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FILIPPOV LEMMA FOR MATRIX FOURTH ORDER DIFFERENTIAL INCLUSIONS

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Abstract. In the paper we give an analogue of the Filippov Lemma for the fourth order differential inclusions

$$\mathcal{D}y = y'''' - (A^2 + B^2)y'' + A^2B^2y \in F(t, y), \tag{*}$$

with the initial conditions

$$y(0) = y'(0) = y''(0) = y'''(0) = 0, \tag{**}$$

where the matrices $A, B \in \mathbb{R}^{d\times d}$ are commutative and the multifunction $F : [0, 1] \times \mathbb{R}^d \rightsquigarrow \mathrm{cl}(\mathbb{R}^d)$ is Lipschitz continuous in y with a t-independent constant $l < \|A\|^2\|B\|^2$.

Main Theorem. *Assume that $F : [0, 1] \times \mathbb{R}^d \rightsquigarrow \mathrm{cl}(\mathbb{R}^d)$ is measurable in t and integrably bounded. Let $y_0 \in W^{4,1}$ be an arbitrary function satisfying (**) and such that*

$$d_H(\mathcal{D}y_0(t), F(t, y_0(t))) \le p_0(t) \text{ a.e. in } [0, 1],$$

where $p_0 \in L^1[0, 1]$. Then there exists a solution $y \in W^{4,1}$ of (*) with (**) such that

$$|\mathcal{D}y(t) - \mathcal{D}y_0(t)| \le p_0(t) + l(Y_4(\,\cdot\,, \alpha, \beta) * p_0)(t)$$
$$|y(t) - y_0(t)| \le (Y_4(\,\cdot\,, \alpha, \beta) * p_0)(t) \text{ a.e. in } [0, 1],$$

where

$$Y_4(x, \alpha, \beta) = \frac{\alpha^{-1}\sinh(\alpha x) - \beta^{-1}\sinh(\beta x)}{\alpha^2 - \beta^2}$$

and α, β depend on $\|A\|$, $\|B\|$ and l.

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1. Introduction. The theory of ordinary differential operators $\mathcal{D}$ has been developed for many years and most of the facts are already discovered. However, the properties of the solution sets of differential inclusions

$$\mathcal{D}y \in F(t,y) \tag{1}$$

in many situations have to be examined. Lately, we observe an increase of interest in this field, especially in the field of ordinary differential inclusions of higher order. In particular, there have been examined boundary value problems [21], [20], of the differential inclusions of Sturm–Liouville type [3], of Schrödinger type [4], of the n-th order of the form $y^{(n)} - \lambda y \in F(t,y)$ in [11] and of the second order $\mathcal{D}y = y'' - A^2 y \in F(t,y)$ [7].

In all known results concerning the existence and properties of solution sets to differential inclusions

$$y' \in F(t,y) \tag{2}$$

with the Lipschitz continuous right-hand sides, analogues of the Filippov Lemma (cf. [1], [2], [8], [9], [12], [10], [13], [14], [15], [17], [19], [22], [23], [24]) play the crucial role.

In this paper the attention is focused on the differential inclusions

$$\mathcal{D}y = y'''' - (A^2+B^2)y'' + A^2B^2y \in F(t,y), \tag{3}$$

where $F : [0,1]\times\mathbb{R}^d \rightsquigarrow \mathbb{R}^d$ is a multifunction and $\mathcal{D}y = y'''' - (A^2+B^2)y'' + A^2B^2y$ is a matrix differential operator with nondegenerated and commutative matrices $A, B \in \mathbb{R}^{d\times d}$. For (3) we impose the initial conditions

$$y(0) = y'(0) = y''(0) = y'''(0) = 0. \tag{4}$$

By a solution of (3) with initial conditions (4) we mean a function $y \in W^{4,1}[0,1]$ satisfying (3) and (4). Our considerations are quite standard and elementary and the methods used are based on the differential equation (see [16])

$$\mathcal{D}y = y'''' - (A^2+B^2)y'' + A^2B^2y = f,$$

where $f \in L^1([0,1],\mathbb{R}^d)$. We present them in Section 2, while in Section 3 we formulate and demonstrate an analogue of the Filippov Lemma for the IVP (3) with (4).

