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A SHARP ITERATION PRINCIPLE FOR HIGHER-ORDER SOBOLEV EMBEDDINGS

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Abstract. We survey results from the paper [CPS] in which we developed a new sharp iteration method and applied it to show that the optimal Sobolev embeddings of any order can be derived from isoperimetric inequalities. We prove thereby that the well-known link between first-order Sobolev embeddings and isoperimetric inequalities translates to embeddings of any order, a fact that had not been known before. We show a general reduction principle that reduces Sobolev type inequalities of any order involving arbitrary rearrangement-invariant norms on open sets in $\mathbb{R}^n$, possibly endowed with a measure density and satisfying an isoperimetric inequality of fairly general type, to considerably simpler one-dimensional inequalities for suitable integral operators depending on the isoperimetric function of the relevant sets. As a direct application of the reduction principle we determine the optimal target space in the relevant Sobolev embeddings both in standard and in non-standard classes of function spaces and underlying measure spaces. In particular, the results apply to any-order Sobolev embedding on regular (John) domains, on Maz'ya classes of (possibly irregular) Euclidean domains described in terms of their isoperimetric function, and on families of product probability spaces, of which the Gauss space and the exponential measure space are classical instances.

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1. Introduction. Sobolev spaces are certain specific Banach spaces containing weakly differentiable functions of several variables that arise in connection with problems in mathematical physics and PDEs. Their key importance for solving such problems has been known since about late 1930's, when the first contribution of S. L. Sobolev to their study appeared ([So2]). Sobolev spaces constitute an indispensable tool in applications, but they are also interesting on their own, as a very particular mathematical structure with unique properties. There is a vast literature available nowadays on Sobolev spaces, including monographs [RAA, AF, Ma3, Ma4, KJF] and more.

The most important feature of Sobolev spaces is how they embed into other function spaces. The primary role is played by Lebesgue spaces but there are other function spaces which are also of interest. Sometimes the class of Lebesgue spaces is not rich enough to enable one to describe all the important characteristics in a satisfactory way, and in such situations other function spaces come handy. Depending on the nature of the problem, the first call then usually goes either for Orlicz spaces or for Lorentz spaces. Orlicz spaces present a convenient replacement for Lebesgue spaces especially in cases when certain limiting growth of functions involved is needed, either more rapid or more slow than the power functions can offer. Lorentz spaces and their likes, on the other hand, turn out to be a very precise tool for fine-tuning and tightening of the results.

The central question behind the Sobolev spaces reads as follows. Given an information about the gradient of a scalar function of several real variables, what can we say about the function itself? For example, if the gradient belongs to certain Lebesgue space, will the function itself belong to the same space? Or to a different Lebesgue space? And, if there are more possibilities, which of these spaces gives the strongest result? The answer is usually formulated in form of some *Sobolev inequality* or *Sobolev embedding.*

For example, let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, having a Lipschitz boundary. The most classical form of the *Sobolev inequality* concerns Lebesgue spaces. It asserts that, given $1 \leq p < n$ and setting $p^* = \frac{np}{n-p}$, there exists $C > 0$ such that

$$\left(\int_\Omega |u(x)|^{p^*}\,dx\right)^{1/p^*} \leq C\left(\int_\Omega \left(|(\nabla u)(x)|^p + |u(x)|^p\right)dx\right)^{1/p}$$

for every weakly-differentiable function u for which the right hand side is finite. (Here and throughout, C denotes a constant independent of important quantities, not necessarily the same at each occurrence.)

It is useful to restate the Sobolev inequality in the form of certain specific relation between function spaces. Given two normed linear spaces $X(\Omega)$ and $Y(\Omega)$ which both are subsets of the set of all Lebesgue-measurable real-valued functions defined on Ω, we say that $X(\Omega)$ is (*continuously*) *embedded* into $Y(\Omega)$ if $X(\Omega) \subset Y(\Omega)$ in the set-theoretical sense and, moreover, the identity operator is continuous from $X(\Omega)$ into $Y(\Omega)$, that is, there exists a positive constant C such that for every $u \in X(\Omega)$ one has $\|u\|_{Y(\Omega)} \leq C\|u\|_{X(\Omega)}$. We shall write $X(\Omega) \to Y(\Omega)$ to denote that $X(\Omega)$ is embedded into $Y(\Omega)$.

Given $p \in [1,\infty)$, we define the *Sobolev space* $W^{1,p}(\Omega)$ as the collection of all weakly-differentiable real-valued functions u on Ω such that $u \in L^p(\Omega)$ and $|\nabla u| \in L^p(\Omega)$, where

∇u is the weak gradient of u. It can be shown that the set $W^{1,p}(\Omega)$, endowed with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)},$$

is a Banach space. We also introduce the Sobolev space of functions vanishing at the boundary of Ω, namely $W_0^{1,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ (the functions having derivatives of all orders and a compact support in Ω) in $W^{1,p}(\Omega)$.

The above Sobolev inequality thus translates to an embedding between function spaces, namely

$$W^{1,p}(\Omega) \to L^{p^*}(\Omega), \quad 1 \le p < n. \tag{1}$$

The relation (1) is called the *Sobolev embedding*. The space $L^p(\Omega)$ will be considered as the *domain* space, while the space L^{p^*} will be considered as the *range* (or *target*) space in the Sobolev embedding. We shall also say that the Sobolev space $W^{1,p}(\Omega)$ is *built upon* the Lebesgue space $L^p(\Omega)$.

The main object of our study will be the question of how *sharp* (or *optimal*) is the range space in the Sobolev embedding. By calling the target function space in an embedding "sharp" within some category we mean that this space cannot be replaced by an essentially smaller space from the same category without violating the embedding.

We study the sharpness (or optimality) in the context of the so-called *rearrangement-invariant spaces* (or *r.i. spaces* for short). The key ingredient in the investigation of optimality of a given function space is the *reduction principle*, a result that enables one to reduce a Sobolev embedding to a considerably simpler one-dimensional inequality.

A typical example of the reduction principle is the following result (see [Ci1] for the case of Orlicz spaces, and [EKP] for the general case). If Ω is a Lipschitz domain in $\mathbb{R}^n$ and $X(\Omega)$, $Y(\Omega)$ are r.i. spaces, then the first-order Sobolev embedding

$$V^1X(\Omega) \to Y(\Omega) \tag{2}$$

holds if and only if the inequality

$$\left\| \int_t^1 f(s)s^{-1+1/n}\, ds \right\|_{Y(0,1)} \le C\|f\|_{X(0,1)} \tag{3}$$

holds for some constant C and every nonnegative $f \in X(0,1)$. (Here the spaces $X(0,1)$, $Y(0,1)$ are the *representation spaces* of $X(\Omega)$, $Y(\Omega)$.)

