Euclidean metric on $\mathbb{R}^n$).

1.5. *Estimates for Lipschitz vector fields.* Let v be a Lipschitz vector field on $\mathbb{R}^n$, with a Lipschitz constant C; its flow χ^v_λ satisfies the equation

$$\chi^v_\lambda(x) = x + \int_0^\lambda v\big(\chi^v_s(x)\big)\, ds.$$

A standard calculation based on this formula gives

$\chi^v_\lambda(x) = x + u^v_\lambda(x)$, u^v_λ is Lipschitz with a Lipschitz constant $e^{C\lambda} - 1$;

$e^{-C\lambda}|x_1 - x_2| \le |\chi^v_\lambda(x_1) - \chi^v_\lambda(x_2)| \le e^{C\lambda}|x_1 - x_2|$.

Suppose now that $p(\mu)$, $q(\mu)$ are two curves in $\mathbb{R}^n$; let $|p(\mu) - q(\mu)| \simeq \mu^\gamma$. Suppose that v_μ is a Lipschitz family of vector fields on $\mathbb{R}^n$ with a Lipschitz constant C. Put

$$\widetilde{p}(\mu) = \chi_1^{v_\mu}(p(\mu)), \quad \widetilde{q}(\mu) = \chi_1^{v_\mu}(q(\mu))$$

(they need not be subanalytic). Then

$$e^{-C}|p(\mu) - q(\mu)| \le |\widetilde{p}(\mu) - \widetilde{q}(\mu)| \le e^{C}|p(\mu) - q(\mu)| \tag{1.1}$$

so $|\widetilde{p}(\mu) - \widetilde{q}(\mu)| \simeq \mu^\gamma$.

It follows that if v_μ preserve $A \in \mathfrak{A}_i$ $(i = 1, \dots, 5)$, $p(\mu)$ is a curve in $\mathbb{R}^n$ and $\widetilde{p}(\mu) = \chi_1^{v_\mu}(p(\mu))$, then

$$d(p(\mu), A) \simeq d(\widetilde{p}(\mu), A).$$

In fact, to prove $\lesssim$, we take a (subanalytic) curve $q(\mu)$ in $\overline{A}$ such that $|p(\mu) - q(\mu)| = d(p(\mu), A)$; then for $\widetilde{q}(\mu) = \chi_1^{v_\mu}(q(\mu))$ we have: $\widetilde{q}(\mu) \in A$ for $\mu > 0$ and $|\widetilde{p}(\mu) - \widetilde{q}(\mu)| \simeq \mu^\gamma \simeq d(p(\mu), A)$.

If $\gtrsim$ were wrong, there would exist a sequence $\mu_\nu \searrow 0$ such that

$$d(\widetilde{p}(\mu_\nu), A)/d(p(\mu_\nu), A) \longrightarrow 0.$$

Let $a_\nu \in A$ be points such that

$$|a_\nu - \widetilde{p}(\mu_\nu)| \le 2d(\widetilde{p}(\mu_\nu), A);$$

if $a^*_\nu = \chi_1^{-v_\mu}(a_\nu)$, then $a^*_\nu \in A$ and

$$|a^*_\nu - a_\nu|/d(p(\mu_\nu), A) \longrightarrow 0$$

which is impossible.

1.6. *Derivatives of subanalytic functions.* Let $f : U \longrightarrow \mathbb{R}$ be a subanalytic function, $U \in \mathfrak{A}_1$ open, $f \in C^\infty(U)$, and $|f| \lesssim 1$. Then for every integer $k > 0$ there exists a $Y \in \mathfrak{A}_1$, $\dim Y < n$, such that for all $x \in U$

$$|D^\alpha f(x)| \lesssim d_Y(x)^{-|\alpha|}, \tag{1.2}$$

$|\alpha| \le k$. If f is semialgebraic, then Y can be chosen semialgebraic. A proof is given in [2].

1.7. *Lipschitz functions with denominators.* In principle this notion will not be used, but we hope it may be helpful.

Let X be a metric space with distance denoted by $|x-y|$. Let $\varrho : X \longrightarrow \mathbb{R}^+$ be a bounded Lipschitz function, where $\mathbb{R}^+$ is the set of nonnegative reals.

DEFINITION. $f \in \mathrm{Lip}(X, \varrho, C)$ if $f : X \longrightarrow \mathbb{R}$ is bounded and for all $x, y \in X$,

$$|f(x) - f(y)| \leq C|x-y|/\min(\varrho(x), \varrho(y)).$$

We shall write $f \in \mathrm{Lip}(X, \varrho)$ if either the value of C is clear or $f \in \mathrm{Lip}(X, \varrho, C)$ for some C. In particular, $f \in \mathrm{Lip}(X, 1)$ means that f is bounded and Lipschitz.

If $f : X \longrightarrow \mathbb{R}^k$, $f = (f_1, \ldots, f_k)$, we shall write $f \in \mathrm{Lip}(X, \varrho)$ if all components f_i are in $\mathrm{Lip}(X, \varrho)$.

We list some obvious properties of the class $\mathrm{Lip}(X, \varrho)$ of scalar valued functions.

1° if $f, g \in \mathrm{Lip}(X, \varrho)$, $|g| \lesssim \varrho$, then $fg \in \mathrm{Lip}(X, 1)$; in particular, if $f \in \mathrm{Lip}(X, \varrho)$, then $\varrho f \in \mathrm{Lip}(X, 1)$;

2° if $f, g \in \mathrm{Lip}(X, \varrho)$, then $fg \in \mathrm{Lip}(X, \varrho)$;

3° if $f \in \mathrm{Lip}(X, 1)$, $|f| \lesssim \varrho$, then $f/\varrho \in \mathrm{Lip}(X, \varrho)$,

4° if $h : Y \longrightarrow X$ is Lipschitz, $f \in \mathrm{Lip}(X, \varrho)$, then $f \circ h \in \mathrm{Lip}(Y, \varrho \circ h)$.

1.8. *Lipschitz stratifications.* We refer to [6] for review of the subject. A stratification $\mathfrak{X} = \{X^j\}$ of $\mathbb{R}^n$ is Lipschitz if it has the following extension property of Lipschitz vector fields: *there exists a constant C such that for every compact K, $X^{l-1} \subset K \subset X^l$ for some l, and every Lipschitz vector field v, defined on K, with a Lipschitz constant M_1, bounded by M_2 (i.e. $|v(x)| \leq M_2$ for all $x \in K$), tangent to $\mathfrak{X}$, there exists a Lipschitz extension $\widetilde{v}$, defined on $\mathbb{R}^n$, with a Lipschitz constant $C(M_1 + M_2)$.*

This definition was first introduced in [5]. There is a simple way of constructing $\widetilde{v}$. To describe it, define for every $x \in \mathring{X}^l$

$$P_x : \mathbb{R}^n = T_x\mathbb{R}^n \longrightarrow T\mathring{X}^l \subset \mathbb{R}^n$$

as orthogonal projection.

Using Kirszbraun's theorem, we extend v to a Lipschitz vector field V, defined on $\mathbb{R}^n$; of course it need not be tangent to $\mathfrak{X}$. Put, for $x \in X^l$,

$$\widetilde{v}_l(x) = \begin{cases} v(x): & x \in K \\ P_x V(x): & x \in \mathring{X}^l. \end{cases}$$

For Lipschitz stratifications this formula gives a Lipschitz vector field $\widetilde{v}_l$. We can proceed further in a similar way. Extend $\widetilde{v}_l$ to a Lipschitz vector field V_1 defined on $\mathbb{R}^n$, and put

$$\widetilde{v}_{l+1}(x) = \begin{cases} \widetilde{v}_l(x): & x \in X^l \\ P_x V_1(x): & x \in \mathring{X}^{l+1} \end{cases}$$

etc. At the end we get $\widetilde{v}_n = \widetilde{v}$.

We note that original definition of a Lipschitz stratification ((1.6,k),(1.7,k) in [4], Def. 1.1 in [5], Def. 1.1 in [6]), equivalent to the above one (as proved in [5]), consisted of a big system of estimates on angles between tangent spaces to strata; this system guarantees that the above construction produces Lipschitz vector fields at every step.

It is known that for every set $A \in \mathfrak{A}_i$ in $\mathbb{R}^n$ there exists a Lipschitz stratification of $\mathbb{R}^n$, compatible with A, with skeletons in $\mathfrak{A}_i$; in general this stratification is not unique.

Remark. The above construction gives also a similar extension property for Lipschitz families of vector fields. Let $\mathfrak{X} = \{X^j\}$ be a Lipschitz stratification of $\mathbb{R}^n$. Let K be compact, $X^l \subset K \subset X^{l+1}$, and v_μ a Lipschitz family of vector fields of uniformly bounded length, tangent to $\mathfrak{X}$, defined for all $x \in K$; then there exists a Lipschitz family of vector fields $\widetilde{v}_\mu$, extending v_μ, defined for all $x \in \mathbb{R}^n$, tangent to $\mathfrak{X}$. The construction of $\widetilde{v}_\mu$ is as above; suppose namely that $\widetilde{v}_{\mu,k}$ is an extension defined for all $x \in X^k$; first we extend $\widetilde{v}_{\mu,k}$ to a Lipschitz family V_μ, defined for all $x \in \mathbb{R}^n$, and then we put

$$\widetilde{v}_{\mu,k+1}(x) = \begin{cases} \widetilde{v}_{\mu,k}(x): & x \in X^k \\ P_x V_\mu(x): & x \in \mathring{X}^{k+1}. \end{cases}$$

Again, the estimates of the original definition of a Lipschitz stratification, mentioned above, imply that $\widetilde{v}_{\mu,k+1}$ is a Lipschitz family of vector fields.

Example. Let $\mathring{X}^j$ be a stratum of a Lipschitz stratification. Let $\varrho : \mathring{X}^j \longrightarrow \mathbb{R}$ be the distance to X^{j-1}. Then the matrix-valued function $\mathring{X}^j \ni x \longmapsto P_x$ is in the class $\operatorname{Lip}(\mathring{X}^j; \varrho)$ as the estimates of the original definition show.

1.9. *L-regular sets.* They are well-known cylinders with an extra property introduced by A. Parusiński in [5].

A subanalytic set $A \subset \mathbb{R}^n$ is a k-dimensional L-regular set ($k \le n$) if, possibly after a linear change of coordinates, it is of the following form:

1° if $k = n$, then

$$A = \{(x', x_n) : x' = (x_1, \dots, x_{n-1}) \in A',\ \varphi(x') < x_n < \psi(x')\} \tag{1.3}$$

where A' is an $(n-1)$-dimensional L-regular set in $\mathbb{R}^{n-1}$, and φ, ψ are subanalytic functions on A' (or semialgebraic, semianalytic), smooth, bounded together with their first derivatives:

$$|\varphi|, |\psi|, \left|\frac{\partial \varphi}{\partial x_\alpha}\right|, \left|\frac{\partial \psi}{\partial x_\alpha}\right| \lesssim 1, \qquad \alpha = 1, \dots, n-1,$$

and $\varphi < \psi$ on A'.

2° if $k < n$, then A is the graph of F, where

$$F : A' \longrightarrow \mathbb{R}^{n-k}$$

is bounded subanalytic (or semialgebraic, or semianalytic) smooth function on an L-regular set $A' \subset \mathbb{R}^k$ of dimension k, and the first derivatives of F are bounded: $\left|\frac{\partial F}{\partial x_\alpha}\right| \lesssim 1$. Of course $\mathbb{R}^k$ is identified with the subspace $\{(x_1, \dots, x_k, 0, \dots, 0)\} \subset \mathbb{R}^n$ and $\mathbb{R}^{n-k}$ with $\{(0, \dots, 0, x_{k+1}, \dots, x_n)\} \subset \mathbb{R}^n$.

Remark. If we drop the condition of boundedness of first derivatives, we get the familiar notion of a cylinder. However, L-regular sets have very useful properties which cylinders in general do not have; some of them we shall mention below.

A basic fact ([5], [7]) states that *every set in $\mathfrak{A}_i$, $i = 1, 2, 3$, can be decomposed into a finite union of L-regular sets in $\mathfrak{A}_i$; these sets can be chosen to be disjoint.*

Every L-regular set A has Whitney's property with exponent 1, i.e. every pair of points $x, y \in A$ can be joined by a piecewise C^1 curve in A of length $\lesssim |x-y|$.