2. An IVP for matrix fourth order ODE. Let $(\mathbb{R}^d, |\cdot|)$ be a finite-dimensional normed space. By $L^1([0,1],\mathbb{R}^d)$ we mean the Banach space of Lebesgue integrable functions $u : [0,1] \to \mathbb{R}^d$ with the norm $\|u\|_1 = \int_{[0,1]} |u(t)|\,dt$ and by

$$V = \left\{ y \in W^{4,1}([0,1],\mathbb{R}^d) : y(0) = y'(0) = y''(0) = y'''(0) = 0 \right\}$$

with the norm

$$\|y\|_V = \|y''''\|_1.$$

Suppose we are given the function $f \in L^1([0,1],\mathbb{R}^d)$ and matrices $A, B \in \mathbb{R}^{d\times d}$. Through our paper we shall assume that the matrices $A, B \in \mathbb{R}^{d\times d}$ are nonsingular, commutative with nonsingular $A^2 - B^2$.

Let us consider the differential equation

$$\mathcal{D}y = y'''' - (A^2 + B^2)y'' + A^2B^2y = f, \tag{5}$$

with initial conditions

$$y(0) = y'(0) = y''(0) = y'''(0) = 0. \tag{6}$$

By a solution we mean a function $y \in V$ satisfying (5) almost everywhere (a.e.) in $[0, 1]$. Let

$$\Phi(t) = \Phi(t, A, B) = (A^2 - B^2)^{-1}\big(A^{-1}\sinh(tA) - B^{-1}\sinh(tB)\big) = \sum_{n=1}^{\infty} \frac{t^{2n+1}}{(2n+1)!}\bigg(\sum_{k=0}^{n-1} A^{2k}B^{2n-2k-2}\bigg).$$

The convergence of the above series is understood in the norm

$$\|A\| = \max_i \Bigg[\sum_{j=1}^{d} |a_{ij}|\Bigg],$$

where the matrix $A = [a_{ij}] \in \mathbb{R}^{d\times d}$.

Since

$$\begin{aligned}
\Phi'(t) &= (A^2 - B^2)^{-1}(\cosh(tA) - \cosh(tB)),\\
\Phi''(t) &= (A^2 - B^2)^{-1}(A\sinh(tA) - B\sinh(tB)),\\
\Phi'''(t) &= (A^2 - B^2)^{-1}(A^2\cosh(tA) - B^2\cosh(tB)),\\
\Phi''''(t) &= (A^2 - B^2)^{-1}(A^3\sinh(tA) - B^3\sinh(tB)),
\end{aligned}$$

we have

$$\begin{aligned}
&\Phi''''(t) - (A^2 + B^2)\Phi''(t) + A^2B^2\Phi(t)\\
&\quad = (A^2 - B^2)^{-1}\Big(A^3\sinh(tA) - B^3\sinh(tB) - (A^2 + B^2)(A\sinh(tA) - B\sinh(tB))\\
&\qquad\qquad + A^2B^2(A^{-1}\sinh(tA) - B^{-1}\sinh(tB))\Big) = 0.
\end{aligned}$$

Moreover,

$$\Phi(0) = \Phi'(0) = \Phi''(0) = 0, \qquad \Phi'''(0) = I.$$

Therefore the function Φ is the Cauchy function for homogeneous (3). Using that function we obtain the following:

PROPOSITION 1. *The solution of* (5) *with the IC* (6) *is in convolution form*

$$y(t) = (\mathcal{R}f)(t) = (\mathcal{R}(A, B)f)(t) = (\Phi * f)(t) = \int_0^t \Phi(t - x)f(x)\,dx.$$

Proof. For each function $\varphi \in C^1([0, 1], \mathbb{R}^{d\times d})$ and $f \in L^1([0, 1], \mathbb{R}^d)$ we have the formula

$$(\varphi * f)'(t) = (\varphi' * f)(t) + \varphi(0)f(t). \tag{7}$$

Thus evaluating the derivatives we obtain

$$
\begin{aligned}
y'(t) &= (\Phi'(x) * f)(t) = (A^2 - B^2)^{-1}\big((\cosh(xA) - \cosh(xB)) * f\big)(t),\\
y''(t) &= (\Phi''(x) * f)(t) = (A^2 - B^2)^{-1}\big((A\sinh(xA) - B\sinh(xB)) * f\big)(t),\\
y'''(t) &= (\Phi'''(x) * f)(t) = (A^2 - B^2)^{-1}\big((A^2\cosh(xA) - B^2\cosh(xB)) * f\big)(t)\\
y''''(t) &= (\Phi''''(x) * f)(t) + f(t)\\
&\qquad\qquad = (A^2 - B^2)^{-1}\big((A^3\sinh(xA) - B^3\sinh(xB)) * f\big)(t) + f(t).
\end{aligned}
$$

