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# THE EULER NUMBER OF THE NORMALIZATION OF AN ALGEBRAIC THREEFOLD WITH ORDINARY SINGULARITIES

SHOJI TSUBOI

*Department of Mathematics and Computer Science*
*Kagoshima University*
*Kourimoto 1-21-35, 890-0065 Kagoshima, Japan*
*E-mail: tsuboi@sci.kagoshima-u.ac.jp*

**Abstract.** By a classical formula due to Enriques, the Euler number $\chi(X)$ of the non-singular normalization $X$ of an algebraic surface $S$ with ordinary singularities in $P^3(\mathbf{C})$ is given by $\chi(X) = n(n^2 - 4n + 6) - (3n - 8)m + 3t - 2\gamma$, where $n$ is the degree of $S$, $m$ the degree of the double curve (singular locus) $D_S$ of $S$, $t$ is the cardinal number of the triple points of $S$, and $\gamma$ the cardinal number of the cuspidal points of $S$. In this article we shall give a similar formula for an algebraic threefold with ordinary singularities in $P^4(\mathbf{C})$ which is free from quadruple points (Theorem 4.1).

**1. Preliminaries.** We begin with recalling some definitions.

Definition 1 ([1]). An irreducible hypersurface $S$ in the complex projective 3-space $P^3(\mathbf{C})$ is called an *algebraic surface with ordinary singularities* if it is locally isomorphic to one of the following germs of hypersurface at the origin of the complex 3-space $\mathbf{C}^3$ at every point of $S$:

(i) $z = 0$ (simple point)
(ii) $yz = 0$ (ordinary double point)
(iii) $xyz = 0$ (ordinary triple point)
(iv) $xy^2 - z^2 = 0$ (cuspidal point),

where $(x, y, z)$ is the coordinate on $\mathbf{C}^3$.

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Definition 2 ([6]). An irreducible hypersurface $T$ in the complex projective 4-space $P^4(\mathbf{C})$ is called an *algebraic threehold with ordinary singularities* if it is locally isomorphic to one of the following germs of hypersurface at the origin of the complex 4-space $\mathbf{C}^4$ at every point of $T$:

(i) $w = 0$ (simple point)

(ii) $zw = 0$ (ordinary double point)

(iii) $yzw = 0$ (ordinary triple point)

(iv) $xyzw = 0$ (ordinary quadruple point)

(v) $xy^2 - z^2 = 0$ (cuspidal point)

(vi) $w(xy^2 - z^2) = 0$ (stationary point),

where $(x, y, z, w)$ is the coordinate on $\mathbf{C}^4$.

It is known that every complex projective surface (*resp.* threefold) is birationally equivalent to an algebraic surface (*resp.* threefold) with ordinary singularities.

Next we give the definition of the *polar classes* of an $r$-dimensional subvariety $X^r$ in a complex projective space $P^n(\mathbf{C})$. Denote by $U$ the open subset of $X^r$ consisting of all simple points of $X$. For a given linear $(n-r+k-2)$-dimensional subspace $L_{(k)}$ of $P^n(\mathbf{C})$, we let $M_k(U)$ denote the locus of points $x \in U$ such that the tangent space $T_xX$ of $X$ at $x$ intersects $L_{(k)}$ in a space at least $k-1$ dimension.

Definition 3. The closure $M_k$ of $M_k(U)$ in $X$ is called the *$k$-th polar locus* of $X$.

$M_k$ has a natural reduced scheme structure and, for a general $L_{(k)}$, $M_k$ has codimension $k$ in $X$. Moreover, for such $L_{(k)}$, the rational equivalent class of the cycle defined by $M_k$ does not depend on $L_{(k)}$ (cf. [5]). This class is denoted by $[M_k]$.

Definition 4. The class $[M_k]$ is called the *$k$-th polar class* of $X$. The degree $\mu_k$ of $M_k$ is called the *$k$-th class*. The top class $\mu_r$ is called the *class* of $X$.

Now we give the definition of the *Segre class* of a closed subscheme $X$ of a scheme $Y$. We denote by $\mathcal{I}$ the ideal sheaf of $X$ in $Y$ and put

$$S^{\cdot} := \sum_{k=0}^{\infty} \mathcal{I}^k/\mathcal{I}^{k+1},$$

which is a graded sheaf of $\mathcal{O}_X$-algebras on $X$. To $S^{\cdot}$ we associate two schemes over $X$: the *cone* of $S^{\cdot}$

$$C := \mathrm{Spec}(S^{\cdot}), \qquad \pi : C \to X;$$

and the *projective cone* $P(C)$ to $X$ in $Y$ by

$$P(C) := \mathrm{Proj}(S^{\cdot}), \qquad p : P(C) \to X.$$

$C$ is called the *normal cone* to $X$ in $Y$, denoted by $C_XY$, and $P(C)$ the *projective normal cone* to $X$ in $Y$. On $P(C)$ there is a canonical line bundle, denoted by $\mathcal{O}_C(1)$. Let $z$ be a variable, $S^{\cdot}[z]$ the graded algebra whose $n$-th graded piece $(S^{\cdot}[z])^n$ is

$$S^n \oplus S^{n-1}z \oplus \ldots \oplus S^1 z^{n-1} \oplus S^0 z^n.$$

The corresponding cone is denoted by $C \oplus 1$. The cone

$$P(C \oplus 1) = \mathrm{Proj}(S^{\cdot}[z]), \qquad q : P(C \oplus 1) \to X$$

is called the *projective completion* of $C$. The element $z$ in $(S^{\cdot}[z])^1$ determines a regular section of $\mathcal{O}_{C\oplus 1}(1)$ on $P(C \oplus 1)$ whose zero-scheme is canonically isomorphic to $P(C)$. The complement to $P(C)$ in $P(C \oplus 1)$ is canonically isomorphic to $C$.

DEFINITION 5. The *Segre class* of $X$ in $Y$, denoted by $s(X, Y)$, is the class in the graded Chow group $A_*X$ of $X$ defined by the formula

$$s(X, Y) := q_*\Big(\sum_{i\geq 0} c_1\big(\mathcal{O}_{C\oplus 1}(1)\big)^i \cap \big[P(C \oplus 1)\big]\Big).$$

Note that $s(X, Y)$ is a birational invariant, which means that if $f : Y' \to Y$ is a morphism of pure-dimensional schemes, $X \subset Y$ a closed subscheme, $X' = f^{-1}(X)$ the inverse image scheme, then the Segre class of $X'$ in $Y'$ pushes forward to the Segre class of $X$ in $Y$. If the normal cone $C_XY$ is a vector bundle $N$, then $s(X, Y) = c(N)^{-1} \cap [X]$ where $c(N)^{-1}$ denotes the total inverse Chern class of $N$ (cf. [2], Chapter 4).

Finally, we give the definitions of *regular embeddings* and *local complete intersection morphisms* of schemes.

DEFINITION 6. We say a closed embedding $\iota : X \to Y$ of schemes is a *regular embedding of codimension* $d$ if every point in $X$ has an affine neighborhood $U$ in $Y$ such that if $A$ is the coordinate ring of $U$, $I$ the ideal of $A$ defining $X$, then $I$ is generated by a regular sequence of length $d$.

If this is the case, the conormal sheaf $\mathcal{I}/\mathcal{I}^2$, where $\mathcal{I}$ is the ideal sheaf of $X$ in $Y$, is a locally free sheaf of rank $d$. The *normal bundle* to $X$ in $Y$, denoted by $N_XY$, is the vector bundle on $X$ whose sheaf of sections is dual to $\mathcal{I}/\mathcal{I}^2$. Note that the normal bundle $N_XY$ is canonically isomorphic to the normal cone $C_XY$ for a (closed) regular embedding $\iota : X \to Y$ since the canonical map from $\mathrm{Sym}(\mathcal{I}/\mathcal{I}^2)$ to $S^{\cdot} := \sum_{k=0}^{\infty} \mathcal{I}^k/\mathcal{I}^{k+1}$ is an isomorphism (cf. [2], Appendix B, B.7).

DEFINITION 7. A morphism $f : X \to Y$ is called a *local complete intersection morphism of codimension* $d$ if $f$ factors into a (closed) regular embedding $\iota : X \to Y$ of some constant codimension $e$, followed by a smooth morphism $p : P \to Y$ of constant relative dimension $d + e$.