A procedure which is by now standard (see [EKP, KP]) enables us to use the reduction theorem to carry out a construction of the optimal range partner for a given fixed domain space in the Sobolev embedding. In the case of the Lipschitz domain, this result reads as follows. Given an r.i. space $X(\Omega)$, define the space $Y'(\Omega)$ as the collection of all measurable functions u on Ω such that

$$\|u\|_{Y'(\Omega)} = \left\| t^{1/n-1} \int_0^t u^*(s)\, ds \right\|_{X'(0,1)} < \infty,$$

where $X'(0,1)$ is the *associate space* to $X(0,1)$. Then the space $Y(\Omega)$, whose associate space is $Y'(\Omega)$, is the desired optimal range partner for $X(\Omega)$. The space $Y(\Omega)$ can be evaluated explicitly in customary situations. For example, if $X(\Omega)$ is the Lebesgue space

$L^p(\Omega)$, then the above procedure yields $Y(\Omega) = L^{p^*,p}(\Omega)$, a *two-parameter Lorentz space*. This means that the Sobolev embedding

$$W^{1,p}(\Omega) \to L^{p^*,p}(\Omega) \tag{4}$$

holds, and the range space is *optimal* in the sense that whenever $Z(\Omega)$ is a rearrangement-invariant space such that

$$W^{1,p}(\Omega) \to Z(\Omega),$$

then, necessarily, $L^{p^*,p}(\Omega) \to Z(\Omega)$. Of course one has, in particular, $L^{p^*,p}(\Omega) \to L^{p^*}(\Omega)$, and this embedding is strict, so (4) indeed is a nontrivial improvement of (1).

It turns out that the Lipschitz property of the underlying domain Ω is not necessary for such results. Both the reduction principle and the optimal range construction work with no changes when Ω is only required that its isoperimetric function $I_\Omega(s)$ satisfies

$$I_\Omega(s) \approx s^{1/n'} \tag{5}$$

near 0, where $n' = \frac{n}{n-1}$. This condition is fulfilled in the class of the so-called *John domains*, which are more general than the Lipschitz ones. (Here, and in what follows, the notation $\approx$ means that the two sides are bounded by each other up to multiplicative constants.)

Some specific problems however entail a further generalization of the underlying domain. The study of elliptic PDEs with degenerating coefficients require underlying domains, whose isoperimetric function decays to 0 faster than in (5). The study of generalized hypercontractivity theory and integrability properties of the associated heat semigroups entails one to investigate Sobolev embeddings on $\mathbb{R}^n$, $n \geq 1$, endowed with some probability measures, of which the Gaussian measure and the exponential measure are typical examples.

The intimate connection of Sobolev spaces to isoperimetric inequalities was first observed by Maz'ya [Ma1, Ma2], who proved that quite general Sobolev inequalities are equivalent to either isoperimetric or isocapacitary inequalities. Before the two problems had been investigated separately, by Sobolev ([So1, So2]), Gagliardo ([Ga]) and Nirenberg ([Ni]) on the one hand, and by De Giorgi ([DeG]) on the other. Independently, Federer and Fleming [FF] also exploited De Giorgi's isoperimetric inequality to exhibit the best constant in the special case of the Sobolev inequality for functions whose gradient is integrable with power 1 in $\mathbb{R}^n$ via De Giorgi's isoperimetric inequality. These advances paved the way to an extensive research, along diverse directions, on the interplay between isoperimetric and Sobolev inequalities, and to a number of remarkable applications, e.g. by Moser [Mo], Talenti [Ta], Aubin [Au], or Brezis and Lieb [BrL]. The contributions to this field now constitute the corpus of a vast literature, which includes the papers [AFT, BCR1, BWW, BH, BoL, BK1, BK2, CK, Che, Ci1, Ci2, CP, EKP, Gr, HK, HS1, HS2, KM, Kl, Ko, LPT, LYZ, Mi, Zh] and the monographs [BZ, CDPT, Cha, He, Ma3, Sa].

The approach to Sobolev embeddings via isoperimetric inequalities has some considerable advantages. First it covers a fairly broad range of situations, including Lipschitz domains, John domains, Maz'ya domains, probability measure spaces, and more. Second, it typically leads to sharp results. The drawback is that the existing literature is often restricted to the first-order derivatives, except perhaps for a very few quite specific instances while many important problems require higher order of the derivatives involved. This is mainly caused by the techniques used such as truncation, symmetrization, Pólya–Szegö principles etc., which do not allow a direct generalization to the higher-order case. Alternative methods which can be employed to handle higher-order Sobolev inequalities such as representation formulas, potential estimates, Fourier transforms or atomic decomposition are not flexible enough to provide us with optimal results in sufficient generality.

Recently, in [CPS], we adopted a new approach which shows that in fact isoperimetric inequalities do imply optimal higher-order Sobolev embeddings in a very general framework. Moreover, we employ Sobolev-type spaces built upon any rearrangement-invariant Banach function spaces. The central ingredient of our approach is the combination of the first-order reduction principle with an iteration method, which is sharp enough to produce optimal higher-order reduction principles from the first-order ones. As a consequence, optimal results within the class of rearrangement-invariant spaces are obtained for Sobolev embeddings of any order. More precisely, we can characterize the best possible target for arbitrary-order Sobolev embeddings, in the class of all rearrangement-invariant Banach function spaces. A key step of the method is the development of a sharp iteration method involving subsequent applications of optimal Sobolev embeddings. We consider this method of independent interest for its possible use in different problems.

Let Ω be a domain in $\mathbb{R}^n$, $n \geq 1$, equipped with a finite measure ν which is absolutely continuous with respect to the Lebesgue measure, with density ω. Namely, $d\nu(x) = \omega(x)\,dx$, where ω is a Borel function such that $\omega(x) > 0$ a.e. in Ω. We assume, for simplicity, that $\nu(\Omega) = 1$. The basic case when ν is the Lebesgue measure will be referred to as Euclidean. Sobolev embeddings of arbitrary order for functions defined in Ω, with unconstrained values on $\partial\Omega$, will be considered.

The isoperimetric function $I_{\Omega,\nu}$ is precisely known only in some rather rare situations. This is so e.g. when Ω is an Euclidean ball [Ma3] or when it is the entire $\mathbb{R}^n$ equipped with the Gauss measure [Bor]. For our purpose, however, only the asymptotic behavior of $I_{\Omega,\nu}$ near 0 is needed, and that can be evaluated for various classes of domains, for example for Lipschitz domains, John domains or for the space $\mathbb{R}^n$ equipped with product probability measures.

