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SINGULARITIES OF NON-DEGENERATE n-RULED $(n+1)$-MANIFOLDS IN EUCLIDEAN SPACE

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Abstract. The objective of this paper is to study singularities of n-ruled $(n+1)$-manifolds in Euclidean space. They are one-parameter families of n-dimensional affine subspaces in Euclidean space. After defining a non-degenerate n-ruled $(n+1)$-manifold we will give a necessary and sufficient condition for such a map germ to be right-left equivalent to the cross cap $\times$ interval. The behavior of a generic n-ruled $(n+1)$-manifold is also discussed.

1. Introduction. The study of ruled surfaces in $\mathbf{R}^3$ is a classical subject in differential geometry and its generalizations in higher dimensions have also been studied by many authors. The ruled surfaces and its generalizations have singularities in general and their generic singularities have been studied in [5], [4] and [7].

In this paper, we give a necessary and sufficient condition for an n-ruled $(n+1)$-manifold germ in $\mathbf{R}^{2n}$ to be right-left equivalent to the cross cap $\times$ interval. It is a generalization of the case of 2-ruled hypersurfaces in $\mathbf{R}^4$ [7]. Furthermore, we show that generic singularities of n-ruled $(n+1)$-manifolds in $\mathbf{R}^{2n}$ coincide with those of C^∞-maps of $(n+1)$-manifolds into $\mathbf{R}^{2n}$.

The paper is organized as follows. Throughout this paper we suppose $N \geq 2n$. In Section 2 we define non-degenerate n-ruled $(n+1)$-manifolds in $\mathbf{R}^N$ as an analogue of classical noncylindrical ruled surfaces. Classical noncylindrical ruled surfaces are those whose rulings always change directions and non-degenerate n-ruled $(n+1)$-manifolds in $\mathbf{R}^N$ are defined in the same way. Then we present the main theorem (Theorem 4) using the notion of a striction curve. In Section 3 we define the striction curve of a non-degenerate n-ruled $(n+1)$-manifold in $\mathbf{R}^N$ as a special base curve. Moreover, we show that the singular points of a non-degenerate n-ruled $(n+1)$-manifold in $\mathbf{R}^N$ are

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contained in the image of the striction curve. In particular, the set of singular points of a non-degenerate n-ruled $(n+1)$-manifold in $\mathbf{R}^{2n}$ coincide with the image of the striction curve. In Section 4 the proof of our main theorem is completed. In Section 5 we discuss generic n-ruled $(n+1)$-manifolds in $\mathbf{R}^{2n}$. We show that the generic singularities of n-ruled $(n+1)$-manifolds in $\mathbf{R}^{2n}$ coincide with those of C^∞-maps of $(n+1)$-manifolds into $\mathbf{R}^{2n}$.

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2. Preliminaries. In this section we give the definition of n-ruled $(n+1)$-manifolds. Let S^{N-1} be the unit sphere of $\mathbf{R}^N$ and $I, J_1, J_2, \dots, J_n$ open intervals.

DEFINITION 1. An *n-ruled $(n+1)$-manifold in* $\mathbf{R}^N$ means (the image of) a map $F_{(\gamma,\delta_1,\delta_2,\dots,\delta_n)} : I \times J_1 \times J_2 \times \dots \times J_n \longrightarrow \mathbf{R}^N$ of the form

$$F_{(\gamma,\delta_1,\delta_2,\dots,\delta_n)}(t, u_1, u_2, \dots, u_n) = \gamma(t) + u_1\delta_1(t) + u_2\delta_2(t) + \dots + u_n\delta_n(t),$$

where $\gamma : I \longrightarrow \mathbf{R}^N$, $\delta_1, \delta_2, \dots, \delta_n : I \longrightarrow S^{N-1}$ are smooth maps. We assume that the dimension of the vector space $\langle \delta_1(t), \delta_2(t), \dots, \delta_n(t)\rangle$ spanned by $\delta_1(t), \delta_2(t), \dots, \delta_n(t)$ is always equal to n for any $t \in I$. We call γ a *base curve* and n curves $\delta_1, \delta_2, \dots, \delta_n$ *director curves.* The n-planes $(u_1, u_2, \dots, u_n) \mapsto \gamma(t) + u_1\delta_1(t) + u_2\delta_2(t) + \dots + u_n\delta_n(t)$ are called *rulings at t.*

$(S^{N-1})^n$ denotes $S^{N-1} \times S^{N-1} \times \dots \times S^{N-1}$, where the number of S^{N-1} is equal to n and $\mathcal{P}_{(N,n)}$ denotes the set

$$\big\{(\delta_1, \delta_2, \dots, \delta_n) \in C^\infty\big(I, (S^{N-1})^n\big) \mid \langle \delta_1(t), \delta_2(t), \dots, \delta_n(t)\rangle = n \text{ for any } t \in I\big\}.$$

We consider $(\gamma, \delta_1, \delta_2, \dots, \delta_n) \in C^\infty(I, \mathbf{R}^N) \times \mathcal{P}_{(N,n)} \subset C^\infty(I, \mathbf{R}^N) \times C^\infty\big(I, (S^{N-1})^n\big) = C^\infty\big(I, \mathbf{R}^N \times (S^{N-1})^n\big)$ and we regard $C^\infty\big(I, \mathbf{R}^N \times (S^{N-1})^n\big)$ equipped with the Whitney C^∞-topology. Put $RM_n(I, \mathbf{R}^N) = C^\infty(I, \mathbf{R}^N) \times \mathcal{P}_{(N,n)}$ equipped with the Whitney C^∞-topology.

DEFINITION 2. Two n-ruled $(n+1)$-manifolds F_1 and $F_2 \in RM_n(I, \mathbf{R}^N)$ are *equivalent* if the ruling of F_1 at t coincides with the ruling of F_2 at t as a subset of $\mathbf{R}^N$, for any $t \in I$. The difference between F_1 and F_2 is the choice of director curves and also of base curve.

We regard $\mathcal{RM}_n(I, \mathbf{R}^N) = RM_n(I, \mathbf{R}^N)/\!\sim$ as the *space of n-ruled $(n+1)$-manifolds.* Let $[F]$ denote the equivalence class containing $F \in RM_n(I, \mathbf{R}^N)$. However, we will usually omit the brackets when discussing ruled manifolds.

A non-degenerate n-ruled $(n+1)$-manifold in $\mathbf{R}^N$ satisfies a condition analogous to that of a noncylindrical ruled surface in $\mathbf{R}^3$ (see [3] for example). Throughout this paper we suppose $N \geq 2n$.

DEFINITION 3.

(I) An n-ruled $(n+1)$-manifold

$$F_{(\gamma,\delta_1,\delta_2,\dots,\delta_n)}(t, u_1, u_2, \dots, u_n) = \gamma(t) + u_1\delta_1(t) + u_2\delta_2(t) + \dots + u_n\delta_n(t)$$

is said to be *non-degenerate at $t \in I$*, if

$$\dim\langle \delta_1(t), \delta_1'(t), \delta_2(t), \delta_2'(t), \dots, \delta_n(t), \delta_n'(t)\rangle = 2n.$$

(II) An n-ruled $(n+1)$-manifold $F_{(\gamma,\delta_1,\delta_2,\dots,\delta_n)}(t,u_1,u_2,\dots,u_n) = \gamma(t) + u_1\delta_1(t) + u_2\delta_2(t)+\dots+u_n\delta_n(t)$ is said to be *non-degenerate on I*, if $F_{(\gamma,\delta_1,\delta_2,\dots,\delta_n)}(t,u_1,u_2,\dots,u_n)$ is globally non-degenerate, that is, if it is non-degenerate at any $t \in I$.