Hence for all $t \in [0,1]$ we get

$$
\begin{aligned}
&y''''(t) - (A^2 + B^2)y''(t) + A^2B^2y(t)\\
&= (\Phi''''(x) * f)(t) + f(t) - (A^2 + B^2)(\Phi''(x) * f)(t) + A^2B^2(\Phi(x) * f)(t)\\
&= \big((\Phi''''(x) - (A^2 + B^2)\Phi''(x) + A^2B^2\Phi(x)) * f\big)(t) + f(t) = f(t).
\end{aligned}
$$

Checking the IC is straightforward. ■

For $d = 1$ we obtain a particular case of (5). In this situation we have an IVP

$$\mathcal{D}_0 y = y'''' - (a^2 + b^2)y'' + (a^2b^2)y = f, \tag{8}$$

with

$$y(0) = y'(0) = y''(0) = y'''(0) = 0. \tag{9}$$

We assume that $a, b \neq 0$ and $a^2 \neq b^2$. Extending the idea of M. Kourensky [18] we take principal fundamental solutions of (8) given by

$$Y_4(t) = Y_4(t,a,b) = \frac{a^{-1}\sinh(at) - b^{-1}\sinh(bt)}{a^2 - b^2}\,, \tag{10}$$

$$Y_3(t) = Y_3(t,a,b) = Y_4'(t) = \frac{\cosh(at) - \cosh(bt)}{a^2 - b^2}\,, \tag{11}$$

$$Y_2(t) = Y_2(t,a,b) = Y_4''(t) = \frac{a\sinh(at) - b\sinh(bt)}{a^2 - b^2}\,, \tag{12}$$

$$Y_1(t) = Y_1(t,a,b) = Y_4'''(t) = \frac{a^2\cosh(at) - b^2\cosh(bt)}{a^2 - b^2}\,. \tag{13}$$

Observe that $Y_4(t)$ has the Taylor expansion

$$Y_4(t) = \sum_{n=1}^{\infty} \frac{\left(\sum_{k=0}^{n-1} a^{2k}b^{2n-2k-2}\right)}{(2n+1)!}\, t^{2n+1} \geq 0$$

and the latter means that $Y_4(t)$ is an analytic and nonnegative solution of $\mathcal{D}_0 y = 0$ with (9). Therefore the solution of (8) with (9) is

$$\mathcal{R}_0 f = Y_4 * f.$$

Since $Y_4(t) = Y_4(t,a,b)$ is nonnegative, then the operator $\mathcal{R}_0 = \mathcal{R}_0(a,b)$ is positive; i.e. $\mathcal{R}_0 f \geq 0$ for $f \geq 0$.

As a conclusion of the previous considerations we obtain the following properties:

Proposition 2. *Let* $a = \|A\|$ *and* $b = \|B\|$. *Then for* $\mathcal{R} = \mathcal{R}(A, B)$ *and* $\mathcal{R}_0 = \mathcal{R}_0(a, b)$ *we have the relations*:

1. $|\mathcal{R}f| \le \mathcal{R}_0|f| \le \mathcal{R}_0 p$ *for* $|f| \le p$;
2. $|(\mathcal{R}f)'(t)| \le (\mathcal{R}_0|f|)'(t)$,
   $|(\mathcal{R}f)''(t)| \le (\mathcal{R}_0|f|)''(t)$,
   $|(\mathcal{R}f)'''(t)| \le (\mathcal{R}_0|f|)'''(t)$.