Let $m \in \mathbb{N}$ and let $X(\Omega,\nu)$ be a rearrangement-invariant space. We define the m-th order Sobolev space $V^m X(\Omega,\nu)$ as

$$V^m X(\Omega,\nu) = \big\{u : u \text{ is } m\text{-times weakly differentiable in } \Omega, \text{ and } |\nabla^m u| \in X(\Omega,\nu)\big\}. \tag{6}$$

We also define the subspace $V^m_\perp X(\Omega,\nu)$ of $V^m X(\Omega,\nu)$ as

$$V^m_\perp X(\Omega,\nu) = \Big\{u \in V^m X(\Omega,\nu) : \int_\Omega \nabla^k u \, d\nu(x) = 0 \ \text{ for } k = 0,\dots,m-1\Big\}. \tag{7}$$

Here, $\nabla^m u$ denotes the vector of all m-th order weak derivatives of u. Let us notice that in the definition of $V^m X(\Omega,\nu)$ it is only required that the derivatives of the highest order m of u belong to $X(\Omega,\nu)$. This assumption does not entail, in general, that also u and its derivatives up to the order $m-1$ belong to $X(\Omega,\nu)$, and even to $L^1(\Omega,\nu)$. Thus, it may happen that $V^m X(\Omega,\nu) \not\subseteq V^k X(\Omega,\nu)$ for $m > k$.

As already mentioned, our approach to reduction principles for higher-order Sobolev embeddings relies on the iteration of first-order results. A natural worry here is of course the question of whether optimality is preserved under iteration. Examples show that when optimality is considered within standard families of function spaces, this is not necessarily so, even in the basic setting of Euclidean domains with Lipschitz boundaries. For instance, for a regular domain Ω in $\mathbb{R}^2$, one has

$$V^2L^1(\Omega) \to L^\infty(\Omega). \tag{8}$$

On the other hand, iteration of two consequent first-order Sobolev embeddings, optimal within Lebesgue spaces, only gives

$$V^2L^1(\Omega) \to V^1L^2(\Omega) \to L^q(\Omega) \tag{9}$$

for every $q < \infty$. The space $V^1L^2(\Omega)$ contains unbounded functions, hence there must be a loss of information in the iteration process.

One might relate the loss of optimality in the chain of embeddings (9) to the lack of an optimal Lebesgue target space for the first-order Sobolev embedding of $V^1L^2(\Omega)$ when $n = 2$. However, similar loss can happen in situations where the optimal first-order target spaces do exist. Consider, for example, Euclidean Sobolev embeddings involving Orlicz spaces. In this setting the optimal target space always exists, and it can be explicitly determined [Ci1, Ci4]. Indeed, if Ω is a regular domain in $\mathbb{R}^n$ and $1 \le m < n$, then

$$V^mL^{n/m}(\Omega) \to \exp L^{n/(n-m)}(\Omega) \tag{10}$$

[Yu, Po, St]; see also [Tr] for $m = 1$. Here, $\exp L^\alpha(\Omega)$, with $\alpha > 0$, denotes the Orlicz space associated with the Young function given by $e^{t^\alpha} - 1$ for $t \ge 0$. The target space in (10) is known to be optimal in the class of all Orlicz spaces [Ci1, Ci4]. Assume, for example, that $n \ge 3$ and $m = 2$. Then (10) reduces to

$$V^2L^{n/2}(\Omega) \to \exp L^{n/(n-2)}(\Omega). \tag{11}$$

However, iterating first-order embeddings, optimal in Orlicz spaces, one gets only

$$V^2L^{n/2}(\Omega) \to V^1L^n(\Omega) \to \exp L^{n/(n-1)}(\Omega) \supsetneq \exp L^{n/(n-2)}(\Omega). \tag{12}$$

Thus, subsequent applications of first-order optimal Sobolev embeddings even in the class of Orlicz spaces, where optimal target space always exists, need not result in optimal higher-order Sobolev embeddings.

The underlying idea behind the method that we shall develop is that such a loss of optimality of the target space under iteration does not occur, provided that first-order (in fact, any-order) Sobolev embeddings whose targets are *optimal among all rearrangement-invariant spaces are iterated.* We thus proceed via a two-step argument, which can be outlined as follows. Firstly, given any r.i. space $X(\Omega,\nu)$ and the isoperimetric function $I_{\Omega,\nu}$ of (Ω,ν), the optimal target among all rearrangement-invariant function norms for the first-order Sobolev space $V^1X(\Omega,\nu)$ is characterized; secondly, first-order Sobolev embeddings with an optimal target are iterated to derive optimal targets in arbitrary-order Sobolev embeddings.

Returning back to our examples, we may recall that the optimal r.i. range partner for $V^1L^1(\Omega)$ is not the Lebesgue space $L^2(\Omega)$ but the (essentially smaller) Lorentz space $L^{2,1}(\Omega)$, and then we get

$$V^2L^1(\Omega) \to V^1L^{2,1}(\Omega) \to L^\infty(\Omega), \tag{13}$$

with no loss of information or optimality. Similarly, iterating the (optimal) embeddings

$$V^1L^{n/2}(\Omega) \to V^1L^{n,n/2}(\Omega) \quad \text{and} \quad V^1L^{n,n/2}(\Omega) \to L^{\infty,n/2;-1}(\Omega),$$

where $L^{\infty,n/2;-1}(\Omega)$ is the so-called *Lorentz–Zygmund space*, we get

$$V^2L^1(\Omega) \to L^{\infty,n/2;-1}(\Omega),$$

which, thanks to the (strict) embedding $L^{\infty,n/2;-1}(\Omega) \to \exp L^{n/(n-2)}(\Omega)$ is even better result than (11), we see that, again, no information has been lost.

2. Preliminaries. We denote by $\mathcal{M}(\Omega,\nu)$ the class of real-valued ν-measurable functions on Ω and by $\mathcal{M}_+(\Omega,\nu)$ the class of nonnegative functions in $\mathcal{M}(\Omega,\nu)$. Given $f \in \mathcal{M}(\Omega,\nu)$, its *non-increasing rearrangement* is defined by

$$f^*(t) = \inf\bigl\{\lambda > 0 : \nu\bigl(\{x \in \Omega : |f(x)| > \lambda\}\bigr) \le t\bigr\}, \quad t \in [0,\infty).$$

We define $\mathcal{M}_0(\Omega,\nu) = \{u \in \mathcal{M}(\Omega,\nu) : u \text{ is finite a.e. in } \Omega\}$. It will be handy to introduce the *maximal non-increasing rearrangement* of f by

$$f^{**}(t) = t^{-1}\int_0^t f^*(s)\,ds, \quad t \in (0,\infty).$$

We say that a functional $\|\cdot\|_{X(0,1)} : \mathcal{M}_+(0,1) \to [0,\infty]$ is a *function norm*, if, for all f, g and $\{f_j\}_{j\in\mathbb{N}}$ in $\mathcal{M}_+(0,1)$, and every $\lambda \ge 0$, the following properties hold:

(P1) $\|f\|_{X(0,1)} = 0$ if and only if $f = 0$; $\|\lambda f\|_{X(0,1)} = \lambda\|f\|_{X(0,1)}$; $\|f+g\|_{X(0,1)} \le \|f\|_{X(0,1)} + \|g\|_{X(0,1)}$;
(P2) $f \le g$ a.e. implies $\|f\|_{X(0,1)} \le \|g\|_{X(0,1)}$;
(P3) $f_j \nearrow f$ a.e. implies $\|f_j\|_{X(0,1)} \nearrow \|f\|_{X(0,1)}$;
(P4) $\|1\|_{X(0,1)} < \infty$;
(P5) $\int_0^1 f(x)\,dx \le C\|f\|_{X(0,1)}$ for some constant C independent of f.