It is obvious that non-degeneracy condition does not depend on the choice of representative among the equivalence class given in Definition 2. Since rank of the non-degeneracy condition is that if $N = 2n$, $2n \times 2n$ matrix (δ, δ') drops by 1 in codimension 1, but if $N > 2n$ the rank drops in codimension 1, the non-degeneracy condition is not generic in the usual sense in the case of $N = 2n$ and it is generic in the case of $N > 2n$. The generic condition in the case $N = 2n$ will be discussed in Section 5.

Recall that $x \in X$ is a *singular point* of a differentiable map $f : X \longrightarrow Y$ between manifolds if $\operatorname{rank}(df)_x < \min\{\dim X, \dim Y\}$. Set $S(f) = \{x \in X \mid x \text{ is a singular point of } f\}$. The image of a singular point of an n-ruled $(n+1)$-manifold will also be called a singular point of an n-ruled $(n+1)$-manifold.

Singular points of non-degenerate n-ruled $(n+1)$-manifolds in $\mathbf{R}^N$ are characterized by the following main theorem, by using the notion of the striction curve σ which will be defined in the next section.

Theorem 4 (Main Theorem). *Let us put $N = 2n$. Let $F = F_{(\sigma,\delta_1,\delta_2,\dots,\delta_n)}$ be the map germ of a non-degenerate n-ruled $(n+1)$-manifold with striction curve $\sigma(t)$ at $(t_0, u_{10}, u_{20}, \dots, u_{n0})$.*

(I) *The point $p_0 = F(t_0, u_{10}, u_{20}, \dots, u_{n0})$ does not lie on the striction curve (i.e., $(u_{10}, u_{20}, \dots, u_{n0}) \neq (0, \dots, 0)$) if and only if the map germ F at $(t_0, u_{10}, u_{20}, \dots, u_{n0})$ is regular.*

(II) *If p_0 lies on the striction curve (i.e., $(u_{10}, u_{20}, \dots, u_{n0}) = (0, \dots, 0)$), then the following two conditions are equivalent.*

(a) *The striction curve $\sigma(t)$ is an immersion near $t = t_0$.*

(b) *The map germ F at $(t_0, u_{10}, u_{20}, \dots, u_{n0})$ is right-left equivalent to the cross cap$_n$ $\times$ interval.*

Here a *cross cap$_n$ $\times$ interval* means the map germ at the origin of the map defined by

$$f(x_1, x_2, \dots, x_n, x_{n+1}) = (x_1^2, x_2, x_3, \dots, x_n, x_{n+1}, x_1x_2, x_1x_3, \dots, x_1x_n),$$

and right-left equivalence is defined as follows. Let $f_i : (X_i, x_i) \longrightarrow (Y_i, y_i)$, $i = 1, 2$, be C^∞-map germs. We say that f_1 and f_2 are *right-left equivalent* if there exist diffeomorphism germs $\phi : (X_1, x_1) \longrightarrow (X_2, x_2)$ and $\psi : (Y_1, y_1) \longrightarrow (Y_2, y_2)$ such that $\psi \circ f_1 = f_2 \circ \phi$ holds.

If we suppose that $n = 1$ and $N = 2$ then the following theorem holds.

Theorem 5. *Let $n = 1$, $N = 2$. Let $F = F_{(\sigma,\delta)}$ be the map germ of a non-degenerate 1-ruled 2-manifold with striction curve $\sigma(t)$ at (t_0, u_0).*

The following two conditions are equivalent.

(c) *The striction curve $\sigma(t)$ satisfies $\sigma'(t_0) = 0$ and $\sigma''(t_0) \neq 0$.*

(d) *The map germ F at (t_0, u_0) is right-left equivalent to the cusp.*

Furthermore, in this case the striction curve has a $(2,3)$-cusp singularity at $t = t_0$.

Here a *cusp* means the map germ at the origin of the map defined by

$$(x_1, x_2) \longrightarrow (x_1, x_2^3 - x_1 x_2),$$

and a (2,3)-*cusp* means the map germ at the origin of the map defined by

$$t \longrightarrow (t^2, t^3).$$

A cross $\mathrm{cap}_1 \times$ interval is also called a *fold*.

3. Striction curve of a non-degenerate n-ruled $(n+1)$-manifold. We will define the striction curve of a non-degenerate n-ruled $(n+1)$-manifold after preparing Lemmas 6 and 7.

LEMMA 6. *For any n-ruled $(n+1)$-manifold $F_{(\gamma,\delta_1,\delta_2,\dots,\delta_n)}(t, u_1, u_2, \dots, u_n) = \gamma(t) + u_1\delta_1(t) + u_2\delta_2(t) + \ldots + u_n\delta_n(t)$, we can find director curves ε_i $(i = 1, 2, \dots, n)$ such that $F_{(\gamma,\delta_1,\delta_2,\dots,\delta_n)}$ is equivalent to $F_{(\gamma,\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n)}$ and not only $\|\varepsilon_i\| = 1$, but also $\varepsilon_i \cdot \varepsilon_j = 0$ for $i \neq j$ and $\varepsilon_i' \cdot \varepsilon_j = 0$ for all i and j hold for any $t \in I$.*

We say that the director curves $\delta_1, \delta_2, \dots, \delta_n$ are *constrictively adapted* if they satisfy the above conditions.

Proof. We may assume that the director curves $\delta_1, \delta_2, \dots, \delta_n$ satisfy the conditions that $\|\delta_i\| = 1$ $(i = 1, 2, \dots, n)$ and $\delta_i \cdot \delta_j = 0$ $(i \neq j)$. Now, we put

$$\begin{pmatrix} \varepsilon_1(t) \\ \varepsilon_2(t) \\ \vdots \\ \varepsilon_n(t) \end{pmatrix} = A(t) \begin{pmatrix} \delta_1(t) \\ \delta_2(t) \\ \vdots \\ \delta_n(t) \end{pmatrix} \tag{1}$$

for a smooth map $A : I \longrightarrow O(n)$.

Then we have $\|\varepsilon_i\| = 1$ $(i = 1, 2, \dots, n)$ and $\varepsilon_i \cdot \varepsilon_j = 0$ $(i \neq j)$. On the other hand, we have

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{pmatrix} (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)' = A \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_n \end{pmatrix} \Big((\delta_1, \delta_2, \dots, \delta_n)\, {}^t A' + (\delta_1', \delta_2', \dots, \delta_n')^t A \Big).$$

Since $\delta_i \cdot \delta_i = 1$ and $\delta_i \cdot \delta_j = 0$ $(i \neq j)$, any solution A of the ordinary differential equation

$${}^t A' A = - \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_n \end{pmatrix} (\delta_1', \delta_2', \dots, \delta_n')$$

gives a desired n-ple $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ of director curves. ∎

LEMMA 7. *Let*

$$\begin{aligned} F(t, u_1, u_2, \dots, u_n) &= F_{(\gamma,\delta_1,\delta_2,\dots,\delta_n)}(t, u_1, u_2, \dots, u_n) \\ &= \gamma(t) + u_1\delta_1(t) + u_2\delta_2(t) + \ldots + u_n\delta_n(t), \end{aligned}$$

$t \in I$, be a non-degenerate n-ruled $(n+1)$-manifold whose director curves $\delta_1, \delta_2, \ldots, \delta_n$ are constrictively adapted.