We shall now make an observation concerning the distance to the boundary ∂A of an L-regular set.

Suppose that $A \subset \mathbb{R}^n$ is n-dimensional,

$$A = \{(x', x_n) : x' \in A',\ \varphi(x') < x_n < \psi(x')\}$$

as before. Let $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-1}$ be the canonical projection. Then

$$\partial A = (\pi^{-1}(\partial A') \cap \overline{A}) \cup \operatorname{graph} \varphi \cup \operatorname{graph} \psi.$$

Put, for every $x = (x', x_n) \in A$

$$\operatorname{hordist}(x, \partial A) = d(x', \partial A') = d(\pi(x), \partial A'), \tag{1.4}$$

$$\operatorname{vertdist}(x, \partial A) = \min(\psi(x') - x_n, x_n - \varphi(x')) . \tag{1.5}$$

Clearly

$$d(x, \partial A) \simeq \min(\operatorname{hordist}(x, \partial A), \operatorname{vertdist}(x, \partial A));$$

in particular, $d(x, \partial A) \lesssim d(\pi(x), \partial A')$.

If A is k-dimensional, $k < n$, then, after a coordinate change, A is the graph of $F : A' \longrightarrow \mathbb{R}^{n-k}$, as before. Let $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ be the standard projection. Then

$$\partial A = \pi^{-1}(\partial A') \cap \overline{A},$$

and, for $x \in A$,

$$d(x, \partial A) \simeq d(\pi(x), \partial A'). \tag{1.6}$$

Finally we make a remark concerning tangent spaces to k-dimensional L-regular sets A in $\mathbb{R}^n$. Suppose $\mathfrak{X} = \{X^j\}$ is a Lipschitz stratification of $\mathbb{R}^n$ and A is an open subset of $\mathring{X}^k$ which is the graph of F as above. Let again $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ be the standard projection. Let π_A be the restriction of π to A. Then the norms of the differentials $(\pi_A)_{*x}, x \in A$, are bounded. The vector fields

$$e_\alpha = (\pi_A)_*^{-1}(\partial/\partial x_\alpha), \quad \alpha = 1, \ldots, k,$$

constitute a basis of tangent vector fields to A and

$$e_\alpha \in \operatorname{Lip}(A, \varrho), \qquad \varrho = d_{X^{k-1}}.$$

This follows again from the estimates of the original definition.

2. Liftings of vector fields in Lipschitz stratifications. In this section we shall work with the product space $\mathbb{R}_t^m \times B_y^N$, where B_y^N is the closed unit ball in $\mathbb{R}_y^n$, centred at 0; let $\pi : \mathbb{R}_t^m \times B_y^N \longrightarrow \mathbb{R}_t^m$ be the standard projection.

Let $\mathfrak{Z} = \{Z^j\}$ be a Lipschitz stratification of $\mathbb{R}_t^m \times B_y^N$ with skeletons in $\mathfrak{A}_i$, $i = 1, 3$. Let $\mathfrak{T} = \{T^j\}$ be any Lipschitz stratification of $\mathbb{R}_t^m$ compatible with all $\pi(Z^j)$, with skeletons in the same $\mathfrak{A}_i$ (it is important here to exclude semi-analytic sets). Very often we shall identify $\mathbb{R}_t^m$ with $\mathbb{R}_t^m \times 0 \subset \mathbb{R}_t^m \times B_y^N$; remark that then every stratum of $\mathfrak{T}$ is a submanifold of some stratum of $\mathfrak{Z}$.

If v is a vector field defined on a subset of $\mathbb{R}^m_t$, then a lift of v is a vector field $\widehat{v} = \widehat{v}(t, y)$, defined on a subset of $\mathbb{R}^m_t \times \mathbb{R}^N_y$ such that

$$\pi_* \widehat{v}(t, y) = v(t)$$

for all (t, y) where both sides are defined. In other words, identifying a vector field on $\mathbb{R}^m_t \times \mathbb{R}^N_y$ with a mapping with values in $\mathbb{R}^m_t \times \mathbb{R}^N_y$ and similarly on $\mathbb{R}^m_t$, we may say that $\widehat{v}(t, y)$ is a lift of $v(t)$ if $\widehat{v}$ is of the form

$$\widehat{v}(t, y) = \big(v(t), V(t, y)\big).$$

Now fix a stratum $\mathring{T}^j \subset \mathbb{R}^m_t$. For every $\varepsilon_0 > 0$ put

$$U_{\varepsilon_0}(\mathring{T}^j) = \big\{(t, y) : t \in \mathring{T}^j,\ |y| < \varepsilon_0 d_{j-1}(t)\big\},$$

where, as in Section 1.3, $d_{j-1}(t) = d_{T^{j-1}}(t)$.

Remark that

$$d_{Z^{j-1}}(t, 0) \geq d_{T^{j-1}}(t, 0) = d_{j-1}(t) \tag{2.1}$$

because

$$d_{Z^{j-1}}(t, 0) \geq d_{\pi(Z^{j-1})}(t) \geq d_{j-1}(t).$$

It follows that for $(t, y) \in U_{\varepsilon_0}(\mathring{T}^j)$

$$d_{Z^{j-1}}(t, y) \simeq d_{j-1}(t)$$

provided that $\varepsilon_0 < 1/2$ as we shall further suppose; more exactly the ratio of these distances is between $1/2$ and 2.

The aim of this section is the following proposition.

Proposition 3. *There exists an ε_0 such that every Lipschitz vector field v on $\mathbb{R}^m_t$, tangent to $\mathfrak{T}$, lifts to a Lipschitz vector field $\widehat{v}$, defined on $U_{\varepsilon_0}(\mathring{T}^j)$, tangent to $\mathfrak{Z}$.*

Remark. We may consider $U^j_{\varepsilon_0}(\mathring{T}^j)$ as a subanalytic neighbourhood of $\mathring{T}^j \times 0$ in $\pi^{-1}(\mathring{T}^j)$. More generally, for every rational $\rho > 0$, the sets

$$U_{\varepsilon_0,\rho}(\mathring{T}^j) = \big\{(t, y) : t \in \mathring{T}^j,\ |y| < \varepsilon_0 d^{\rho}_{j-1}(t)\big\}$$

are also subanalytic neighbourhoods of $\mathring{T}^j \times 0$ in $\pi^{-1}(\mathring{T}^j)$. So it is worth noticing that a lifting $\widehat{v}$ exists not only on some subanalytic neighbourhood of $\mathring{T}^j \times 0$ in $\pi^{-1}(\mathring{T}^j)$ but on a neighbourhood "with exponent" $\rho = 1$.

We shall start the proof with a slight strengthening of a lemma of A. Parusiński [8].

Quite generally, consider a Lipschitz stratification $\mathfrak{X} = \{X^j\}$ in $\mathbb{R}^n$ with skeletons in any $\mathfrak{A}_i$. We shall say that Lipschitz vector fields $e_0, \ldots, e_{j-1}$, defined on $\mathbb{R}^n$, tangent to $\mathfrak{X}$, *satisfy condition $P(C, \varepsilon)$ at a point $q \in \mathring{X}^j$* if there exist a $k < j$ and a point $q' \in \mathring{X}^k$ such that $|q - q'| = d_k(q)$ and

1° $e_0, \ldots, e_{j-1}$ are orthonormal in $B(q, \varepsilon d_k(q))$,

2° for every i, e_i has $C/d_i(q)$ as a Lipschitz constant,

3° $e_0, \ldots, e_{k-1}$ satisfy $P(C, \varepsilon)$ at q'.

LEMMA 2.1. *There exist C, ε, depending only on the stratification, such that for every j and every $q \in \mathring{X}^j$ there exist vector fields $e_0, \dots, e_{j-1}$ which satisfy $P(C, \varepsilon)$ at q.*

REMARK. The index k appearing in the definition of condition $P(C, \varepsilon)$ will be chosen at the beginning of the proof of the lemma; this choice will be also used in the proof of Lemma 2.2 below.

Proof of the lemma. We shall use increasing induction with respect to j; if $j = 0$ or 1 the lemma is obvious. Clearly it is enough to prove the lemma with the constants C, ε depending on j; for if the lemma is true for $C = C(j)$, $\varepsilon = \varepsilon(j)$, we may put at the end $C = \max C(j)$, $\varepsilon = \min \varepsilon(j)$.

By [8], there exists a C_2, depending only on the stratification, such that for every $q \in \mathring{X}^j$ there exist vector fields $e_0^*, \dots, e_{j-1}^*$, tangent to $\mathfrak{X}$, orthonormal at q and e_i^* has $C_2/d_i(q)$ as a Lipschitz constant.

Let C_1, ε_1 be constants such that the conclusion of the lemma holds with C_1 and ε_1 instead of C, ε for all $q \in \mathring{X}^l$, $l < j$.

Let A be any constant such that

$$A > 1, \qquad 2/(A-1) < \varepsilon_1.$$

Define k as the smallest number such that $k < j$ and

$$d_i(q) \le A d_{i+1}(q) \quad \text{for all } i, \qquad k \le i \le j-1.$$

Let $q' \in \mathring{X}^k$ realise the distance of q to X^k:

$$|q - q'| = d_k(q).$$

By induction hypothesis, there exist vector fields $e_0, \dots, e_{k-1}$ which satisfy $P(C_1, \varepsilon_1)$ at q'; in particular, they are orthonormal in $B\big(q', \varepsilon_1 d_l(q')\big)$ for some $l < k$.

Remark that

$$d_l(q') \ge d_l(q) - |q - q'| = d_l(q) - d_k(q) \ge (A-1) d_k(q);$$

thus if $|x - q| \le d_k(q)$, then

$$|x - q'| \le |x - q| + |q - q'| \le 2 d_k(q) \le \frac{2}{A-1}\, d_l(q) \le \varepsilon_1 d_l(q'),$$

so $B\big(q, d_k(q)\big) \subset B(q', \varepsilon_1 d_l(q'))$ and therefore $e_0, \dots, e_{k-1}$ are orthogonal in $B\big(q, d_k(q)\big)$. We have to add to them suitably chosen fields $e_k, \dots, e_{j-1}$.

Replacing the vector fields e_i^* by $\sum a_{ij} e_j^*$, where (a_{ij}) is a suitable orthogonal matrix (with constant entities) we may assume that

$$e_0(q), \dots, e_{k-1}(q), e_k^*(q), \dots, e_{j-1}^*(q)$$

are orthonormal. We have, for all $x \in B(q, d_i(q)/2C_2)$

$$\big|e_i^*(x) - e_i^*(q)\big| \le C_2 \frac{|x - q|}{d_i(q)} \le \frac{1}{2}\,.$$

For $i \le k$ we have $d_i(q) \ge d_k(q)$; for $i \ge k$ we have $d_i(q) \ge A^{-n} d_k(q)$. Thus for all i

$$d_i(q) \ge A^{-n} d_k(q)$$

and therefore

$$B\Big(q, \frac{d_i(q)}{2C_2}\Big) \supset B\Big(q, \frac{d_k(q)}{2C_2 A^n}\Big). \tag{2.2}$$

Remark that $e_0, \dots, e_{k-1}, e_k^*, \dots, e_{j-1}^*$ are all Lipschitz with a Lipschitz constant

$$\frac{L}{d_k(q)}, \qquad L = A^n(C_1 + C_2)$$

(this is, of course, a very rough estimate); we may assume that $C_1 + C_2 \geq 1$.

Put

$$M = (100n)^n L, \qquad \varepsilon = \frac{1}{100nM}$$

(again these choices are very far from the best).