If, in addition,

(P6) $\|f\|_{X(0,1)} = \|g\|_{X(0,1)}$ whenever $f^* = g^*$,

we say that $\|\cdot\|_{X(0,1)}$ is a *rearrangement-invariant function norm.*

With any rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$, there is associated another functional on $\mathcal{M}_+(0,1)$, denoted by $\|\cdot\|_{X'(0,1)}$, and defined, for $g \in \mathcal{M}_+(0,1)$, as

$$\|g\|_{X'(0,1)} = \sup_{\substack{f\ge 0\\ \|f\|_{X(0,1)}\le 1}} \int_0^1 f(s)g(s)\,ds.$$

It turns out that $\|\cdot\|_{X'(0,1)}$ is also a rearrangement-invariant function norm, which is called the *associate function norm* of $\|\cdot\|_{X(0,1)}$.

Given a rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$, the space $X(\Omega,\nu)$ is defined as the collection of all functions $u \in \mathcal{M}(\Omega,\nu)$ such that the expression

$$\|u\|_{X(\Omega,\nu)} = \|u^*\|_{X(0,1)} \tag{14}$$

is finite. Such an expression defines a norm on $X(\Omega,\nu)$, and the latter is a Banach space endowed with this norm, called a rearrangement-invariant space. Moreover, $X(\Omega,\nu) \subset \mathcal{M}_0(\Omega,\nu)$ for any rearrangement-invariant space $X(\Omega,\nu)$. The space $X(0,1)$ is called the *representation space* of $X(\Omega,\nu)$.

Let $0 < p,q \leq \infty$ and $\alpha \in \mathbb{R}$. Then the *Lorentz–Zygmund space* $L^{p,q;\alpha}(\Omega,\nu)$ is the collection of all $f \in \mathcal{M}(\Omega,\nu)$ such that $\|f\|_{L^{p,q;\alpha}(\Omega,\nu)} < \infty$, where

$$\|f\|_{L^{p,q;\alpha}(\Omega,\nu)} := \|t^{1/p-1/q}\log^{\alpha}\big(\tfrac{e}{t}\big)f^*(t)\|_{L^q(0,1)}.$$

Occasionally we will have to work with a modification of Lorentz–Zygmund spaces in which f^* is replaced by f^{**}. We denote such a space by $L^{(p,q;\alpha)}(\Omega)$, hence

$$\|f\|_{L^{(p,q;\alpha)}(\Omega)} := \|t^{1/p-1/q}\log^{\alpha}\big(\tfrac{e}{t}\big)f^{**}(t)\|_{L^q(0,1)}.$$

If one of the following conditions

$$\begin{cases} 1 < p < \infty,\ 1 \leq q \leq \infty,\ \alpha \in \mathbb{R}; \\ p = 1,\ q = 1,\ \alpha \geq 0; \\ p = \infty,\ q = \infty,\ \alpha \leq 0; \\ p = \infty,\ 1 \leq q < \infty,\ \alpha + 1/q < 0, \end{cases} \tag{15}$$

is satisfied, then $L^{p,q;\alpha}(\Omega)$ is equivalent to a rearrangement-invariant Banach function space.

Given any *Young function* $A : [0,\infty) \to [0,\infty)$, namely a convex increasing function vanishing at 0, the *Orlicz space* $L^A(\Omega,\nu)$ is defined as the collection of all ν-measurable functions u on Ω such that $\|u\|_{L^A(\Omega,\nu)} < \infty$, where

$$\|u\|_{L^A(\Omega,\nu)} = \inf\Big\{\lambda > 0 : \int_\Omega A\Big(\frac{|u(x)|}{\lambda}\Big)\,d\nu(x) \leq 1\Big\}.$$

The assumptions on A guarantee that the functional $\|u\|_{L^A(\Omega,\nu)}$ is a norm with respect to which the Orlicz space $L^A(\Omega,\nu)$ is a Banach space. The norm $\|u\|_{L^A(\Omega,\nu)}$ is called the *Luxemburg norm.*

3. Main results and examples. The isoperimetric inequality relative to (Ω,ν) tells us that

$$P_\nu(E,\Omega) \geq I_{\Omega,\nu}(\nu(E)), \tag{16}$$

where E is any measurable set $E \subset \Omega$, and $P_\nu(E,\Omega)$ stands for its perimeter in Ω with respect to ν. Moreover, $I_{\Omega,\nu}$ denotes the largest non-decreasing function in $[0,\frac{1}{2}]$ for which (16) holds, called the isoperimetric function (or isoperimetric profile) of (Ω,ν), which was introduced in [Ma1].

The results concerning optimality of function spaces in Sobolev embeddings depend only on a lower bound for the isoperimetric function $I_{\Omega,\nu}$ of (Ω,ν) in terms of some other non-decreasing function $I : [0,1] \to [0,\infty)$; precisely, on the existence of a positive constant c such that

$$I_{\Omega,\nu}(s) \geq cI(cs) \quad \text{for } s \in [0, \tfrac{1}{2}]. \tag{17}$$

First, it can be observed that if $I_{\Omega,\nu}(s)$ does not decay at 0 faster than linearly, namely if there exists a positive constant C such that

$$I_{\Omega,\nu}(s) \geq Cs \quad \text{for } s \in [0, \tfrac{1}{2}], \tag{18}$$

then any function $u \in V^m X(\Omega,\nu)$ does at least belong to $L^1(\Omega,\nu)$, together with all its derivatives up to the order $m-1$. In the light of (18), we can safely assume that

$$\inf_{t\in(0,1)} \frac{I(t)}{t} > 0. \tag{19}$$

The next theorem is our main general reduction principle.

Theorem 3.1. *Assume that (Ω,ν) fulfils (17) for some non-decreasing function I satisfying (19). Let $m \in \mathbb{N}$, and let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. If there exists a constant C_1 such that*

$$\left\| \int_t^1 \frac{f(s)}{I(s)} \Big(\int_t^s \frac{dr}{I(r)} \Big)^{m-1} ds \right\|_{Y(0,1)} \leq C_1 \|f\|_{X(0,1)} \tag{20}$$

for every nonnegative $f \in X(0,1)$, then

$$V^m X(\Omega,\nu) \to Y(\Omega,\nu), \tag{21}$$

and there exists a constant C_2 such that

$$\|u\|_{Y(\Omega,\nu)} \leq C_2 \|\nabla^m u\|_{X(\Omega,\nu)} \tag{22}$$

for every $u \in V_\perp^m X(\Omega,\nu)$.

It turns out that inequality (20) holds for every nonnegative $f \in X(0,1)$ if and only if it just holds for every nonnegative and non-increasing $f \in X(0,1)$.