(I) *Then there exists a smooth curve $\sigma : I \longrightarrow \mathbf{R}^N$ such that $F_{(\gamma,\delta_1,\delta_2,\ldots,\delta_n)}$ is equivalent to $F_{(\sigma,\delta_1,\delta_2,\ldots,\delta_n)}$ and $\sigma' \cdot \delta_i' = 0$ $(i = 1, 2, \ldots, n)$.*

(II) *$\sigma(t)$ does not depend on choice of base curves, that is, the choice of representative among the equivalence class given in Definition 2.*

Proof. (I) Let $M(t)$ be the matrix of functions $\left(\delta_i'(t) \cdot \delta_j'(t)\right)$. Since $\delta_1', \delta_2', \ldots, \delta_n'$ are linearly independent by the non-degeneracy of the n-ruled $(n+1)$-manifold $F_{(\gamma,\delta_1,\delta_2,\ldots,\delta_n)}$, we see that $\det M(t) \neq 0$. So, we can put

$$\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} = M(t)^{-1} \begin{pmatrix} -\gamma' \cdot \delta_1' \\ -\gamma' \cdot \delta_2' \\ \vdots \\ -\gamma' \cdot \delta_n' \end{pmatrix}.$$

Then, $\sigma(t) = \gamma(t) + \sum_{i=1}^{n} f_i(t)\delta_i(t)$ satisfies the conditions $\sigma' \cdot \delta_i' = 0$ $(i = 1, 2, \ldots, n)$.

We can prove part (II) using a similar argument in the proof of [3, Lemma 17.8]. ∎

The curve $\sigma(t)$ which satisfies the condition in Lemma 7 is called a *striction curve* of the given non-degenerate n-ruled $(n+1)$-manifold $F_{(\gamma,\delta_1,\delta_2,\ldots,\delta_n)}(t, u_1, u_2, \ldots, u_n)$. Now we give a lemma concerning the relation between the singular locus and the striction curve.

LEMMA 8. *Let $F_{(\sigma,\delta_1,\delta_2,\ldots,\delta_n)}(t, u_1, u_2, \ldots, u_n) = \sigma(t) + u_1\delta_1(t) + u_2\delta_2(t) + \ldots + u_n\delta_n(t)$ be a non-degenerate n-ruled $(n+1)$-manifold with the striction curve $\sigma(t)$.*

(I) *If $N = 2n$, then the set of singular points of the n-ruled $(n+1)$-manifold coincides with the image of the striction curve $\sigma(t)$. In fact, we have*

$$S(F_{(\sigma,\delta_1,\delta_2,\ldots,\delta_n)}) = I \times \{0\} \times \ldots \times \{0\}.$$

(II) *If $N > 2n$, then every singular point of a non-degenerate n-ruled $(n+1)$-manifold in $\mathbf{R}^N$ is contained in the image of the striction curve σ. Moreover, at every singular point $p_0 = F_{(\sigma,\delta_1,\delta_2,\ldots,\delta_n)}(t_0, u_{10}, u_{20}, \ldots, u_{n0})$, the ruling through $\sigma(t_0)$ of $F_{(\sigma,\delta_1,\delta_2,\ldots,\delta_n)}$ is tangent to σ.*

Proof. By the definition, $(t, u_1, u_2, \ldots, u_n)$ is a singular point of $F = F_{(\sigma,\delta_1,\delta_2,\ldots,\delta_n)}$ if and only if the Jacobian matrix

$$\left(\frac{\partial F}{\partial t}, \frac{\partial F}{\partial u_1}, \frac{\partial F}{\partial u_2}, \ldots, \frac{\partial F}{\partial u_n}\right)(t, u_1, u_2, \ldots, u_n) = \left(\sigma'(t) + \sum_{i=1}^{n} u_i\delta_i'(t), \delta_1(t), \delta_2(t), \ldots, \delta_n(t)\right)$$

of F is not of full rank.

(I) Since $N = 2n$, we have $\sigma' \in \langle \delta_1, \delta_2, \ldots, \delta_n \rangle$ and the $2n$ vectors $\delta_1, \delta_1', \delta_2, \delta_2', \ldots, \delta_n$ and δ_n' are linearly independent. Then, we see easily that the above matrix is not of full rank if and only if $u_1 = u_2 = \ldots = u_n = 0$.

(II) If the above matrix is not of full rank, then we have $u_1 = u_2 = \ldots = u_n = 0$ and $\sigma' \in \langle \delta_1, \delta_2, \ldots, \delta_n \rangle$. Note that we assume always $N \geq 2n$. ■

4. Proof of the main theorem. Throughout this section, we suppose $N = 2n$. Let $f : (\mathbf{R}^{n+1}, 0) \longrightarrow (\mathbf{R}^{2n}, 0)$ be a smooth map germ and we consider the Thom-Boardman singularity set $\Sigma^{1,0} \subset J^2(n+1, 2n)$ and $\Sigma^{1,1,0} \subset J^3(2,2)$ defined in [1]. Morin [6] proved the following lemma.

LEMMA 9 ([6], Théorème). *Let $f : (\mathbf{R}^{n+1}, 0) \longrightarrow (\mathbf{R}^{2n}, 0)$ be a smooth map germ.*

(I) *The following two conditions are equivalent.*

(i) *$j^2 f(0) \in \Sigma^{1,0}$ and the map germ $j^2 f : (\mathbf{R}^{n+1}, 0) \longrightarrow J^2(n+1, 2n)$ is transverse to $\Sigma^{1,0}$ at $j^2 f(0)$.*

(ii) *f is right-left equivalent to the cross cap$_n$ $\times$ interval, that is, there exist local coordinates $(x_1, x_2, \ldots, x_{n+1})$ of $\mathbf{R}^{n+1}$ around 0 and local coordinates $(y_1, y_2, \ldots, y_{2n})$ of $\mathbf{R}^{2n}$ around 0, such that $f = (y_1 \circ f, y_2 \circ f, \ldots, y_{2n} \circ f)$ is expressed as*

$$f(x_1, x_2, \ldots, x_n, x_{n+1}) = (x_1^2, x_2, x_3, \ldots, x_n, x_{n+1}, x_1x_2, x_1x_3, \ldots, x_1x_n).$$

(II) *Furthermore, if $n = 1$, the following two conditions are also equivalent.*

(i) *$j^3 f(0) \in \Sigma^{1,1,0}$ and the map germ $j^3 f : (\mathbf{R}^2, 0) \longrightarrow J^2(2,2)$ is transverse to $\Sigma^{1,1,0}$ at $j^3 f(0)$.*

(ii) *f is right-left equivalent to the cusp, that is, there exist local coordinates (x_1, x_2) of $\mathbf{R}^2$ around 0 and local coordinates (y_1, y_2) of $\mathbf{R}^2$ around 0, such that $f = (y_1 \circ f, y_2 \circ f)$ is expressed as*

$$f(x_1, x_2) = (x_1, x_2^3 - x_1x_2).$$

Furthermore, he rewrote the above conditions as follows. We use the notation

$$\begin{aligned} &f(x_1, x_2, \ldots, x_n, x_{n+1}) \\ &= \big(f_1(x_1, x_2, \ldots, x_n, x_{n+1}), f_2(x_1, x_2, \ldots, x_n, x_{n+1}), \ldots, f_N(x_1, x_2, \ldots, x_n, x_{n+1})\big). \end{aligned}$$

LEMMA 10 ([6], Lemme).