**2. The existence of a good linear pencil of hyperplane sections.** Throughout this section we denote by $X$ an algebraic threefold with ordinary singularities of degree $n$ in the complex projective 4-space $P^4(\mathbf{C})$, by $D$ the double surface of $X$, i.e., the singular locus of $X$, by $T$ the triple points locus of $X$, by $C$ the cuspidal point locus of $X$, by $\sum s$ the stationary point locus of $X$. Let $m$, $t$, $\gamma$ be the degrees of $D$, $T$, $C$, respectively. Let $P_\infty$ be a two-dimensional linear subspace of $P^4(\mathbf{C})$ such that $C_\infty := P_\infty \cap X$ is an irreducible curve with ordinary double points in $P_\infty \simeq P^2(\mathbf{C})$. Let $P$ be a one-dimensional linear subspace of $P^4(\mathbf{C})$ situated in twisted position with respect to $P_\infty$, i.e., the linear subspace $L(P_\infty, P)$ generated by $P_\infty$ and $P$ is equal to $P^4(\mathbf{C})$. Let $\pi : X \setminus C_\infty \to P$ be the linear projection with center $C_\infty$, i.e., $\pi(x) := H_x \cap P$ for $x \in X \setminus C_\infty$, where

$H_x = L(x, P_\infty)$ is the hyperplane generated by $x$ and $P_\infty$. We put $X_\lambda := H_\lambda \cap X$ for $\lambda \in P$ and $\mathcal{L} := \bigcup_{\lambda \in P} X_\lambda$. Then $\mathcal{L}$ is a linear system on $X$ with the base point locus $Bs(\mathcal{L}) = C_\infty$. Let $f : X_1 \to X$ be the normalization map and $\widetilde{\mathcal{L}} := \bigcup_{\lambda \in P} \widetilde{X_\lambda}$ the pull-back of $\mathcal{L}$ to $X_1$.

THEOREM 2.1. *If we take $P_\infty$ sufficiently general, there exists a finite set $\{\lambda_1, \dots, \lambda_c\}$ of points of $P$ such that*

(i) *$\widetilde{X_\lambda}$ is non-singular for $\lambda$ with $\lambda \neq \lambda_i$ $(1 \leq i \leq c)$, and*

(ii) *$\widetilde{X_{\lambda_i}}$ is a surface with only one isolated ordinary double point which is contained in $X_1 \setminus f^{-1}(C_\infty)$ for any $i$ with $1 \leq i \leq c$,*

*where $c$ is the class of $X$.*

*Proof.* We consider the *Gauss map*

$$\Phi : X \to P^4(\mathbf{C})^\vee$$

defined by

$$\Phi(p) = \Big[\frac{\partial F}{\partial x_0}(p) : \frac{\partial F}{\partial x_1}(p) : \frac{\partial F}{\partial x_2}(p) : \frac{\partial F}{\partial x_3}(p) : \frac{\partial F}{\partial x_4}(p)\Big] \tag{2.1}$$

for $p \in X$, where $F$ is the homogeneous polynomial defining $X$ in $P^4(\mathbf{C})$, $[x_0 : x_1 : x_2 : x_3 : x_4]$ the homogeneous coordinate on $P^4(\mathbf{C})$, and $P^4(\mathbf{C})^\vee$ the dual projective space of $P^4(\mathbf{C})$. $\Phi$ is a rational map, which is not defined on the singular locus $D$ of $X$. Let $\overline{X}$ be the closure in $X \times P^4(\mathbf{C})^\vee$ of the graph of $\Phi$. We denote by $\pi_1 : \overline{X} \to X$ the morphism induced by the projection to the first factor, and $\pi_2 : \overline{X} \to P^4(\mathbf{C})^\vee$ the one induced by the projection to the second factor. We call $\pi_1 : \overline{X} \to X$ the *Nash blow-up* of $X$. Note that the rational map $\Phi$ can be extended to $\overline{X}$ and $\overline{X}$ is minimal among the varieties with such property. In our case, since $X$ is a hypersurface, $\overline{X}$ coincides with the blow-up of the Jacobian ideal of $X$ ([4], Remark 2, p. 300). We denote by $X^\vee$ the image of $\overline{X}$ by $\pi_2 : \overline{X} \to P^4(\mathbf{C})^\vee$, and call it the *dual variety* of $X$. The dimension of $X^\vee$ is not less than 1, nor greater than 3 ([3], Example 15.22, p. 196).

We are now going to define an algebraic subset $B$ in $P^4(\mathbf{C})^\vee$, whose points correspond to hyperplanes in $P^4(\mathbf{C})$ being in *bad* positions in some sense at their intersecting points with the cuspidal point locus $C$, or stationary point locus $\sum s$ of $X$. Let $p$ be a point of $C$, or $\sum s$. Then there is an open neighborhood $U$ of $p$ and a complex analytic local coordinates $(x, y, z, w)$ with center $p$ such that the defining equation of $X$ is given by one of the following:

$$xy^2 - z^2 = 0 \tag{2.2}$$

$$w(xy^2 - z^2) = 0. \tag{2.3}$$

Let $(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ be a linear affine coordinate with center $p$, and $H$ a hyperplane passing through $p$, defined by the equation

$$\sum_{i=1}^{4} a_i \zeta_i = 0 \qquad (a_i \in \mathbf{C},\ 1 \leq i \leq 4). \tag{2.4}$$

We say $H$ is in a *bad* position at the point $p$, if the coefficients of the equation (2.4) satisfy the following two conditions:

$$\sum_{i=1}^{4} a_i \frac{\partial \zeta_i}{\partial y}(0) = 0, \tag{2.5}$$

$$\sum_{i=1}^{4} a_i \frac{\partial \zeta_i}{\partial w}(0) = 0. \tag{2.6}$$

We define $B_p$ to be the algebraic subset of $P^4(\mathbf{C})^\vee$ consisting of all points which correspond to hyperplanes in $P^4(\mathbf{C})$ passing through $p$ and being in a bad position at $p$ in the sense defined above. We define an algebraic subset $B$ of $P^4(\mathbf{C})^\vee$ by

$$B := \bigcup_{p \in C} B_p. \tag{2.7}$$

Let us note that the stationary points are included in $C$, and since $\dim B_p = 1$, the codimension of $B$ is greater than 1. We choose a line $L^*$ in $P^4(\mathbf{C})^\vee$ which satisfies all of the following conditions:

(2.8) $L^* \cap \{X^\vee \setminus \Phi(X_{\mathrm{sm}})\} = \emptyset,$

(2.9) $L^* \cap (X^\vee)_{\mathrm{sing}} = \emptyset,$

(2.10) $L^* \cap B = \emptyset,$

(2.11) $L^*$ intersects transversely with $\Phi(X_{\mathrm{sm}}) \setminus (X^\vee)_{\mathrm{sing}},$

where $X_{\mathrm{sm}}$ denotes $X \setminus D$, the simple point locus of $X$, and $(X^\vee)_{\mathrm{sing}}$ the singular point locus of $X^\vee$. This is always possible because all the codimensions of $X^\vee \setminus \Phi(X_{\mathrm{sm}})$, $(X^\vee)_{\mathrm{sing}}$ and $B$ are greater than 1 in $P^4(\mathbf{C})^\vee$. Note that the cardinal number of the set $L^* \cap \{\Phi(X_{\mathrm{sm}}) \setminus (X^\vee)_{\mathrm{sing}}\}$ is nothing but the *class* of $X$. We denote by $H_\lambda$ the hyperplane in $P^4(\mathbf{C})$ corresponding to each $\lambda \in L^*$. We put $X_\lambda := X \cap H_\lambda$ and consider the linear pencil

$$\mathcal{L} = \bigcup_{\lambda \in L^*} X_\lambda$$

of hyperplane sections of $X$. We are now going to show that the assertions (i) and (ii) of the theorem hold for the pull-back $\widetilde{\mathcal{L}} = \bigcup_{\lambda \in L^*} \widetilde{X_\lambda}$ of $\mathcal{L}$ to the normal model $X_1$ of $X$ by the normalization map $f : X_1 \to X$.