Proof. Since

$$\Phi(t) = \sum_{n=1}^{\infty}\Big(\sum_{k=0}^{n-1} A^{2k}B^{2n-2k-2}\Big)\frac{t^{2n+1}}{(2n+1)!},$$

we have

$$(\mathcal{R}f)(t) = \sum_{n=1}^{\infty}\Big(\sum_{k=0}^{n-1} A^{2k}B^{2n-2k-2}\Big)\frac{1}{(2n+1)!}(x^{2n+1} * f)(t).$$

Therefore

$$|(\mathcal{R}f)(t)| \le \sum_{n=1}^{\infty}\frac{1}{(2n+1)!}(x^{2n+1} * |f|)(t)\Big(\sum_{k=0}^{n-1}\|A\|^{2k}\|B\|^{2n-2k-2}\Big)$$
$$= (Y_4 * |f|)(t) = (\mathcal{R}_0|f|)(t).$$

Using the same arguments we obtain

$$(\mathcal{R}f)'(t) = \sum_{n=1}^{\infty}\frac{1}{(2n)!}\Big(\sum_{k=0}^{n-1} A^{2k}B^{2n-2k-2}\Big)(x^{2n} * f)(t)$$

and thus

$$|(\mathcal{R}f)'(t)| \le \sum_{n=1}^{\infty}\frac{1}{(2n)!}(x^{2n} * |f|)(t)\Big(\sum_{k=0}^{n-1}\|A\|^{2k}\|B\|^{2n-2k-2}\Big)$$
$$= (Y_3 * |f|)(t) = (\mathcal{R}_0|f|)'(t).$$

Similar calculation yields analogous inequalities for $|(\mathcal{R}f)''(t)|$ and $|(\mathcal{R}f)'''(t)|$. ∎

For $u \in L^1([0,1])$ let us define recursively

$$u^{(*1)} = u, \quad u^{(*n)} = u * u^{(*(n-1))}, \quad n = 2, 3, \dots .$$

Lemma 3. *Let* $f \in L^1([0,1])$ *and* $Y_4(t,a,b) = \frac{a^{-1}\sinh(at) - b^{-1}\sinh(bt)}{a^2-b^2}$, *where* $a, b \ne 0$ *and* $a^2 \ne b^2$. *Then for each* $c \in \mathbb{R}$ *with* $|c| < ab$ *the series* $\sum_{n=1}^{\infty}|c|^{2n}\big(Y_4^{(*n)}(\,\cdot\,, a, b) * f\big)(t)$ *is absolutely and uniformly convergent and for*

$$z(t) = \sum_{n=1}^{\infty}|c|^{2n}\big(Y_4^{(*n)}(\,\cdot\,, a, b) * |f|\big)(t)$$

we have a uniform estimate

$$z(t) \le c^2\big(Y_4(\,\cdot\,, \alpha, \beta) * |f|\big)(t),$$

where

$$\alpha = \frac{\sqrt{a^2+b^2+2\sqrt{a^2b^2-c^2}}+\sqrt{a^2+b^2-2\sqrt{a^2b^2-c^2}}}{2},$$
$$\beta = \frac{\sqrt{a^2+b^2+2\sqrt{a^2b^2-c^2}}-\sqrt{a^2+b^2-2\sqrt{a^2b^2-c^2}}}{2}.$$

Proof. Consider for each $k=1,2,\ldots$ the k-th partial sum

$$z_k(t) = \sum_{n=1}^{k} c^{2n}\big(Y_4^{(*n)}(\,\cdot\,,a,b) * |f|\big)(t).$$

Then we have

$$\begin{aligned} z_k(t) &= c^2\Big(Y_4(\,\cdot\,,a,b) * \Big(\sum_{n=1}^{k-1} c^{2n}\big(Y_4^{(*n)}(\,\cdot\,,a,b) * |f|\big) + |f|\Big)\Big)(t) \\ &= c^2\Big(Y_4(\,\cdot\,,a,b) * (z_k - c^{2k}\big(Y_4^{(*k)}(\,\cdot\,,a,b) * |f|\big) + |f|)\Big)(t) \end{aligned}$$

with

$$z_k(0) = z_k'(0) = z_k''(0) = z_k'''(0) = 0.$$

Hence

$$\begin{aligned} & z_k''''(t) - (a^2+b^2)z_k''(t) + (a^2b^2)z_k(t) \\ &\qquad = c^2\Big(z_k - c^{2k}\big(Y_4^{(*k)}(\,\cdot\,,a,b) * |f|\big) + |f|\Big)(t) \le c^2 z_k(t) + c^2|f(t)|. \end{aligned}$$

In other words

$$z_k''''(t) - (a^2+b^2)z_k''(t) + (a^2b^2-c^2)z_k(t) = g(t) \le c^2|f(t)|. \tag{14}$$

