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ON THE DUAL SPACE OF THE TJURINA ALGEBRA ATTACHED TO A SEMI-QUASIHOMOGENEOUS ISOLATED SINGULARITY

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Abstract. A dual space of the Tjurina algebra attached to a non-quasihomogeneous unimodal or bimodal singularity is considered. It is shown that almost every algebraic local cohomology class, belonging to the dual space, can be characterized as a solution of a holonomic system of first order differential equations.

1. Introduction. Let f be a holomorphic function with an isolated singularity at the origin. Let $\mathcal{H}_M$ denote the set of algebraic local cohomology classes with support at the origin annihilated by partial derivatives of the holomorphic function f. The set $\mathcal{H}_M$ is the dual space, via the Grothendieck local duality, of the Milnor algebra associated with the isolated singularity. In this paper, local cohomology classes in $\mathcal{H}_M$ are considered in the context of D-modules.

In [8], T. Yano investigated b-functions by using algebraic local cohomology classes. In [6], the authors of the present paper showed that if a given holomorphic function f is quasihomogeneous, an algebraic local cohomology class σ which generates $\mathcal{H}_M$ over $\mathcal{O}_{X,O}$ is characterized as a solution of a simple holonomic system of first order linear partial differential equations. We also proved that, in the non-quasihomogeneous cases, none of the generators σ can be characterized uniquely by means of first order system of linear partial differential equations (see also [3], [2]).

In this paper, we mainly consider the algebraic local cohomology classes belonging

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to the dual space of the Tjurina algebra, that is, the algebraic local cohomology classes in $\mathcal{H}_M$ annihilated by multiplying the function f. The aim is to show that every local cohomology class τ in the dual space of Tjurina algebra attached to semi-quasihomogeneous unimodal and bimodal isolated singularities, except the bimodal Z type singularity cases, has a characterization as the solution of a simple holonomic system of first order partial differential equations. The proofs of these results involve case by case computations, which are too lengthy to include. Thus, instead of presenting a full account of proof, we give the main idea and our strategy of the proof.

In Section 2, we consider non-quasihomogeneous cases and analyze an algebraic local cohomology $\sigma \in \mathcal{H}_M$ which is not annihilated by the function f. We introduce an ideal $\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\sigma)$, in the ring $\mathcal{D}_X$ of linear partial differential operators, generated by partial differential operators of order at most one which annihilate $\sigma \in \mathcal{H}_M$. We show that, by adopting the same approach as in [3], [2], such local cohomology class σ is not characterized uniquely as a solution of the holonomic system $\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\sigma)$. The statement amounts to say that the system $\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\sigma)$ is not simple as a $\mathcal{D}_X$-module.

In Section 3, we introduce a dual space $\mathcal{H}_T$ of Tjurina algebra as a subspace of $\mathcal{H}_M$. We give two main theorems of the present paper concerning an algebraic local cohomology class τ in $\mathcal{H}_T$ attached to an exceptional unimodal or bimodal singularity.

In Section 4, we analyze properties of first order partial differential operators which annihilate a zero-dimensional algebraic local cohomology class in general. We provide a method to describe the solution space of $\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\sigma)$. After that, we consider semi-quasihomogeneous cases. We present a criterion for the simplicity of the holonomic system $\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\sigma)$ which is effectively used in proving the main results.

In Section 5, we explain the main idea and describe a strategy of proofs. We present some results of computations including bimodal Z type cases.

2. Local cohomology classes in the dual space of Milnor algebra. Let X be an open neighborhood of the origin O in the complex n-dimensional affine space $\mathbb{C}^n$. Let $\mathcal{O}_X$ be the sheaf on X of holomorphic functions. For a holomorphic function $f = f(z_1, \dots, z_n) \in \mathcal{O}_{X,O}$ with an isolated singularity at the origin O, let $\mathcal{I}$ denote the ideal in $\mathcal{O}_{X,O}$ generated by the partial derivatives $f_j = \dfrac{\partial f}{\partial z_j}$ $(j = 1, \dots, n)$ of f:

$$\mathcal{I} = \langle f_1, \dots, f_n \rangle_O.$$

Let $\mathcal{H}^n_{[O]}(\mathcal{O}_X)$ be the n-th algebraic local cohomology group with support at the origin O. Denote by $\mathcal{H}_M$ the vector space consisting of algebraic local cohomology classes in $\mathcal{H}^n_{[O]}(\mathcal{O}_X)$ which are annihilated by every element in $\mathcal{I}$:

$$\mathcal{H}_M = \{\sigma \in \mathcal{H}^n_{[O]}(\mathcal{O}_X) \,|\, g\sigma = 0,\ g \in \mathcal{I}\}.$$

Let $\mathcal{D}_X$ be the sheaf on X of linear partial differential operators. The algebraic local cohomology group $\mathcal{H}^n_{[O]}(\mathcal{O}_X)$ has a structure of coherent $\mathcal{D}_X$-module. We denote by $\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\sigma)$ the left ideal in $\mathcal{D}_X$ generated by differential operators of order at most one which annihilate an algebraic local cohomology class $\sigma \in \mathcal{H}_M$. The $\mathcal{D}_X$-module

$\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\sigma)$ becomes a holonomic system supported at the origin. As a generalization of results in [6], we have the following theorem.

THEOREM 2.1. *Assume that the function f defines a non-quasihomogeneous singularity at the origin. For any class σ in $\mathcal{H}_M$ which is not annihilated by f,*

$$\dim \mathrm{Hom}_{\mathcal{D}_X}\big(\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\sigma), \mathcal{H}^n_{[O]}(\mathcal{O}_X)\big) \geq 2.$$

Proof. Let us denote by $F \in \mathcal{D}_X$ the multiplication operator defined by $F = f$. Let $P = \sum_{j=1}^n a_j(z)\,\dfrac{\partial}{\partial z_j} + a_0(z)$ be an annihilator of the cohomology class σ in $\mathcal{H}_M$. Then

$$\begin{aligned} P(f\sigma) &= PF\sigma \\ &= (PF - FP)\sigma + FP\sigma \\ &= \sum_{j=1}^n a_j(z)\,\frac{\partial f}{\partial z_j}\,\sigma. \end{aligned}$$

Since $\sum_{j=1}^n a_j(z) f_j \in \mathcal{I}$, $P(f\sigma) = 0$ holds. As σ and $f\sigma (\neq 0)$ are linearly independent, we have

$$\dim \mathrm{Hom}_{\mathcal{D}_X}\big(\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\sigma), \mathcal{H}^n_{[O]}(\mathcal{O}_X)\big) \geq 2. \qquad \blacksquare$$

COROLLARY 2.1. *Under the same assumption as in Theorem* 2.1*, the multiplicity of the holonomic system $\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\sigma)$ at the origin is greater than or equal to two.*

Let $\mathcal{A}nn_{\mathcal{D}_X}(\sigma)$ be the annihilator in $\mathcal{D}_X$ of $\sigma \in \mathcal{H}_M$ consisting of linear partial differential operators which annihilate σ. The following holds.