In $B_2 = B(q, 2\varepsilon d_k(q))$ we have, for all $i \geq k$,

$$\big|e_i^*(x) - e_i^*(q)\big| \leq \frac{1}{2}.$$

Let

$$B_1 = B(q, \varepsilon d_k(q)), \qquad D = \mathbb{R}^n \setminus B_2, \qquad \varphi = \frac{d_D}{d_D + d_{B_1}},$$

where, for every subset $A \subset \mathbb{R}^n$, d_A is, as before, the distance to A. Clearly φ has the following properties:

$\varphi = 0$ outside B_2 (i.e. on D), $\varphi = 1$ on B_1, $0 \leq \varphi \leq 1$,

φ is Lipschitz with the constant

$$\frac{1}{\varepsilon d_k(q)} + \frac{2}{\varepsilon d_k(q)} = \frac{3}{\varepsilon d_k(q)}.$$

Now we apply the Gram-Schmidt orthonormalisation procedure to $e_k^*, \dots, e_{j-1}^*$ in the following way: we put

$$\breve{e}_k = e_k^* - \sum_{i=0}^{k-1} \langle e_k^*, e_i\rangle\, e_i, \quad \widetilde{e}_k = \frac{\breve{e}_k}{|\breve{e}_k|},$$

$$\breve{e}_{k+1} = e_{k+1}^* - \sum_{i=0}^{k-1} \big\langle e_{k+1}^*, e_i\big\rangle\, e_i - \big\langle e_{k+1}^*, \widetilde{e}_k\big\rangle\, \widetilde{e}_k, \quad \widetilde{e}_{k+1} = \frac{\breve{e}_{k+1}}{|\breve{e}_{k+1}|},$$

etc.

By induction on m (where $m \geq k$) we shall prove that in B_2

1° $\widetilde{e}_m$ has

$$\frac{(100n)^{m-k}}{d_k(q)}\, L$$

as a Lipschitz constant,

2° $$\frac{1}{2} \leq |\breve{e}_m| \leq \frac{3}{2};$$

this of course implies that all $\widetilde{e}_i$ are defined in B_2 and have $M/d_k(q)$ as a Lipschitz constant.

We shall treat only the induction step, since it covers also the first step; assume thus that 1° and 2° hold for $\widetilde{e}_{m-1}$.

Clearly both $\langle e_m^*, e_i\rangle\, e_i$ (for $i < k$) and $\langle e_m^*, \widetilde{e}_i\rangle\, \widetilde{e}_i$ (for $k \le i \le m-1$) are Lipschitz with a Lipschitz constant

$$2 \cdot \frac{3}{2}\,\frac{(100n)^{m-k-1}L}{d_k(q)} + \frac{L}{d_k(q)} \le \frac{4 \times 100^{m-k-1} n^{m-k-1} L}{d_k(q)}\,,$$

so for a Lipschitz constant of $\breve{e}_m$ we may take

$$L_m = n\,\frac{4 \times 100^{m-k-1} n^{m-k-1} L}{d_k(q)} = \frac{4 \times 100^{m-k-1} n^{m-k} L}{d_k(q)}\,.$$

Since $|\breve{e}_m(q)| = |e_m^*(q)| = 1$ and $\varepsilon L_m < 1/2 d_k(q)$, it follows that in B_2

$$\frac{1}{2} \le |\breve{e}_m| \le \frac{3}{2}$$

as claimed. Thus for a Lipschitz constant of $\widetilde{e}_m$ we may take

$$\frac{L_m}{\frac{1}{2}} + \frac{3}{2}\,\frac{L_m}{(\frac{1}{2})^2} < 10 L_m < \frac{(100n)^{m-k}L}{d_k(q)}\,.$$

1° and 2° are thus proved, and with them we know, as remarked above, that $\widetilde{e}_m$ are defined in B_2, Lipschitz with a Lipschitz constant $M/d_k(q)$.

Finally we put, for $k \le i \le j-1$,

$$e_i = \varphi \widetilde{e}_i + (1-\varphi)e_i^* \text{ in } B_2, \qquad 0 \text{ outside of } B_2.$$

In B_1 all $e_0, \ldots, e_{j-1}$ are orthonormal and the Lipschitz constant of every e_i is, for $i \ge k$,

$$\frac{3}{\varepsilon d_k(q)} + \frac{M}{d_k(q)} + \frac{3}{\varepsilon d_k(q)} \sup_{B_2} |e_i^*| + \frac{C_1}{d_k(q)}\,,$$

so is of the form $\frac{\text{const.}}{d_k(q)}$, where const. depends only on the stratification.

The lemma is proved. ∎

Let v be a Lipschitz vector field on $\mathbb{R}^n$, tangent to $\mathfrak{X}$. We shall study now its components λ_i in an orthonormal basis that satisfies condition $P(C, \varepsilon)$.

Lemma 2.2. *Let $q \in \mathring{X}^j$ and e_i satisfy $P(C,\varepsilon)$ at q; then in $B\big(q, \varepsilon d_k(q)\big) \cap X^j$ we may write*

$$v = \sum \lambda_i e_i, \qquad \lambda_i = \langle v, e_i\rangle\,,$$

and λ_i have the following properties:

1° $|\lambda_i(x)| \le K d_i(q)$ *for all* $x \in B\big(q, \varepsilon d_k(q)\big)$ *and* $i = 0, \ldots, j-1$;

2° λ_i *is Lipschitz*;

moreover, K and the Lipschitz constant of λ_i depend only on the Lipschitz constant of v, C and ε.

Proof. To prove the first estimate we use induction on j; we choose k, l and q' as in the proof of the previous lemma and we may assume that the statement is correct for $i < k$ in $B\big(q', \varepsilon_1 d_l(q')\big)$. Since the latter ball contains $B\big(q, \varepsilon d_k(q)\big)$ (see p. 190) and $d_i(q') \simeq d_i(q)$ [recall that $|d_i(q') - d_i(q)| \le |q' - q| = d_k(q)$ and $d_i(q) \ge A d_k(q)$, $A > 1$], our statement is true for $i < k$.

Let $i \geq k$. For $x \in B(q, \varepsilon d_k(q))$

$$\begin{aligned}|\lambda_i(x)| = |\langle v(x), e_i(x)\rangle| &\leq |\langle v(x) - v(q'), e_i(x)\rangle| + |\langle v(q'), e_i(x)\rangle| \\ &\lesssim |x - q'| + \sum_{s<k} |\lambda_s(q')|\, |\langle e_s(q'), e_i(x)\rangle| \\ &\lesssim d_k(q) + \sum_{s<k} d_s(q)\, |\langle e_s(q') - e_s(x), e_i(x)\rangle + \langle e_s(x), e_i(x)\rangle| \,.\end{aligned}$$

Since $s < k$ and $i \geq k$, $\langle e_s(x), e_i(x)\rangle = 0$. From

$$|e_s(q') - e_s(x)| \leq C\, \frac{|q' - x|}{d_s(q)}$$

we get

$$|\lambda_i(x)| \lesssim d_k(q) \simeq d_i(q), \quad i \geq k.$$

To prove that λ_i are Lipschitz in $B\big(q, \varepsilon d_k(q)\big) \cap X^j$ we write, for $x, x' \in B\big(q, \varepsilon d_k(q)\big)$,

$$\begin{aligned}|\lambda_i(x) - \lambda_i(x')| &= |\langle v(x), e_i(x)\rangle - \langle v(x'), e_i(x')\rangle| \\ &\leq |\langle v(x) - v(x'), e_i(x)\rangle| + |\langle v(x'), e_i(x) - e_i(x')\rangle| \,.\end{aligned}$$

The first summand is $\lesssim |x - x'|$. We write the second as

$$\Big|\sum_{s=0}^{j-1} \lambda_s(x')\, \langle e_s(x'), e_i(x) - e_i(x')\rangle\Big|.$$

Since $\langle e_s, e_i\rangle = \delta_{si}$ in $B\big(q, \varepsilon d_k(q)\big)$, we have

$$\begin{aligned}0 &= \langle e_s(x), e_i(x)\rangle - \langle e_s(x'), e_i(x')\rangle \\ &= \langle e_s(x) - e_s(x'), e_i(x)\rangle - \langle e_s(x'), e_i(x') - e_i(x)\rangle \,.\end{aligned}$$

Thus for every s we have

$$\begin{aligned}|\lambda_s(x')\, \langle e_s(x'), e_i(x) - e_i(x')\rangle| &= |\lambda_s(x')|\, |\langle e_s(x) - e_s(x'), e_i(x)\rangle| \\ &\lesssim d_s(q)\, |e_s(x) - e_s(x')| \lesssim |x - x'| \,.\end{aligned}$$

The lemma is proved. ∎

We return now to the situation of the beginning of this section; thus we have the spaces $\mathbb{R}^m_t \times B^N_y$, $\mathbb{R}^m_t$, with stratifications $\mathfrak{Z}$ and $\mathfrak{T}$, respectively, and the projection $\pi : \mathbb{R}^m_t \times B^N_y \longrightarrow \mathbb{R}^m_t$. We shall apply Lemmas 2.1 and 2.2 to $\mathbb{R}^n = \mathbb{R}^m_t$, $\mathfrak{X} = \mathfrak{T}$; we shall write t_0 instead of q.

Take a stratum $\mathring{T}^j$ and a point $t_0 \in \mathring{T}^j$; let $e_0, \dots, e_{j-1}$ be the vector fields on $\mathbb{R}^m_t$ constructed in Lemma 2.1 which satisfy $P(C, \varepsilon)$ at t_0. The symbol $d_i(t_0)$ denotes, as before, $d_{T^i}(t_0)$.

Lemma 2.3. *The vector fields e_i extend from $B\big(t_0, \varepsilon d_k(t_0)\big) \subset \mathbb{R}^m_t$ to Lipschitz vector fields $E_i(t, y)$, defined on $\mathbb{R}^m_t \times B^N_y$, tangent to $\mathfrak{Z}$, such that the Lipschitz constant of E_i is $C_0/d_i(t_0)$, where C_0 depends only on the stratifications $\mathfrak{Z}$ and $\mathfrak{T}$.*

Proof. We keep the notation of Lemma 2.1; in particular the index k and the constants A and ε have the same meaning. By an induction argument on j we may assume that

$e_0, \dots, e_{k-1}$ extend as stated. Consider $e_k, \dots, e_{j-1}$; their Lipschitz constant does not exceed $C/d_{j-1}(t_0)$.

Consider the vector fields $e'_k, \dots, e'_{j-1}$ defined on $B(t_0, \varepsilon d_k(t_0)) \cup Z^{j-1}$ by the formula: $e'_i = e_i$ on $B\big(t_0, \varepsilon d_k(t_0)\big)$, $e'_i = 0$ on Z^{j-1}. We shall show that the Lipschitz constant of e'_i is

$$\frac{\max(C, 2A^n)}{d_k(t_0)}\,.$$

To prove this estimate it is enough to show that for every $t \in B\big(t_0, \varepsilon d_k(t_0)\big)$

$$d(t, Z^{j-1}) \geq \frac{d_k(t_0)}{2A^n}\,;$$

of course we are identifying t with $(t, 0) \in \mathbb{R}^m_t \times B^N_y$.

Recall that $d_{j-1}(t_0) \geq A^{-n} d_k(t_0)$, so

$$B\big(t_0, \varepsilon d_k(t_0)\big) \subset B\big(t_0, \varepsilon A^n d_{j-1}(t_0)\big) \subset B\Big(t_0, \frac{d_{j-1}(t_0)}{2}\Big);$$

also $d(t, Z^{j-1}) \geq d(t, T^{j-1})$ as remarked in (2.1). Therefore

$$B\big(t_0, \varepsilon d_k(t_0)\big) \subset B\Big((t_0, 0), \frac{d(t_0, Z^{j-1})}{2}\Big)$$

which implies the desired estimate.