Assertion (i). Let $\{\lambda_1, \ldots, \lambda_c\}$ be all of the distinct points of $L^* \cap \{\Phi(X_{\mathrm{sm}}) \setminus (X^\vee)_{\mathrm{sing}}\}$, and $\lambda$ a point $L^*$ with $\lambda \neq \lambda_i$ $(1 \leq i \leq c)$. Then $\lambda \notin X^\vee$. This means that $H_\lambda$ is not tangent to $X$ at any point of $X_{\mathrm{sm}}$, and not a limit of tangent hyperplanes to $X_{\mathrm{sm}}$. Hence we infer that $\widetilde{X_\lambda}$ is non-singular at every point of $X_1 \setminus f^{-1}(C)$. Therefore what we have to do is to show that $\widetilde{X_\lambda}$ is non-singular at $f^{-1}(p)$ for any point $p \in H_\lambda \cap C$. In the subsequence we shall show this fact only when $p$ is a stationary point, since the proof for a cuspidal point is more easy. Assume $p$ is a cuspidal point of $X$ and $p \in H_\lambda$. We take a complex analytic local coordinate $(x, y, z, w)$ with center $p$ such that the defining equation of $X$ is given by the equation (2.3). We also take a linear affine coordinate $(\zeta_0, \zeta_1, \zeta_2, \zeta_3)$ with center $p$ and assume that the defining equation of $H_\lambda$ is given by the equation (2.4). We rewrite the equation (2.4) as

$$Ax + By + Cz + Dw = 0, \tag{2.12}$$

where $A, B, C$ and $D$ are complex analytic functions defined in a neighborhood of $p$. $f^{-1}(p)$ consists of two points, say $q_1, q_2$, where the normalization map $f : X_1 \to X$ is given as follows:

$$f_1 : (u_1, v_1, t_1) \to (u_1^2, v_1, u_1 v_1, t_1) = (x, y, z, w),$$
$$f_2 : (u_2, v_2, t_2) \to (u_2, v_2, t_2, 0) = (x, y, z, w).$$

Here $(u_i, v_i, t_i)$ $(i = 1, 2)$ is a complex analytic local coordinate with center $q_i$. Then the pull-backs of the defining equation of $H_\lambda$ in (2.12) by $f_i$ $(i = 1, 2)$ are given by

$$A_1^* u_1^2 + B_1^* v_1 + C_1^* u_1 v_1 + D_1^* t_1 = 0, \tag{2.13}$$

and

$$A_2^* u_1 + B_2^* v_2 + C_2^* t_2 = 0 \tag{2.14}$$

where $A_i^*$, $B_i^*$, $C_i^*$ and $D_i^*$ $(i = 1, 2)$ are the pull-backs of $A$, $B$, $C$ and $D$ by the map $f_i$. The equations above give the defining equations of $\widetilde{X_\lambda}$ at $q_1$ and $q_2$, respectively. Concerning the equation (2.13), if $B_1^*(0) \neq 0$ or $D_1^*(0) \neq 0$, then $\widetilde{X_\lambda}$ is non-singular at $q_1$. Assume $B_1^*(0) = D_1^*(0) = 0$ to the contrary, then $B(0) = D(0) = 0$. Since

$$A(0)x + B(0)y + C(0)z + D(0)w = 0$$

is the equation of the embedded tangent space to $H_\lambda$ at $p$ in terms of the local coordinate $(x, y, z, w)$, and since $H_\lambda$ is defined by the equation 2.4, we have

$$\sum_{i=0}^{4} a_i \frac{\partial \zeta_i}{\partial y}(0) = B(0) = 0 \qquad \text{and} \qquad \sum_{i=0}^{4} a_i \frac{\partial \zeta_i}{\partial w}(0) = D(0) = 0.$$

On the other hand, since $\lambda \notin B$, this is because of the condition (2.10), we have

$$\sum_{i=0}^{4} a_i \frac{\partial \zeta_i}{\partial y}(0) \neq 0 \qquad \text{or} \qquad \sum_{i=0}^{4} a_i \frac{\partial \zeta_i}{\partial w}(0) \neq 0.$$

This is a contradiction. Therefore we conclude that $B_1^*(0) \neq 0$ or $D_1^*(0) \neq 0$, and so $\widetilde{X_\lambda}$ is non-singular at $q_1$. Concerning the equation (2.14), if $A_2^*(0) = B_2^*(0) = C_2^*(0) = 0$, then $A(0) = B(0) = C(0) = 0$. This means the equation of the embedded tangent space to $H_\lambda$ at $p$ with respect to the local coordinate $(x, y, z, w)$ is $w = 0$, that is, $H_\lambda$ is tangent to the hypersurface $w = 0$ at $p$. But this is a contradiction, because, since $\lambda \notin X^\vee$, $H_\lambda$ is not a limit of tangent hyperplanes to $X$ in $P^4(\mathbf{C})$ at simple points of $X$. Therefore we conclude that at least one of $A_2^*(0)$, $B_2^*(0)$ and $C_2^*(0)$ is not zero, and so $\widetilde{X_\lambda}$ is non-singular at $q_2$.

Assertion (ii). From the same reasoning as in the proof of assertion (i) it follows that $\widetilde{X_{\lambda_i}}$ is non-singular at every point of $f^{-1}(D_{\lambda_i})$ where $D_{\lambda_i} = X_{\lambda_i} \cap D$. Hence it suffices to show that $X_{\lambda_i}$ has only one isolated ordinary double point on $X_{\lambda_i} \cap X_{\mathrm{sm}}$. By the manner of choosing the line $L^*$ in $P^4(\mathbf{C})^\vee$, the hyperplane $H_{\lambda_i}$ is tangent to $X$ at only one point, say $q$, of $X_{\mathrm{sm}}$. Therefore $X_{\lambda_i}$ is non-singular at all but one point $q$ of $X_{\lambda_i} \cap X_{\mathrm{sm}}$. To prove that $X_{\lambda_i}$ has an isolated ordinary double point at $q$, we assume that the homogeneous coordinate $[x_0 : x_1 : x_2 : x_3 : x_4]$ of $q$ is $[1 : 0 : 0 : 0 : 0]$ and $H_{\lambda_i}$ is defined by $x_4 = 0$.

We put $\zeta_i = x_i/x_0$ $(1 \leq i \leq 4)$, and use this linear affine coordinate $(\zeta_1, \dots, \zeta_4)$ in the subsequent arguments. Then $X$ is defined by $F(1, \zeta_1, \zeta_2, \zeta_3, \zeta_4) = 0$, $q$ is the origin $(0, \dots, 0)$, and $H_{\lambda_i}$ is defined by $\zeta_4 = 0$. Since the tangent hyperplane to $X$ at $q$ is the hyperplane $H_{\lambda_i} : \zeta_4 = 0$, we have

$$\frac{\partial F}{\partial \zeta_i}(1, 0, \dots, 0) = 0 \qquad (1 \leq i \leq 3) \tag{2.15}$$

$$\frac{\partial F}{\partial \zeta_4}(1, 0, \dots, 0) \neq 0. \tag{2.16}$$

Because of (2.16), there is an analytic function $\phi(\zeta_1, \zeta_2, \zeta_3)$ of the variables $\zeta_1, \zeta_2, \zeta_3$ defined in a neighborhood of the origin, which satisfies the following:

$$\phi(0, 0, 0) = 0, \tag{2.17}$$

$$F\big(1, \zeta_1, \zeta_2, \zeta_3, \phi(\zeta_1, \zeta_2, \zeta_3)\big) \equiv 0 \quad \text{(locally)}. \tag{2.18}$$

This means that the defining equation of $X$ in a neighborhood of $q$ is given by

$$\zeta_4 = \phi(\zeta_1, \zeta_2, \zeta_3) \tag{2.19}$$

By the same reasoning as before, we have

$$\frac{\partial \phi}{\partial \zeta_i}(0, 0, 0) = 0 \qquad (1 \leq i \leq 3) \tag{2.20}$$

Hence $\phi$ is expressed as

$$\phi = \sum_{1 \leq i, j \leq 3} \frac{\partial^2 \phi}{\partial \zeta_i \, \partial \zeta_j}(0) \, \zeta_i \zeta_j + O(|\zeta|^3). \tag{2.21}$$

If we regard $(\zeta_1, \zeta_2, \zeta_3)$ as a local coordinate on $H_{\lambda_i}$, $X_{\lambda_i}$ is defined by $\phi(\zeta_1, \zeta_2, \zeta_3) = 0$ in $H_{\lambda_i}$. Therefore, if we prove

$$\det\Big(\frac{\partial^2 \phi}{\partial \zeta_i \, \partial \zeta_j}(0)\Big) \neq 0 \tag{2.22}$$

then we can conclude that, after suitable change of local coordinates, the defining equation of $X_{\lambda_i}$ will become

$$\zeta_1^2 + \zeta_2^2 + \zeta_3^2 = 0$$

in a neighborhood of the origin in $H_{\lambda_i}$. This proves that assertion (ii) holds. To prove (2.22), we evaluate the Hessian $\det\big(\partial^2 F/\partial x_i \, \partial x_j\big)$ of the homogeneous polynomial $F$ at $q = [1 : 0 : 0 : 0 : 0]$.