COROLLARY 2.2. *Under the same assumption as in Theorem* 2.1*, $\mathcal{A}nn_{\mathcal{D}_X}(\sigma)$ is not equal to $\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\sigma)$.*

Proof. Since the holonomic system $\mathcal{D}_X/\mathcal{A}nn_{\mathcal{D}_X}(\sigma)$ is simple at the origin, the dimension of the solution space $\mathrm{Hom}_{\mathcal{D}_X}\big(\mathcal{D}_X/\mathcal{A}nn_{\mathcal{D}_X}(\sigma), \mathcal{H}^n_{[0]}(\mathcal{O}_X)\big)$ is one. Therefore, if $\mathcal{A}nn_{\mathcal{D}_X}(\sigma) = \mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\sigma)$, then $\dim \mathrm{Hom}_{\mathcal{D}_X}\big(\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\sigma), \mathcal{H}^n_{[O]}(\mathcal{O}_X)\big) = 1$, which is a contradiction. $\blacksquare$

REMARK. We have recently shown ([4]) that if f defines an exceptional unimodal singularity and σ is a generator of $\mathcal{H}_M$ over $\mathcal{O}_{X,O}$, then

$$\dim \mathrm{Hom}_{\mathcal{D}_X}\big(\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\sigma), \mathcal{H}^n_{[O]}(\mathcal{O}_X)\big) = 2.$$

We have also verified that $\mathcal{A}nn_{\mathcal{D}_X}(\sigma)$ is generated by partial differential operators of order at most two in this case. Please refer to [3].

3. Tjurina local cohomologies attached to exceptional unimodal and bimodal singularities. We define $\mathcal{H}_T$ to be the subspace of $\mathcal{H}_M$ consisting of algebraic local cohomology classes which are annihilated by f:

$$\mathcal{H}_T = \{\tau \in \mathcal{H}_M \mid f\tau = 0\} \subseteq \mathcal{H}_M.$$

Note that the Grothendieck local duality between $\mathcal{O}_{X,O}/\mathcal{I}$ and $\mathcal{H}_M$ naturally induces a duality between $\mathcal{O}_{X,O}/\langle \mathcal{I}, f\rangle$ and the space $\mathcal{H}_T$, where $\langle f, \mathcal{I}\rangle$ is the ideal generated by f and $\mathcal{I}$. We call elements in $\mathcal{H}_T$ Tjurina local cohomology classes.

Let us consider the holonomic system $\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\tau)$ attached to a Tjurina local cohomology class $\tau \in \mathcal{H}_T$ and the solution space $\mathrm{Hom}_{\mathcal{D}_X}\big(\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\tau), \mathcal{H}^n_{[O]}(\mathcal{O}_X)\big)$. We present a simple example for illustration.

EXAMPLE 1. The function $f = x^4 + y^5 + ax^2y^3$ is the normal form of W_{12} unimodal singularity at the origin. The Gröbner basis with the lexicographic ordering $x \succ y$ of $\mathcal{I}$ is $\{y^6, y^4x, 5y^4 + 3ay^2x^2, 2x^3 + ay^3x\}$.

Let us consider a local cohomology class $\sigma = \big[-2\frac{1}{x^3y^4} + \frac{6}{5}a\frac{1}{xy^6} + a\frac{1}{x^5y}\big]$ in $\mathcal{H}_M$. Since $f\sigma \neq 0$, Theorem 2.1 yields $\dim \mathrm{Hom}_{\mathcal{D}_X}\big(\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\sigma), \mathcal{H}^2_{[0]}(\mathcal{O}_X)\big) \geq 2$. Indeed, the ideal $\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\sigma)$, generated by

$$\begin{aligned}&yx\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + 7y,\\ &5ay^3\frac{\partial}{\partial x} - 10yx\frac{\partial}{\partial y} + (-3a^2y - 40)x,\\ &y^6,\ y^4x,\ 5y^4 + 3ay^2x^2,\ 2x^3 + ay^3x,\end{aligned}$$

defines the holonomic system $\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\sigma)$ which is not simple. In fact, a direct calculation yields that $\mathrm{Hom}_{\mathcal{D}_X}\big(\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\sigma), \mathcal{H}^2_{[0]}(\mathcal{O}_X)\big) = \mathrm{Span}\{\sigma, [1/xy]\}$, which implies $\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\sigma) \neq \mathcal{A}nn_{\mathcal{D}_X}(\sigma)$.

Now let τ be a Tjurina local cohomology class $\big[\frac{1}{x^3y^3} - \frac{3}{5}a\frac{1}{xy^5}\big]$ in $\mathcal{H}_T$. $\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\tau)$ is generated by a first order differential operator $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 6$ and multiplication operators defined by y^5, y^3x, $5y^3 + 3ayx^2$ and x^3. The solution space of the holonomic system $\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\tau)$ is spanned by τ. Thus, $\mathcal{A}nn_{\mathcal{D}_X}(\tau) = \mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\tau)$.

For Tjurina local cohomology classes attached to an exceptional unimodal or bimodal singularity at the origin, we have the following results.

THEOREM 3.1. *Let $f \in \mathcal{O}_{X,O}$ be a holomorphic function defining an exceptional unimodal singularity at the origin. For a cohomology class $\tau \in \mathcal{H}_T$,*

$$\dim \mathrm{Hom}_{\mathcal{D}_X}\big(\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\tau), \mathcal{H}^n_{[0]}(\mathcal{O}_X)\big) = 1.$$

THEOREM 3.2. *Let $f \in \mathcal{O}_{X,O}$ be a holomorphic function defining an exceptional bimodal singularity at the origin. For a class $\tau \in \mathcal{H}_T$, the following holds.*

$$\dim \mathrm{Hom}_{\mathcal{D}_X}\big(\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\tau), \mathcal{H}^n_{[O]}(\mathcal{O}_X)\big) \begin{cases} \leq 2 & \textit{if } f \textit{ is of type } Z,\\ = 1 & \textit{otherwise.}\end{cases}$$

Note that Z type singularities consist of three cases. These normal forms are given by $Z_{17} : x^3y + y^8 + (a + by)xy^6$, $Z_{18} : x^3y + xy^6 + (a + by)y^9$, $Z_{19} : x^3y + y^9 + (a + by)xy^7$.