Clearly every e'_i is tangent (where defined) to strata of dimension not exceeding j in $\mathfrak{Z}$. Thus it extends, by the basic property of Lipschitz stratifications, to a Lipschitz vector field E_i, defined on $\mathbb{R}^m_t \times B^N_y$, tangent to $\mathfrak{Z}$, with the Lipschitz constant

$$C_2\,\frac{\max(C, 2A^n)}{d_k(t_0)}\,,$$

where C_2 depends only on $\mathfrak{Z}$. This proves the lemma with $C_0 = C_2 \max(C, 2A^n)$. ■

We keep the previous notation; we have thus $t_0 \in \mathring{T}^j$ with vector fields e_i which satisfy $P(C, \varepsilon)$ at t_0. Let

$$U_{\varepsilon_0}(\mathring{T}^j, t_0) = U_{\varepsilon_0}(t_0) = \big\{(t, y) : t \in B\big(t_0, \varepsilon d_k(t_0)\big) \cap \mathring{T}^j,\ |y| < \varepsilon_0 d_{j-1}(t_0)\big\};$$

alternatively,

$$U_{\varepsilon_0}(t_0) = U_{\varepsilon_0}(\mathring{T}^j) \cap \pi^{-1}\big(B(t_0, \varepsilon d_k(t_0))\big).$$

Lemma 2.4. *There exists an ε_0, depending only on the stratifications, such that for every $t_0 \in \mathring{T}^j$, every e_i has a Lipschitz lifting $\widehat{e}_i(t, y)$, defined on $U_{\varepsilon_0}(t_0)$, tangent to $\mathfrak{Z}$, with a Lipschitz constant $C_1/d_i(t_0)$, where C_1 depends only on $\mathfrak{Z}$ and $\mathfrak{T}$.*

Proof. Let $E_i(t, y)$ be the extensions of $e_i(t)$ constructed in Lemma 2.3; the constant C_0 has the same meaning as in Lemma 2.3. We may assume that $C_0 \geq C$. Let

$$E'_i(t, y) = \pi_* E_i(t, y) \quad \text{for } (t, y) \in U_{\varepsilon_0}(t_0).$$

Since $E'_i(t, 0) = e_i(t)$, we have in $U_{\varepsilon_0}(t_0)$

$$|E'_i(t, y) - e_i(t)| \leq C_0 \varepsilon_0\, \frac{d_{j-1}(t_0)}{d_i(t_0)}\,.$$

Thus we may write, for $i = 0, \dots, j-1$,

$$E_i'(t,y) = e_i(t) + \sum_{p=0}^{j-1} a_{ip}(t,y) e_p(t),$$

$$|a_{ip}(t,y)| \le C_0\varepsilon_0 \frac{d_{j-1}(t_0)}{d_i(t_0)} \le C_0\varepsilon_0$$

for all p.

Obviously

$$e_i(t) = \sum b_{ip}(t,y) E_p'(t,y),$$

where $b_{ip}(t,y)$ are elements of the matrix $(I+A)^{-1}$, where $A = (a_{ip}(t,y))$. The fields

$$\widehat{e}_i(t,y) = \sum b_{ip}(t,y) E_p'(t,y)$$

are liftings of e_i, tangent to $\mathfrak{Z}$.

It remains to prove that for ε_0 sufficiently small $I+A$ is invertible and to estimate the Lipschitz constant of every $\widehat{e}_i$.

The first fact is obvious: since $d_{j-1} \le d_i$ for all $i \le j$,

$$|a_{ip}(t,y)| \le C_0\varepsilon_0,$$

so if ε_0 is small enough (for instance if $C_0\varepsilon_0 < 1/(2m)$ as we shall further suppose), $\|A\| \le 1/2$, and

$$(I+A)^{-1} = \sum_{s=0}^{\infty} (-A)^s.$$

We shall now prove that every $a_{ip}(t,y)$ is Lipschitz in $U_{\varepsilon_0}(t_0)$ with the Lipschitz constant $C_0(2 + C\varepsilon_0)/d_i(t_0)$. In fact, writing z for (t,y) and z' for (t',y'), we have

$$\begin{aligned} |a_{ip}(z) - a_{ip}(z')| &= |\langle E_i'(z) - e_i(t), e_p(t)\rangle - \langle E_i'(z') - e_i(t'), e_p(t')\rangle| \\ &\le |\langle E_i'(z) - E_i'(z') - e_i(t) + e_i(t'), e_p(t)\rangle| + |\langle E_i'(z') - e_i(t'), e_p(t) - e_p(t')\rangle| \\ &\le \frac{2C_0}{d_i(t_0)} |z'-z| + |E_i'(z') - e_i(t')| \frac{C|t'-t|}{d_p(t_0)} \\ &\le \frac{2C_0 + CC_0\varepsilon_0}{d_i(t_0)} |z'-z| = \frac{C_0(2+C\varepsilon_0)}{d_i(t_0)} |z'-z|. \end{aligned}$$

Let $a_{ip}^{(s)}(z)$ be the elements of the matrix $(-A)^s$. We shall prove by induction on s that for every $s > 0$ and $z \in U_{\varepsilon_0}(t_0)$

$$\left|a_{ip}^{(s)}(z)\right| \le \frac{mC_0\varepsilon_0}{2^{s-1}} \frac{d_{j-1}(t_0)}{d_i(t_0)} \le \frac{1}{2^s}$$

and that $a_{ip}^{(s)}(z)$ is Lipschitz with the Lipschitz constant

$$\frac{sm^2}{2^{s-1}} \frac{C_0(2+C\varepsilon_0)}{d_i(t_0)}.$$

In fact, first of all

$$\left|a_{ip}^{(s)}\right| = \left|\sum_q a_{iq} a_{qp}^{(s-1)}\right| \le m \frac{C_0\varepsilon_0 d_{j-1}(t_0)}{d_i(t_0)} \|A\|^{s-1} \le \frac{mC_0\varepsilon_0}{2^{s-1}} \frac{d_{j-1}(t_0)}{d_i(t_0)}.$$

Then, $a_{ip}^{(s)}(z) - a_{ip}^{(s)}(z')$ are elements of the matrix

$$(-A)^s(z) - (-A)^s(z') = \sum_{k+l=s-1} (-A)^k(z)\big[(-A)(z) - (-A)(z')\big](-A)^l(z')$$

so

$$\begin{aligned}
&\big|a_{ip}^{(s)}(z) - a_{ip}^{(s)}(z')\big| \\
&\qquad \le \sum_{k+l=s-1} \sum_{q,r} \Big|a_{iq}^{(k)}(z)\big(a_{qr}(z) - a_{qr}(z')\big)a_{rp}^{(l)}(z')\Big| \\
&\qquad \le \sum_{k+l=s-1} \sum_{q,r} \frac{mC_0\varepsilon_0}{2^{k-1}} \frac{d_{j-1}(t_0)}{d_i(t_0)} \frac{C_0(2+C\varepsilon_0)}{d_q(t_0)} \frac{mC_0\varepsilon_0}{2^{l-1}} \frac{d_{j-1}(t_0)}{d_r(t_0)} |z-z'| \\
&\qquad \le \sum\sum \frac{1}{2^k} \frac{C_0(2+C\varepsilon_0)}{d_i(t_0)} \frac{1}{2^l} |z-z'| \le \frac{sm^2}{2^{s-1}} \frac{C_0(2+C\varepsilon_0)}{d_i(t_0)} |z-z'|.
\end{aligned}$$

After summing over s we deduce that b_{ip} are Lipschitz with a Lipschitz constant $K/d_i(t_0)$, where K depends only on the stratifications. It follows that

$$|b_{ip}| \le K \frac{d_{j-1}(t_0)}{d_i(t_0)} \text{ on } U_{\varepsilon_0}(t_0).$$

It is now easy to prove that $\widehat{e}_i$ are Lipschitz; in fact, remembering that on $U_{\varepsilon_0}(t_0)$

$$|E_p(z)| \le 1 + C_0 \frac{d_{j-1}(t_0)}{d_i(t_0)} \le 1 + C_0,$$

$$\begin{aligned}
|\widehat{e}_i(z) - \widehat{e}_i(z')| &\le \sum |b_{ip}(z) - b_{ip}(z')|\,|E_p(z')| + \sum |b_{ip}(z')|\,|E_p(z) - E_p(z')| \\
&\le \sum \bigg(\frac{K|z-z'|}{d_i(t_0)} |E_p(z')| + K \frac{d_{j-1}(t_0)}{d_i(t_0)} C_0 \frac{|z-z'|}{d_p(t_0)} \bigg) \\
&\le (1+2C_0)K \frac{|z-z'|}{d_i(t_0)}
\end{aligned}$$

and the lemma is proved with $C_1 = (1+2C_0)K$. ∎

Corollary 2. *Let v be a Lipschitz vector field on $\mathbb{R}^m_t$, tangent to $\mathfrak{T}$. Then, for every $t_0 \in \mathring{T}^j$, v has a lift $\widehat{v}_{t_0}$, defined on $U_{\varepsilon_0}(t_0)$, Lipschitz and tangent to $\mathfrak{Z}$.*

Proof. We write $v = \sum \lambda_i e_i$ in $B\big(t_0, \varepsilon d_k(t_0)\big) \cap T^j$, $\lambda_i = \langle v, e_i \rangle$. Let $\widehat{e}_i(t,y) = \widehat{e}_i(z)$ be the lifts of e_i constructed in Lemma 2.4. The field

$$\widehat{v}_{t_0}(t,y) = \sum \lambda_i(t)\widehat{e}_i(t,y)$$

is clearly a lift of v, tangent to $\mathfrak{Z}$. The estimates of Lemmas 2.2 and 2.4 imply that $\widehat{v}_{t_0}$ is Lipschitz. ∎

To prove Proposition 3 we shall glue together the $\widehat{v}_{t_0}$'s by means of a partition of unity. The following lemma is similar to Lemma 3.1 in [3]; the latter treats only the case $\alpha = 2$, but the proof requires almost no change.

Lemma 2.5. *Let $K \subset \mathbb{R}^n$ be compact, $\alpha > 0$. There exist numbers $M_0, M_1 > 0$ and a family of functions $\varphi_i \ge 0$ $(i \in I)$ with the following properties*:

1° *the family of all supports* $\operatorname{supp}\varphi_i \cap K = \varnothing$ *for all* i, *and for every* $x \in \mathbb{R}^n \setminus K$ *there exist at most* M_0 *functions* φ_i *such that* $x \in \operatorname{supp}\varphi_i$,

2° $\sum \varphi_i = 1$ *on* $\mathbb{R}^n \setminus K$,

3° *for every* $i \in I$, $\operatorname{diam}(\operatorname{supp}\varphi_i) \leq \alpha d(K, \operatorname{supp}\varphi_i)$,

4° *every* φ_i *is Lipschitz with a Lipschitz constant*

$$\frac{M_1}{d(K, \operatorname{supp}\varphi_i)}\,.$$

Proof (after [3]). For every $p = 0, 1, 2, \ldots$ let C_p be the family of all cubes obtained by cutting $\mathbb{R}^n$ by all hyperplanes $x_i = m/2^p$, $m \in \mathbb{Z}$. The diameter of every cube in C_p is of course $\sqrt{n}/2^p$. Let K_0 be the family of all $S \in C_0$ such that

$$d(S, K) \geq \frac{2\sqrt{n}}{\alpha}\,.$$

Inductively, let K_p be the family of all $S \in C_p$ such that

$$d(S, K) \geq \frac{\sqrt{n}}{2^{p-1}\alpha} \text{ and } S \not\subseteq \bigcup_{j<p} K_j\,.$$

For every $S \in I$ we have, obviously, $d(S, K) \geq 2\operatorname{diam}(S)/\alpha$.

Let x_S be the centre of S and let S' be the cube centred at x_S with $\operatorname{diam}(S') = \lambda \operatorname{diam}(S)$, where $\lambda = (2+\alpha)/(1+\alpha)$; then

$$\operatorname{diam}(S') \leq \alpha d(S', K).$$

In fact,

$$\begin{aligned} d(S', K) \geq d(S, K) - (\lambda - 1)\operatorname{diam}(S) &\geq \frac{2}{\alpha}\operatorname{diam}(S) - (\lambda - 1)\operatorname{diam}(S) \\ &= \Big(\frac{2}{\alpha} - \lambda + 1\Big)\lambda^{-1}\operatorname{diam}(S') = \alpha^{-1}\operatorname{diam}(S'). \end{aligned}$$

For every $S \in I$ let $f_S(x) = d(x, S)$, $g_S(x) = d(x, \mathbb{R}^n \setminus S')$,

$$\psi_S = \frac{g_S}{f_S + g_S}\,, \qquad \varphi_S = \frac{\psi_S}{\sum_{T \in I}\psi_T}\,.$$

The family φ_S, $S \in I$, satisfies all the requirements of the lemma.

We shall now apply this lemma taking $\mathbb{R}^m_t$ instead of $\mathbb{R}^n$ and $Z^{j-1} \cap \mathbb{R}^m_t$ instead of K; for α we take ε.