(I) *Let $f : (\mathbf{R}^{n+1}, 0) \longrightarrow (\mathbf{R}^{2n}, 0)$ be a smooth map germ. Then $j^2 f(0) \in \Sigma^{1,0}$ and the map germ $j^2 f : (\mathbf{R}^{n+1}, 0) \longrightarrow J^2(n+1, 2n)$ is transverse to $\Sigma^{1,0}$ at $j^2 f(0)$ if and only if for some local coordinates, called adapted, $(x_1, x_2, \ldots, x_n, x_{n+1})$ of $\mathbf{R}^{n+1}$ and $(z_1, z_2, \ldots, z_{2n})$ of $\mathbf{R}^{2n}$ satisfying $f_i(x_1, x_2, \ldots, x_n, x_{n+1}) = x_i$ $(i = 2, 3, \ldots, n+1)$,*

$$\frac{\partial f_1}{\partial x_j}(0, 0, \ldots, 0) = 0 \quad (j = 1, 2, \ldots, n+1)$$

and

$$\frac{\partial f_{n+i}}{\partial x_j}(0, 0, \ldots, 0) = 0 \quad (i = 2, 3, \ldots, n,\ j = 1, 2, \ldots, n+1),$$

we have

(i) $\dfrac{\partial^2 f}{\partial x_1^2}(0,0,0) \neq 0,$

and

(ii) $\operatorname{rank}\left(\dfrac{\partial^2 f}{\partial x_1^2}, \dfrac{\partial^2 f}{\partial x_1\, \partial x_2}, \dfrac{\partial^2 f}{\partial x_1\, \partial x_3}, \ldots, \dfrac{\partial^2 f}{\partial x_1\, \partial x_{n+1}}\right)(0, 0, \ldots, 0) = n.$

(II) *Let* $f : (\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}^2, 0)$ *be a smooth map germ. Then* $j^3 f(0) \in \Sigma^{1,1,0}$ *and the map germ* $j^3 f : (\mathbf{R}^2, 0) \longrightarrow J^3(2, 2)$ *is transverse to* $\Sigma^{1,1,0}$ *at* $j^3 f(0)$ *if and only if for some adapted local coordinates* (x_1, x_2) *of* $\mathbf{R}^2$ *and* (z_1, z_2) *of* $\mathbf{R}^2$ *satisfying* $f_2(x_1, x_2) = x_2$ *and* $\dfrac{\partial f_1}{\partial x_1}(0) = 0$, *we have*

(i) $\dfrac{\partial^2 f}{\partial x_1^2}(0) = 0,$

(ii) $\dfrac{\partial^2 f}{\partial x_1\,\partial x_2}(0) \neq 0,$

(iii) $\dfrac{\partial^3 f}{\partial x_1^3}(0) \neq 0.$

Proof of Theorem 4. The statement (I) follows directly from Lemma 8. So we prove (II) here.

Let $F_{(\sigma,\delta_1,\delta_2,\dots,\delta_n)}(t, u_1, u_2, \dots, u_n) = \sigma(t) + u_1\delta_1(t) + u_2\delta_2(t) + \dots + u_n\delta_n(t)$ be a non-degenerate n-ruled $(n+1)$-manifold with the striction curve $\sigma(t)$. For any $t_0 \in I$, the point p_0 denotes $F_{(\sigma,\delta_1,\delta_2,\dots,\delta_n)}(t_0, 0, 0, \dots, 0)$. We put $F = F_{(\sigma,\delta_1,\delta_2,\dots,\delta_n)}$ and suppose that the director curves $\delta_1, \delta_2, \dots, \delta_n$ are constrictively adapted.

First, changing the coordinates $(z_1, z_2, \dots, z_{2n})$ of $\mathbf{R}^{2n}$ by an orthogonal transformation if necessary, we may assume

$$\delta_i(t_0) = (\underbrace{0, \dots, 0}_{i \text{ times}}, 1, 0, \dots, 0) \quad (i = 1, 2, \dots, n).$$

Let us define the new coordinates $(x_1, x_2, \dots, x_n, x_{n+1})$ of $\mathbf{R}^{n+1}$ by

$$\begin{aligned} x_1 &= t - t_0, \\ x_2 &= \big(F(t, u_1, u_2, \dots, u_n) - F(t_0, 0, \dots, 0)\big) \cdot \delta_1(t_0), \\ x_3 &= \big(F(t, u_1, u_2, \dots, u_n) - F(t_0, 0, \dots, 0)\big) \cdot \delta_2(t_0), \\ &\ \vdots \\ x_{n+1} &= \big(F(t, u_1, u_2, \dots, u_n) - F(t_0, 0, \dots, 0)\big) \cdot \delta_n(t_0). \end{aligned} \tag{2}$$

Then we get

$$\begin{cases} \dfrac{\partial F}{\partial x_1}(0, 0, \dots, 0) = 0, \\ \dfrac{\partial F}{\partial x_2}(0, 0, \dots, 0) = \delta_1(t_0), \\ \dfrac{\partial F}{\partial x_3}(0, 0, \dots, 0) = \delta_2(t_0), \\ \qquad \vdots \\ \dfrac{\partial F}{\partial x_{n+1}}(0, 0, \dots, 0) = \delta_n(t_0). \end{cases} \tag{3}$$

So, the coordinates $(x_1, x_2, \dots, x_{n+1})$ and $(z_1, z_2, \dots, z_{2n})$ are adapted coordinate systems in the sense of Lemma 10. Using the notation: $\sigma_0' = \sigma'(t_0)$, $\delta_{i0} = \delta_i(t_0)$ and

$\delta'_{i0} = \delta'_i(t_0)$, we have

$$\begin{aligned}\sigma''(t_0) &= \frac{\partial^2 F}{\partial t^2}(t_0, 0, \dots, 0)\\ &= \Big(\frac{\partial^2 F}{\partial x_1^2} + \sum_{i=1}^{n}\sum_{j=1}^{n}(\sigma'_0 \cdot \delta_{i0})(\sigma'_0 \cdot \delta_{i0})\frac{\partial^2 F}{\partial x_{i+1}\,\partial x_{j+1}}\\ &\quad + \sum_{i=1}^{n} 2(\sigma'_0 \cdot \delta_{i0})\frac{\partial^2 F}{\partial x_1\,\partial x_{i+1}} + \sum_{i=1}^{n}(\sigma''(t_0)\cdot\delta_{i0})\delta_{i0}\Big)(0,0,\dots,0),\end{aligned} \tag{4}$$

$$\begin{aligned}\delta'_i(t_0) &= \frac{\partial^2 F}{\partial t\,\partial u_i}(t_0,0,\dots,0)\\ &= \Big(\frac{\partial^2 F}{\partial x_1\,\partial x_i} + \sum_{j=1}^{n}(\sigma'_0\cdot\delta_{j0})\frac{\partial^2 F}{\partial x_i\,\partial x_{j+1}}\Big)(0,0,\dots,0) \quad (i=1,2,\dots,n)\end{aligned} \tag{5}$$

and

$$0 = \frac{\partial^2 F}{\partial u_i\,\partial u_j}(t_0,0,\dots,0) = \frac{\partial^2 F}{\partial x_{i+1}\,\partial x_{j+1}}(0,0,\dots,0) \quad (i,j=1,2,\dots,n). \tag{6}$$

Since $\dim\langle \delta_1(t), \delta'_1(t), \dots, \delta_n(t), \delta'_n(t)\rangle = 2n$ and $\sigma'(t)\cdot\delta'_i(t) = 0$ $(i=1,2,\dots,n)$ for any $t \in I$, we have

$$\sigma'(t) = \sum_{i=1}^{n}\big(\sigma'(t)\cdot\delta_i(t)\big)\delta_i(t)$$

and hence

$$\sigma''(t) - \sum_{i=1}^{n}\big(\sigma''(t)\cdot\delta_i(t)\big)\delta_i(t) = \sum_{i=1}^{n}\big(\sigma'(t)\cdot\delta_i(t)\big)\delta'_i(t). \tag{7}$$

By the equations (4), (5) and (6), we get

$$\sigma''(t_0) = \frac{\partial^2 F}{\partial x_1\,\partial x_i} + \sum_{i=1}^{n} 2\delta_{i0}(\sigma'_0\cdot\delta_{i0}) + \sum_{i=1}^{n}(\sigma''(t_0)\cdot\delta_{i0})\delta_{i0}. \tag{8}$$

Then by (7) and (8), we obtain

$$\frac{\partial^2 F}{\partial x_1^2}(0,0,\dots,0) = -\sum_{i=1}^{n}(\sigma'_0\cdot\delta_{i0})\delta'_{i0}. \tag{9}$$