A major feature of Theorem 3.1 is the difference occurring in (20) between the first-order case ($m=1$) and the higher-order case ($m>1$). Indeed, the integral operator appearing in (20) when $m=1$ is just a weighted Hardy-type operator, namely a primitive of f times a weight, whereas, in the higher-order case, a genuine kernel, with a more complicated structure, comes into play. In fact, this seems to be the first known instance where such a kernel operator is needed in a reduction principle for Sobolev-type embeddings.

As we shall see, the Sobolev embedding (21) (or the Poincaré inequality (22)) and inequality (20) are actually equivalent in customary families of measure spaces (Ω,ν). This fact allows us to determine the optimal rearrangement-invariant target spaces in Sobolev embeddings for these measure spaces. Incidentally, let us mention that when $m=1$, this is the case whenever the geometry of (Ω,ν) allows the construction of a family of trial functions u in (21) or (22) characterized by the following properties: the level sets of u are isoperimetric (or almost isoperimetric) in (Ω,ν); $|\nabla u|$ is constant (or almost constant) on the boundary of the level sets of u. If $m>1$, then the latter requirement has to be complemented by requiring that the derivatives of u up to the order m restricted to the boundary of the level sets satisfy certain conditions depending on I.

Such conditions have, however, a technical nature, and it is not worth to state them explicitly. In fact, heuristically speaking, properties (20), (22) and (21) turn out to be equivalent for every $m \geq 1$ on the same measure spaces (Ω, ν) as for $m = 1$. Such an equivalence certainly holds in any customary, non-pathological situation, including the three frameworks to which our results will be applied, namely John domains, Euclidean domains from Maz'ya classes and product probability spaces in $\mathbb{R}^n$ extending the Gauss space. In all these cases, we can characterize optimal arbitrary-order rearrangement-invariant target spaces.

Now we are in a position to characterize the space which is the optimal rearrangement-invariant target space in the Sobolev embedding (21). Such an optimal space is the one associated with the rearrangement-invariant function norm $\|\cdot\|_{X_{m,I}(0,1)}$, whose associate norm is defined as

$$\|f\|_{X'_{m,I}(0,1)} = \left\| \frac{1}{I(s)} \int_0^s \left(\int_t^s \frac{dr}{I(r)} \right)^{m-1} f^*(t)\, dt \right\|_{X'(0,1)} \tag{23}$$

for $f \in \mathcal{M}_+(0,1)$.

Theorem 3.2. *Assume that (Ω, ν), m, I and $\|\cdot\|_{X(0,1)}$ are as in Theorem 3.1. Then the functional $\|\cdot\|_{X'_{m,I}(0,1)}$, given by (23), is a rearrangement-invariant function norm, whose associate norm $\|\cdot\|_{X_{m,I}(0,1)}$ satisfies*

$$V^m X(\Omega, \nu) \to X_{m,I}(\Omega, \nu), \tag{24}$$

and there exists a constant C such that

$$\|u\|_{X_{m,I}(\Omega,\nu)} \leq C \|\nabla^m u\|_{X(\Omega,\nu)} \tag{25}$$

for every $u \in V_\perp^m X(\Omega, \nu)$.

Moreover, if (Ω, ν) is such that (21), or equivalently (22), implies (20), and hence (20), (21) and (22) are equivalent, then the function norm $\|\cdot\|_{X_{m,I}(0,1)}$ is optimal in (24) and (25) among all rearrangement-invariant norms.

An important special case of Theorems 3.1 and 3.2 is enucleated in the following corollary.

Corollary 3.3. *Assume that (Ω, ν), m, I and $\|\cdot\|_{X(0,1)}$ are as in Theorem 3.1. If*

$$\left\| \frac{1}{I(s)} \left(\int_0^s \frac{dr}{I(r)} \right)^{m-1} \right\|_{X'(0,1)} < \infty, \tag{26}$$

then

$$V^m X(\Omega, \nu) \to L^\infty(\Omega, \nu), \tag{27}$$

and there exists a constant C such that

$$\|u\|_{L^\infty(\Omega,\nu)} \leq C \|\nabla^m u\|_{X(\Omega,\nu)} \tag{28}$$

for every $u \in V_\perp^m X(\Omega, \nu)$.

Moreover, if (Ω, ν) is such that (21), or equivalently (22), implies (20), and hence (20), (21) and (22) are equivalent, then (26) is necessary for (27) or (28) to hold.

If (Ω,ν) is such that (21), or equivalently (22), implies (20), and hence (20), (21) and (22) are equivalent, then (27) cannot hold, whatever $\|\cdot\|_{X(0,1)}$ is, if I decays so fast at 0 that

$$\int_0 \frac{dr}{I(r)} = \infty.$$

We shall now point out the preservation of optimality in targets among all rearrangement-invariant spaces under iteration of Sobolev embeddings of arbitrary order.

Theorem 3.4. *Assume that (Ω,ν), I and $\|\cdot\|_{X(0,1)}$ are as in Theorem* 3.1. *Let $k,h\in\mathbb{N}$. Then*

$$(X_{k,I})_{h,I}(\Omega,\nu) = X_{k+h,I}(\Omega,\nu), \tag{29}$$

up to equivalent norms.

In many instances in practice, the function I satisfies the estimate

$$\int_0^s \frac{dr}{I(r)} \approx \frac{s}{I(s)} \quad \text{for } s\in(0,1). \tag{30}$$

We note that (30) is not true for every relevant case. It holds for instance for John domains and for domains from Maz'ya classes $\mathcal{J}_\alpha$ with $\alpha<1$, but it does not hold for domains in $\mathcal{J}_1$ or for the Gaussian space.

It is useful to treat the cases for which (30) holds separately because then the results of Theorems 3.1, 3.2 and 3.4 can be considerably simplified. For example, the reduction theorem then reads as follows.

Corollary 3.5. *Let (Ω,ν), m, I, $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be as in Theorem* 3.1. *Assume, in addition, that I fulfils* (30). *If there exists a constant C_1 such that*

$$\left\|\int_t^1 f(s)\,\frac{s^{m-1}}{I(s)^m}\,ds\right\|_{Y(0,1)} \le C_1\|f\|_{X(0,1)} \tag{31}$$

for every nonnegative $f\in X(0,1)$, then

$$V^m X(\Omega,\nu) \to Y(\Omega,\nu), \tag{32}$$

and there exists a constant C_2 such that

$$\|u\|_{Y(\Omega,\nu)} \le C_2\|\nabla^m u\|_{X(\Omega,\nu)} \tag{33}$$

for every $u\in V_\perp^m X(\Omega,\nu)$.

Let us now deal with specific situations separately.