COROLLARY 3.1. *For exceptional families of unimodal and bimodal singularities except Z type bimodal singularities (i.e., Z_{17}, Z_{18} and Z_{19}), the annihilator $\mathcal{A}nn_{\mathcal{D}_X}(\tau)$ coincides with $\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\tau)$ for any $\tau \in \mathcal{H}_T$.*

4. The first order differential operators. Let σ be an algebraic local cohomology class in $\mathcal{H}^n_{[0]}(\mathcal{O}_X)$ supported at the origin. Let $\mathcal{I}(\sigma)$ denote the annihilator in $\mathcal{O}_{X,O}$ of σ. Let $\mathcal{H}(\sigma)$ denote the set of algebraic local cohomology classes generated by σ over $\mathcal{O}_{X,O}$, i.e., $\mathcal{H}(\sigma) = \mathcal{O}_{X,O}\sigma$, which is equal to $\big(\mathcal{O}_{X,O}/\mathcal{I}(\sigma)\big)\sigma$. Let $\mathcal{L}(\sigma)$ be the set of linear partial differential operators of order at most one which annihilate σ:

$$\mathcal{L}(\sigma) = \Big\{P = \sum_{j=1}^{n} a_j(z)\,\frac{\partial}{\partial z_j} + a_0(z)\,\big|\,P\sigma = 0,\ a_j(z) \in \mathcal{O}_{X,O},\ j = 0, 1, \dots, n\Big\}.$$

Then $\mathcal{D}_X\mathcal{L}(\sigma)$ gives rise to the left ideal $\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\sigma)$.

Lemma 4.1. *Let P be a first order linear partial differential operator in $\mathcal{L}(\sigma)$. Then $P(\mathcal{H}(\sigma)) \subseteq \mathcal{H}(\sigma)$.*

Proof. By the definition of $\mathcal{H}(\sigma)$, any class $\eta \in \mathcal{H}(\sigma)$ is written in the form $\eta = h\sigma$ with some holomorphic function $h \in \mathcal{O}_{X,O}$. Let v_P be the first order part $\sum_{j=1}^{n} a_j(z)\,\dfrac{\partial}{\partial z_j}$ of an operator $P = \sum_{j=1}^{n} a_j(z)\,\dfrac{\partial}{\partial z_j} + a_0(z)$ in $\mathcal{L}(\sigma)$. Then

$$\begin{aligned} P(\eta) &= P(h\sigma) \\ &= (Ph - hP)\sigma + hP\sigma \\ &= v_P(h)\sigma \in \mathcal{H}(\sigma). \end{aligned}$$

Thus we have $P(\mathcal{H}(\sigma)) \subseteq \mathcal{H}(\sigma)$. ∎

Let $\mathcal{V}(\sigma)$ denote the set of differential operators of the form $\sum_{j=1}^{n} a_j(z)\,\dfrac{\partial}{\partial z_j}$ acting on $\mathcal{H}(\sigma)$. Then we have

$$\mathcal{V}(\sigma) = \Big\{v = \sum_{j=1}^{n} a_j(z)\,\frac{\partial}{\partial z_j}\,\Big|\,vg \in \mathcal{I}(\sigma) \text{ for all } g \in \mathcal{I}(\sigma),\ a_j(z) \in \mathcal{O}_{X,O},\ j = 1, \dots, n\Big\}.$$

Lemma 4.2. *The mapping from $\mathcal{L}(\sigma)$ to $\mathcal{V}(\sigma)$ which associates the first order part $v_P \in \mathcal{V}(\sigma)$ to $P \in \mathcal{L}(\sigma)$ is surjective.*

It follows immediately from the definition of $\mathcal{V}(\sigma)$ that any element $v \in \mathcal{V}(\sigma)$ induces a linear map, denoted by $\bar{v}$, acting on $\mathcal{O}_{X,O}/\mathcal{I}(\sigma)$:

$$\bar{v} : \mathcal{O}_{X,O}/\mathcal{I}(\sigma) \to \mathcal{O}_{X,O}/\mathcal{I}(\sigma).$$

We define $\mathcal{K}(\sigma)$ to be

$$\mathcal{K}(\sigma) = \big\{h \in \mathcal{O}_{X,O}/\mathcal{I}(\sigma)\,|\,\bar{v}h = 0 \text{ for all } v \in \mathcal{V}(\sigma)\big\}.$$

We have the following result ([6]).

Theorem 4.1. *For an algebraic local cohomology class $\sigma \in \mathcal{H}^n_{[O]}(\mathcal{O}_X)$,*

$$\mathrm{Hom}_{\mathcal{D}_X}\big(\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\sigma), \mathcal{H}^n_{[O]}(\mathcal{O}_X)\big) = \mathrm{Span}\{h\sigma\,|\,h \in \mathcal{K}(\sigma)\}.$$

Proof. Regarding each element of $\mathcal{I}(\sigma)$ as a multiplication operator, namely a linear partial differential operator of order zero, we have $\mathcal{D}_X\mathcal{I}(\sigma) \subset \mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\sigma)$. This implies

$$\mathrm{Hom}_{\mathcal{D}_X}\big(\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\sigma), \mathcal{H}^n_{[O]}(\mathcal{O}_X)\big) \subset \mathcal{H}(\sigma).$$

Thus any solution in $\mathcal{H}^n_{[0]}(\mathcal{O}_X)$ of the holonomic system $\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\sigma)$ can be written in the form $h\sigma$ with some $h \in \mathcal{O}_{X,O}/\mathcal{I}(\sigma)$. If $P \in \mathcal{L}(\sigma)$, then $P(h\sigma) = (v_P h)\sigma = 0$. Hence an algebraic local cohomology class $h\sigma$ is in $\mathrm{Hom}_{\mathcal{D}_X}\big(\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\sigma), \mathcal{H}^n_{[O]}(\mathcal{O}_X)\big)$ if and only if $\bar{v}_P h = 0$, for all $P \in \mathcal{L}(\sigma)$, which is equivalent to $h \in \mathcal{K}(\sigma)$. This completes the proof. ∎

Corollary 4.1. *The multiplicity of the holonomic system $\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\sigma)$ at the origin is equal to the dimension of $\mathcal{K}(\sigma)$.*

Let us return to the semi-quasihomogeneous isolated singularity cases. Let $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{N}^n$ be a weight vector of the quasihomogeneous part of the semi-quasihomogeneous function f with respect to a coordinate system $z = (z_1, \ldots, z_n) \in X$.

Here we introduce a notion of the weighted degree to algebraic local cohomology classes with respect to the quasi-weight $\mathbf{w}$.

Definition. For a cohomology class $\eta = \left[\sum_{\mathbf{k}\in\Lambda} c_{\mathbf{k}} \dfrac{1}{z^{\mathbf{k}}}\right] \in \mathcal{H}^n_{[O]}(\mathcal{O}_X)$, we define its *degree* $d_{\mathbf{w}}(\eta)$ to be the smallest degree of classes $\left[\dfrac{1}{z^{\mathbf{k}}}\right]$ in η:

$$d_{\mathbf{w}}(\eta) = \min\big\{-\langle \mathbf{w}, \mathbf{k}\rangle = -(w_1k_1 + \ldots + w_nk_n) \,|\, \mathbf{k} \in \Lambda\big\},$$

where Λ is a set of all exponents $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$ of non-zero term $c_{\mathbf{k}} \dfrac{1}{z^{\mathbf{k}}}$ in the above expression of the cohomology class η.

We also define the weighted degree $d_{\mathbf{w}}(R)$ of a differential operator $R = \sum_{j=1}^{n} a_j(z) \dfrac{\partial}{\partial z_j}$ to be $\min_{1\le j\le n}\{d_{\mathbf{w}}(a_j(z)) - w_j\}$, where j runs $1 \le j \le n$ with non-zero coefficient $a_j(z)$ and $d_{\mathbf{w}}(a_j(z))$ is the weighted degree of the function $a_j(z)$. With the aid of the notion of the weighted degree, one immediately obtains the following lemma.