Hence

$$\begin{aligned}&\Big(\frac{\partial^2 F}{\partial x_1^2}, \frac{\partial^2 F}{\partial x_1\,\partial x_2}, \frac{\partial^2 F}{\partial x_1\,\partial x_3}, \dots, \frac{\partial^2 F}{\partial x_1\,\partial x_{n+1}}\Big)(0,0,\dots,0)\\ &\qquad = \Big(-(\sigma'\cdot\delta_1)\delta'_1 - (\sigma'\cdot\delta_2)\delta'_2 - \dots - (\sigma'\cdot\delta_n)\delta'_n, \delta'_1, \delta'_2, \dots, \delta'_n\Big)(t_0).\end{aligned} \tag{10}$$

This means that condition (ii) of Lemma 10 (I) is always satisfied for F. Furthermore, condition (i) is equivalent to

$$\frac{\partial^2 F}{\partial x_1^2}(0,0,0) = -\sum_{i=1}^{n}(\sigma'_0\cdot\delta_{i0})\delta'_{i0} \neq 0,$$

that is, either $\sigma_0' \cdot \delta_{10} \neq 0$, $\sigma_0' \cdot \delta_{20} \neq 0$, $\dots$, $\sigma_0' \cdot \delta_{(n-1)0} \neq 0$ or $\sigma_0' \cdot \delta_{n0} \neq 0$. Since $\sigma' \in \langle \delta_1, \delta_2, \dots , \delta_n \rangle$, this condition is equivalent to $\sigma' \neq 0$ at $t = t_0$. This completes the proof. ∎

Proof of Theorem 5. Let $F = F_{(\sigma,\delta)}(t, u) = \sigma(t) + u\delta(t)$ be a non-degenerate 1-ruled 2-manifold with the striction curve $\sigma(t)$. For any $t_0 \in I$, the point p_0 denotes $F(t_0, 0)$. The director curve δ is constrictively adapted.

We take the new coordinates (x_1, x_2) as in (2). Then we get

$$\begin{aligned} &\frac{\partial^2 F}{\partial x_1^2}(0,0) = -\delta_0'(\sigma_0' \cdot \delta_0), \\ &\frac{\partial^2 F}{\partial x_1\, \partial x_2}(0,0) = -\delta_0' \\ \text{and} \quad &\frac{\partial^2 F}{\partial x_2^2}(0,0) = 0 \end{aligned} \tag{11}$$

by the same calculation as in the proof of Theorem 4. Since $\dim\langle \delta(t), \delta'(t) \rangle = 2$ and $\sigma'(t) \cdot \delta'(t) = 0$ for any $t \in I$, we have the formulas

$$\begin{aligned} \sigma'(t) &= (\sigma'(t) \cdot \delta(t))\delta(t), \\ \sigma''(t) &= (\sigma''(t) \cdot \delta(t))\delta(t) + (\sigma'(t) \cdot \delta(t))\delta'(t) \end{aligned}$$

and

$$\begin{aligned} \sigma'''(t) = \big(\sigma'''(t) \cdot \delta(t)\big)\delta(t) &+ \big(\sigma''(t) \cdot \delta'(t)\big)\delta(t) \\ &+ 2\big(\sigma''(t) \cdot \delta(t)\big)\delta'(t) + \big(\sigma'(t) \cdot \delta(t)\big)\delta''(t) \end{aligned} \tag{12}$$

for any $t \in I$. We will calculate the third derivative of F. By a direct calculation, we get

$$\begin{aligned} \delta''(t_0) = \frac{\partial^3 F}{\partial t^2\, \partial u} &= \left(\frac{\partial^3 F}{\partial x_1^2\, \partial x_2} + \frac{\partial^3 F}{\partial x_1\, \partial x_2^2}(\sigma_0' \cdot \delta_0)\right)(0,0), \\ 0 = \frac{\partial^3 F}{\partial t\, \partial u^2} &= \left(\frac{\partial^3 F}{\partial x_1\, \partial x_2^2}\right)(0,0), \\ 0 = \frac{\partial^3 F}{\partial u^3} &= \left(\frac{\partial^3 F}{\partial x_2^3}\right)(0,0) \end{aligned} \tag{13}$$

and

$$\begin{aligned} \sigma'''(t_0) &= \frac{\partial^3 F}{\partial t^3} \\ &= \Bigg(\frac{\partial^3 F}{\partial x_1^3} + \frac{\partial^3 F}{\partial x_1^2\, \partial x_2}(\sigma_0' \cdot \delta_0) + 2\left(\frac{\partial^3 F}{\partial x_1^2\, \partial x_2} + \frac{\partial^3 F}{\partial x_1\, \partial x_2^2}(\sigma_0' \cdot \delta_0)\right)(\sigma_0' \cdot \delta_0) \\ &\quad + 2\frac{\partial^2 F}{\partial x_1\, \partial x_2}(\sigma_0'' \cdot \delta_0) + \left(\frac{\partial^3 F}{\partial x_1\, \partial x_2^2} + \frac{\partial^3 F}{\partial x_2^3}(\sigma_0' \cdot \delta_0)\right)(\sigma_0' \cdot \delta_0)^2 \\ &\quad + 2\frac{\partial^2 F}{\partial x_2^2}(\sigma_0'' \cdot \delta_0)(\sigma_0' \cdot \delta_0) + \left(\frac{\partial^2 F}{\partial x_1\, \partial x_2} + \frac{\partial^2 F}{\partial x_2^2}(\sigma_0' \cdot \delta_0)\right)(\sigma_0'' \cdot \delta_0) \\ &\quad + \frac{\partial F}{\partial x_2}(\sigma'''(t_0) \cdot \delta_0)\Bigg)(0,0). \end{aligned} \tag{14}$$

Here, $\sigma_0' = \sigma'(t_0)$, $\sigma_0'' = \sigma''(t_0)$, $\delta_0 = \delta(t_0)$, $\delta_0' = \delta'(t_0)$ and $\delta_0'' = \delta''(t_0)$. By (13), we see that

$$\sigma'''(t_0) = \frac{\partial^3 F}{\partial x_1^3} + 3(\sigma_0' \cdot \delta_0)\delta_0'' + 3(\sigma_0'' \cdot \delta_0)\delta_0' + \big(\sigma'''(t_0) \cdot \delta_0\big)\delta_0 \tag{15}$$

and by (12), we obtain

$$\frac{\partial^3 F}{\partial x_1^3} = -2(\sigma_0' \cdot \delta_0)\delta_0'' + \big(\sigma_0'' \cdot \delta'(t_0)\big)\delta_0 - (\sigma_0'' \cdot \delta_0)\delta_0'. \tag{16}$$

So,

$$\frac{\partial^2 F}{\partial x_1^2}(0,0) = 0, \qquad \frac{\partial^2 F}{\partial x_1\, \partial x_2}(0,0) \neq 0 \quad \text{and} \quad \frac{\partial^3 F}{\partial x_1^3}(0,0) \neq 0$$

is equivalent to

$$\sigma_0' \cdot \delta_0 = 0, \quad \delta_0' \neq 0 \quad \text{and} \quad -2(\sigma_0' \cdot \delta_0)\delta_0'' + (\sigma_0'' \cdot \delta_0')\delta_0 - (\sigma_0'' \cdot \delta_0)\delta_0' \neq 0.$$

Since $\sigma'(t_0) \cdot \delta'(t_0) = 0$ and $\delta(t_0)$, $\delta'(t_0)$ are linearly independent, we see it is equivalent to

$$\sigma'(t_0) = 0 \text{ and } \sigma''(t_0) \neq 0.$$

Now from Lemma 10 (II), we have the theorem. ■

5. Singularities of generic n-ruled $(n+1)$-manifolds. Throughout this section we suppose $N = 2n$. We consider singularities of generic n-ruled $(n+1)$-manifolds in $\mathbf{R}^{2n}$. Non-degenerate n-ruled $(n+1)$-manifolds in $\mathbf{R}^{2n}$ are not generic in the usual sense. We will define *almost non-degenerate* n-ruled $(n+1)$-manifolds in $\mathbf{R}^{2n}$, which are generic in the usual sense. They have *exceptional rulings* where the striction curve diverges and no singular points are contained. We name Theorem 13 which characterizes the singularities of generic n-ruled $(n+1)$-manifolds.