We say that a bounded open set Ω in $\mathbb{R}^n$ is called a *John domain* if there exist a constant $c\in(0,1)$ and a point $x_0\in\Omega$ such that for every $x\in\Omega$ there exists a rectifiable curve $\varpi:[0,l]\to\Omega$, parameterized by the arc length and such that $\varpi(0)=x$, $\varpi(l)=x_0$, and

$$\operatorname{dist}(\varpi(r),\partial\Omega)\ge cr \qquad \text{for } r\in[0,l].$$

The class of John domains includes other more classical families of domains, such as Lipschitz domains, and domains with the cone property. The John domains arise in connection with the study of holomorphic dynamical systems and quasiconformal mappings.

John domains are known to support a first-order Sobolev inequality with the same exponents as in the standard Sobolev inequality [Bo, HK, KM]. In fact, being a John domain is a necessary condition for such a Sobolev inequality to hold in the class of two-dimensional simply connected open sets, and in quite general classes of higher dimensional domains [BK1]. We note that, as a consequence of (5), the corresponding function I now satisfies (30). The reduction principle thus takes the following form.

Theorem 3.6. *Let $n \in \mathbb{N}$, $n \geq 2$, and let $m \in \mathbb{N}$. Assume that Ω is a John domain in $\mathbb{R}^n$. Let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Then the following assertions are equivalent.*

(i) *The Hardy type inequality*

$$\Big\|\int_t^1 f(s)s^{-1+m/n}\,ds\Big\|_{Y(0,1)} \leq C_1\|f\|_{X(0,1)} \tag{34}$$

holds for some constant C_1, and for every nonnegative $f \in X(0,1)$.

(ii) *The Sobolev embedding*

$$V^m X(\Omega) \to Y(\Omega) \tag{35}$$

holds.

(iii) *The Poincaré inequality*

$$\|u\|_{Y(\Omega)} \leq C_2\|\nabla^m u\|_{X(\Omega)} \tag{36}$$

holds for some constant C_2 and every $u \in V^m_\perp X(\Omega)$.

Given an r.i. space $X(\Omega)$ where Ω is a John domain, the corresponding optimal range partner in the Sobolev embedding on a John domain is then the space $X_{m,\mathrm{John}}(\Omega)$, whose associate space has norm

$$\|f\|_{X'_{m,\mathrm{John}}(\Omega)} = \Big\|s^{-1+m/n}\int_0^s f^*(r)\,dr\Big\|_{X'(0,1)}, \quad f \in \mathcal{M}_+(\Omega). \tag{37}$$

Our next set of instances will be *Maz'ya classes* of Euclidean domains. Given $\alpha \in [\frac{1}{n'},1]$, we denote by $\mathcal{J}_\alpha$ the *Maz'ya class* of all Euclidean domains Ω satisfying (17), with $I(s) = s^\alpha$ for $s \in [0,\frac{1}{2}]$, namely domains Ω in $\mathbb{R}^n$ such that

$$I_\Omega(s) \geq Cs^\alpha \quad \text{for } s \in [0,\tfrac{1}{2}], \tag{38}$$

for some positive constant C. Thanks to (5), any John domain belongs to the class $\mathcal{J}_{1/n'}$.

The reduction theorem in the class $\mathcal{J}_\alpha$ takes the following form.

Theorem 3.7. *Let $n \in \mathbb{N}$, $n \geq 2$, $m \in \mathbb{N}$, and $\alpha \in [\frac{1}{n'},1]$. Let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Assume that either $\alpha \in [\frac{1}{n'},1)$ and there exists a constant C_1 such that*

$$\Big\|\int_t^1 f(s)s^{-1+m(1-\alpha)}\,ds\Big\|_{Y(0,1)} \leq C_1\|f\|_{X(0,1)} \tag{39}$$

for every nonnegative $f \in X(0,1)$, or $\alpha = 1$ and there exists a constant C_1 such that

$$\Big\|\int_t^1 f(s)\frac{1}{s}\Big(\log\frac{s}{t}\Big)^{m-1}\,ds\Big\|_{Y(0,1)} \leq C_1\|f\|_{X(0,1)} \tag{40}$$

for every nonnegative $f \in X(0,1)$. Then the Sobolev embedding

$$V^m X(\Omega) \to Y(\Omega) \tag{41}$$

holds for every $\Omega \in \mathcal{J}_\alpha$ and, equivalently, the Poincaré inequality

$$\|u\|_{Y(\Omega)} \le C_2 \|\nabla^m u\|_{X(\Omega)} \tag{42}$$

holds for every $\Omega \in \mathcal{J}_\alpha$, for some constant C_2 and every $u \in V_\perp^m X(\Omega)$.

Conversely, if the Sobolev embedding (41)*, or, equivalently, the Poincaré inequality* (42)*, holds for every $\Omega \in \mathcal{J}_\alpha$, then either inequality* (39)*, or* (40) *holds, according to whether $\alpha \in [\frac{1}{n'}, 1)$ or $\alpha = 1$.*

Given an r.i. space $X(\Omega)$ where $\Omega \in \mathcal{J}_\alpha$, the corresponding optimal range partner in the Sobolev embedding on a Maz'ya domain is then the space $X_{m,\alpha}(\Omega)$, whose associate space has norm

$$\|f\|_{X'_{m,\alpha}(\Omega)} = \begin{cases} \left\| s^{-1+m(1-\alpha)} \int_0^s f^*(r)\,dr \right\|_{X'(0,1)} & \text{if } \alpha \in [\frac{1}{n'}, 1), \\ \left\| \frac{1}{s} \int_0^s \left(\log \frac{s}{r}\right)^{m-1} f^*(r)\,dr \right\|_{X'(0,1)} & \text{if } \alpha = 1, \quad f \in \mathcal{M}_+(\Omega). \end{cases} \tag{43}$$

We shall now apply the general results to some concrete function spaces. We shall mainly focus on Lebesgue, Lorentz and Orlicz spaces.

Theorem 3.8. *Let $n \in \mathbb{N}$, $n \ge 2$, and let $\Omega \in \mathcal{J}_\alpha$ for some $\alpha \in [\frac{1}{n'}, 1)$. Let $m \in \mathbb{N}$ and $p \in [1, \infty]$. Then*

$$V^m L^p(\Omega) \to \begin{cases} L^{p/(1-mp(1-\alpha))}(\Omega) & \textit{if } m(1-\alpha) < 1 \textit{ and } 1 \le p < \frac{1}{m(1-\alpha)}, \\ L^r(\Omega) \textit{ for any } r \in [1,\infty) & \textit{if } m(1-\alpha) < 1 \textit{ and } p = \frac{1}{m(1-\alpha)}, \\ L^\infty(\Omega) & \textit{otherwise.} \end{cases} \tag{44}$$

Moreover, in the first and the third cases, the target spaces in (44) *are optimal among all Lebesgue spaces, as Ω ranges in $\mathcal{J}_\alpha$.*

Although the target spaces in (44) cannot be improved in the class of Lebesgue spaces, the conclusions of (44) can be strengthened if more general rearrangement-invariant spaces are employed. Such a strengthening can be obtained as a special case of a Sobolev embedding for Lorentz spaces which reads as follows.