Let $S \in I$ and let t_S be its centre (denoted before by x_S); let S' be the cube defined in the proof of Lemma 2.5. We note that

$$S' \subset B\big(t_S, \varepsilon d_k(t_S)\big).$$

In fact, to prove it one has to know that

$$\operatorname{diam}(S') < \varepsilon d_k(t_S) = \varepsilon d(t_S, K).$$

This follows at once from

$$\operatorname{diam}(S') = \lambda \operatorname{diam}(S) < \frac{1}{2}\,\lambda\varepsilon d(S, K) \leq \varepsilon d(t_S, K).$$

Now the required lifting $\widehat{v}$ of v is given by

$$\widehat{v}(t,y)=\sum_{S\in I}\varphi_S(t)\widehat{v}_{T_S}(t,y),\qquad (t,y)\in U_{\varepsilon_0}.$$

It is obvious that $\widehat{v}$ is a lifting of v and that $\widehat{v}$ is tangent to $\mathfrak{Z}$. To prove that $\widehat{v}$ is Lipschitz, it is enough to write

$$\widehat{v}(t,y)=v(t)+\sum_{S\in I}\varphi_S(t)\big(\widehat{v}_{T_S}(t,y)-v(t)\big)$$

and to recall that $v,\widehat{v}$ are Lipschitz, $|\widehat{v}(t,y)-v(t)|\lesssim d_{j-1}(t_S)$ on $U_{\varepsilon_0}(t_S)$, and that the Lipschitz constant of φ_S is $\lesssim 1/d_{j-1}(t_S)$ since

$$d(\operatorname{supp}\varphi_S,K)\ge d(S',K)\ge(1-\varepsilon)d_{j-1}(t_S).$$

Proposition 3 is thus proved. ∎

A minor generalisation of it is a version for Lipschitz families of vector fields.

PROPOSITION 3′. *There exists an ε_0 such that every Lipschitz family v_μ on $\mathbb{R}^m_t$, tangent to $\mathfrak{T}$, lifts to a Lipschitz family $\widehat{v}_\mu$ of vector fields on $U_{\varepsilon_0}(\mathring{T}^j)$, tangent to $\mathfrak{Z}$.*

3. Proof of Proposition 1. As mentioned on p. 181, we shall prove it only for one family $X\longrightarrow T$. Proposition 1′ can be proved along the same lines, using Proposition 3′ instead of Proposition 3.

We start with the given family

$$\begin{array}{lcl} X & \subset & \mathbb{R}^n_t\times\mathbb{R}^n_x\\ \downarrow & & \quad\downarrow\pi\\ T & \subset & \quad\mathbb{R}^m_t,\end{array}$$

$X,T\in\mathfrak{A}_i$, $i=1,3,4,5$. Let $\mathbb{R}^1_s$ be a copy of $\mathbb{R}$ and we introduce the family $CX\subset\mathbb{R}^m_t\times\mathbb{R}^1_s\times\mathbb{R}^n_x$ of cones over X:

$$CX=\big\{(t,s,sx):t\in T,\ s\in\mathbb{R}^1_s,\ x\in X_t\big\}.$$

We shall consider CX as family over T:

$$\begin{array}{lcl} CX & \subset & \mathbb{R}^m_t\times\mathbb{R}^1_s\times\mathbb{R}^n_x\\ \downarrow & & \qquad\downarrow\pi\\ T & \subset & \qquad\mathbb{R}^m_t.\end{array}$$

Thus the fibre $(CX)_t$ is the cone over the fibre X_t. T imbeds in CX in the obvious way: $t\longmapsto(t,0,0)$; of course $(t,0,0)$ is the vertex of the cone $(CX)_t$.

Also X imbeds in CX: $(t,x)\longmapsto(t,1,x)$.

We put $y=(s,x)$, $\mathbb{R}^1_s\times\mathbb{R}^n_x=\mathbb{R}^N_y$. Let B^N_y be the closed unit ball in $\mathbb{R}^N_y$ centred at 0. Let $\mathfrak{Z}=\{Z^j\}$ be a Lipschitz stratification of $\mathbb{R}^m_t\times B^N_y$ compatible with CX with skeletons in $\mathfrak{A}_i$; as in the previous section, let $\mathcal{T}$ be any Lipschitz stratification of $\mathbb{R}^m_t$ compatible with T and all $\pi(Z^j)$.

Let v be any Lipschitz vector field on $\mathbb{R}^m_t$, tangent to $\mathcal{T}$. Fix a stratum $\mathring{T}^j$ of T. By Proposition 3, there exists an ε_0 such that v lifts to a Lipschitz vector field $\widehat{v}$, tangent to $\mathfrak{Z}$, defined on $U_{\varepsilon_0}(T^j)$. Put

$$(3.1)\qquad \begin{gathered} d_{j-1}(t) = \varrho(t) \\ \widehat{v}(t,s,x) = \big(v(t), V(t,s,x)\big), \text{ where } V(t,s,x) \in T_{(s,x)}\left(\mathbb{R}^1_s \times \mathbb{R}^n_x\right) \end{gathered}$$

for simplicity of notation, and denote the flow of $\widehat{v}$ by

$$\lambda \longmapsto (\chi^v_\lambda, \varphi_\lambda, h_\lambda),$$

i.e. the image of a point (t,s,x) after time λ is

$$(\chi^v_\lambda(t), \varphi_\lambda(t,s,x), h_\lambda(t,s,x)).$$

We make three remarks.

1. Observe that for $|s|$ sufficiently small and all $t \in \mathring{T}^j$, $x \in X_t$ and $\lambda \in [0,1]$ (actually any finite interval would do, for the price of choosing an appropriate constant appearing implicitly in the signs $\lesssim, \simeq$ below)

$$(3.2)\qquad |\varphi_\lambda(t,s,x)| \simeq |s|, \quad |h_\lambda(t,s,x)| \lesssim |s|.$$

In fact, the flow of $\widehat{v}$ is bi-Lipschitz and preserves the family of vertices of cones $(CX)_t$, i.e. $T \times \{0\} \times \{0\}$; clearly for $(s,x) \in (CX)_t$

$$|(s,x)| = \text{distance of } (s,x) \text{ to the vertex of } (CX)_t \simeq |s|.$$

2. Recall (1.1) that for $\lambda \in [0,1]$

$$(3.3)\qquad w_\lambda(t,s,x) = \varphi_\lambda(t,s,x) - s, \quad u_\lambda(t,s,x) = h_\lambda(t,s,x) - x$$

have Lipschitz constant $C\lambda$, where C depends only on $\widehat{v}$.

3. $\varrho(\chi^v_\lambda(t)) \simeq \varrho(t)$ for $t \in \mathring{T}^j$, $\lambda \in [0,1]$. In fact, if $t' \in T^{j-1}$ is one of the closest points in $\mathring{T}^{j-1}$ to t, then $\varrho(t) = |t - t'|$; since χ^v_λ preserves T^{j-1}, $\chi^v_\lambda(t') \in \mathring{T}^j$ and

$$\varrho(\chi^v_\lambda(t)) \le |\chi^v_\lambda(t) - \chi^v_\lambda(t')| \lesssim |t - t'| = \varrho(t).$$

To prove the converse inequality it is enough to reverse the direction of "time" λ.

It follows that if ε_1 is sufficiently small, then for all $t \in \mathring{T}^j$, all x such that $|x| \le 1$, all s such that $|s| < \varepsilon_1 \varrho(t)$, the trajectory of (t,s,sx) under the flow of $\widehat{v}$ stays in $U_{\varepsilon_0}(T^j)$ for time λ in $[0,1]$.

Now define a map $\widetilde{H}_\lambda$ by the formula

$$\widetilde{H}_\lambda(t_0,x) = \frac{h_\lambda\big(t_0, \varepsilon_1\varrho(t_0), \varepsilon_1\varrho(t_0)x\big)}{\varphi_\lambda\big(t_0, \varepsilon_1\varrho(t_0), \varepsilon_1\varrho(t_0)x\big)},$$

where $t_0 \in \mathring{T}^j$, $x \in X_{t_0}$, $\lambda \in [0,1]$.

Remark that $\widetilde{H}_\lambda$ is well defined: $\varrho(t_0) \neq 0$ and, by our first remark above,

$$\varphi_\lambda\big(t_0, \varepsilon_1\varrho(t_0), \varepsilon_1\varrho(t_0)x\big) \simeq \varepsilon_1\varrho(t_0),$$

so the denominator does not vanish. Obviously $\widetilde{H}_\lambda(t_0,x)$ is a continuous function of $(\lambda, t_0, x) \in [0,1] \times \mathring{T}^j \times X_{t_0}$; clearly $\widetilde{H}_0(t_0,x) = x$.

It is easy to see that $\widetilde{H}_\lambda(t_0,x) \in X_{\chi^v_\lambda(t_0)}$. In fact, $x \in X_{t_0}$, so

$$\big(\varepsilon_1\varrho(t_0), \varepsilon_1\varrho(t_0)x\big) \in (CX)_{t_0}$$

and therefore

$$\left(\varphi_\lambda\left(\varepsilon_1\varrho(t_0),\varepsilon_1\varrho(t_0)x\right),h_\lambda\left(\varepsilon_1\varrho(t_0),\varepsilon_1\varrho(t_0)x\right)\right)\in(CX)_{\chi^v_\lambda(t_0)}$$

because the flow of $\widehat{v}$ preserves CX. Thus, writing

$$h_\lambda\big(\varepsilon_1\varrho(t_0),\varepsilon_1\varrho(t_0)x\big)=\varphi_\lambda\big(\varepsilon_1\varrho(t_0),\varepsilon_1\varrho(t_0)x\big)\widetilde{H}_\lambda(t_0,x)$$

we get $\widetilde{H}_\lambda(t_0,x)\in X_{\chi^v_\lambda(t_0)}$.

We shall now prove that $\widetilde{H}_\lambda(t_0,x)-x$ is Lipschitz with respect to x with a constant $K\lambda$, where K is independent of (λ,t_0). We may write

$$\widetilde{H}_\lambda(t_0,x)=\frac{x+\frac{u_\lambda(t_0,\varepsilon_1\varrho(t_0),\varepsilon_1\varrho(t_0)x)}{\varepsilon_1\varrho(t_0)}}{1+\frac{w_\lambda(t_0,\varepsilon_1\varrho(t_0),\varepsilon_1\varrho(t_0)x)}{\varepsilon_1\varrho(t_0)}}\,.$$

It is enough to prove that both

$$\frac{u_\lambda\left(t_0,\varepsilon_1\varrho(t_0),\varepsilon_1\varrho(t_0)x\right)}{\varepsilon_1\varrho(t_0)}\,,\qquad\frac{w_\lambda\left(t_0,\varepsilon_1\varrho(t_0),\varepsilon_1\varrho(t_0)x\right)}{\varepsilon_1\varrho(t_0)}$$

are Lipschitz with respect to x with a constant $C\lambda$. So let $x,x'\in X_{t_0}$; we have

$$\left|\frac{u_\lambda\left(t_0,\varepsilon_1\varrho(t_0),\varepsilon_1\varrho(t_0)x\right)}{\varepsilon_1\varrho(t_0)}-\frac{u_\lambda\left(t_0,\varepsilon_1\varrho(t_0),\varepsilon_1\varrho(t_0)x'\right)}{\varepsilon_1\varrho(t_0)}\right|$$
$$\le\frac{C\lambda\varepsilon_1\varrho(t_0)|x-x'|}{\varepsilon_1\varrho(t_0)}=C\lambda|x-x'|\,,$$

and similarly for $w_\lambda/\varepsilon_1\varrho(t_0)$.