Lemma 4.3. *Let f be a semi-quasihomogeneous holomorphic function with an isolated singularity at the origin, τ an algebraic local cohomology class in $\mathcal{H}_T$. Assume that there is an operator v in $\mathcal{V}(\tau)$ satisfying $d_{\mathbf{w}}(v) = 0$. Then, $\mathcal{K}(\tau) = \mathrm{Span}\{1\}$.*

We call an operator with the weighted degree zero *Euler operator.*

Combining Theorem 4.1 and Lemma 4.3, we have the following criterion.

Proposition 4.1. *Let τ be an algebraic local cohomology in $\mathcal{H}_T$ attached to semi-quasihomogeneous isolated singularity. If there are Euler operators in $\mathcal{V}(\tau)$, then*

$$\mathrm{Hom}_{\mathcal{D}_X}\big(\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\tau), \mathcal{H}^n_{[O]}(\mathcal{O}_X)\big) = \mathrm{Span}\{\tau\}.$$

We give an example to illustrate the above results. Actual computations are carried out on the polynomial ring $\mathbb{Q}[z]$ over the field $\mathbb{Q}$ of rational numbers and in the Weyl algebra $A_n := \mathbb{Q}[z_1, \ldots, z_n]\langle \partial/\partial z_1, \ldots, \partial/\partial z_n\rangle$. For a polynomial $f \in \mathbb{Q}[z]$, let I be the primary component of the ideal $\langle \partial f/\partial z_1, \ldots, \partial f/\partial z_n\rangle$ in $\mathbb{Q}[z]$ corresponding to the origin. Let H_M be the dual space in $H^n_{[O]}(\mathcal{O}_X) = \Gamma\big(X, \mathcal{H}^n_{[O]}(\mathcal{O}_X)\big)$ of $\mathbb{Q}[z]/I$. For a cohomology class $\sigma \in H_M$, let $I(\sigma)$ and $\mathrm{Ann}_{A_n}(\sigma)$ denote the annihilator of σ in $\mathbb{Q}[z]$ and in A_n

respectively. Let $V(\sigma)$ denote a finite dimensional vector space defined by

$$V(\sigma)=\Big\{v=\sum_{j=1}^{n}a_j(z)\,\frac{\partial}{\partial z_j}\,\Big|\,a_j(z)\in\mathbb{Q}[z]/I(\sigma),\ vh(z)\in I(\sigma)\text{ for all }h\in I(\sigma)\Big\}.$$

Notice that, in the definition above, all the coefficients $a_j(z)$ of v in $V(\sigma)$ are taken from the quotient space $\mathbb{Q}[z]/I(\sigma)$. Put

$$K(\sigma)=\{h(z)\in\mathbb{Q}[z]/I(\sigma)\,|\,vh(z)\in I(\sigma)\text{ for all }v\in V(\sigma)\}$$

and $H(\sigma)=\mathbb{Q}[z]\sigma$. The results for $\mathcal{V}(\sigma)$, $\mathcal{K}(\sigma)$ and $\mathcal{H}(\sigma)$ presented in this section are also valid for $V(\sigma)$, $K(\sigma)$ and $H(\sigma)$. Put $H_T=\{\sigma\in H_M\,|\,f\sigma=0\}\subset H_M$.

Example 2 (W_{12} singularity). Let us consider the polynomial $f=x^4+y^5+ax^2y^3$ again. The primary decomposition of the ideal $\langle\partial f/\partial x,\partial f/\partial y\rangle$ is $I\cap\langle 3a^2y-10,27a^5x^2+500\rangle$ where $I=\langle y^6,y^4x,5y^4+3ay^2x^2,2x^3+ay^3x\rangle$. Put $\sigma_{11}=\big[-2\frac{1}{x^3y^4}+a\frac{1}{x^5y}+\frac{6}{5}a\frac{1}{xy^6}\big]$. The cohomology class σ_{11} is in $H_M\setminus H_T$ and there are no Euler operators in $V(\sigma_{11})$. The operator with the smallest degree in $V(\sigma_{11})$ is $v=yx\partial_x+y^2\partial_y$ with $d_{(5,4)}(v)=4>0$. We find $K(\sigma_{11})=\mathrm{Span}\{1,y^5\}$ and thus, by Theorem 3.1,

$$\dim\mathrm{Hom}_{\mathcal{D}_X}\big(\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\sigma_{11}),\mathcal{H}^n_{[O]}(\mathcal{O}_X)\big)=2$$

(cf. Example 1).

H_M consists of linear combinations of $\tau_0=\big[\frac{1}{xy}\big]$, $\tau_1=\big[\frac{1}{x^2y}\big]$, $\tau_2=\big[\frac{1}{x^3y}\big]$, $\tau_3=\big[\frac{1}{xy^2}\big]$, $\tau_4=\big[\frac{1}{x^2y^2}\big]$, $\tau_5=\big[\frac{1}{x^3y^2}\big]$, $\tau_6=\big[\frac{1}{xy^3}\big]$, $\tau_7=\big[\frac{1}{x^2y^3}\big]$, $\tau_8=\big[\frac{1}{x^3y^3}-\frac{3}{5}a\frac{1}{xy^5}\big]$, $\tau_9=\big[\frac{1}{xy^4}\big]$, $\tau_{10}=\big[-2\frac{1}{x^2y^4}+a\frac{1}{x^4y}\big]$ and σ_{11}.

Let τ be the cohomology class $\tau=\tau_8+p_7\tau_7\in H_T$ with a parameter p_7. τ_7 satisfies $\tau_7=x\tau_8$. We have $I(\tau)=I(\tau_8)=\langle y^4,2x^2+ay^3\rangle$ and an Euler operator $3x\partial_x+2y\partial_y$ in $V(\tau)=V(\tau_8)$. From Lemma 4.3, $K(\tau)=\mathrm{Span}\{1\}$ and thus

$$\dim\mathrm{Hom}_{\mathcal{D}_X}\big(\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\tau),\mathcal{H}^n_{[O]}(\mathcal{O}_X)\big)=1.$$

Put $\tau=p_9\tau_9+p_6\tau_6+p_4\tau_4\in H_T$ with parameters p_4, p_6 and p_9 satisfying $p_9\neq 0$. Then $I(\tau)=\langle y^4,p_9^2x+p_6p_4y^3-p_9p_4y^2\rangle$ and $V(\tau)=\mathrm{Span}\{y^3\partial_y,2p_4y^3\partial_x+p_9y^2\partial_y,(-3p_6p_4y^3+2p_9p_4y^2)\partial_x+p_9^2y\partial_y\}$. We use the lexicographical order $x\succ y$ in computations and we identify $\mathbb{Q}[x,y]/I(\tau)$ with $\mathrm{Span}\{1,y,y^2,y^3\}$. The operator $v=(-3p_6p_4y^3+2p_9p_4y^2)\partial_x+p_9^2y\partial_y$ of weighted degree 0 is an Euler operator in $V(\tau)$. Now let G denote the zeroth order linear partial differential operator defined to be $G=p_9^2x+p_6p_4y^3-p_9p_4y^2\in I(\tau)$. The first order part of the differential operator $v+3\partial_xG$ becomes $(-p_9p_4y^2+3p_9^2x)\partial_x+p_9^2y\partial_y$ which also acts on $\mathbb{Q}[x,y]/I(\tau)$. Notice that the term "Euler operators" in this paper is consistent with the classical one.