Theorem 3.9. *Let $n \in \mathbb{N}$, $n \ge 2$, and let $\Omega \in \mathcal{J}_\alpha$ for some $\alpha \in [\frac{1}{n'}, 1)$. Let $m \in \mathbb{N}$ and $p, q \in [1, \infty]$. Assume that one of the conditions in* (15) *with $\alpha = 0$ holds. Then*

$$V^m L^{p,q}(\Omega) \to \begin{cases} L^{p/(1-mp(1-\alpha)),q}(\Omega) & \textit{if } m(1-\alpha) < 1 \textit{ and } 1 \le p < \frac{1}{m(1-\alpha)}, \\ L^{\infty,q;-1}(\Omega) & \textit{if } m(1-\alpha) < 1,\ p = \frac{1}{m(1-\alpha)} \textit{ and } q > 1, \\ L^\infty(\Omega) & \textit{otherwise,} \end{cases} \tag{45}$$

Moreover, the target spaces in (45) *are optimal among all rearrangement-invariant spaces, as Ω ranges in $\mathcal{J}_\alpha$.*

The particular choice of parameters $p = q$, $1 \le p < \frac{1}{m(1-\alpha)}$ in Theorem 3.9 shows that

$$V^m L^p(\Omega) \to L^{p/(1-mp(1-\alpha)),p}(\Omega).$$

This is a non-trivial strengthening of the first embedding in (44), since

$$L^{p/(1-mp(1-\alpha)),p}(\Omega) \subsetneqq L^{p/(1-mp(1-\alpha))}.$$

Likewise, the choice $m(1-\alpha) < 1$ and $p = q = \frac{1}{m(1-\alpha)}$ shows that also the second embedding in (44) can be in fact essentially improved to

$$V^m L^p(\Omega) \to L^{\infty,p;-1}(\Omega).$$

Assume now that $\alpha = 1$. The embedding theorem in Lebesgue spaces takes the following form.

THEOREM 3.10. *Let $n \in \mathbb{N}$, $n \ge 2$, and let $\Omega \in \mathcal{J}_1$. Let $m \in \mathbb{N}$ and $p \in [1,\infty]$. Then*

$$V^m L^p(\Omega) \to \begin{cases} L^p(\Omega) & \text{if } 1 \le p < \infty, \\ L^r(\Omega) & \text{for any } r \in [1,\infty), \text{ if } p = \infty. \end{cases} \tag{46}$$

In the former case, the target space is optimal in (46) *among all Lebesgue spaces, as Ω ranges in $\mathcal{J}_1$.*

Optimal embeddings for Lorentz–Sobolev spaces are provided in the next theorem.

THEOREM 3.11. *Let $n \in \mathbb{N}$, $n \ge 2$, and let $\Omega \in \mathcal{J}_1$. Let $m \in \mathbb{N}$ and $p, q \in [1,\infty]$. Assume that one of the conditions in* (15) *with $\alpha = 0$ holds. Then*

$$V^m L^{p,q}(\Omega) \to \begin{cases} L^{p,q}(\Omega) & \text{if } 1 \le p < \infty, \\ \exp L^{1/m}(\Omega) & \text{if } p = q = \infty. \end{cases} \tag{47}$$

The target spaces are optimal in (47) *among all rearrangement-invariant spaces, as Ω ranges in $\mathcal{J}_1$.*

Our next application concerns Orlicz–Sobolev spaces. Let $n \in \mathbb{N}$, $n \ge 2$, $m \in \mathbb{N}$, $\alpha \in [\frac{1}{n'}, 1)$, and let A be a Young function. We may assume, without loss of generality, $m(1-\alpha) < 1$ and

$$\int_0 \Big(\frac{t}{A(t)}\Big)^{(m(1-\alpha))/(1-m(1-\alpha))} dt < \infty. \tag{48}$$

Indeed, the function A can be modified near 0, if necessary, in such a way that (48) is fulfilled, on leaving the space $V^m L^A(\Omega)$ unchanged (up to equivalent norms).

If $m < \frac{1}{1-\alpha}$ and

$$\int^\infty \Big(\frac{t}{A(t)}\Big)^{(m(1-\alpha))/(1-m(1-\alpha))} dt = \infty, \tag{49}$$

we define the function $H_{m,\alpha} : [0,\infty) \to [0,\infty)$ as

$$H_{m,\alpha}(s) = \bigg(\int_0^s \Big(\frac{t}{A(t)}\Big)^{(m(1-\alpha))/(1-m(1-\alpha))} dt\bigg)^{1-m(1-\alpha)} \qquad \text{for } s \ge 0, \tag{50}$$

and the Young function $A_{m,\alpha}$ as

$$A_{m,\alpha}(t) = A(H_{m,\alpha}^{-1}(t)) \qquad \text{for } t \ge 0. \tag{51}$$

THEOREM 3.12. *Assume that $n \in \mathbb{N}$, $n \geq 2$, $m \in \mathbb{N}$, $\alpha \in [\frac{1}{n'}, 1)$ and $\Omega \in \mathcal{J}_\alpha$. Let A be a Young function fulfilling* (48). *Then*

$$V^m L^A(\Omega) \to \begin{cases} L^{A_{m,\alpha}}(\Omega) & \text{if } m < \frac{1}{1-\alpha} \text{ and (49) holds,} \\ L^\infty(\Omega) & \text{if either } m \geq \frac{1}{1-\alpha} \text{ or } m < \frac{1}{1-\alpha} \text{ and (49) fails.} \end{cases} \tag{52}$$

Moreover, the target spaces in (52) *are optimal among all Orlicz spaces, as Ω ranges in $\mathcal{J}_\alpha$.*

EXAMPLE 3.13. Consider the case when

$A(t) \approx t^p (\log t)^\beta$ near infinity, where either $p > 1$ and $\beta \in \mathbb{R}$, or $p = 1$ and $\beta \geq 0$.

Hence, $L^A(\Omega) = L^p \log^\beta(\Omega)$. An application of Theorem 3.12 tells us that

$$V^m L^p \log^\beta(\Omega) \to \begin{cases} L^{p/(1-pm(1-\alpha))} \log^{\beta/(1-pm(1-\alpha))}(\Omega) & \text{if } mp(1-\alpha) < 1, \\ \exp L^{1/(1-(1+\beta)m(1-\alpha))}(\Omega) & \text{if } mp(1-\alpha) = 1 \text{ and } \beta < \frac{1-m(1-\alpha)}{m(1-\alpha)}, \\ \exp\exp L^{1/(1-m(1-\alpha))}(\Omega) & \text{if } mp(1-\alpha) = 1 \text{ and } \beta = \frac{1-m(1-\alpha)}{m(1-\alpha)}, \\ L^\infty(\Omega) & \text{if either } mp(1-\alpha) > 1, \\ & \text{or } mp(1-\alpha) = 1 \text{ and } \beta > \frac{1-m(1-\alpha)}{m(1-\alpha)}. \end{cases} \tag{53}$$

Moreover, the target spaces in (53) are optimal among all Orlicz spaces, as Ω ranges in $\mathcal{J}_\alpha$.