Let

$$\alpha = \frac{\sqrt{a^2+b^2+2\sqrt{a^2b^2-c^2}}+\sqrt{a^2+b^2-2\sqrt{a^2b^2-c^2}}}{2},$$
$$\beta = \frac{\sqrt{a^2+b^2+2\sqrt{a^2b^2-c^2}}-\sqrt{a^2+b^2-2\sqrt{a^2b^2-c^2}}}{2}$$

be solutions of the system

$$\begin{cases} \alpha^2+\beta^2 = a^2+b^2, \\ \alpha^2\beta^2 = a^2b^2-c^2. \end{cases}$$

Hence (14) can be read as

$$z_k''''(t) - (\alpha^2+\beta^2)z_k''(t) + (\alpha^2\beta^2)z_k(t) = g(t) \le c^2|f(t)|.$$

Therefore for each $k=1,2,\ldots$ we have

$$z_k(t) = (Y_4(\,\cdot\,,\alpha,\beta) * g)(t) \le c^2\big(Y_4(\,\cdot\,,\alpha,\beta) * |f|\big)(t).$$

Passing to the limit with $k\to\infty$ we obtain the required estimate. ∎

COROLLARY 4. *The series* $\sum_{n=1}^{\infty} c^{2n}(\Phi(\,\cdot\,,A,B) * f)$ *is for* $|c| < \|A\|\|B\|$ *strongly convergent in* V.

3. A Filippov Lemma. Consider an IVP problem

$$\mathcal{D}y \in F(t,y), \tag{15}$$

$$y(0) = y'(0) = y''(0) = y'''(0) = 0. \tag{16}$$

By a solution of (15) with initial conditions (16) we mean a function $y \in V$ satisfying (15). We shall pose the following assumptions on $F : [0,1] \times \mathbb{R}^d \rightsquigarrow c(\mathbb{R}^d)$, where $c(\mathbb{R}^d)$ stands for the family of all nonempty compact subsets of $\mathbb{R}^d$:

Condition 1. *For every $y \in \mathbb{R}^d$ the multifunction $F(\,\cdot\,, y)$ is Lebesgue measurable in t.*

Condition 2. *The multifunction $F(t,\,\cdot\,)$ is Lipschitz continuous in y with a constant l, i.e. for every $y_1, y_2 \in \mathbb{R}^n$ the inequality*

$$d_H\big(F(t,y_1), F(t,y_2)\big) \le l|y_1 - y_2|$$

holds for a.a. $t \in [0,1]$, where $d_H(K,L)$ stands for the Hausdorff distance between sets $K, L \in c(\mathbb{R}^d)$.

Condition 3. *The multivalued mapping $t \mapsto F(t,0)$ is integrably bounded by a function $\gamma \in L^1[0,1]$, i.e.*

$$\sup\{|z| : z \in F(t,0)\} \le \gamma(t) \text{ a.e. in } [0,1].$$

The main result of the paper is the following.

Theorem 5. *Assume that $F : [0,1] \times \mathbb{R}^d \rightsquigarrow c(\mathbb{R}^d)$ satisfies Conditions 1, 2 and 3 with $l < \|A\|^2\|B\|^2$. Let $y_0 \in V$ be an arbitrary function with $y_0(0) = y_0'(0) = y_0''(0) = y_0'''(0) = 0$ such that*

$$\operatorname{dist}\big(\mathcal{D}y_0(t), F(t,y_0(t))\big) \le p_0(t) \text{ a.e. in } [0,1],$$

where $p_0 \in L^1[0,1]$. Then there exists a solution $y \in V$ of (15) such that

$$|\mathcal{D}y(t) - \mathcal{D}y_0(t)| \le p_0(t) + l\big(Y_4(\,\cdot\,,\alpha,\beta) * p_0\big)(t) \text{ a.e. in } [0,1] \tag{17}$$

and

$$|y(t) - y_0(t)| \le \big(Y_4(\,\cdot\,,\alpha,\beta) * p_0\big)(t) \text{ a.e. in } [0,1], \tag{18}$$

where

$$\begin{aligned} \alpha &= \frac{\sqrt{\|A\|^2 + \|B\|^2 + 2\sqrt{\|A\|^2\|B\|^2 - l}} + \sqrt{\|A\|^2 + \|B\|^2 - 2\sqrt{\|A\|^2\|B\|^2 - l}}}{2} \\ \beta &= \frac{\sqrt{\|A\|^2 + \|B\|^2 + 2\sqrt{\|A\|^2\|B\|^2 - l}} - \sqrt{\|A\|^2 + \|B\|^2 - 2\sqrt{\|A\|^2\|B\|^2 - l}}}{2}. \end{aligned} \tag{19}$$