5. A strategy of proofs and examples. Theorem 4.1 says that the determination of the local cohomology solution space of $\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\sigma)$ amounts to the computation of the space $K(\sigma)$. It follows immediately from the definition that the space $K(\sigma)$ depends only on the annihilator $I(\sigma)$, namely we have the following.

Lemma 5.1. *Let σ and σ' be two algebraic local cohomology classes in $\mathcal{H}^n_{[0]}(\mathcal{O}_X)$. Assume that $I(\sigma)=I(\sigma')$. Then $K(\sigma)=K(\sigma')$.*

The observation above yields the following strategy to examine the holonomic system $\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\sigma)$.

Step 1. Classify $I(\sigma)$.

Step 2. Compute $V(\sigma)$.

Step 3. Look for Euler operators.
If there are Euler operators, we have

$$\dim \mathrm{Hom}_{\mathcal{D}_X}\big(\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\sigma), \mathcal{H}^n_{[O]}(\mathcal{O}_X)\big) = 1,$$

else, compute explicitly $K(\sigma)$. Then

$$\dim \mathrm{Hom}_{\mathcal{D}_X}\big(\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\sigma), \mathcal{H}^n_{[O]}(\mathcal{O}_X)\big) = \dim K(\sigma).$$

We executed the above procedure for the exceptional family of unimodal and bimodal singularity cases.

Unimodal singularities. There are 14 types of exceptional unimodal singularities. Computing $V(\sigma)$ for each normal forms of exceptional unimodal singularities, we have found Euler operators. Thus we arrived at Theorem 3.1.

In order to illustrate the procedure presented above, we give an example.

Example 3. The function $f = x^3 + xy^5 + ay^8$ defines E_{13} unimodal singularity at the origin ($a \neq 0$). A basis of the space H_T is given by $\tau_0, \tau_2, \dots, \tau_{11}$ where $\tau_0 = \big[\frac{1}{xy}\big]$, $\tau_1 = \big[\frac{1}{x^2y}\big]$, $\tau_2 = \big[\frac{1}{xy^2}\big]$, $\tau_3 = \big[\frac{1}{x^2y^2}\big]$, $\tau_4 = \big[\frac{1}{xy^3}\big]$, $\tau_5 = \big[\frac{1}{x^2y^3}\big]$, $\tau_6 = \big[\frac{1}{xy^4}\big]$, $\tau_7 = \big[\frac{1}{x^2y^4}\big]$, $\tau_8 = \big[\frac{1}{xy^5}\big]$, $\tau_9 = \big[-3\frac{1}{xy^6} + \frac{1}{x^3y}\big]$, $\tau_{10} = \big[-3\frac{1}{xy^7} + \frac{1}{x^3y^2}\big]$, $\tau_{11} = \big[-3\frac{1}{xy^8} + \frac{1}{x^3y^3} + \frac{24}{5}a\frac{1}{x^2y^5}\big]$.
According to the form of the annihilator $I(\tau)$, Tjurina local cohomology classes, represented by linear combinations of τ_j with parameters p_j ($j = 0, 1, \dots, 11$), fall into the following 17 cases listed below. There is an Euler operator (i.e., an operator of the weighted degree 0) in $V(\tau)$ for each case (i)–(xvii). This yields

$$\dim \mathrm{Hom}_{\mathcal{D}_X}\big(\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\tau), \mathcal{H}^n_{[O]}(\mathcal{O}_X)\big) = 1$$

for any $\tau \in H_T$. The computation is carried out with the lexicographical order $x \succ y$.

Classification of local cohomology classes in H_T.

(i) $\tau = \sum_{j=0}^{11} p_j\tau_j$ with $p_{11} \neq 0$.
$H(\tau) = \mathbb{Q}[z](p_{11}\tau_{11} + p_{10}\tau_{10} + p_7\tau_7)$.
$I(\tau) = \big\langle y^8, -15p_{11}y^3x + (24p_{10}a - 5p_7)y^7 - 24p_{11}ay^6, 3x^2 + y^5\big\rangle$.
$$\begin{aligned}V(\tau) = \mathrm{Span}\big\{&y^7\partial_x, y^6\partial_x, y^5\partial_x, y^2x\partial_y, yx\partial_y, (-44ay^2x + 5y^4)\partial_x + 5x\partial_y, y^7\partial_y, y^6\partial_y,\\ &y^5\partial_y, y^4\partial_y, 5y^2x\partial_x + 2y^3\partial_y, ((192a^2y^2 + 125y)x - 40ay^4)\partial_x + 50y^2\partial_y,\\ &\big(((-36864p_{11}a^4 - 19200p_{10}a^2 + 4000p_7a)y^2 + 4800p_{11}a^2y + 3125p_{11})x\\ &+ (7680p_{11}a^3 + 3000p_{10}a - 625p_7)y^4 - 1000p_{11}ay^3\big)\partial_x + 1250p_{11}y\partial_y\big\}.\end{aligned}$$

(ii) $\tau = \sum_{j=0}^{10} p_j\tau_j$ with $p_{10} \neq 0$.
$H(\tau) = \mathbb{Q}[z](p_{10}\tau_{10} + p_9\tau_9 + p_7\tau_7 + p_5\tau_5)$.
$I(\tau) = \big\langle y^7, 3p_{10}^2y^2x + (-p_7p_9 + p_5p_{10})y^6 + p_7p_{10}y^5, 3x^2 + y^5\big\rangle$.

$$V(\tau) = \mathrm{Span}\{y^6\partial_x, y^5\partial_x, yx\partial_y, 2y^4\partial_x + 3x\partial_y, y^6\partial_y, y^5\partial_y, y^4\partial_y, y^3\partial_y, (15p_{10}yx - p_7y^4)\partial_x + 6p_{10}y^2\partial_y, \big((3p_7^2p_{10}y + 45p_{10}^3)x + (9p_7p_{10}p_9 - 9p_5p_{10}^2 + p_7^3)y^4 - 3p_7p_{10}^2y^3\big)\partial_x + 18p_{10}^3y\partial_y\}.$$

(iii) $\tau = \sum_{j=0}^{9} p_j\tau_j$ with $p_9 \neq 0, p_7 \neq 0$.
$H(\tau) = \mathbb{Q}[z](p_9\tau_9 + p_8\tau_8 + p_7\tau_7 + p_5\tau_5)$.
$I(\tau) = \langle y^6, 9p_9^2y^2x + (3p_5p_9 + p_8p_7)y^5 + 3p_7p_9y^4, 3x^2 + y^5\rangle$.
$V(\tau) = \mathrm{Span}\{y^5\partial_x, y^4\partial_x, yx\partial_y, -4p_7yx\partial_x + 3p_9x\partial_y, y^5\partial_y, y^4\partial_y, y^3\partial_y, 2yx\partial_x + y^2\partial_y, \big(((9p_9^3 + 6p_7p_5p_9 + 2p_8p_7^2)y + 12p_7^2p_9)x + 3p_7p_9^2y^3\big)\partial_x + 6p_7^2p_9y\partial_y\}$.