First we mention some remarks about $\det\big(\partial^2 F/\partial x_i \, \partial x_j(1, 0)\big)$, where and in what follows we write $(1, 0)$ instead of $(1, 0, 0, 0, 0)$ for short. From the Euler identity

$$\sum_{i=0}^{4} x_i \, \frac{\partial F}{\partial x_i} = nF \qquad (n = \deg F), \tag{2.23}$$

it follows that

$$\sum_{j=0}^{4} x_j \, \frac{\partial^2 F}{\partial x_i \, \partial x_j} = (n-1) \, \frac{\partial F}{\partial x_i} \qquad (0 \leq i \leq 4). \tag{2.24}$$

If $x_0 \neq 0$, by use of (2.24) and (2.23), we can derive

$$\det\Big(\frac{\partial^2 F}{\partial x_i\,\partial x_j}\Big) = \Big(\frac{n-1}{x_0}\Big)^2 \begin{vmatrix} \frac{n}{n-1}F & \frac{\partial F}{\partial x_1} & \cdots & \frac{\partial F}{\partial x_4} \\ \frac{\partial F}{\partial x_1} & \frac{\partial^2 F}{\partial x_1^2} & \cdots & \frac{\partial^2 F}{\partial x_1\,\partial x_4} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F}{\partial x_4} & \frac{\partial^2 F}{\partial x_4\,\partial x_1} & \cdots & \frac{\partial^2 F}{\partial x_4^2} \end{vmatrix}. \tag{2.25}$$

Therefore, since $F(1,0) = 0$ and $(\partial F/\partial x_i)(1,0) = 0$ $(1 \leq i \leq 3)$ (cf. (2.15)), we have

$$\det\Big(\frac{\partial^2 F}{\partial x_i\,\partial x_j}(1,0)\Big) = (n-1)^2\Big(\frac{\partial F}{\partial x_4}(1,0)\Big)^2 \det\Big(\frac{\partial^2 F}{\partial x_i\,\partial x_j}(1,0)\Big)_{1\leq i,j\leq 3} \tag{2.26}$$

Here we need to recall that $\Phi(q) = \lambda$ does not belong to $(X^\vee)_{\text{sing}}$ because of condition (2.9). This means the Gauss map $\Phi$ defined by (2.1) gives a biregular morphism between $X$ and $X^\vee$ in a neighborhood of $q$. Therefore the right-hand side of (2.26) is not zero, and so we have

$$\det\Big(\frac{\partial^2 F}{\partial x_i\,\partial x_j}(1,0)\Big)_{1\leq i,j\leq 3} \neq 0 \tag{2.27}$$

since $(\partial F/\partial x_4)(1,0) \neq 0$ (cf. (2.16)). On the other hand, derivating the equation (2.18) twice with respect to the variables $\zeta_1, \zeta_2, \zeta_3$ and substituting 0 for all $\zeta_i$, we have

$$\det\Big(\frac{\partial^2 F}{\partial x_i\,\partial x_j}(1,0)\Big)_{1\leq i,j\leq 3} = -\Big(\frac{\partial F}{\partial x_4}(1,0)\Big)^3 \det\Big(\frac{\partial^2 \phi}{\partial \zeta_i\,\partial \zeta_j}(1,0)\Big) \tag{2.28}$$

Since $(\partial F/\partial x_4)(1,0) \neq 0$, by (2.28) and (2.27) we have

$$\det\Big(\frac{\partial^2 \phi}{\partial \zeta_i\,\partial \zeta_j}(1,0)\Big) \neq 0$$

as required. This completes the proof of the theorem. ■

In what follows we assume that $P_\infty$ is sufficiently general so that Theorem 2.1 holds.

LEMMA 2.2. *With the notation from Theorem* 2.1, *we have the following*:

(i) $\widetilde{C_\infty} := f^{-1}(C_\infty)$ *is a non-singular curve,*

(ii) $\widetilde{\mathcal{L}} := \bigcup_{\lambda\in P} \widetilde{X_\lambda}$ *is a linear system on* $X_1$ *with the base point locus* $B_s(\widetilde{\mathcal{L}}) = \widetilde{C_\infty}$,

(iii) *for* $\lambda, \mu \in P$ *with* $\lambda \neq \mu$, $\widetilde{X_\lambda}$ *and* $\widetilde{X_\mu}$ *intersect transversely along* $\widetilde{C_\infty}$.

*Proof.* We take an affine coordinate neighborhood $U$ of $P^4(\mathbf{C})$ with $U \cap P_\infty \neq \emptyset$, and work on this neighborhood. Let $(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ be a linear affine coordinate on $U$. We may assume that

(2.29)
(a) $P_\infty \cap T = \emptyset$ and $P_\infty \cap C = \emptyset$,
(b) $P_\infty$ and $X$ intersect transversely at every non-singular point of $X$, and
(c) $P_\infty$ and $D$ intersect transversely.

Let $P_\infty = H_0 \cap H_1$ where $H_0$ and $H_1$ are hyperplanes in $P^4(\mathbf{C})$, and let $\varphi_i$ be a linear function which defines $H_i$ on $U$ for $i = 1, 2$. Note that the linear system $\widetilde{\mathcal{L}} := \bigcup_{\lambda\in P} \widetilde{X_\lambda}$ is defined by $\alpha f^*\varphi_0 + \beta f^*\varphi_1$ $(\alpha, \beta \in \mathbf{C})$ where $f^*\varphi_i$ $(i = 1, 2)$ denotes the pull-back of $\varphi_i$ by the normalization map $f : X_1 \to X$. Therefore assertion (ii) is trivial. By the assumption

(2.29b) the assertions (i) and (iii) also trivially hold at $q = f^{-1}(p)$ for a non-singular point $p$ of $X$, so we will prove that the assertions (i) and (iii) hold at $q \in f^{-1}(p)$ for $p \in D \cap U$. We assume that $\overline{X}$ is defined by $XY = 0$ with respect to some complex analytic local coordinate $(X, Y, Z, W)$ with center $p$, and assume that the normalization map $f$ is given by

$$(u, v, t) \mapsto (0, u, v, t) = (X, Y, Z, W),$$

where $(u, v, t)$ is a complex analytic local coordinate with center $q := f^{-1}(p)$. The Jacobian matrix of $f^*\varphi_0, f^*\varphi_1$ with respect to $(u, v, t)$ at $q$ is given as follows:

$$\frac{\partial(f^*\varphi_0, f^*\varphi_1)}{\partial(u,v,t)}(q)$$

$$= \begin{pmatrix} \sum_{i=1}^{4} \frac{\partial\zeta_i}{\partial Y}(p)\frac{\partial\varphi_0}{\partial\zeta_i}(p), & \sum_{i=1}^{4} \frac{\partial\zeta_i}{\partial Z}(p)\frac{\partial\varphi_0}{\partial\zeta_i}(p), & \sum_{i=1}^{4} \frac{\partial\zeta_i}{\partial W}(p)\frac{\partial\varphi_0}{\partial\zeta_i}(p) \\ \sum_{i=1}^{4} \frac{\partial\zeta_i}{\partial Y}(p)\frac{\partial\varphi_1}{\partial\zeta_i}(p), & \sum_{i=1}^{4} \frac{\partial\zeta_i}{\partial Z}(p)\frac{\partial\varphi_1}{\partial\zeta_i}(p), & \sum_{i=1}^{4} \frac{\partial\zeta_i}{\partial W}(p)\frac{\partial\varphi_1}{\partial\zeta_i}(p) \end{pmatrix}. \tag{2.30}$$

On the other hand, by the assumption (2.29c),

$$\begin{vmatrix} \frac{\partial\varphi_0}{\partial Z}(p) & \frac{\partial\varphi_0}{\partial W}(p) \\ \frac{\partial\varphi_1}{\partial Z}(p) & \frac{\partial\varphi_1}{\partial W}(p) \end{vmatrix} \neq 0.$$

Hence

$$\begin{vmatrix} \sum_{i=1}^{4} \frac{\partial\zeta_i}{\partial Z}(p)\frac{\partial\varphi_0}{\partial\zeta_i}(p), & \sum_{i=1}^{4} \frac{\partial\zeta_i}{\partial W}(p)\frac{\partial\varphi_0}{\partial\zeta_i}(p) \\ \sum_{i=1}^{4} \frac{\partial\zeta_i}{\partial Z}(p)\frac{\partial\varphi_1}{\partial\zeta_i}(p), & \sum_{i=1}^{4} \frac{\partial\zeta_i}{\partial W}(p)\frac{\partial\varphi_1}{\partial\zeta_i}(p) \end{vmatrix} \neq 0. \tag{2.31}$$