Proof. Recall that from [7] it follows that for any $y \in V$ the multivalued mapping $t \mapsto F(t,y(t))$ is measurable with compact values and integrably bounded by $\gamma(t) + l|y(t)|$, i.e.

$$\sup\big\{|z| : z \in F(t,y(t))\big\} \le \gamma(t) + l|y(t)| \text{ a.e. in } [0,1]. \tag{20}$$

Take any $u \in L^1([0,1],\mathbb{R}^d)$ and let

$$y = \mathcal{R}u = \mathcal{R}(A,B)u = \Phi * u,$$

where

$$\Phi(t) = (A^2 - B^2)^{-1}\big(A^{-1}\sinh(tA) - B^{-1}\sinh(tB)\big),$$

be the unique solution $y \in V$ of

$$\mathcal{D}y = u$$

with homogeneous initial conditions. Let

$$\mathcal{K}(u) = \big\{f \in L^1([0,1],\mathbb{R}^d) : f(t) \in F(t,(\mathcal{R}u)(t)) \text{ a.e. in } [0,1]\big\}.$$

Since $\mathcal{R}u \in V \subset L^1([0,1],\mathbb{R}^d)$, by (20) each $\mathcal{K}(u)$ is nonempty. Moreover, for every $u, v \in L^1([0,1],\mathbb{R}^d)$ and any $f \in \mathcal{K}(u)$ there is a $g \in \mathcal{K}(v)$ such that

$$\begin{aligned} |f(t) - g(t)| &= d_H\big(F(t,(\mathcal{R}u)(t)), F(t,(\mathcal{R}v)(t))\big) \\ &\le l(|\mathcal{R}u - \mathcal{R}v|)(t) \le l\big(\mathcal{R}_0(\|A\|,\|B\|)|u-v|\big)(t) \\ &\le l\big(\mathcal{R}_0(\|A\|,\|B\|)p_0\big)(t) \text{ a.e. in } [0,1]. \end{aligned} \tag{21}$$

In what follows we shall adopt the Filippov technique with necessary changes. Starting with $y_0 = \mathcal{R}u_0$ we can choose such $u_1 \in \mathcal{K}(u_0)$ that

$$|u_0(t) - u_1(t)| = \big|(\mathcal{D}y_1)(t) - (\mathcal{D}y_0)(t)\big| \le p_0(t) \text{ a.e. in } [0,1],$$

where $y_1 = \mathcal{R}u_1$. Hence for all $t \in [0,1]$ we have

$$|y_0(t) - y_1(t)| = \big|(\Phi * (u_0 - u_1))(t)\big| \le \big(Y_4(\,\cdot\,,\|A\|,\|B\|) * p_0\big)(t).$$

Now the relation (21) yields

$$d_H\big((\mathcal{D}y_1)(t), F(t,y_1(t))\big) \le l\big(Y_4(\,\cdot\,,\|A\|,\|B\|) * p_0\big)(t) \text{ a.e. in } [0,1].$$

We now may pick up $y_2 = \mathcal{R}u_2 \in V$ such that $u_2 = \mathcal{D}y_2 \in \mathcal{K}(u_1)$ and

$$\big|(\mathcal{D}y_2)(t) - (\mathcal{D}y_1)(t)\big| \le l\big(Y_4(\,\cdot\,,\|A\|,\|B\|) * p_0\big)(t) \text{ a.e. in } [0,1].$$

Therefore for all $t \in [0,1]$

$$|y_2(t) - y_1(t)| \le l\big(Y_4^{(*2)}(\,\cdot\,,\|A\|,\|B\|) * p_0\big)(t).$$

The latter together with (21) yields

$$d_H\big((\mathcal{D}y_2)(t), F(t,y_2(t))\big) \le l^2\big(Y_4^{(*2)}(\,\cdot\,,\|A\|,\|B\|) * p_0\big) \text{ a.e. in } [0,1].$$