DEFINITION 11. An n-ruled $(n+1)$-manifold $F_{(\gamma,\delta_1,\delta_2,\dots,\delta_n)}(t,u_1,u_2,\dots,u_n) = \gamma(t) + u_1\delta_1(t) + u_2\delta_2(t) + \dots + u_n\delta_n(t)$, $t \in I$, is said to be *almost non-degenerate on I*, if there exists a discrete subset $D \subset I$ such that the following four conditions hold.

(A1) $F_{(\gamma,\delta_1,\delta_2,\dots,\delta_n)}$ is non-degenerate at any $t \notin D$.

(A2) $\dim\langle \delta_1(t), \delta_1'(t), \delta_2(t), \delta_2'(t), \dots, \delta_n(t), \delta_n'(t)\rangle = 2n-1$ for any $t_k \in D$.

(A3) Let A_t denote $\det\big(\delta_1(t), \delta_1'(t), \delta_2(t), \delta_2'(t), \dots, \delta_n(t), \delta_n'(t)\big)$. Then $\left.\dfrac{dA_t}{dt}\right|_{t=t_k} \neq 0$ for any $t_k \in D$.

(A4) $\gamma'(t_k) \notin \langle \delta_1(t_k), \delta_1'(t_k), \delta_2(t_k), \delta_2'(t_k), \dots, \delta_n(t_k), \delta_n'(t_k)\rangle$ for any $t_k \in D$.

It is easy to check that condition (A4) does not depend on the choice of the base curve γ. For an almost non-degenerate n-ruled $(n+1)$-manifold the rulings

$$(u_1, u_2, \dots, u_n) \mapsto \gamma(t_k) + \sum_{i=1}^{n} u_i\delta_i(t_k) \quad (t_k \in D)$$

are called *exceptional rulings*. Note that condition (A4) implies that $F_{(\gamma,\delta_1,\delta_2,\dots,\delta_n)}$ is non-singular at any point in the exceptional rulings.

The following lemma shows that an almost non-degenerate n-ruled $(n+1)$-manifold is generic in the usual sense.

LEMMA 12. *The set*

$$\mathcal{R} = \{(\gamma, \delta_1, \delta_2, \dots, \delta_n) \mid F_{(\gamma,\delta_1,\delta_2,\dots,\delta_n)} \text{ is an almost non-degenerate } n\text{-ruled manifold}\}$$

is open and dense in $\mathcal{RM}_n(I, \mathbf{R}^N)$ with respect to the quotient Whitney C^∞-topology.

Proof. This lemma follows from an analogous proof of [7, Lemma 5.2]. ∎

Now, we prove the following Theorems 13 and 14, which show that the generic singularities of n-ruled $(n+1)$-manifolds in $\mathbf{R}^{2n}$ are the cross cap$_n$ × interval (the case $n \geq 2$), the fold and the cusp (the case $n = 1$). Since any singularity of a generic smooth map germ of an $(n+1)$-manifold into $\mathbf{R}^{2n}$ is of the same kind, the following theorems assert that the generic singularities of n-ruled $(n+1)$-manifolds are the same as those of generic C^∞-maps of $(n+1)$-manifolds into $\mathbf{R}^N$, although the set of n-ruled $(n+1)$-manifolds is a thin subset in the space of all C^∞-maps.

THEOREM 13. *If $n \geq 2$ then there exists an open and dense subset*

$$\mathcal{O} \subset \mathcal{RM}_n(I, \mathbf{R}^N)$$

with respect to the quotient Whitney C^∞-topology such that for any $(\gamma, \delta_1, \delta_2, \dots, \delta_n) \in \mathcal{O}$ the n-ruled $(n+1)$-manifold germ $F_{(\gamma,\delta_1,\delta_2,\dots,\delta_n)}$ is an immersion germ or is right-left equivalent to the cross cap$_n$ × interval at any point $(t, u_1, u_2, \dots, u_n)$.

Proof. We define three subsets Q_l as follows.

$$\begin{aligned} Q_1 = \{&j^2(\gamma, \delta_1, \delta_2, \dots, \delta_n)(t) \in J^1(I, \mathbf{R}^{2n} \times (S^{2n-1})^n) \mid \\ &\qquad \dim\langle \delta_1(t), \delta_2(t), \dots, \delta_n(t)\rangle = n-1,\ t \in I\}, \\ Q_2 = \{&j^2(\gamma, \delta_1, \delta_2, \dots, \delta_n)(t) \in J^1(I, \mathbf{R}^{2n} \times (S^{2n-1})^n) \setminus Q_1 \mid \\ &\qquad \dim\langle \delta_1(t), \delta_1'(t), \delta_2(t), \delta_2'(t), \dots, \delta_n(t), \delta_n'(t)\rangle = 2n-2,\ t \in I\}, \\ Q_3 = \{&j^2(\gamma, \delta_1, \delta_2, \dots, \delta_n)(t) \in X \mid \\ &\qquad \dim\langle \delta_1(t), \delta_1'(t), \delta_2(t), \delta_2'(t), \dots, \delta_n(t), \delta_n'(t)\rangle = 2n-1,\ t \in I\}. \end{aligned}$$

In the definition of Q_3 we use a notation of an open submanifold

$$X = J^1\big(I, \mathbf{R}^{2n} \times (S^{2n-1})^n\big) \setminus (Q_1 \cup Q_2)$$

of $J^1\big(I, \mathbf{R}^{2n} \times (S^{2n-1})^n\big)$.

We take $(\gamma, \delta_1, \delta_2, \dots, \delta_n) \in C^\infty\big(I, \mathbf{R}^{2n} \times (S^{2n-1})^n\big)$ such that

$$j^2(\gamma, \delta_1, \delta_2, \dots, \delta_n)(t_0) \in J^2\big(I, \mathbf{R}^{2n} \times (S^{2n-1})^n\big) \setminus (Q_1 \cup Q_2 \cup Q_3).$$

Then $(\gamma, \delta_1, \delta_2, \dots, \delta_n)$ gives a non-degenerate n-ruled $(n+1)$-manifold near t_0. Since $F_{(\gamma,\delta_1,\delta_2,\dots,\delta_n)}$ is non-degenerate at t_0, there exists a striction curve $\sigma(t)$ near t_0. We shall rewrite the condition $\sigma'(t_0) = 0$.

We replace $\delta_1, \delta_2, \dots, \delta_n$ with constrictively adapted director curves $\bar{\delta}_1, \bar{\delta}_2, \dots, \bar{\delta}_n$ by using Lemma 6. Then,

$$\sigma'(t_0) = 0 \Longleftrightarrow \sigma'(t_0) \cdot \bar{\delta}_i(t_0) = 0 \quad (i = 1, 2, \dots, n).$$

Note that $G_i(t) = \sigma'(t) \cdot \bar{\delta}_i(t)$ are C^∞-functions of the partial derivatives at $t = t_0$ of the components of γ and δ_i $(i = 1, 2, \dots, n)$ of order at most two. We define a C^∞-map

$$\Phi : J^2\big(I, \mathbf{R}^{2n} \times (S^{2n-1})^n\big) \setminus (Q_1 \cup Q_2 \cup Q_3) \longrightarrow \mathbf{R}^n$$

by

$$\Phi\big(j^2(\gamma, \delta_1, \delta_2, \dots, \delta_n)(t_0)\big) = (G_1, G_2, \dots, G_n).$$

To determine the rank of the Jacobian matrix of Φ at $j^2(\gamma, \delta_1, \delta_2, \dots, \delta_n)(t_0)$, we calculate the derivative of Φ with respect to the coordinates of $J^2\big(I, \mathbf{R}^{2n} \times (S^{2n-1})^n\big)$ corresponding to the second order derivatives of the n components of γ. Then the derivatives of G_i $(i = 1, 2, \dots, n)$ coincide with the n components of

$$\big(\bar{\delta}'_k \cdot \bar{\delta}'_l\big)^{-1} \begin{pmatrix} -\bar{\delta}'_1 \\ -\bar{\delta}'_2 \\ \vdots \\ -\bar{\delta}'_n \end{pmatrix}.$$

So the rank of the Jacobian matrix of Φ at $j^2(\gamma, \delta_1, \delta_2, \dots, \delta_n)(t_0)$ is equal to n. Hence $(0, 0, \dots, 0) \in \mathbf{R}^n$ is a regular value of Φ and $T = \Phi^{-1}(0, 0, \dots, 0)$ is a closed submanifold of $J^2\big(I, \mathbf{R}^{2n} \times (S^{2n-1})^n\big) \setminus (Q_1 \cup Q_2 \cup Q_3)$ of codimension $n \geq 2$.