Now it is easy to construct H_λ of Proposition 1. Recall that $H_\lambda(t_0,\cdot)$ should be defined on $\mathbb{R}^n_x$ while $\widetilde{H}_\lambda(t_0,\cdot)$ is defined only on X_{t_0}. Choose an integer N so big that for all $t_0\in\mathring{T}^j$ and $\lambda\in[0,1/N]$, the Lipschitz constant with respect to x of $\widetilde{H}_\lambda(t_0,x)-x$ is smaller than $1/2\sqrt{n}$. We may write, for $\lambda\in[0,1/N]$,

$$\widetilde{H}_\lambda(t_0,x)=x+\widetilde{G}_\lambda(t_0,x),$$

where the Lipschitz constant of $\widetilde{G}_\lambda$ with respect to x is smaller than $1/2\sqrt{n}$. By Kirszbraun's theorem we can extend $\widetilde{G}_\lambda$ to a function $G_\lambda(t_0,x)$, defined for all $x\in\mathbb{R}^n_x$, $t_0\in\mathring{T}^j$, $\lambda\in[0,1/N]$, continuous with respect to all variables and Lipschitz with respect to x with a Lipschitz constant $\frac{1}{2}$. Put

$$H^*_\lambda(t_0,x)=x+G_\lambda(t_0,x),\quad x\in\mathbb{R}^n_x,\ \lambda\in[0,1/N];$$

then $H^*_\lambda:\mathbb{R}^n_x\longrightarrow\mathbb{R}^n_x$ is bi-Lipschitz.

Finally, for $\lambda\in[0,1]$ and any $x\in\mathbb{R}^n_x$, $t_0\in\mathring{T}^j$, we put for $i=1,\ldots,N$, $x=x_0$,

$$t_{i+1}=\chi^v_{\lambda/N}(t_i),\quad x_{i+1}=H^*_{\lambda/N}(t_i,x_i)$$

and

$$H_\lambda(t_0,x)=x_N.$$

Proposition 1 is proved.

4. Proof of Proposition 2

4.1. *Notation*

1° If $v_1, \dots, v_N$ are Lipschitz vector fields on $\mathbb{R}^n$, we define the "joint flow" $\chi_\lambda^{\underline{v}}$ of $\underline{v} = (v_1, \dots, v_N)$. Let $x_0 \in \mathbb{R}^n$; we put, inductively,

$$x_{i+1} = \chi_1^{v_i}(x_i), \quad i = 1, \dots, N.$$

For $\lambda \in [0, N]$ define

$$\widetilde{\chi}_\lambda^{\underline{v}}(x_0) = \chi_{\lambda - i}^{v_{i+1}}(x_i)$$

if $i \le \lambda < i + 1$, $0 \le i \le N - 1$,

$$\widetilde{\chi}_N^{\underline{v}}(x_0) = x_N.$$

Thus for $\lambda \in [i, i+1]$, the curve $\lambda \longmapsto \widetilde{\chi}_\lambda^{\underline{v}}(x_0)$ is a trajectory of v_{i+1}. Finally we normalise λ:

$$\chi_\lambda^{\underline{v}}(x_0) = \widetilde{\chi}_{N\lambda}^{\underline{v}}(x_0), \quad \lambda \in [0, 1].$$

The map $(x_0, \lambda) \longmapsto \chi_\lambda^{\underline{v}}(x_0)$, $\lambda \in [0, 1]$, is the *joint flow* of $\underline{v}$.

2° Let $p, q \in \mathbb{R}^n$. We shall say that $\underline{v} = (v_1, \dots, v_N)$ *moves* p *to* q if $q = \chi_1^{\underline{v}}(p)$. We shall say that $\underline{v}$ *moves* p *to* q *regularly* if, moreover, the map

$$\lambda \longmapsto h_\lambda^{\underline{v}}(p), \quad \lambda \in [0, 1], \tag{4.1}$$

is a bi-Lipschitz homeomorphism onto its image, i.e. for some C

$$C^{-1}|\lambda_1 - \lambda_2| \le |h_{\lambda_1}^{\underline{v}}(p) - h_{\lambda_2}^{\underline{v}}(p_2)| \le C|\lambda_1 - \lambda_2|; \tag{4.2}$$

of course the last inequality is superfluous since it follows at once from the assumption that all v_i's are Lipschitz.

If $\underline{v}$ moves p to q regularly, then the length of the curve $\lambda \longmapsto \chi_\lambda^{\underline{v}}(p)$ is of order $|p - q|$.

We shall say that $\underline{v}$ *moves* p *to* q *in a controlled way* if, for some constant K,

$$|v_i(x)| \le K|p - q| \tag{4.3}$$

for all $x \in \mathbb{R}^n$ and $i = 1, \dots, N$; this condition is a tautology here, but will become significant in 3°.

We shall say that p *can be moved to* q (regularly, in a controlled way) if there exists a $\underline{v}$ which moves p to q (regularly, in a controlled way).

3° We shall now replace points p, q in 2° by (subanalytic) curves in $\mathbb{R}^n$, Lipschitz vector fields by one-parameter Lipschitz families of vector fields and repeat definitions of 2° in a parametrised way.

Let $p(\mu), q(\mu)$ be curves, $\underline{v}_\mu = (v_{1,\mu}, \dots, v_{N,\mu})$, where $v_{i,\mu}$ are Lipschitz families of vector fields on $\mathbb{R}^n$.

We shall say that $\underline{v}_\mu$ *moves* $p(\mu)$ *to* $q(\mu)$ if, for all $\mu > 0$,

$$\chi_1^{\underline{v}_\mu}(p(\mu)) = q(\mu).$$

$\underline{v}_\mu$ *moves* $p(\mu)$ *to* $q(\mu)$ *regularly* if, moreover, (4.2) holds for all $\mu > 0$ with a constant C independent of μ:

$$C^{-1}|\lambda_1 - \lambda_2| \le \left|\chi_{\lambda_1}^{\underline{v}_\mu}(p(\mu)) - \chi_{\lambda_2}^{\underline{v}_\mu}(p(\mu))\right| \le C|\lambda_1 - \lambda_2|; \tag{4.2$'$}$$

again the second inequality follows at once from the fact that the Lipschitz constants of every $v_{i,\mu}$ are independent of μ.

$\underline{v}_\mu$ *moves* $p(\mu)$ *to* $q(\mu)$ *in a controlled way* if (4.3) holds for all x and μ, with K independent of μ:

$$|v_{i,\mu}(x)| \le K|p(\mu) - q(\mu)| \tag{4.3$'$}$$

for all $x \in \mathbb{R}^n$, $\mu > 0$. This condition implies that the lengths of the curves

$$[0,1] \ni \lambda \longmapsto \chi_\lambda^{\underline{v}_\mu}(p(\mu))$$

are of order of $|p(\mu) - q(\mu)|$.

4° In the introduction we defined the notion of Lipschitz homogeneity (*LH*); we shall now define a related notion.

A subset $A \subset \mathbb{R}^n$ is *WLH* (weakly Lipschitz homogeneous) if for every pair of curves $p(\mu)$, $q(\mu)$ in A such that, for some C,

$$|p(\mu) - q(\mu)| \le C d_{\partial A}\big(\{p(\mu), q(\mu)\}\big), \tag{4.4}$$

$p(\mu)$ can be moved to $q(\mu)$, regularly, in a controlled way, by some $\underline{v}_\mu = (v_{1,\mu}, \ldots, v_{N,\mu})$, such that all $v_{i,\mu}$ vanish on ∂A and every $v_{i,\mu}$ preserves A: $\chi_\lambda^{v_{i,\mu}}(A) \subset A$ for all $\lambda \in [0,1]$ and all $\mu > 0$. Of course $d_{\partial A}\big(\{p(\mu), q(\mu)\}\big) = \min\big(d_{\partial A}(p(\mu)), d_{\partial A}(q(\mu))\big)$.

Remark. If $p(\mu), q(\mu)$ are in A and $p(\mu)$ can be moved to $q(\mu)$ by a $\underline{v}_\mu$ such that all $v_{i,\mu}$ vanish on ∂A then $d_{\partial A}(p(\mu)) \simeq d_{\partial A}(q(\mu))$.

4.2. *A homogeneity property*

Proposition 4. *Every set* A *in* $\mathfrak{A}_i$, $i = 1, 3$, *is a finite union of not necessarily disjoint WLH sets* B_j *in the same class* $\mathfrak{A}_i$.

Remarks.

1° As follows from proofs below, the number N which appears in the definition of *WLH* sets is bounded by n for every B_j.

2° We shall prove also that B_j are smooth (non-compact if $\dim B_j > 0$) manifolds and $v_{i,\mu}$ are smooth on them.

Proof of Proposition 4. We use induction on the dimension of the set; the proposition is obvious for 0-dimensional sets. Since every A is a finite union of L-regular sets, we can at once assume that A is L-regular of dimension m, i.e.

$$A = \operatorname{graph}(F : A' \longrightarrow \mathbb{R}^k),$$

where $A' \subset \mathbb{R}^m$ is open L-regular, $m + k = n$, F is smooth and bounded on A' together with its first derivatives. Let $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be as usual the standard projection.

Let $Z' \subset \overline{A'}$ satisfy:

$$\dim Z' < m, \quad |D^2 F| \lesssim 1/d_{Z'}, \quad Z' \in \mathfrak{A}_i,$$

as in (1.2).

Decompose $A' \setminus Z'$ into a union of L-regular sets A'_β; for each of them fix a projection $\pi'_\beta : \mathbb{R}^m \longrightarrow \mathbb{R}^{m-1}$ such that A'_β is a cylinder over $A''_\beta = \pi'_\beta(A'_\beta)$ as in (1.3).

Since A is the union of all $(\pi'_\beta \pi)^{-1}(A''_\beta) \cap A$, it is enough to prove Proposition 4 for each of the latter sets instead of A. Take any one of them. To simplify notation, let us omit the index β; thus we are in the following situation: we have the standard projections

$$\mathbb{R}^n \xrightarrow{\pi} \mathbb{R}^m \xrightarrow{\pi'} \mathbb{R}^{m-1},$$

L-regular sets A, A', A'', surjections

$$A \xrightarrow{\pi} A' \xrightarrow{\pi'} A''$$

and

$$A' = \{(x'', x_m) : x'' \in A'',\ \varphi(x'') < x_m < \psi(x'')\}, \tag{4.5}$$

φ, ψ are smooth, bounded on A'' together with $|D\varphi|$, $|D\psi|$,

$$A = \operatorname{graph}(F : A' \longrightarrow \mathbb{R}^k),$$

F smooth, bounded on A' together with $|DF|$, and

$$|D^2 F| \lesssim 1/d_{\partial A'}. \tag{4.6}$$

Let $Y''_1 \subset \overline{A''}$ satisfy:

$$|D^2\varphi|, |D^2\psi| \lesssim 1/d_{Y''_1}, \quad \dim Y''_1 < m-1, \quad Y''_1 \in \mathfrak{A}_i. \tag{4.7}$$

It follows that

$$\left|D^2(\psi - \varphi)\right| \lesssim 1/d_{Y''_1}. \tag{4.8}$$

Consider A' as a family $A' \xrightarrow{\pi'} A''$ over A'' with one-dimensional fibres. Let $\widehat{A}'$ be the family obtained from A' by replacing its fibres (i.e. the open intervals $(\varphi(x''), \psi(x''))$) by their closures $[\varphi(x''), \psi(x'')]$. We apply Proposition $1'$ to this family; thus we put $T = A''$, $X = \widehat{A}'$. Let $\mathcal{T}$ be a stratification of $\mathbb{R}^{m-1}$ which satisfies the conclusion of this proposition and let Y''_2 be its skeleton of dimension $m-2$ (more precisely, union of all strata of dimension smaller than $m-1$). Let

$$Y'' = Y''_1 \cup Y''_2.$$

The following observation, which is a special case of the conclusion of Proposition 1, is basic for the proof.

Let v be any Lipschitz vector field on $\mathbb{R}^{m-1}$ vanishing on Y''; then there exists a family of maps

$$H_\lambda : (A'' \setminus Y'') \times \mathbb{R} \longrightarrow \mathbb{R},$$

where $\mathbb{R}$ is the x_m-axis, and this family satisfies all requirements of Proposition 1; in particular 3° reads:

$$H_\lambda(x''_0, \cdot) : \left[\varphi(x''), \psi(x'')\right] \longrightarrow \left[\varphi(\chi^v_\lambda(x''_0)), \psi(\chi^v_\lambda(x''_0))\right]$$

is bi-Lipschitz for $\lambda \in [0,1]$. It follows that for some C, independent of $x_0'' \in A'' \setminus Y''$ and $\lambda \in [0,1]$,

$$C^{-1}\big|(\psi-\varphi)(x_0'')\big| \le \big|(\psi-\varphi)(\chi_\lambda^v(x_0''))\big| \le C\big|(\psi-\varphi)(x_0'')\big|; \tag{4.9}$$

intuitively: the intervals $\big[\varphi(x''),\psi(x'')\big]$, $\big[\varphi(\chi_\lambda^v(x_0'')),\psi(\chi_\lambda^v(x_0''))\big]$ are of comparable length.