By (2.30) and (2.31), we conclude $\{\partial(f^*\varphi_0, f^*\varphi_1)/\partial(u,v,t)\}(p)$ has the maximal rank. From this it follows that $\widetilde{C_\infty}$ is non-singular at $q$. Furthermore, if $[\alpha : \beta] \neq [\alpha' : \beta']$ as a point of $\mathrm{P}^1(\mathbf{C})$, then $\alpha\beta' - \alpha'\beta \neq 0$, so

$$\frac{\partial(f^*\varphi_0, f^*\varphi_1)}{\partial(u,v,t)}(q) \qquad \text{and} \qquad \frac{\partial(\alpha f^*\varphi_0 + \beta f^*\varphi_1,\, \alpha' f^*\varphi_0 + \beta' f^*\varphi_1)}{\partial(u,v,t)}(q)$$

have the same rank. Hence $\{\partial(\alpha f^*\varphi_0 + \beta f^*\varphi_1,\, \alpha' f^*\varphi_0 + \beta' f^*\varphi_1)/\partial(u,v,t)\}(q)$ also has the maximal rank. This shows that assertion (iii) holds at $q$ as required. This completes the proof of the lemma. ∎

Let $\sigma : \widehat{X_1} \to X_1$ be the blowing-up along $\widetilde{C_\infty} := f^{-1}(C_\infty)$, and $\widehat{\mathcal{L}} := \bigcup_{\lambda \in P} \widehat{X_\lambda}$ the proper inverse of $\widetilde{\mathcal{L}} := \bigcup_{\lambda \in P} \widetilde{X_\lambda}$. Then $\widehat{\mathcal{L}}$ gives a fibering of $\widehat{X_1}$ over $P \simeq P^1(\mathbf{C})$, which we denote by $\pi : \widehat{X_1} \to P$. Let $S = \{\lambda_1, \dots, \lambda_c\}$ and $\widehat{X_1}^* = \widehat{X_1} - \pi^{-1}(S)$. From the exact Borel-Moore homology sequence determined by the space, the closed subspace, and its complement, it follows that

$$\chi(\widehat{X_1}) = \chi(\widehat{X_1}^*) + \chi\big(\pi^{-1}(S)\big). \tag{2.32}$$

It is clear that

$$\chi\big(\pi^{-1}(S)\big) = \sum_{i=1}^{c} \chi(\widehat{X_{\lambda_i}}). \tag{2.33}$$

Since $\widehat{X_1}^* \to P - S$ is locally trivial (as a differential fiber space), it follows from the spectral sequence of Leray for this fiber space that

$$\chi(\widehat{X_1}^*) = \chi(\widehat{X_\lambda})\chi(P - S), \tag{2.34}$$

where $\widehat{X_\lambda}$ denote a generic fiber of $\widehat{X_1}^* \to P - S$. By the same reason as before, we have

$$\chi(P) = \chi(P - S) + c. \tag{2.35}$$

Comparing (2.32), (2.33), (2.34) and (2.35), we have

$$\chi(\widehat{X_1}) = \chi\big(P^1(\mathbf{C})\big)\chi(\widehat{X_\lambda}) + \sum_{j=1}^{c}\big(\chi(\widehat{X_{\lambda_j}}) - \chi(\widehat{X_\lambda})\big) = 2\chi(\widehat{X_\lambda}) - c.$$

The second equality above follows from the fact that a topological 2-cycle vanishes when $\lambda \to \lambda_j$ for $j = 1, \dots, c$. We put $\widehat{E} := \sigma^{-1}(\widetilde{C_\infty})$. Then, since $\widehat{X_1} \setminus \widehat{E} \simeq X_1 \setminus \widetilde{C_\infty}$,

$$\chi(\widehat{X_1}) - \chi(X_1) = \chi(\widehat{E}) - \chi(\widetilde{C_\infty}) = \chi\big(P^1(\mathbf{C})\big)\chi(\widetilde{C_\infty}) - \chi(\widetilde{C_\infty}) = \chi(\widetilde{C_\infty}).$$

Hence,

$$\chi(X_1) = \chi(\widehat{X_1}) - \chi(\widetilde{C_\infty}) = 2\chi(\widehat{X_\lambda}) - \chi(\widetilde{C_\infty}) - c = 2\chi(\widetilde{X_\lambda}) - \chi(\widetilde{C_\infty}) - c. \tag{2.36}$$

Since $C_\infty$ is a curve whose degree is equal to $n$ with $m$ ordinary double points in $P_\infty \simeq P^2(\mathbf{C})$, the genus $g(\widetilde{C_\infty})$ is given by

$$g(\widetilde{C_\infty}) = \frac{1}{2}(n-1)(n-2) - m.$$

Hence,

$$\chi(\widetilde{C_\infty}) = 2 - 2g(\widetilde{C_\infty}) = 2 - (n-1)(n-2) + 2m. \tag{2.37}$$

Note that $X_\lambda$ is a surface with ordinary singularities in $H_\lambda \simeq P^3(\mathbf{C})$ of degree $n$, whose numerical characteristics related to its singularities are as follows:

- the degree of its double curve $D_\lambda = m$,
- $\#\{\text{triple points of } X_\lambda\} = t$,
- $\#\{\text{cuspidal points of } X_\lambda\} = \gamma$.

Therefore, by the classical formula,

$$\chi(\widetilde{X_\lambda}) = n(n^2 - 4n + 6) - (3n - 8)m + 3t - 2\gamma. \tag{2.38}$$

By (2.36), (2.37) and (2.38), we have the following:

Proposition 2.3.

$$\begin{aligned}\chi(X_1) &= 2n(n^2 - 4n + 6) - 2(3n - 8)m + 6t - 4\gamma - 2 + (n-1)(n-2) - 2m - c \\ &= n(2n^2 - 7n + 9) - 2(3n - 7)m + 6t - 4\gamma - c.\end{aligned}$$

**3. The computation of the class of an algebraic threefold with ordinary singularities in $P^4(\mathbf{C})$.** Throughout this section we denote a rational equivalence class of an algebraic cycle, say $\alpha$, by $[\alpha]$, and denote the intersection class of two algebraic cycle classes, say $[\alpha]$ and $[\beta]$, by $\alpha \cdot \beta$. We refer to the following theorem from [5].

THEOREM 3.1 ([5], Theorem (2.3)). *Let $X^n$ be a hypersurface of degree $d$ in $P^{n+1}$. Then its $k$-th polar class is given by*

$$[M_k] = \big[(d-1)c_1(L)\big]^k \cap [X] - \sum_{i=0}^{k-1} \binom{k}{i} \big[(d-1)c_1(L)\big]^i \cap s_{n-k+i}(J,X) \qquad (0 \le k \le n)$$

*where $L = \mathcal{O}_{P^n}(1)$ and $s(J,X) = \sum_{k=0}^{n} s_k(J,X)$, ($s_k(J,X) \in A_k(J)$) denotes the Segre class of the singular subscheme $J$ of $X$.*

In what follows, using the theorem above, we shall compute the *class* $c$ of an algebraic threefold with ordinary singularities in the complex projective 4-space $P^4(\mathbf{C})$ for the case where the threefold is free from quadruple points. First we fix the notation as follows:

| | |
|---|---|
| $Y = P^4(\mathbf{C})$: | the complex projective 4-space, |
| $\overline{X}$: | an algebraic threefold with ordinary singularities in $Y$, which is free from quadruple points, |
| $\overline{J}$: | the singular subscheme of $\overline{X}$ defined by the Jacobian ideal of $\overline{X}$, |
| $\overline{D}$: | the singular locus of $\overline{X}$, |
| $\overline{T}$: | the triple point locus of $\overline{X}$, which is equal to the singular locus of $\overline{D}$, |
| $\overline{C}$: | the cuspidal point locus of $\overline{X}$, precisely, its closure, since we always consider $\overline{C}$ contains the stationary points, |
| $\sum \overline{s}$: | the stationary point locus of $\overline{X}$, |
| $n_{\overline{X}} : X \to \overline{X}$: | the normalization of $\overline{X}$, |
| $f : X \to Y$: | the composite of the normalization map $n_{\overline{X}}$ and the inclusion $\iota : \overline{X} \hookrightarrow Y$, |
| $J$: | the scheme-theoretic inverse of $\overline{J}$ by $f$, |
| $D$, $T$, $C$: | the inverse images of $\overline{D}$, $\overline{T}$, $\overline{C}$ by $f$, respectively, |
| $\sum s = T \cap C$: | the intersection of $T$ and $C$. |

Note that $\overline{T}$ and $\overline{C}$ are non-singular curves, intersecting transversely at $\sum \overline{s}$, and that the normalization $X$ of $\overline{X}$ is also non-singular. Calculating by use of local coordinates, we can easily see the following:

(i) $J$ contains $D$, and the *residual scheme* (cf. [2], Definition 9.2.1, p. 160) to $D$ in $J$ is the reduced scheme $C$;

(ii) $D$ is a surface with ordinary singularities, free from triple points, whose singular locus is $T$,

(iii) $D$ is the *double point* locus of the map $f : X \to Y$, i.e., the closure of the set $\{q \in X \,|\, \# f^{-1}(f(q)) \ge 2\}$;

(iv) the map $f_{|D} : D \to \overline{D}$ is generically two to one, simply ramified at $C$;

(v) the map $f_{|T} : T \to \overline{T}$ is generically three to one, simply ramified at $\sum s$.