Therefore, the set

$$\begin{aligned} \bar{\mathcal{O}} = \big\{(\gamma, \delta_1, \delta_2, \dots, \delta_n) \in RM_n(I, \mathbf{R}^N) \mid \\ F_{(\gamma, \delta_1, \delta_2, \dots, \delta_n)} \text{ is an almost non-degenerate } n\text{-ruled } (n+1)\text{-manifold} \\ \text{and the striction curve is an immersion}\big\} \end{aligned}$$

coincides with the set

$$\begin{aligned} \bar{\mathcal{O}}' = \big\{(\gamma, \delta_1, \delta_2, \dots, \delta_n) \in C^\infty\big(I, \mathbf{R}^{2n} \times (S^{2n-1})^n\big) \mid \\ j^2(\gamma, \delta_1, \delta_2, \dots, \delta_n) \text{ is transverse to } Q_1, Q_2, Q_3, S \text{ and } T\big\}, \end{aligned}$$

where a codimension 1 submanifold S is defined by

$$\begin{aligned} S = \big\{j^2(\gamma, \delta_1, \delta_2, \dots, \delta_n)(t) \in Q_3 \mid \\ \gamma'(t) \in \langle \delta_1(t), \delta'_1(t), \delta_2(t), \delta'_2(t), \dots, \delta_n(t), \delta'_n(t)\rangle,\ t \in I\big\}. \end{aligned}$$

By Thom's jet transversality theorem, the set $\bar{\mathcal{O}}'$ is dense in $C^\infty\big(I, \mathbf{R}^{2n} \times (S^{2n-1})^n\big)$. Hence $\bar{\mathcal{O}}$ is dense in $RM_n(I, \mathbf{R}^N)$. So, $\mathcal{O} = \bar{\mathcal{O}}/\!\sim$ is dense in $\mathcal{RM}_n(I, \mathbf{R}^N)$.

On the other hand, we define a map $F_\sharp : \mathcal{RM}_n(I, \mathbf{R}^N) \longrightarrow C^\infty(I \times J_1 \times J_2 \times \dots \times J_n, \mathbf{R}^N)$ by $F_\sharp(\gamma, \delta_1, \delta_2, \dots, \delta_n) = F_{(\gamma, \delta_1, \delta_2, \dots, \delta_n)}$. Then, $F_\sharp$ is continuous. Furthermore, it is easy to check that the set

$$\begin{aligned} \mathcal{S} = \{f \in C^\infty(I \times J_1 \times J_2 \times \dots \times J_n, \mathbf{R}^N) \mid f \text{ is an immersion or is the right-left} \\ \text{equivalent to the cross cap}_n \times \text{interval at any point of } I \times J_1 \times J_2 \times \dots \times J_n\} \end{aligned}$$

is an open set.

Hence the set $F_\sharp^{-1}(\mathcal{S}) \cap \mathcal{R}$ is an open subset of $\mathcal{RM}_n(I, \mathbf{R}^N)$. By Theorem 4, it is clear that $\mathcal{O} = F_\sharp^{-1}(\mathcal{S}) \cap \mathcal{R}$. So, $\mathcal{O}$ is an open set of $\mathcal{RM}_n(I, \mathbf{R}^N)$. Therefore, $\mathcal{O}$ is an open and dense subset of $\mathcal{RM}_n(I, \mathbf{R}^N)$. This completes the proof. ∎

For the case $n = 1$ and $N = 2$ we have a slightly different result.

THEOREM 14. *There exists an open and dense subset*

$$\mathcal{O}_1 \subset \mathcal{RM}_1(I, \mathbf{R}^2)$$

with respect to the quotient Whitney C^∞-topology such that for any $(\gamma, \delta) \in \mathcal{O}_1$ the 1-ruled 2-manifold germ $F_{(\gamma,\delta)}$ is an immersion germ or is right-left equivalent to the fold or the cusp at any point (t, u).

Proof. We take $(\gamma, \delta) \in C^\infty(I, \mathbf{R}^2 \times S^1)$ such that

$$j^3(\gamma, \delta)(t_0) \in J^3(I, \mathbf{R}^2 \times S^1) \setminus Q_3,$$

Q_3 is the submanifold defined in the proof of Theorem 13. Then, since $F_{(\gamma,\delta)}$ is non-degenerate at t_0, there exists a striction curve $\sigma(t)$ near t_0.

We put

$$a(t) = \left(\gamma'(t) \cdot \delta(t) - \left(\frac{\gamma'(t) \cdot \delta'(t)}{\delta'(t) \cdot \delta'(t)}\right)'\right).$$

Then $\sigma(t) = 0$ if and only if $a(t) = 0$. Also $\sigma'(t) = \sigma''(t) = 0$ if and only if $a(t) = a'(t) = 0$. We define C^∞-functions

$$A_1 : J^3(I, \mathbf{R}^2 \times S^1) \setminus Q_3 \longrightarrow \mathbf{R} \quad \text{and} \quad A_2 : J^3(I, \mathbf{R}^2 \times S^1) \setminus Q_3 \longrightarrow \mathbf{R}^2$$

by

$$A_1\big(j^3(\gamma, \delta)(t_0)\big) = a \quad \text{and} \quad A_2\big(j^3(\gamma, \delta)(t_0)\big) = (a, a').$$

Then we see that $0 \in \mathbf{R}$ and $(0, 0) \in \mathbf{R}^2$ are regular values of A_1 and A_2. So, $A_1^{-1}(0)$ is a submanifold of $J^3(I, \mathbf{R}^2 \times S^1) \setminus Q_3$ of codimension 1 and $A_2^{-1}(0, 0)$ is a submanifold of $J^3(I, \mathbf{R}^2 \times S^1) \setminus Q_3$ of codimension 2.

Hence

$$\bar{\mathcal{O}}_1 = \big\{(\gamma, \delta) \in RM_1(I, \mathbf{R}^2) \,|\, j^3(\gamma, \delta) \text{ is transverse to } Q_3, \tilde{S}, A_1^{-1}(0) \text{ and } A_2^{-1}(0, 0)\big\}$$

is dense in $RM_1(I, \mathbf{R}^2)$ with respect to the Whitney C^∞-topology and we can easily check that for any $(\gamma, \delta) \in \bar{\mathcal{O}}_1$ and for any $(t, u) \in I \times J$, $F_{(\gamma,\delta)}$ at (t, u) is right-left equivalent to a fold or a cusp by the Theorem 5. Hence $\mathcal{O}_1 = \bar{\mathcal{O}}_1/\!\sim$ is dense.