By induction hypothesis $A'' \setminus Y''$ is a finite union of *WLH* sets:

$$A'' \setminus Y'' = \bigcup A''_\alpha. \tag{4.10}$$

The following lemma implies Proposition 4:

LEMMA 4.1. *Every* $A_\alpha = (\pi'\pi)^{-1}(A''_\alpha) \cap A$ *is WLH.*

In fact, $A = \bigcup A''_\alpha \cup \big[\pi^{-1}(Y'') \cap A\big]$ and $\pi^{-1}(Y'') \cap A$ is of dimension smaller than m.

To simplify notation, we omit α and write A instead of A_α, A' instead of $\pi(A)$ and A'' instead of A''_α.

Proof of Lemma 4.1. Let $p(\mu)$, $q(\mu)$ be curves in A which satisfy (4.4); we have to prove that $p(\mu)$ is moved to $q(\mu)$ by a $\underline{v}_\mu$ such that all $v_{i,\mu} = 0$ on ∂A and their flows preserve A.

Let $p'(\mu)$, $q'(\mu)$ (resp. $p''(\mu)$, $q''(\mu)$) be the projections of $p(\mu)$, $q(\mu)$ under π (resp. $\pi'\pi$). Then, by (1.6), $p''(\mu)$, $q''(\mu)$ satisfy (4.4) with A'' instead of A, and similarly $p'(\mu)$, $q'(\mu)$.

Since A'' is *WLH*, there is a $\underline{v}''_\mu$ which moves $p''(\mu)$ in $q''(\mu)$, $\chi_\lambda^{\underline{v}''_\mu}$ preserves A'' and $\underline{v}''_{i,\mu}$ vanish on $\partial A''$.

Step 1. We shall prove that $p'(\mu)$ can be moved to $q'(\mu)$ by a $\underline{v}'_\mu$ with similar properties. We shall do it as follows. First we shall find, for every $\mu > 0$, a continuous piecewise C^1 curve Γ'_μ joining $p'(\mu)$ to $q'(\mu)$, and then we shall show that the tangent vector fields to C^1 segments of Γ'_μ extend to Lipschitz vector fields $v_{i,\mu}$, defined on $\mathbb{R}^n$, with desired properties.

Let us write

$$p'(\mu) = \big(p''(\mu), p_m(\mu)\big), \quad q'(\mu) = \big(q''(\mu), q_m(\mu)\big).$$

For every $\mu > 0$ we have a curve

$$\Gamma''_\mu(\lambda) = \chi_\lambda^{\underline{v}''_\mu}(p''(\mu))$$

joining $p''(\mu)$ with $q''(\mu)$; it consists of segments $\Gamma''_{i,\mu}$, $i = 1, \ldots, N$, which are integral curves of $\underline{v}''_{i,\mu}$; the total length of Γ''_μ is $\simeq |p''(\mu) - q''(\mu)|$ (cf. Remark 1° after Proposition 4) and the mapping $\lambda \longmapsto \Gamma''_\mu(\lambda)$ is bi-Lipschitz homeomorphism onto its image.

It is convenient to write

$$\Gamma''_\mu = p''(\mu; \lambda).$$

The curve Γ'_μ joining $p'(\mu)$ and $q'(\mu)$ consists of $N+1$ C^1 segments $\Gamma'_{i,\mu}$, $i = 1, \ldots, N+1$. The first N segments are liftings of $\Gamma''_{i,\mu}$ constructed as follows.

We may write

$$p_m(\mu) = \varphi\big(p''(\mu)\big) + \theta(\mu)(\psi - \varphi)\big(p''(\mu)\big),$$

where $\theta(\mu)$ takes values in $[0, 1]$.

We lift Γ''_μ to a curve $\widetilde{\Gamma}'_\mu$ in $\mathbb{R}^m$ by the formula

$$\widetilde{\Gamma}'_\mu : p' = p'(\mu; \lambda) = \big(p''(\mu; \lambda), \varphi(p''(\mu; \lambda))\big) + \theta(\mu)(\psi - \varphi)\big(p''(\mu; \lambda)\big).$$

Of course $\widetilde{\Gamma}'_\mu$ is piecewise C^1; its C^1 segments $\widetilde{\Gamma}'_{i,\mu}$ project on $\Gamma''_{i,\mu}$.

The curve $\widetilde{\Gamma}'_\mu$ does not join in general $p'(\mu)$ with $q'(\mu)$. Let $\widetilde{q}'(\mu)$ be its end, i.e.

$$\widetilde{q}'(\mu) = p'(\mu; 1).$$

This point projects under π' into $q''(\mu)$; the point $q'(\mu)$ has also the same property. So $\widetilde{q}'(\mu)$ and $q'(\mu)$ are joined by a segment parallel to the x_m-axis. We take this "vertical" segment for $\Gamma'_{N+1,\mu}$; the curve Γ'_μ is defined as the curve consisting of $\widetilde{\Gamma}'_\mu$ and the added segment $\Gamma'_{N+1,\mu}$.

The curve Γ'_μ is supposed to be parametrised by the unit interval. So on $\widetilde{\Gamma}'_\mu$ we change the parametrisation by λ above into the parametrisation by $\lambda^* = \frac{N}{N+1}\lambda$. The vertical interval $\Gamma'_{N+1,\mu}$ is parametrised linearly by $\lambda^* \in \left[\frac{N}{N+1}, 1\right]$. Thus finally we have the curve $\Gamma'_\mu(\lambda^*)$ joining $p'(\mu)$ and $q'(\mu)$ consisting of $N+1$ C^1 segments $\Gamma'_{i,\mu}$, $i = 1, \ldots, N+1$.

Now we shall show that the tangent vector field to every segment $\Gamma'_{i,\mu}$ extends to a Lipschitz family of vector fields $v'_{i,\mu}$, vanishing on $\partial A'$.

I. We start with the segments $\Gamma'_{i,\mu}$, $i \le N$. Of course, for the existence of $v'_{i,\mu}$ the reparametrisation $\lambda \longmapsto \lambda^*$ does not matter, and we shall use the parameter λ. The tangent vector field to $\Gamma'_{i,\mu}$ is given (component-wise, as on p. 184) by

$$\bar{t}_\mu(\lambda) = \big(v'_{i,\mu}(p''(\mu; \lambda)), \big(d\varphi + \theta(\mu) d(\psi - \varphi)\big)(v'_{i,\mu})(p''(\mu; \lambda))\big).$$

We claim that it is enough to prove the following two statements:

1° $\bar{t}_\mu(\lambda)$ is Lipschitz on $\widetilde{\Gamma}'_{i,\mu}$ (with a Lipschitz constant independent of μ) and continuous with respect to λ, μ;

2° $|\bar{t}_\mu(\lambda)| \lesssim d_{\partial A'}(\widetilde{\Gamma}'_{i,\mu}(\lambda))$, with a constant appearing implicitly in the sign $\lesssim$ independent of λ, μ.

In fact, if 1° and 2° hold, we may define a Lipschitz family of vector fields on $\partial A' \cup \widetilde{\Gamma}'_{i,\mu}$ by putting 0 on $\partial A'$ and $\bar{t}_\mu(\lambda)$ on $\widetilde{\Gamma}'_{i,\mu}$. By Kirszbraun's theorem it extends to a family $v'_{i,\mu}$ we are looking for.

ad 1°. Continuity of $\bar{t}_\mu(\lambda)$ with respect to λ, μ is obvious. To prove the Lipschitz estimate it is enough to bound the derivative with respect to λ of the m-th component of $\bar{t}_\mu(\lambda)$ (recall, Remark 2° after Proposition 4, that $v''_{i,\mu}$ are smooth on A'').

Let v''_j be the components of $v''_{i,\mu}$, i.e.

$$v''_{i,\mu} = \sum_{j \le m-1} v''_j \, \partial/\partial x_j.$$

We have

$$\frac{d}{d\lambda}\big\{\big[d\varphi+\theta(\mu)d(\psi-\phi)\big](v''_{i,\mu})(p''(\mu;\lambda))\big\}$$
$$=\sum_{j,k\le m-1}\frac{\partial^2}{\partial x_j\,\partial x_k}\big[\varphi+\theta(\mu)(\psi-\phi)\big]v''_j v''_k$$
$$+\sum_{j\le m-1}\frac{\partial}{\partial x_j}\big[\varphi+\theta(\mu)(\psi-\phi)\big]\frac{d(v''_j)}{d\lambda},$$

where, of course, the right-hand side is evaluated at $p''(\mu;\lambda)$.

The second term on the right-hand side is bounded since $|D\varphi|$, $|D\psi|$ are bounded and $v''_{i,\mu}$ is Lipschitz.

To bound the first term we use (4.7), (4.8) and the estimates

$$|v''_{i,\mu}|\lesssim d_{\partial A''}\le d_{Y''_1};$$

the first one follows from the fact that $v''_{i,\mu}$ are Lipschitz and vanish on $\partial A''$, and the second from the inclusion $Y''_1\subset\partial A''$ which follows from (4.10).

ad 2°. Since $|D\varphi|$, $|D(\psi-\varphi)|$ are bounded,

$$|\overline{t}_\mu(\lambda)|\lesssim\big|v''_{i,\mu}(p''(\mu;\lambda))\big|. \tag{4.11}$$

Since $p''(\mu)$ is moved by $\underline{v}''_\mu$ to $q''(\mu)$ regularly and in a controlled way,

$$|v''_{i,\mu}|\lesssim|p''(\mu)-q''(\mu)|\le|p'(\mu)-\widetilde{q}'(\mu)|.$$

It is thus enough to show that for all $x'\in\widetilde{\Gamma}'_{i,\mu}$, $i\le N$,

$$|p'(\mu)-\widetilde{q}'(\mu)|\lesssim d_{\partial A'}(x').$$

This is true not only on the segment $\widetilde{\Gamma}'_{i,\mu}$, but on the whole curve $\widetilde{\Gamma}'_\mu$, i.e. for all points x' of the form $p'(\mu;\lambda)$.

In fact,

$$\operatorname{hordist}(p'(\mu;\lambda),\partial A')=d_{\partial A''}(p''(\mu;\lambda))\simeq d_{\partial A''}(p''(\mu))$$

since, for every μ, the curve $\lambda\longmapsto p''(\mu;\lambda)$ consists of segments being integral curves of Lipschitz vector fields preserving $\partial A''$. Further,

$$\operatorname{vertdist}(p'(\mu;\lambda),\partial A')=\theta(\mu)\big|(\psi-\phi)(p''(\mu;\lambda))\big|$$
$$\simeq\theta(\mu)\,|(\psi-\phi)(p''(\mu))|\simeq\operatorname{vertdist}(p'(\mu),\partial A'),$$

as follows from (4.9) after taking for x_0 the end-points of successive segments $\Gamma''_{i,\mu}$ and taking $v''_{i,\mu}$ for v. Thus for all $x'\in\Gamma'_\mu$

$$d_{\partial A'}(x')\simeq d_{\partial A'}(p'(\mu))\ge d_{\partial A''}(p''(\mu))\gtrsim|p''(\mu)-q''(\mu)|\simeq|p'(\mu)-\widetilde{q}'(\mu)|.$$

The case of segments $\widetilde{\Gamma}'_{i,\mu}$, $i\le N$, is finished.

II. Now we consider the last segment $\Gamma'_{N+1,\mu}$. The tangent vector field to it is given by

$$(N+1)[q_m(\mu)-\widetilde{q}_m(\mu)]\,\partial/\partial x_m, \tag{4.12}$$

where, of course, $q_m(\mu)$, $\widetilde{q}_m(\mu)$ are the m-coordinates of the points $q'_m(\mu)$, $\widetilde{q}'_m(\mu)$. We claim that for the existence of $v'_{N+1,\mu}$ it is enough to prove that

$$|q_m(\mu) - \widetilde{q}_m(\mu)| \lesssim d_{\partial A'}\big(\{\widetilde{q}'(\mu), q(\mu)\}\big). \tag{4.13}$$

In fact, if (4.13) holds, then the family of vector fields equal to (4.12) on $\Gamma'_{N+1,\mu}$ and 0 on $\partial A'$ is Lipschitz (and of course continuous with respect to μ), so, by Kirszbraun's theorem, it extends to a Lipschitz family of vector fields.