(iv) $\tau = \sum_{j=0}^{9} p_j\tau_j$ with $p_9 \neq 0$, $p_7 = 0$.
$H(\tau) = \mathbb{Q}[z](p_9\tau_9 + p_8\tau_8 + p_5\tau_5 + p_3\tau_3)$.
$I(\tau) = \langle y^6, 9p_9^2yx + (3p_3p_9 + p_8p_5)y^5 + 3p_5p_9y^4, 3x^2 + y^5\rangle$.
$V(\tau) = \mathrm{Span}\{y^5\partial_x, y^4\partial_x + 3x\partial_y, y^5\partial_y, y^4\partial_y, y^3\partial_y, p_5y^4\partial_x - p_9y^2\partial_y, \big(-15p_9^2x + (3p_3p_9 + p_8p_5)y^4 + p_5p_9y^3\big)\partial_x - 6p_9^2y\partial_y\}$.

(v) $\tau = \sum_{j=0}^{8} p_j\tau_j$ with $p_8 \neq 0$, $p_7 \neq 0$.
$H(\tau) = \mathbb{Q}[z](p_8\tau_8 + p_7\tau_7)$.
$I(\tau) = \langle y^5, p_8y^3x - p_7y^4, x^2\rangle$.
$V(\tau) = \mathrm{Span}\{y^2x\partial_x, yx\partial_x, y^4\partial_x, y^2x\partial_y, yx\partial_y, 4p_7x\partial_x + p_8x\partial_y, y^4\partial_y, y^3\partial_y, y^2\partial_y, x\partial_x + y\partial_y\}$.

(vi) $\tau = \sum_{j=0}^{8} p_j\tau_j$ with $p_8 \neq 0$, $p_7 = 0$, $p_5 \neq 0$.
$H(\tau) = \mathbb{Q}[z](p_8\tau_8 + p_6\tau_6 + p_5\tau_5 + p_3\tau_3)$.
$I(\tau) = \langle y^5, p_8^2yx + (-p_8p_3 + p_6p_5)y^4 - p_8p_5y^3, x^2\rangle$.
$V(\tau) = \mathrm{Span}\{y^4\partial_x, 3p_5^2y^3\partial_x + p_8^2x\partial_y, y^4\partial_y, y^3\partial_y, 2p_5y^3\partial_x + p_8y^2\partial_y, \big(2p_8^2x + (p_8p_3 - p_6p_5)y^3\big)\partial_x + p_8^2y\partial_y\}$.

(vii) $\tau = \sum_{j=0}^{8} p_j\tau_j$ with $p_8 \neq 0$, $p_7 = p_5 = 0$.
$H(\tau) = \mathbb{Q}[z](p_8\tau_8 + p_6\tau_6 + p_3\tau_3 + p_1\tau_1)$.
$I(\tau) = \langle y^5, p_8^2x + (-p_8p_1 + p_6p_3)y^4 - p_8p_3y^3\rangle$.
$V(\tau) = \mathrm{Span}\{y^4\partial_y, y^3\partial_y, 3p_3y^4\partial_x + y^2\partial_y, \big((4p_8p_1 - 4p_6p_3)y^4 + 3p_8p_3y^3\big)\partial_x + p_8^2y\partial_y\}$.

(viii) $\tau = \sum_{j=0}^{7} p_j\tau_j$ with $p_7 \neq 0$. $H(\tau) = \mathbb{Q}[z](\tau_7)$.
$I(\tau) = \langle y^4, x^2\rangle$.
$V(\tau) = \mathrm{Span}\{y^3x\partial_x, y^2x\partial_x, yx\partial_x, x\partial_x, y^3x\partial_y, y^2x\partial_y, yx\partial_y, y^3\partial_y, y^2\partial_y, y\partial_y\}$.

(ix) $\tau = \sum_{j=0}^{6} p_j\tau_j$ with $p_6 \neq 0$, $p_5 \neq 0$.
$H(\tau) = \mathbb{Q}[z](p_6\tau_6 + p_5\tau_5)$.
$I(\tau) = \langle y^4, p_6y^2x - p_5y^3, x^2\rangle$.
$V(\tau) = \mathrm{Span}\{yx\partial_x, y^3\partial_x, yx\partial_y, 3p_5x\partial_x + p_6x\partial_y, y^3\partial_y, y^2\partial_y, x\partial_x + y\partial_y\}$.

(x) $\tau = \sum_{j=0}^{6} p_j\tau_j$ with $p_6 \neq 0$, $p_5 = 0$.
$H(\tau) = \mathbb{Q}[z](p_6\tau_6 + p_4\tau_4 + p_3\tau_3 + p_1\tau_1)$.

$I(\tau) = \langle y^4, p_6^2 x + (-p_6p_1 + p_4p_3)y^3 - p_6p_3y^2\rangle$.
$V(\tau) = \mathrm{Span}\{y^3\partial_y, 2p_3y^3\partial_x + p_6y^2\partial_y, ((3p_6p_1 - 3p_4p_3)y^3 + 2p_6p_3y^2)\partial_x + p_6^2y\partial_y\}$.

(xi) $\tau = \sum_{j=0}^{5} p_j\tau_j$ with $p_5 \neq 0$.
$H(\tau) = \mathbb{Q}[z](\tau_5)$.
$I(\tau) = \langle y^3, x^2\rangle$.
$V(\tau) = \mathrm{Span}\{y^2x\partial_x, yx\partial_x, x\partial_x, y^2x\partial_y, yx\partial_y, y^2\partial_y, y\partial_y\}$.

(xii) $\tau = \sum_{j=0}^{4} p_j\tau_j$ with $p_4 \neq 0$, $p_3 \neq 0$.
$H(\tau) = \mathbb{Q}[z](p_4\tau_4 + p_3\tau_3)$, $p_4 \neq 0$.
$I(\tau) = \langle y^3, p_4yx - p_3y^2, x^2\rangle$.
$V(\tau) = \mathrm{Span}\{y^2\partial_x, 2p_3x\partial_x + p_4x\partial_y, y^2\partial_y, x\partial_x + y\partial_y\}$.

(xiii) $\tau = \sum_{j=0}^{4} p_j\tau_j$ with $p_4 \neq 0$, $p_3 = 0$.
$H(\tau) = \mathbb{Q}[z](p_4\tau_4 + p_1\tau_1)$.
$I(\tau) = \langle y^3, p_4x - p_1y^2\rangle$.
$V(\tau) = \mathrm{Span}\{y^2\partial_y, 2p_1y^2\partial_x + p_4y\partial_y\}$.