The proof that $\mathcal{O}_1$ is an open set is the same as the proof of Theorem 13. Therefore $\mathcal{O}_1 = \bar{\mathcal{O}}_1/\!\sim$ is an open and dense subset of $\mathcal{RM}_1(I, \mathbf{R}^2)$. This completes the proof. ∎

Before ending the paper, we study the behavior of the striction curve near the exceptional rulings. Let $F_{(\gamma,\delta_1,\delta_2,\dots,\delta_n)}$ be an almost non-degenerate n-ruled manifold in $\mathbf{R}^{2n}$. Then it has a striction curve except for $t_k \in D$ (see Definition 11). Recall that singular points of $F_{(\gamma,\delta_1,\delta_2,\dots,\delta_n)}$ are located only on the striction curve. We take constrictively adapted director curves δ_i $(i = 1, 2, \dots, n)$. Take any point $t_k \in D$. By renumbering δ_i's if necessary, we may assume that

$$\delta_n'(t_k) = v_1\delta_1'(t_k) + v_2\delta_2'(t_k) + \dots + v_{n-1}\delta_{n-1}'(t_k) \tag{17}$$

for some $v_i \in \mathbf{R}$ $(i = 1, 2, \dots, n - 1)$, and

$$\delta_n'(t) = a(t)\gamma(t) - a(t)\sum_{l=1}^{n}\big(\gamma'(t) \cdot \delta_l(t)\big)\delta_l(t) + \sum_{l=1}^{n-1} b_l(t)\delta_l'(t) \tag{18}$$

for t near t_k. From the almost non-degeneracy, we have $a(t) \neq 0$. Recall that the coefficients for the striction curve $\sigma(t) = \gamma(t) + \sum_{i=1}^{n} f_i(t)\delta_i(t)$ $(t \notin D)$ are given by

$$\begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix} = -M(t)^{-1} \begin{pmatrix} \gamma'(t)\cdot\delta_1'(t) \\ \gamma'(t)\cdot\delta_2'(t) \\ \vdots \\ \gamma'(t)\cdot\delta_n'(t) \end{pmatrix}$$

(see Lemma 7). By an elementary calculation of linear algebra, we have

$$f_i = -\det K^i/\det M \quad (i = 1, 2, \dots, n-1)$$

where

$$K^i = \begin{pmatrix} K_1^i & \dots & K_{i-1}^i & \begin{matrix} \gamma'\cdot\delta_1' \\ \gamma'\cdot\delta_2' \\ \vdots \\ \gamma'\cdot\delta_{n-1}' \\ \gamma'\cdot\delta_n' \end{matrix} & K_{i+1}^i & \dots & K_{n-1}^i & K_n^i \end{pmatrix}$$

and $K_j^i = {}^t(\delta_1'\cdot\delta_j', \delta_2'\cdot\delta_j', \dots, \delta_{n-1}'\cdot\delta_j', \delta_n'\cdot\delta_j')$ $(j = 1, 2, \dots, n,\ j \neq i)$. From (18) we get

$$\begin{aligned} \delta_j'\cdot\delta_n' &= a(\gamma'\cdot\delta_j') + \sum_{l=1}^{n-1} b_l(\delta_l'\cdot\delta_j), \\ \delta_n'\cdot\delta_n' &= a^2(\gamma'\cdot\gamma') - a^2\sum_{l=1}^{n}(\gamma'\cdot\delta_l)^2 + \Big(\sum_{l=1}^{n-1} b_l\delta'\Big)\cdot\Big(\sum_{l=1}^{n-1} b_l\delta'\Big) + 2a\sum_{l=1}^{n-1} b_l(\gamma'\cdot\delta_l') \\ \text{and}\quad \gamma'\cdot\delta_n' &= a(\gamma'\cdot\gamma') - \sum_{l=1}^{n} a(\gamma'\cdot\delta_l)^2 + \sum_{l=1}^{n-1} b_l(\gamma'\cdot\delta_l'). \end{aligned}$$

By subtracting b_m times m-column from n-column of the matrix K^i for any $m = 1, 2, \dots, n-1$ we get a simplified matrix $\bar{K^i}$ with $\det\bar{K^i} = \det K^i$. Next, by subtracting b_m times m-row from n-row for any $m = 1, 2, \dots, n-1$, substracting a times i-row from n-row and changing i-row and n-row of the matrix $\bar{K^i}$, we get a simpler matrix L^i with $\det L^i = \det\bar{K^i}$. Here,

$$L^i = \begin{pmatrix} L_1^i & \dots & L_{i-1}^i & \begin{matrix} b_i(\delta_1'\cdot\delta_j') \\ b_i(\delta_2'\cdot\delta_j') \\ \vdots \\ b_i(\delta_{n-1}\cdot\delta_j') \\ ab_i(\gamma'\cdot\delta_i') \end{matrix} & L_{i+1}^i & \dots & L_{n-1}^i & L_n^i \end{pmatrix},$$

$$L_j^i = {}^t\big(\delta_1\cdot\delta_j, \dots, \delta_{n-1}\cdot\delta_j,\ a(\gamma'\cdot\delta_j')\big) \quad (j = 1, 2, \dots, n-1,\ j \neq i)$$

$$\text{and}\quad L_n^i = {}^t\Big(\gamma'\cdot\delta_1', \gamma'\cdot\delta_2', \dots, \gamma'\cdot\delta_{n-1}, a(\gamma'\cdot\gamma') - a\sum_{l=1}^{n-1}(\gamma'\cdot\delta_l)^2\Big).$$

Now, we define another matrix

$$M_1 = \begin{pmatrix} \delta_1' \cdot \delta_1' & \dots & \delta_1' \cdot \delta_{n-1}' & a(\gamma' \cdot \delta_1') \\ \vdots & \ddots & \vdots & \vdots \\ \delta_{n-1}' \cdot \delta_1' & \dots & \delta_{n-1}' \cdot \delta_{n-1}' & a(\gamma' \cdot \delta_{n-1}') \\ a(\gamma' \cdot \delta_1') & \dots & a(\gamma' \cdot \delta_{n-1}') & a^2(\gamma' \cdot \gamma') - a^2 \sum_{l=1}^{n} (\gamma' \cdot \delta_l)^2 \end{pmatrix}.$$

Then by direct calculations we see that $\det M = \det M_1$ and $\det L^i = (b_i/a) \det M_1$. Since $\det M(t) \neq 0$ for $t \neq t_k$, we have $f_i = b_i/a$ $(i = 1, 2, \dots, n-1)$. By the same kind of calculations, we get $f_n = -1/a$. So, the striction curve $\sigma(t)$ can be written as

$$\sigma(t) = \gamma(t) + \sum_{i=1}^{n-1} \frac{b_i}{a} \delta_i - \frac{1}{a} \delta_n = \gamma(t) + f_n(t) \Big(- \sum_{i=1}^{n-1} b_i \delta_i(t) + \delta_n(t) \Big)$$

near t_k.

Since $\delta_n'(t) \to v_1 \delta_1'(t_k) + v_2 \delta_2'(t_k) + \dots + v_{n-1} \delta_{n-1}'(t_k)$ as $t \to t_k$, we have $a(t) \to 0$ and $b_i(t) \to v_i$ as $t \to t_k$. Hence we get

$$\lim_{t \to t_k} |f_n(t)| = \infty \quad \text{and} \quad \lim_{t \to t_k} - \sum_{i=1}^{n-1} b_i(t) \delta_i(t) + \delta_n(t) = - \sum_{i=1}^{n-1} v_i \delta_i(t_k) + \delta_n(t_k).$$

Therefore the striction curve has an asymptotic direction

$$v = - \sum_{i=1}^{n-1} v_i \delta_i(t_k) + \delta_n(t_k)$$

in the exceptional ruling at t_k.

So, two branches of the striction curve approaching to the exceptional ruling from the both sides $\sigma(t)$, $t \in (t_k - \varepsilon, t_k)$ and $\sigma(t)$, $t \in (t_k, t_k + \varepsilon)$ should have the same asymptotic direction v. Moreover, by the condition (A3) of almost non-degeneracy we see that $a'(t_k) \neq 0$, so they diverge to the opposite directions each other.

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