Formula (4.13) is proved as follows:

$$\begin{aligned}|q_m(\mu) - \widetilde{q}_m(\mu)| \le |q'(\mu) - \widetilde{q}'(\mu)| &\le |\widetilde{q}'(\mu) - p'(\mu)| + |p'(\mu) - q'(\mu)| \\ &\lesssim |q''(\mu) - p''(\mu)| + |p'(\mu) - q'(\mu)|,\end{aligned}$$

the inequality $\lesssim$ follows from the fact that the direction of tangents to the segments $\Gamma'_{i,\mu}$, $i \le N$, (i.e. $\bar{t}_\mu(\lambda)$) are bounded away from the vertical direction (i.e. the direction of the x_m-axis) according to (4.11). Thus

$$|q_m(\mu) - \widetilde{q}_m(\mu)| \lesssim |p'(\mu) - q'(\mu)| \lesssim d_{\partial A'}\big(\{p'(\mu), q'(\mu)\}\big),$$

because, as remarked at the beginning of the proof, $p'(\mu)$, $q'(\mu)$ satisfy (4.4). Finally,

$$d_{\partial A'}\left(\widetilde{q}'(\mu)\right) \simeq d_{\partial A'}\left(\widetilde{p}'(\mu)\right),$$

because $p'(\mu)$ is moved to $\widetilde{q}'(\mu)$ by a Lipschitz family of vector fields $(v'_{1,\mu}, \dots, v'_{N,\mu})$ which vanish on $\partial A'$.

Step 1 of the proof of Lemma 4.1 is complete.

Step 2. We shall prove that $p(\mu)$ can be moved to $q(\mu)$ in A regularly and in a controlled way. We lift the curve Γ'_μ to A via π, i.e. we put

$$\Gamma_\mu(\lambda^*) = \pi^{-1}\Gamma'_\mu(\lambda^*), \quad \lambda^* \in [0,1].$$

In other words,

$$\Gamma_\mu(\lambda^*) = \big(\Gamma'_\mu(\lambda^*), F\Gamma'_\mu(\lambda^*)\big)$$

in the splitting $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^k$.

Clearly Γ_μ starts at $p(\mu)$ and ends at $q(\mu)$.

Again, using Kirszbraun's theorem, it is enough to prove that the tangent vector field to $\Gamma_\mu(\lambda^*)$, i.e.

$$\frac{d}{d\lambda^*}\,\Gamma_\mu(\lambda^*)$$

is Lipschitz on Γ_μ and its length is bounded, up to a multiplicative constant, by $d_{\partial A}$.

The latter statement is almost immediate: since $|DF|$ is bounded, we get, by Step 1,

$$\Big|\frac{d}{d\lambda^*}\,\Gamma_\mu(\lambda^*)\Big| \lesssim \Big|\frac{d}{d\lambda^*}\,\Gamma'_\mu(\lambda^*)\Big| \lesssim d_{\partial A'}(\Gamma'_\mu(\lambda^*)) \lesssim d_{\partial A}(\Gamma_\mu(\lambda^*)).$$

To prove the first statement we shall show that

$$\frac{d^2}{d{\lambda^*}^2}\,\Gamma_{i,\mu}(\lambda^*)$$

is bounded on every segment $\Gamma_{i,\mu}$, $i \le N+1$. Of course it is enough to prove that $\frac{d^2F}{d\lambda^{*2}}$, or, in a more exact notation,

$$\frac{d^2}{d\lambda^{*2}} F(\Gamma'_\mu(\lambda^*))$$

is bounded.

Take any of these segments, $\Gamma_{i,\mu}$, and for simplicity of notation put

$$v'_{i,\mu} = v' = \sum_{k\le m} v'_k \, \partial/\partial x_k.$$

Denote by $\partial_w \varphi$ the directional derivative of a function φ, i.e. $\sum \frac{\partial\varphi}{\partial x_k} w_k$, and by $\nabla_w z$ the covariant derivative in the flat (Euclidean) connection. Then we have

$$\frac{d^2}{d\lambda^{*2}} F = \partial_{v'}\,\partial_{v'} F = \sum_{k\le m} \frac{\partial}{\partial x_k}\Big(\sum_{l\le m} \frac{\partial F}{\partial x_l}\, v'_l\Big) v'_k = \partial_{\nabla_{v'}v'} F + \sum_{k,l} \frac{\partial^2 F}{\partial x_k\,\partial x_l}\, v'_k v'_l.$$

The first term of the last expression is bounded because $|DF|$ is bounded and $\nabla_{v'}v'$ is bounded because v' is Lipschitz and $|v'|$ is bounded. The second term is bounded because of (3.6) and $|v'| \lesssim d_{\partial A'}$.

The proof of Proposition 4 is complete. ∎

4.3. *Proof of Proposition* 2. Let $X_s \subset \mathbb{R}^n$ be a given finite family of sets in $\mathfrak{A}_i$, $i = 1,3$. By induction with respect to d we shall prove the existence of a Lipschitz stratification $\mathfrak{Z} = \{Z^j\}$ of $\mathbb{R}^n$, compatible with all X_s, with skeletons Z^j in $\mathfrak{A}_i$, such that every stratum $\mathring{Z}^j$, $j < d$, is a finite union of *LHrel* $\mathfrak{Z}$ sets in $\mathfrak{A}_i$.

The case $d = 1$ is obvious.

For the induction step we start with any stratification $\mathfrak{S} = \{S^j\}$, compatible with all X_s's, $S^j \in \mathfrak{A}_i$. By Proposition 4, $\mathring{S}^d$ is a finite union of *WLH* sets:

$$\mathring{S}^d = \bigcup M_r.$$

Let

$$Y = \bigcup \partial M_r.$$

Let $\mathfrak{Z} = \{Z^j\}$ be any stratification of $\mathbb{R}^n$ compatible with all X_s's and Y, such that every stratum $\mathring{Z}^k$, $k < d$, is a union of *WLH* sets:

$$\mathring{Z}^k = \bigcup_{\beta\in B_k} A^k_\beta, \quad A^k_\beta \in \mathfrak{A}_i, \quad A^k_\beta \text{ is } \textit{LHrel}\,\mathfrak{Z}, \quad B_k \text{ finite.}$$

We shall prove that $\mathring{Z}^d$ also is a union of *LHrel* $\mathfrak{Z}$-sets, and this will end the proof.

For every sequence $\overline{\beta} = \big(\beta(1), \ldots, \beta(d-1)\big)$ such that $\beta(i) \in B_i$ for all i, put

$$Z^d_{r\bar\beta} = \big\{x \in M_r : d_k(x) = d(x, \overline{A}^k_{\beta(k)}) \text{ for all } k < d\big\},$$

where, as usual, $d_k(x) = d(x, Z^k)$. Since $\mathring{Z}^d$ is the union of all $Z^d_{r\bar\beta}$'s, it is enough to prove the following lemma:

LEMMA 4.2. *Every* $Z^d_{r\bar\beta}$ *is LHrel* $\mathfrak{Z}$.

Proof. Let $p(\mu), q(\mu)$ be two curves in $Z^d_{r\bar\beta}$ having distances to skeletons of $\mathfrak{Z}$ of dimension less than d of the same order. Let l be the smallest integer, naturally smaller than d, such that

$$\operatorname{ord} d_l(p(\mu)) = \operatorname{ord} d_{d-1}(p(\mu));$$

in other words,

$$d_l(p(\mu)) \simeq d_{d-1}(p(\mu)),$$
$$d_{l-1}(p(\mu)) \gg d_l(p(\mu))$$

(i.e. $d_l(p(\mu))/d_{l-1}(p(\mu)) \longrightarrow 0$ as $\mu \longrightarrow 0$). By the hypothesis on orders of distances of $p(\mu), q(\mu)$ to skeletons, l is also the smallest integer for which

$$\operatorname{ord} d_l(q(\mu)) = \operatorname{ord} d_{d-1}(q(\mu)).$$

Let us choose curves $p^*(\mu), q^*(\mu)$ in $A^l_{\beta(l)}$ such that

$$|p(\mu) - p^*(\mu)| \le 2d(p(\mu), \overline{A}^l_{\beta(l)}),$$
$$|q(\mu) - q^*(\mu)| \le 2d(q(\mu), \overline{A}^l_{\beta(l)}).$$

Remark that for all $k < l$

$$d_k(p^*(\mu)) \simeq d_k(p(\mu)), \tag{4.14}$$
$$d_k(q^*(\mu)) \simeq d_k(q(\mu)), \tag{4.15}$$

because

$$d_k(p(\mu)) - |p(\mu) - p^*(\mu)| \le d_k(p^*(\mu)) \le d_k(p(\mu)) + |p(\mu) - p^*(\mu)|.$$

By hypothesis, there exists a $\underline{v} = (v_{1,\mu}, \ldots, v_{N,\mu})$ which moves $p^*(\mu)$ to $q^*(\mu)$, and all $v_{i,\mu}$ are Lipschitz families of vector fields, tangent to $\mathfrak{Z}$.

Put

$$\widetilde{q}(\mu) = \chi_1^{\underline{v}_\mu}(p(\mu)).$$

Observe that $\widetilde{q}(\mu) \in M_r$ for $\mu > 0$; in fact, $p(\mu) \in M_r$ and the sets M_r are invariant under the flow of every $v_{i,\mu}$ since $M_r \subset \mathring{Z}^d$ and $\partial M_r \subset Z^{d-1}$.

For all $k < d$,

$$d_k(\widetilde{q}(\mu)) \simeq d_k(p(\mu)) \simeq d_k(q(\mu));$$

this is proved as (4.14) and (4.15).

We claim that

$$|\widetilde{q}(\mu) - q(\mu)| \lesssim d\big(\{q(\mu), \widetilde{q}(\mu)\}, Z^{d-1}\big) \simeq d(q(\mu), Z^{d-1}).$$

In fact,

$$\begin{aligned}|\widetilde{q}(\mu) - q(\mu)| &\le |\widetilde{q}(\mu) - q^*(\mu)| + |q^*(\mu) - q(\mu)| \\ &= \big|\chi_1^{\underline{v}_\mu}(p(\mu)) - \chi_1^{\underline{v}_\mu}(p^*(\mu))\big| + |q^*(\mu) - q(\mu)| \\ &\lesssim |p(\mu) - p^*(\mu)| + |q^*(\mu) - q(\mu)| \\ &\le 2d\big(p(\mu), Z^{d-1}\big) + 2d\big(q(\mu), Z^{d-1}\big) \simeq d\big(q(\mu), Z^{d-1}\big).\end{aligned}$$

Since $\widetilde{q}(\mu), q(\mu)$ lie in the same M_r, it follows that

$$|\widetilde{q}(\mu) - q(\mu)| \lesssim d\big(q(\mu), \partial M_r\big).$$

By hypothesis, M_r is *WLH*, so $\widetilde{q}(\mu)$ can be moved to $q(\mu)$ by a $\underline{w} = (w_{1,\mu}, \ldots, w_{N_1,\mu})$. $w_{i,\mu}$ vanish on ∂M_r, so we may ask that they are defined on $\partial M_r \cup Z^{d-1}$ and take the value 0 on $Z^{d-1} \setminus M_r$. Now we extend every $w_{i,\mu}$ to a Lipschitz family, tangent to $\mathfrak{Z}$.

Now the $N + N_1$ vector fields $(\underline{v}_\mu, \underline{w}_\mu) = (v_{1,\mu}, \ldots, v_{N,\mu}, w_{1,\mu}, \ldots, w_{N_1,\mu})$, after being multiplied by suitable numbers, move $p(\mu)$ to $q(\mu)$; this "normalisation" is similar to the introduction of the parameter λ^* on p. 205, so we omit the details.

Proposition 2 is thus proved.

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