Continuing this procedure by mathematical induction we can find for $n = 1,2,3,\ldots$ a sequence $u_{n+1} \in \mathcal{K}(u_n)$ such that for $y_{n+1} = \mathcal{R}u_{n+1} \in V$

$$|\mathcal{D}y_{n+1}(t) - \mathcal{D}y_n(t)| \le l^n\big(Y_4^{(*n)}(\,\cdot\,,\|A\|,\|B\|) * p_0\big) \text{ a.e. in } [0,1].$$

Hence

$$|y_{n+1}(t) - y_n(t)| \le l^n\big(Y_4^{(*(n+1))}(\,\cdot\,,\|A\|,\|B\|) * p_0\big) \text{ a.e. in } [0,1]$$

and therefore by (21)

$$d_H(u_{n+1}(t), F(t,(\mathcal{R}(u_{n+1}))(t))) \le l^{n+1}(Y_4^{(*(n+1))}(\,\cdot\,,\|A\|,\|B\|) * p_0) \text{ a.e. in } [0,1].$$

Let

$$\varphi(t) = \sum_{n=1}^{\infty} l^n\big(Y_4^{(*n)}(\,\cdot\,,\|A\|,\|B\|) * p_0\big).$$

By Lemma 3 we have an estimate

$$\varphi(t) \le l\big(Y_4(\,\cdot\,,\alpha,\beta) * p_0\big)(t), \tag{22}$$

where α and β are given by formulas (19). Thus φ is continuous and hence integrable. Moreover, for $n = 0, 1, 2, \dots$ and $m = 1, 2, \dots$ we have

$$\begin{aligned} |\mathcal{D}y_{n+m}(t) - \mathcal{D}y_n(t)| &\le \sum_{k=n}^{n+m-1} l^k (Y_4^{(*k)}(\,\cdot\,, \|A\|, \|B\|) * p_0) \\ &\le \sum_{k=n}^{\infty} l^k (Y_4^{(*k)}(\,\cdot\,, \|A\|, \|B\|) * p_0) \le p_0(t) + \varphi(t) \text{ a.e. in } [0,1]. \end{aligned} \tag{23}$$

Hence

$$|y_{n+m}(t) - y_n(t)| \le \sum_{k=n}^{\infty} l^k \big(Y_4^{(*(k+1))}(\,\cdot\,, \|A\|, \|B\|) * p_0\big) \le \frac{\varphi(t)}{l} \text{ a.e. in } [0,1]. \tag{24}$$

Therefore the sequences $\{u_n\} \subset L^1([0,1], \mathbb{R}^n)$ and $\{y_n\} = \{\mathcal{R}u_n\} \subset V$ are convergent pointwisely and, by the Lebesgue Dominated Convergence Theorem, strongly. Let

$$\lim \mathcal{D}y_n = \mathcal{D}y.$$

Thus

$$\lim y_n = y.$$

For each $n = 0, 1, 2, \dots$

$$(\mathcal{D}y_{n+1})(t) \in F(t, y_n(t)) \text{ a.e. in } [0,1]$$

and each $F(t, \cdot\,)$ is Lipschitz continuous, hence y is a solution of (15). We shall check that it is the required one. Indeed, taking $n = 0$ in (23), (24) and passing to the limit with $m \to \infty$ we obtain a.e. in $[0,1]$

$$|\mathcal{D}y(t) - \mathcal{D}y_0(t)| \le p_0(t) + \varphi(t)$$

and

$$|y(t) - y_0(t)| \le \frac{\varphi(t)}{l}\,.$$

Hence, by (22), we have a.e. in $[0,1]$ the estimates

$$\begin{aligned} |\mathcal{D}y(t) - \mathcal{D}y_0(t)| &\le p_0(t) + l(Y_4(\,\cdot\,, \alpha, \beta) * p_0)(t) \text{ a.e. in } [0,1](t) \\ |y(t) - y_0(t)| &\le (Y_4(\,\cdot\,, \alpha, \beta) * p_0)(t), \end{aligned}$$

which ends the proof. ∎

References

[1] J.-P. Aubin, A. Cellina, *Differential Inclusions. Set-Valued Maps and Viability Theory*, Grundlehren Math. Wiss. 264, Springer, Berlin, 1984.