To compute the Segre class $s(J,X)$, the following proposition is useful.

Proposition 3.2 ([2], Proposition 9.2, p. 161). *Let $D \subset W \subset V$ be closed embeddings of schemes, with $V$ a $k$-dimensional variety, and $D$ a Cartier divisor on $V$. Let $R$ be the residual scheme to $D$ in $W$. Then, for all $m$,*

$$s(W,V)_m = s(D,V)_m + \sum_{j=0}^{k-m} \binom{k-m}{j} [-D]^j \cdot s(R,V)_{m+j}$$

*in $A_m(W)$, the $m$-th rational equivalence class group of algebraic cycles on $W$.*

In our case, since $D = f^{-1}(\overline{D})$ is a Cartier divisor, its normal cone $C_D X$ to $D$ in $X$ is isomorphic to $\mathcal{O}_X(D)_{|D}$, the restriction to $D$ of the line bundle $\mathcal{O}_X(D)$ associated to $D$. Therefore,

$$\begin{aligned} s(D,X) &= c(\mathcal{O}_X(D)_{|D})^{-1} \cap [D] \\ &= [D] - c_1(\mathcal{O}_X(D)_{|D}) \cap [D] + c_1(\mathcal{O}_X(D)_{|D})^2 \cap [D] = [D] - [D]^2 + [D]^3. \end{aligned}$$

Since $C$ is non-singular,

$$c(N_{C/X})^{-1} \cap [C] = [C] - c_1(N_{C/X}) \cap [C].$$

Hence, applying Proposition 3.2 for $W = J$, $D = f^{-1}(\overline{D})$ and $R = C$, we have

$$\text{(3.1)} \qquad \begin{cases} s(J,X)_2 = [D] \\ s(J,X)_1 = -[D]^2 + [C] \\ s(J,X)_0 = [D]^3 - c_1(N_{C/X}) \cap [C] - 3D \cdot C \end{cases}$$

Since $s(\overline{J},\overline{X})_2 = f_* s(J,X)_2$, from the first equality above it follows that

$$\text{(3.2)} \qquad s(\overline{J},\overline{X})_2 = 2[\overline{D}].$$

To know $s(\overline{J},\overline{X})_1$, we need to understand $f_*[D]^2$, the push-forward of $[D]^2$ by $f$. For this purpose, we compute $f^*[D]^2$. To compute this, we consider the following fiber square:

$$\text{(3.3)} \qquad \begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle \tau_T}\downarrow & & \downarrow{\scriptstyle \sigma_{\overline{T}}} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

Here

| | |
|---|---|
| $\sigma_{\overline{T}} : Y' \to Y$: | the blowing-up of $Y$ along the triple point locus $\overline{T}$ of $\overline{X}$, |
| $\overline{X'}$: | the proper inverse image of $\overline{X}$ by $\sigma_{\overline{T}}$, |
| $X' := X \times_{\overline{X}} \overline{X'}$: | the fiber product of $X$ and $\overline{X'}$ over $\overline{X}$, |
| $n_{\overline{X'}} : X' \to \overline{X'}$: | the projection to the second factor of $X \times_{\overline{X}} \overline{X'}$, which is nothing but the normalization of $\overline{X'}$, |
| $f' : X' \to Y'$: | the composite of the normalization map $n_{\overline{X'}}$ and the inclusion $\iota' : \overline{X'} \hookrightarrow Y'$, |
| $\tau_T : X' \to X$: | the projection to the first factor of $X \times_{\overline{X}} \overline{X'}$, which is nothing but the blowing-up of $X$ along $T$. |

In what follows, we denote by $\overline{D'}$, $\overline{T'}$ and $\overline{C'}$ the proper inverse images of $\overline{D}$, $\overline{T}$ and $\overline{C}$ by $\sigma_{\overline{T}}$, respectively. We consider the following fiber square:

$$\begin{array}{ccc} E_{\overline{T}} & \xrightarrow{\ \overline{j}\ } & Y' \\ {\scriptstyle \overline{p}}\downarrow & & \downarrow{\scriptstyle \sigma_{\overline{T}}} \\ \overline{T} & \xrightarrow[\ \overline{\imath}\ ]{} & Y, \end{array} \tag{3.4}$$

where $E_{\overline{T}} = P(N_{\overline{T}}Y)$ is the exceptional divisor of the blowing-up $\sigma_{\overline{T}}$, which is a $P^2(\mathbf{C})$-bundle on $\overline{T}$, and $\overline{p} : E_{\overline{T}} \to \overline{T}$ is the projection to the base space of this bundle. We denote by $\mathcal{O}_{N_{\overline{T}}Y}(1)$ the canonical line bundle on $E_{\overline{T}}$. Then the *tautological line bundle* on $E_{\overline{T}}$ is $\mathcal{O}_{N_{\overline{T}}Y}(-1)$, which is a subbundle of $\overline{p}^*N_{\overline{T}}Y$.

LEMMA 3.3. $\sigma_{\overline{T}}^*[\overline{D}]$ *is expressed as*

$$\sigma_{\overline{T}}^*[\overline{D}] = [\overline{D'}] + 3\overline{j}_*[\xi_{\overline{T}}] + \overline{j}_*\overline{p}^*[\alpha_0] \tag{3.5}$$

*where* $[\xi_{\overline{T}}] = c_1\big(\mathcal{O}_{N_{\overline{T}}Y}(1)\big) \cap [E_{\overline{T}}]$ *and* $[\alpha_0]$ *an algebraic* 0-*cycle class on* $\overline{T}$.

*Proof.* By the *blow-up formula* ([2], Theorem 6.7, p. 116),

$$\sigma_{\overline{T}}^*[\overline{D}] = [\overline{D'}] + \overline{j}_*\{c(E) \cap \overline{p}^*s(\overline{T},\overline{D})\}_2 \tag{3.6}$$

where $E = \overline{p}^*N_{\overline{T}}Y/N_{E_{\overline{T}}}Y' = \overline{p}^*N_{\overline{T}}Y/\mathcal{O}_{N_{\overline{T}}Y}(-1)$. Since

$$c_1(E) = \overline{p}^*c_1(N_{\overline{T}}Y) - c_1\big(\mathcal{O}_{N_{\overline{T}}Y}(-1)\big) = \overline{p}^*c_1(N_{\overline{T}}Y) + c_1\big(\mathcal{O}_{N_{\overline{T}}Y}(1)\big),$$

we have

$$\begin{aligned} \{c(E) \cap s(\overline{T},\overline{D})\}_2 &= \overline{p}^*s_0(\overline{T},\overline{D}) + c_1(E) \cap \overline{p}^*s_1(\overline{T},\overline{D}) \\ &= \overline{p}^*\{s_0(\overline{T},\overline{D}) + c_1(N_{\overline{T}}Y) \cap s_1(\overline{T},\overline{D})\} + c_1\big(\mathcal{O}_{N_{\overline{T}}Y}(1)\big) \cap \overline{p}^*s_1(\overline{T},\overline{D}) \end{aligned} \tag{3.7}$$

To compute $s(\overline{T},\overline{D})$, we consider the normalization map $n_{\overline{D}} : \overline{D}^* \to \overline{D}$. Since $\overline{D}^*$ is non-singular, if we put $\overline{T}^* := n_{\overline{D}}^{-1}(\overline{T})$, then

$$s(\overline{T}^*,\overline{D}^*) = c(N_{\overline{T}^*}\overline{D}^*)^{-1} \cap [\overline{T}^*] = (1 - c_1(N_{\overline{T}^*}\overline{D}^*)) \cap [\overline{T}^*] = [\overline{T}^*] - \overline{T}^* \cdot \overline{T}^*.$$

Therefore,

$$s(\overline{T},\overline{D}) = n_{\overline{D}_*}s(\overline{T}^*,\overline{D}^*) = 3[\overline{T}] - n_{\overline{D}_*}(\overline{T}^* \cdot \overline{T}^*),$$

and so,

$$\begin{cases} s_0(\overline{T},\overline{D}) = -n_{\overline{D}_*}(\overline{T}^* \cdot \overline{T}^*) \\ s_1(\overline{T},\overline{D}) = 3[\overline{T}]. \end{cases} \tag{3.8}$$