(xiv) $\tau = \sum_{j=0}^{3} p_j\tau_j$ with $p_3 \neq 0$.
$H(\tau) = \mathbb{Q}[z](\tau_3)$.
$I(\tau) = \langle y^2, x^2\rangle$.
$V(\tau) = \mathrm{Span}\{yx\partial_x, x\partial_x, yx\partial_y, y\partial_y\}$.

(xv) $\tau = \sum_{j=0}^{2} p_j\tau_j$ with $p_2 \neq 0$.
$H(\tau) = \mathbb{Q}[z](p_2\tau_2 + p_1\tau_1)$.
$I(\tau) = \langle y^2, p_2x - p_1y\rangle$.
$V(\tau) = \mathrm{Span}\{p_1y\partial_x + p_2y\partial_y\}$.

(xvi) $\tau = p_1\tau_1 + p_0\tau_0$ with $p_1 \neq 0$.
$H(\tau) = \mathbb{Q}[z](\tau_1)$.
$I(\tau) = \langle y, x^2\rangle$.
$V(\tau) = \mathrm{Span}\{x\partial_x\}$.

(xvii) $\tau = p_0\tau_0$.
$H(\tau) = \mathbb{Q}[z](\tau_0)$.
$I(\tau) = \langle y, x\rangle$.

Bimodal singularities. There are 14 normal forms of exceptional bimodal singularities. For these normal forms except Z type (i.e., Z_{17}, Z_{18}, Z_{19}), we have found Euler operators in $V(\tau)$ and have verified that the dimension of the solution space of $\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\tau)$ equals one. For the case of Z type singularities, we have found that the situation is a little bit different and subtle. We have explicitly computed $K(\tau)$ to determine the dimension of the solution space of $\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\tau)$ and we have verified $\dim K(\tau) \leq 2$ by direct computation. This yields Theorem 3.2.

EXAMPLE 4. The polynomial $f = x^2z + yz^2 + xy^4 + (a + by)y^6$ defines S_{16} bimodal singularity at the origin. The following 14 local cohomology classes constitute a basis of H_T: $\tau_0 = \left[\frac{1}{xyz}\right]$, $\tau_1 = \left[\frac{1}{xyz^2}\right]$, $\tau_2 = \left[\frac{1}{x^2yz}\right]$, $\tau_3 = \left[\frac{1}{xy^2z}\right]$, $\tau_4 = \left[\frac{1}{2}\frac{1}{xy^2z^2} - \frac{1}{x^3yz}\right]$, $\tau_5 = \left[\frac{1}{x^2y^2z}\right]$, $\tau_6 = \left[\frac{1}{xy^3z}\right]$, $\tau_7 = \left[\frac{1}{2}\frac{1}{xy^3z^2} - \frac{1}{x^3y^2z}\right]$, $\tau_8 = \left[\frac{1}{x^2y^3z}\right]$, $\tau_9 = \left[\frac{1}{xy^4z}\right]$, $\tau_{10} = \left[\frac{1}{2}\frac{1}{xy^4z^2} - \frac{1}{x^3y^3z}\right]$, $\tau_{11} = \left[-\frac{1}{xyz^3} + \frac{1}{4}\frac{1}{x^2y^4z}\right]$, $\tau_{12} = \left[2\frac{1}{xy^5z} - \frac{1}{x^2yz^2}\right]$, $\tau_{13} = \left[-\frac{1}{xy^6z} + \frac{1}{2}\frac{1}{x^2y^2z^2} + \frac{3}{2}a\frac{1}{x^2y^4z} - \frac{1}{x^4yz}\right]$.

Cohomology classes in H_T fall into 25 cases and there are Euler operators in each $V(\tau)$.

Put, for instance, $\tau = \sum_{j=0}^{13} p_j\tau_j$, $p_{13} \neq 0$, $p_{10} \neq 0$. Then $I(\tau) = \langle y^6, 4p_{13}y^3x + (6p_{13}a + p_{11})y^5, p_{13}^2yx^2 + (-2p_{10}p_{12} - p_7p_{13})y^5 - p_{10}p_{13}y^4, -x^3 + y^5, x^2 + 2zy, 2zx + y^4, -p_{11}y^5 + p_{13}z^2\rangle$ as the Gröbner basis with the lexicographical order $z \succ x \succ y$. There is an Euler operator

$$\begin{gathered}\big(((-48p_{10}^2p_{13}^3a - 8p_{10}p_{13}^4 - 8p_{10}^2p_{11}p_{13}^2 - 24p_7p_{10}^3p_{13} - 48p_{12}p_{10}^4)y - 72p_{10}^4p_{13})x \\ - 8p_{10}^3p_{13}^2y^2 + (-72p_{10}^3p_{13}^2a - 16p_{10}^2p_{13}^3 - 12p_{10}^3p_{11}p_{13})z\big)\partial_x - 48p_{10}^4p_{13}y\partial_y \\ +\big((12p_{10}^2p_{13}^3a + 8p_{10}^2p_{13}^4 - 46p_{10}^3p_{11}p_{13}^2 + 24p_7p_{10}^3p_{13} + 48p_{12}p_{10}^4)x^2 \\ +\big(((-48p_{10}p_{13}^4 + 144p_{10}^2p_{11}p_{13}^2)a - 12p_{13}^4 + 56p_{10}p_{11}p_{13}^3 - 48p_7p_{10}^2p_{13}^2 \\ +(24p_{10}^2p_{11}^2 - 96p_{12}p_{10}^3)p_{13})y^2 + 8p_{10}^3p_{13}^2y\big)x \\ +\big((-72p_{10}p_{13}^4 + 216p_{10}^2p_{11}p_{13}^2)a^2 + (-18p_{13}^5 + 72p_{10}p_{11}p_{13}^3 - 36p_7p_{10}^2p_{13}^2 \\ +(72p_{10}^2p_{11}^2 - 72p_{12}p_{10}^3)p_{13})a - 3p_{11}p_{13}^4 + 14p_{10}p_{11}^2p_{13}^2 + 42p_7p_{10}^2p_{11}p_{13} \\ +(6p_{10}^2p_{11}^3 + 84p_{12}p_{10}^3p_{11})\big)y^4 + (-8p_{10}^2p_{13}^3 + 48p_{10}^3p_{11}p_{13})y^3 - 96p_{10}^4p_{13}z\big)\partial_z\end{gathered}$$

in $V(\tau)$.

Let us see the cases of Z type bimodal singularities Z_{17}, Z_{18}, Z_{19}. We use the lexicographical order with $x \succ y$.