[2] J.-P. Aubin, H. Frankowska, *Set-Valued Analysis*, System Control Found. Appl. 2, Birkhäuser, Boston, 1990.

[3] G. Bartuzel, A. Fryszkowski, *A topological property of the solution set to the Sturm–Liouville differential inclusions*, Demonstratio Math. 28 (1995), 903–914.

[4] G. Bartuzel, A. Fryszkowski, *Stability of principal eigenvalue of the Schrödinger type problems for differential inclusions*, Topol. Methods Nonlinear Anal. 16 (2000), 181–194.

[5] G. Bartuzel, A. Fryszkowski, *A class of retracts in L^p with some applications to differential inclusion*, Discuss. Math. Differ. Incl. Control Optim. 22 (2001), 213–224.

[6] G. Bartuzel, A. Fryszkowski, *Pointwise comparison principle for clamped Timoshenko beam*, Topol. Methods Nonlinear Anal. 39 (2012), 335–359.

[7] G. Bartuzel, A. Fryszkowski, *Filippov Lemma for certain second order differential inclusions*, Cent. Eur. J. Math. 10 (2012), 1944–1952.

[8] A. Bressan, A. Cellina, A. Fryszkowski, *A class of absolute retracts in spaces of integrable functions*, Proc. Amer. Math. Soc. 112 (1991), 413–418.

[9] A. Cellina, *On the set of solutions to Lipschitzian differential inclusions*, Differential Integral Equations 1 (1988), 495–500.

[10] A. Cellina, A. Ornelas, *Representations of the attainable set for Lipschitzean differential inclusions*, Rocky Mountain J. Math. 22 (1992), 117–124.

[11] A. Cernea, *On the existence of solutions for a higher order differential inclusion without convexity*, Electron. J. Qual. Theory Differ. Equ. 2007, No. 8, 1–8; http://www.math.u-szeged.hu/ejqtde.

[12] R. M. Colombo, A. Fryszkowski, T. Rzeżuchowski, V. Staicu, *Continuous selection of solution sets of Lipschitzean differential inclusions*, Funkcial. Ekvac. 34 (1991), 321–330.

[13] A. F. Filippov, *Classical solutions of differential equations with multivalued right-hand side*, Vestnik Moskov. Univ. Ser. I Mat. Meh. 22 (1967), no. 3, 16-26; English transl.: SIAM J. Control 5 (1967), 609–621.

[14] A. Fryszkowski, *Fixed Point Theory for Decomposable Sets*, Topol. Fixed Point Theory 2, Kluwer, Dordrecht, 2004.

[15] A. Fryszkowski, T. Rzeżuchowski, *Continuous version of Filippov–Ważewski relaxation theorem*, J. Differential Equations 94 (1991), 254–265.

[16] Ph. Hartman, *Ordinary Differential Equations*, Birkhäuser, Boston, 1982.

[17] Sh. Hu, N. S. Papageorgiou, *Handbook of Multivalued Analysis*, vol. I, Math. Appl. 419, Kluwer, Dordrecht, 1997.

[18] M. Kourensky, *Zur Lösung der Differentialgleichung der Biegung des Balkens auf elastischer Unterlage*, Tôhoku Math. J. 39 (1934), 192–199.

[19] O. Naselli Ricceri, *$\mathcal{A}$-Fixed points of multivalued contractions*, J. Math. Anal. Appl. 135 (1988), 406–418.

[20] N. S. Papageorgiou, *Boundary value problems for evolution inclusions*, Comment. Math. Univ. Carolin. 29 (1988), 355–363.

[21] T. Pruszko, *Some applications of the topological degree theory to multivalued boundary value problems*, Dissertationes Math. (Rozprawy Mat.) 229 (1984).

[22] D. Repovš, P. V. Semenov, *Continuous selections of multivalued mappings*, Math. Appl. 455, Kluwer, Dordrecht, 1998.

[23] L. E. Rybiński, *A fixed point approach in the study of solution sets of Lipschitzean functional-differential inclusions*, J. Math. Anal. Appl. 160 (1991), 24–46.

[24] A. A. Tolstogonov, *On the structure of the solution set for differential inclusions in Banach spaces*, Mat. Sb. (N.S.) 118(160) (1982), 3–18; English transl.: Math. USSR-Sb. 46 (1983), 1–15.