By (3.7) and (3.8), if we put $[\alpha_0] := -n_{\overline{D}_*}(\overline{T}^* \cdot \overline{T}^*) + 3c_1(N_{\overline{T}}Y) \cap [\overline{T}]$,

$$\{c(E) \cap s(\overline{T},\overline{D})\}_2 = \overline{p}^*[\alpha_0] + 3[\xi_{\overline{T}}].$$

Consequently, by (3.6), we have the equality in (3.5). ■

PROPOSITION 3.4.

$$[D]^2 = f^*[\overline{X}] \cdot D - f^*[\overline{D}] + [T] - [C]. \tag{3.9}$$

*Proof.* To know $[D]^2$, we compute $f^*[\overline{D}]$. For this purpose, we use the diagram in (3.3). Since $\tau_T : X' \to X$ is a blowing-up, we have $\tau_{T_*}\tau_T^*\alpha = \alpha$ for any algebraic cycle $\alpha \in A_*(X)$. Hence,

$$\tau_{T_*} f'^* \sigma_{\overline{T}}^* [\overline{D}] = \tau_{T_*}\tau_T^* f^*[\overline{D}] = f^*[\overline{D}]. \tag{3.10}$$

Since $\overline{D'}$ is regularly embedded in $Y'$, i.e., $C_{\overline{D'}}Y' \simeq N_{\overline{D'}}Y'$, while $\overline{D}$ is not, we can apply the *excess intersection formula* ([2], Theorem 6.3, p. 102) to $\overline{D'}$. Then, denoting the tangent bundle of a non-singular algebraic variety, say $Z$, by $\mathcal{T}_Z$ we have

$$\begin{aligned} f'^*[\overline{D'}] &= c_1(f'^* N_{\overline{D'}}Y'/N_{D'}X') \cap [D'] \\ &= \{c_1(f'^*\mathcal{T}_{Y'}) - c_1(f'^*\mathcal{T}_{\overline{D'}}) - c_1(\mathcal{T}_{X'}) + c_1(\mathcal{T}_{D'})\} \cap [D'] \\ &= \{c_1(f'^*\mathcal{T}_{Y'}) - c_1(\mathcal{T}_{X'})\} \cap [D'] - C', \end{aligned} \tag{3.11}$$

where the last equality follows from the *ramification formula* ([2], Example 3.2.20, p. 62). On the other hand, by the *double point formula* ([2], Theorem 9.3, p. 166),

$$[D'] = f'^*[\overline{X'}] - \{c_1(f'^*\mathcal{T}_{Y'}) - c_1(\mathcal{T}_{X'})\} \cap [X']. \tag{3.12}$$

By (3.11) and (3.12), we have

$$f'^*[\overline{D'}] = f'^*[\overline{X'}] \cdot D' - [D']^2 - C'. \tag{3.13}$$

Next, in view of Lemma 3.3, we compute $f'^*(3\overline{j}_*[\xi_{\overline{T}}] + \overline{j}_*\overline{p}^*[\alpha_0])$. For this purpose, we consider the following fiber square:

$$\begin{array}{ccc} E_T & \xrightarrow{\ j\ } & X' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \tau_T} \\ T & \xrightarrow[\ \iota\ ]{} & X, \end{array} \tag{3.14}$$

where $E_T = P(N_TX)$ is the exceptional divisor of the blowing-up $\tau_T$, which is a $P^1(\mathbf{C})$-bundle on $T$, and $p : E_T \to T$ is the projection to the base space of this bundle. There is a set of morphisms from the diagram in (3.14) to the one in (3.4) induced by those in the diagram in (3.3). We denote by $g$ and $g'$ the restriction of $f : X \to Y$ to $T$ and that of $f' : X' \to Y'$ to $E_T$, respectively. Note that the morphism $g' : E_T \to E_{\overline{T}}$ maps each fiber of $p : E_T \to T$ to that of $\overline{p} : E_{\overline{T}} \to \overline{T}$, and so $g'^{a}st[\xi_{\overline{T}}] = [\xi_T]$ where $\xi_T = c_1(\mathcal{O}_{N_TX}(1)) \cap [E_T]$. Since $f' : X' \to Y'$ and $g' : E_T \to E_{\overline{T}}$ are local complete intersection morphisms of the same codimension, we can apply Proposition 6.6(c) from [2], p. 113, to the fiber square

$$\begin{array}{ccc} E_T & \xrightarrow{\ g'\ } & E_{\overline{T}} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \overline{j}} \\ X' & \xrightarrow[\ f'\ ]{} & Y'. \end{array} \tag{3.15}$$

Then $f'^*\overline{j}_*[\xi_{\overline{T}}] = j_*g'^*[\xi_{\overline{T}}] = j_*[\xi_T]$ and $f'^*\overline{j}_*\overline{p}^*[\alpha_0] = j_*g'^{a}st\overline{p}^*[\alpha_0] = j_*p^*g^*[\alpha_0]$. Therefore, we have

$$f'^*\big(3\overline{j}_*[\xi_{\overline{T}}] + \overline{j}_*\overline{p}^*[\alpha_0]\big) = 3j_*[\xi_{\overline{T}}] + \overline{j}_*\overline{p}^*g^*[\alpha_0]. \tag{3.16}$$

By (3.5), (3.13) and (3.16), we have

$$f'^*\sigma^*_{\overline{T}}[\overline{D}] = f'^*[\overline{X'}]\cdot D' - [D']^2 - C' + 3j_*[\xi_{\overline{T}}] + j_*\overline{p}^*g^*[\alpha_0].$$

Since $\tau_{T_*}[C'] = [C]$, $\tau_{T_*}j_*[\xi_{\overline{T}}] = T$ and $\tau_{T_*}j_*\overline{p}^*g^*[\alpha_0] = 0$, by the equality above and (3.10),

$$f^*[\overline{D}] = \tau_{T_*}f'^*\sigma^*_{\overline{T}}[\overline{D}] = \tau_{T_*}\big(f'^*[\overline{X'}]\cdot D'\big) - \tau_{T_*}[D']^2 - [C] + 3[T]. \tag{3.17}$$

Since $\tau^*_T[D] = [D'] + 2[E_T]$,

$$\tau_{T_*}\big(f'^*[\overline{X'}]\cdot D'\big) = \tau_{T_*}\big(f'^*[\overline{X'}]\cdot\tau^*_T[D] - 2f'^*[\overline{X'}]\cdot E_T\big). \tag{3.18}$$

On the other hand, since $\sigma^*_{\overline{T}}[\overline{X}] = [\overline{X'}] + 3[E_{\overline{T}}]$,

$$f'^*[\overline{X'}] = f'^*\sigma^*_T[\overline{X}] - 3[E_T].$$

Hence, by the *projection formula*,

$$\tau_{T_*}\big(f'^*[\overline{X'}]\cdot\tau^*_T[D]\big) = \tau_{T_*}\big(f'^*[\overline{X'}]\big)\cdot D = \tau_{T_*}\big(f'^*\sigma^*_{\overline{T}}[\overline{X'}]\big)\cdot D = f^*[\overline{X}]\cdot D, \tag{3.19}$$

and

$$\begin{aligned}\tau_{T_*}\big(f'^*[\overline{X'}]\cdot E_T\big) &= \tau_{T_*}\big(f'^*\sigma^*_{\overline{T}}[\overline{X}]\cdot E_T - 3[E_T]^2\big)\\ &= \tau_{T_*}\big(\tau^*_T f^*[\overline{X}]\cdot E_T\big) + 3\tau_{T_*}j_*[\xi_T] = f^*[\overline{X}]\cdot\tau_{T_*}[E_T] + 3i_*[T] = 3[T]\end{aligned} \tag{3.20}$$

Therefore, by (3.18), (3.19) and (3.20),

$$\tau_{T_*}\big(f'^*[\overline{X'}]\cdot D'\big) = f^*[\overline{X}]\cdot D - 6[T]. \tag{3.21}$$

Furthermore, we have

$$\begin{aligned}\tau_{T_*}[D']^2 &= \tau_{T_*}((\tau^*_T[D] - 2[E_T])^2)\\ &= \tau_{T_*}((\tau^*_T[D])^2 - 4\tau^*_T[D]\cdot[E_T] + 4[E_T]^2)\\ &= \tau_{T_*}(\tau^*_T[D])\cdot D - 4D\cdot\tau_{T_*}[E_T] - 4\tau_{T_*}j_*[\xi_T] = [D]^2 - 4[T].\end{aligned} \tag{3.22}$$