EXAMPLE 5. For the normal form $x^3y + y^8 + (a + by)xy^6$ of bimodal Z_{17} singularity, $I = \langle y^{10}, -289a^2y^6x + 480by^9 - 408ay^8, 3yx^2 + by^7 + ay^6, -289a^2x^3 - 1734a^3y^5x - 3360b^2y^9 + 2856aby^8 - 2312a^2y^7\rangle$. H_T has the following 15 cohomology classes as basis: $\tau_0 = \left[\frac{1}{xy}\right]$, $\tau_1 = \left[\frac{1}{x^2y}\right]$, $\tau_2 = \left[\frac{1}{x^3y}\right]$, $\tau_3 = \left[\frac{1}{xy^2}\right]$, $\tau_4 = \left[\frac{1}{x^2y^2}\right]$, $\tau_5 = \left[\frac{1}{xy^3}\right]$, $\tau_6 = \left[\frac{1}{x^2y^3}\right]$, $\tau_7 = \left[\frac{1}{xy^4}\right]$, $\tau_8 = \left[\frac{1}{x^2y^4}\right]$, $\tau_9 = \left[\frac{1}{xy^5}\right]$, $\tau_{10} = \left[\frac{1}{x^2y^5}\right]$, $\tau_{11} = \left[\frac{1}{xy^6}\right]$, $\tau_{12} = \left[\frac{1}{6}\frac{1}{x^2y^6} - a\frac{1}{x^4y}\right]$, $\tau_{13} = \left[3\frac{1}{xy^7} - a\frac{1}{x^3y^2}\right]$, $\tau_{14} = \left[-3b\frac{1}{xy^7} + 3a\frac{1}{xy^8} - 4\frac{1}{x^2y^6} - a^2\frac{1}{x^3y^3}\right]$.

For each class, $K(\tau_j) = \mathrm{Span}\{1\}$ $(j = 0, \ldots, 14)$. However, for $\tau = \sum_{j=0}^{14} p_j\tau_j$ with $p_{14} \neq 0$ and $p_{12}^2 - 48p_{14}p_{12} + 576p_{14}^2 \neq 0$, we have $K(\tau) = \mathrm{Span}\{1, y^7\}$. Otherwise, $K(\tau) = \mathrm{Span}\{1\}$.

EXAMPLE 6. The polynomial $f = x^3y + xy^6 + (a + by)y^9$ is the normal form of bimodal Z_{18} singularity. Then the following 16 local cohomology classes constitute a basis of H_T: $\tau_0 = \left[\frac{1}{xy}\right]$, $\tau_1 = \left[\frac{1}{x^2y}\right]$, $\tau_2 = \left[\frac{1}{x^3y}\right]$, $\tau_3 = \left[\frac{1}{xy^2}\right]$, $\tau_4 = \left[\frac{1}{x^2y^2}\right]$, $\tau_5 = \left[\frac{1}{xy^3}\right]$, $\tau_6 = \left[\frac{1}{x^2y^3}\right]$, $\tau_7 = \left[\frac{1}{xy^4}\right]$, $\tau_8 = \left[\frac{1}{x^2y^4}\right]$, $\tau_9 = \left[\frac{1}{xy^5}\right]$, $\tau_{10} = \left[\frac{1}{x^2y^5}\right]$, $\tau_{11} = \left[\frac{1}{xy^6}\right]$, $\tau_{12} = \left[\frac{1}{6}\frac{1}{x^2y^6} - \frac{1}{x^4y}\right]$, $\tau_{13} = \left[3\frac{1}{xy^7} - \frac{1}{x^3y^2}\right]$, $\tau_{14} = \left[3\frac{1}{xy^8} - \frac{1}{x^3y^3}\right]$, $\tau_{15} = \left[3\frac{1}{xy^9} - \frac{9}{2}a\frac{1}{x^2y^6} - \frac{1}{x^3y^4}\right]$.

Although the local cohomology class τ_{15} satisfies $f\tau_{15} = 0$, $K(\tau_{15}) = \mathrm{Span}\{1, y^8\}$. Put $\tau = \sum_{j=0}^{15} p_j\tau_j$. If $p_{15} \neq 0$, there are no Euler operators in $V(\tau)$ and $K(\tau) = \mathrm{Span}\{1, y^8\}$. Even if $p_{15} = 0$, if neither p_{12} nor p_{14} are equal to zero, there are no Euler operators in $V(\tau)$ and $K(\tau) = \mathrm{Span}\{1, y^7\}$. The condition $p_{15} = p_{14} = 0$ or $p_{15} = p_{12} = 0$ implies $K(\tau) = \mathrm{Span}\{1\}$ and thus $\dim \mathrm{Hom}_{\mathcal{D}_X}\big(\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\tau), \mathcal{H}^2_{[0]}(\mathcal{O}_X)\big) = 1$ for this case.

EXAMPLE 7. The polynomial $f = x^3y + y^9 + (a+by)xy^7$ is the normal form of bimodal Z_{19} singularity. A basis of H_T is given by the following 17 local cohomology classes. $\tau_0 = \left[\frac{1}{xy}\right]$, $\tau_1 = \left[\frac{1}{x^2y}\right]$, $\tau_2 = \left[\frac{1}{x^3y}\right]$, $\tau_3 = \left[\frac{1}{xy^2}\right]$, $\tau_4 = \left[\frac{1}{x^2y^2}\right]$, $\tau_5 = \left[\frac{1}{xy^3}\right]$, $\tau_6 = \left[\frac{1}{x^2y^3}\right]$, $\tau_7 = \left[\frac{1}{xy^4}\right]$, $\tau_8 = \left[\frac{1}{x^2y^4}\right]$, $\tau_9 = \left[\frac{1}{xy^5}\right]$, $\tau_{10} = \left[\frac{1}{x^2y^5}\right]$, $\tau_{11} = \left[\frac{1}{xy^6}\right]$, $\tau_{12} = \left[\frac{1}{x^2y^6}\right]$, $\tau_{13} = \left[\frac{1}{xy^7}\right]$, $\tau_{14} = \left[\frac{1}{7}\frac{1}{x^2y^7} - a\frac{1}{x^4y}\right]$, $\tau_{15} = \left[3\frac{1}{xy^8} - a\frac{1}{x^3y^2}\right]$, $\tau_{16} = \left[-3b\frac{1}{xy^8} + 3a\frac{1}{xy^9} - \frac{27}{7}\frac{1}{x^2y^7} - a^2\frac{1}{x^3y^3}\right]$.

Put $\tau = \sum_{j=0}^{16} p_j\tau_j$. If $p_{16} \neq 0$ and $729p_{16}^2 - 54p_{14}p_{16} + p_{14}^2 \neq 0$, there are no Euler operators in $V(\tau)$ and $K(\tau) = \mathrm{Span}\{1, y^8\}$. Otherwise,

$$\dim \mathrm{Hom}_{\mathcal{D}_X}\big(\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\tau), \mathcal{H}^2_{[O]}(\mathcal{O}_X)\big) = 1.$$

Remark that the method given in this paper to examine the solution space

$$\mathrm{Hom}_{\mathcal{D}_X}\big(\mathcal{D}_X/\mathcal{A}nn^{(1)}_{\mathcal{D}_X}(\sigma), \mathcal{H}^n_{[O]}(\mathcal{O}_X)\big)$$

is also available for any isolated singularities.

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