Consequently, by (3.17), (3.21) and (3.22),

$$\begin{aligned}f^*[\overline{D}] = f^*[\overline{X}]\cdot D - 6[T] - [D]^2 + 4[T] - [C] + 3[T]\\ = f^*[\overline{X}]\cdot D - [D]^2 - [C] + [T],\end{aligned}$$

from which equality (3.9) follows. ∎

Since $f_*[X] = [\overline{X}]$, $f_*[D] = 2[\overline{D}]$, $f_*[T] = 3[\overline{T}]$ and $f_*[C] = [\overline{C}]$, by Proposition 3.4, we have the following:

COROLLARY 3.5.

$$f_*[D]^2 = \overline{X}\cdot\overline{D} + 3[\overline{T}] - [\overline{C}] \tag{3.23}$$

By Proposition 3.4 and the second equality in (3.1),

$$s(J,X)_1 = -f^*[\overline{X}]\cdot D + f^*[\overline{D}] - [T] + 2[C]$$

and so, by the *projection formula*

$$s(\overline{J},\overline{X})_1 = -\overline{X}\cdot\overline{D} - 3[\overline{T}] + 2[\overline{C}]. \tag{3.24}$$

Now we compute $s(\overline{J},\overline{X})_0$. By Proposition 3.4,

$$[D]^3 = f^*[\overline{X}]\cdot[D]^2 - f^*[\overline{D}]\cdot D + D\cdot T - D\cdot C.$$

Hence, by the third equality in (3.1),

$$s(J,X)_0 = f^*[\overline{X}]\cdot[D]^2 - f^*[\overline{D}]\cdot D + D\cdot T - 4D\cdot C - c_1(N_C X)\cap[C]. \tag{3.25}$$

Since $\overline{T}$ and $\overline{C}$ are regularly embedded in $Y$, we can apply the *excess intersection formula* to them. Then,

$$\begin{aligned} f^*[\overline{T}] &= c_1(f^*N_{\overline{T}}Y/N_T X)\cap[T] \\ &= \{c_1(f^*\mathcal{T}_Y) - c_1(f^*\mathcal{T}_{\overline{T}}) - c_1(\mathcal{T}_X) + c_1(\mathcal{T}_T)\}\cap[T] \\ &= \{c_1(f^*\mathcal{T}_Y) - c_1(\mathcal{T}_X)\}\cap[T] - [\textstyle\sum s] \\ &= f^*[\overline{X}]\cdot T - D\cdot T - [\textstyle\sum s], \end{aligned}$$

where the last but one step follows from the *ramification formula* for $g : T \to \overline{T}$ and the last step from the *double point formula* for $f : X \to Y$. Similarly, since $\overline{C} \simeq C$, we have

$$f^*[\overline{C}] = f^*[\overline{X}]\cdot C - D\cdot C.$$

Therefore we have

$$\begin{cases} D\cdot T = f^*[\overline{X}]\cdot T - f^*[\overline{T}] - [\sum s] \\ D\cdot C = f^*[\overline{X}]\cdot C - f^*[\overline{C}] \end{cases} \tag{3.26}$$

By the *adjunction formula*, the *double point formula* for $f : X \to Y$ and the second equality in (3.26),

$$\begin{aligned} c_1(N_C X)\cap[C] &= -K_X\cdot C + [k_C] \\ &= \left(-f^*[\overline{X} + K_Y] + D\right)\cdot C + [k_C] = -f^*[K_Y]\cdot C - f^*[\overline{C}] + [k_C], \end{aligned} \tag{3.27}$$

where $K_Y$, $K_X$ and $k_C$ are the canonical divisors of $Y$, $X$ and $C$, respectively. Substituting (3.26) and (3.27) into (3.25), we have

$$\begin{aligned} s(J,X)_0 = f^*[\overline{X}]\cdot[D]^2 - f^*[\overline{D}]\cdot D + f^*[\overline{X}]\cdot T - f^*[\overline{T}] - [\textstyle\sum s] \\ - 4f^*[\overline{X}]\cdot C + 4f^*[\overline{C}] + f^*[K_Y]\cdot C + f^*[\overline{C}] - [k_C]. \end{aligned}$$

Consequently, using Corollary 3.5 and the fact that $f_*[X] = [\overline{X}]$, $f_*[D] = 2[\overline{D}]$, $f_*[T] = 3[\overline{T}]$, $f_*[\sum s] = [\sum \overline{s}]$ and $\overline{C} \simeq C$, we have,

$$s(\overline{J},\overline{X})_0 = [\overline{X}]^2\cdot\overline{D} - 2[\overline{D}]^2 + 5\overline{X}\cdot\overline{T} + K_Y\cdot\overline{C} - [k_{\overline{C}}] - [\textstyle\sum \overline{s}].$$

We collect the results obtained till now in the following proposition:

PROPOSITION 3.6. *The Segre classes of the singular subscheme $\overline{J}$, defined by the Jacobian ideal, of an algebraic threefold $\overline{X}$ with ordinary singularities in the four-dimensional projective space $Y = P^4(\mathbf{C})$ are given as follows, if $\overline{X}$ is free from quadruple points:*

$$\begin{cases} s(\overline{J},\overline{X})_2 = 2[\overline{D}] \\ s(\overline{J},\overline{X})_1 = -\overline{X}\cdot\overline{D} - 3\overline{T} + 2\overline{C} \\ s(\overline{J},\overline{X})_0 = [\overline{X}]^2\cdot\overline{D} - 2[\overline{D}]^2 + 5\overline{X}\cdot\overline{T} + K_Y\cdot\overline{C} - [k_{\overline{C}}] - [\sum \overline{s}] \end{cases}$$

*Here $\overline{D}$, $\overline{T}$, $\overline{C}$ and $\sum \overline{s}$ are the singular locus, triple point locus, cuspidal point locus and stationary point locus of $\overline{X}$, respectively. $K_Y$ is the canonical divisor of the projective 4-space $Y$, and $k_{\overline{C}}$ that of $\overline{C}$.*

**4. The Euler number of the normalization of an algebraic threefold with ordinary singularities.** By Theorem 3.1, the top polar class $[M_3]$ of $\overline{X}$ is given by

$$[M_3] = (n-1)^3 h^3 - 3(n-1)^2 h^2 \cap s_2 - 3(n-1)h \cap s_1 - s_0,$$

where $h$ denotes the hyperplane section class and $s_i$ $i$-th Segre class $s(\overline{J}, \overline{X})_i$ $(0 \leq i \leq 2)$ and $n = \deg \overline{X}$, the degree of $\overline{X}$ in $Y$. We put

$$m = \deg \overline{D},\ t = \deg \overline{T},\ \gamma = \deg \overline{C} \text{ and } \#\sum \overline{s} = \text{the cardinal number of } \sum \overline{s}.$$

Then, by Proposition 3.6,

$$\begin{cases} \deg s_2 = 2m \\ \deg s_1 = -nm + 2\gamma - 3t \\ \deg s_0 = n^2 m - 2m^2 + 5nt - 5\gamma - \#\sum \overline{s} - \deg k_{\overline{C}}. \end{cases}$$

Consequently, the *class* $c$ of $\overline{X}$ is given by

$$\begin{aligned} c = \deg[M_3] &= (n-1)^3 \deg \overline{X} - 3(n-1)^2 \deg s_2 - 3(n-1) \deg s_1 - \deg s_0 \\ &= (n-1)^3 n - (4n^2 - 9n - 2m + 6)m + (4n-9)t - (6n-11)\gamma + \#\sum \overline{s} + \deg k_{\overline{C}}. \end{aligned}$$

By this formula together with Proposition 2.3, we have the following:

THEOREM 4.1. *The Euler number $\chi(X)$ of the non-singular normalization $X$ of an algebraic threefold $\overline{X}$ with ordinary singularities in $P^4(\mathbf{C})$ which is free from quadruple points is given by*

$$\begin{aligned} \chi(X) = -n(n^3 - 5n^2 + 10n - 10) + (4n^2 - 15n - 2m + 20)m - (4n-15)t \\ + (6n-15)\gamma - \#\sum \overline{s} - \deg k_{\overline{C}}. \end{aligned}$$

*Here $n = \deg \overline{X}$, $m = \deg \overline{D}$, $t = \deg \overline{T}$ and $\gamma = \deg \overline{C}$ are the degrees of $\overline{X}$, the singular locus, the triple point locus and the cuspidal point locus, respectively. $\#\sum \overline{s}$ is the cardinal number of the stationary point locus $\sum \overline{s}$, and $\deg k_{\overline{C}}$ the degree of the canonical divisor of the cuspidal point locus $\overline{C}$.*

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