The first three embeddings in (53) can be improved on allowing more general rearrangement-invariant target spaces. Indeed, we have

$$V^m L^p \log^\beta(\Omega) \to \begin{cases} L^{p/(1-pm(1-\alpha)),p;\beta/p}(\Omega) & \text{if } mp(1-\alpha) < 1, \\ L^{\infty,1/(m(1-\alpha));m(1-\alpha)\beta-1}(\Omega) & \text{if } mp(1-\alpha) = 1 \text{ and } \beta < \frac{1-m(1-\alpha)}{m(1-\alpha)}, \\ L^{\infty,1/(m(1-\alpha));-m(1-\alpha),-1}(\Omega) & \text{if } mp(1-\alpha) = 1 \text{ and } \beta = \frac{1-m(1-\alpha)}{m(1-\alpha)}, \end{cases} \tag{54}$$

the targets being optimal among all rearrangement-invariant spaces in (54) as Ω ranges among all domains in $\mathcal{J}_\alpha$.

Our final set of examples will concern the *product probability spaces*.

The class of product probability measures in $\mathbb{R}^n$, $n \geq 1$, arises in connection with the study of generalized hypercontractivity theory and integrability properties of the associated heat semigroups. The isoperimetric problem in the corresponding probability spaces was studied in [BCR], see also [BCR1, BH, Le1, Le2].

Assume that $\Phi : [0, \infty) \to [0, \infty)$ is a strictly increasing, twice continuously differentiable convex function in $(0, \infty)$ such that $\sqrt{\Phi}$ is concave, and $\Phi(0) = 0$. Let μ_Φ be the probability measure on $\mathbb{R}$ given by

$$d\mu_\Phi(x) = c_\Phi e^{-\Phi(|x|)}\, dx, \tag{55}$$

where c_Φ is a constant chosen in such a way that $\mu_\Phi(\mathbb{R}) = 1$. The product measure $\mu_{\Phi,n}$

on $\mathbb{R}^n$, $n \geq 1$, generated by μ_Φ, is then defined as

$$\mu_{\Phi,n} = \underbrace{\mu_\Phi \times \ldots \times \mu_\Phi}_{n-\text{times}}. \tag{56}$$

Clearly, $\mu_{\Phi,1} = \mu_\Phi$, and $(\mathbb{R}^n, \mu_{\Phi,n})$ is a probability space for every $n \in \mathbb{N}$.

The main example of a measure μ_Φ is obtained by taking

$$\Phi(t) = \tfrac{1}{2}t^2.$$

This choice yields $\mu_{\Phi,n} = \gamma_n$, the Gauss measure which obeys

$$d\gamma_n(x) = (2\pi)^{-n/2} e^{-|x|^2/2}\, dx. \tag{57}$$

More generally, the *Boltzmann measures* $\gamma_{n,\beta}$ are associated with

$$\Phi(t) = \tfrac{1}{\beta}t^\beta$$

for some $\beta \in [1,2]$. Let $H : \mathbb{R} \to (0,1)$ be defined as

$$H(t) = \int_t^\infty c_\Phi e^{-\Phi(|r|)}\, dr \quad \text{for } t \in \mathbb{R}, \tag{58}$$

and let $F_\Phi : [0,1] \to [0,\infty)$ be given by

$$F_\Phi(s) = c_\Phi e^{-\Phi(|H^{-1}(s)|)} \quad \text{for } s \in (0,1), \quad \text{and} \quad F_\Phi(0) = F_\Phi(1) = 0. \tag{59}$$

Since μ_Φ is a probability measure and $\mu_{\Phi,n}$ is defined by (56), it is easily seen that, for each $i = 1, \ldots, n$,

$$\mu_{\Phi,n}\big(\{(x_1, \ldots, x_n) : x_i > t\}\big) = H(t) \quad \text{for } t \in \mathbb{R}, \tag{60}$$

and

$$P_{\mu_{\Phi,n}}\big(\{(x_1, \ldots, x_n) : x_i > t\}, \mathbb{R}^n\big) = c_\Phi e^{-\Phi(|t|)} = -H'(t) \quad \text{for } t \in \mathbb{R}. \tag{61}$$

Hence, $F_\Phi(s)$ agrees with the perimeter of any half-space of the form $\{x_i > t\}$, whose measure is s.

Next, define $L_\Phi : [0,1] \to [0,\infty)$ as

$$L_\Phi(s) = s\Phi'\big(\Phi^{-1}\big(\log(\tfrac{2}{s})\big)\big) \qquad \text{for } s \in (0,1], \text{ and } \quad L_\Phi(0) = 0. \tag{62}$$

Then the isoperimetric function of $(\mathbb{R}^n, \mu_{\Phi,n})$ satisfies

$$I_{(\mathbb{R}^n,\mu_{\Phi,n})}(s) \approx F_\Phi(s) \approx L_\Phi(s) \quad \text{for } s \in [0, \tfrac{1}{2}] \tag{63}$$

(see [BCR, Proposition 13 and Theorem 15]). Furthermore, half-spaces, whose boundary is orthogonal to a coordinate axis, are "approximate solutions" to the isoperimetric problem in $(\mathbb{R}^n, \mu_{\Phi,n})$ in the sense that there exist constants C_1 and C_2, depending on n, such that, for every $s \in (0,1)$, any such half-space V with measure s satisfies

$$C_1 P_{\mu_{\Phi,n}}(V, \mathbb{R}^n) \leq I_{(\mathbb{R}^n,\mu_{\Phi,n})}(s) \leq C_2 P_{\mu_{\Phi,n}}(V, \mathbb{R}^n).$$

In the special case when $\mu_{\Phi,n} = \gamma_n$, the Gauss measure, equation (63) yields

$$I_{(\mathbb{R}^n,\gamma_n)}(s) \approx s\big(\log \tfrac{2}{s}\big)^{1/2} \quad \text{for } s \in (0, \tfrac{1}{2}]. \tag{64}$$

Note that now (30) is not satisfied. Moreover, any half-space is, in fact, an exact minimizer in the isoperimetric inequality [Bo, ST].

The reduction theorem for Sobolev embeddings in product probability spaces reads as follows.

THEOREM 3.14. *Let $n \in \mathbb{N}$, $m \in \mathbb{N}$, let $\mu_{\Phi,n}$ be the probability measure defined by* (56), *and let $\|\cdot\|_{X(0,1)}$ and $\|\cdot\|_{Y(0,1)}$ be rearrangement-invariant function norms. Then the following facts are equivalent.*

(i) *The inequality*

$$\left\| \int_s^1 f(r) \, \frac{\big(\Phi^{-1}(\log \frac{2}{s}) - \Phi^{-1}(\log \frac{2}{r})\big)^{m-1}}{r\Phi'\big(\Phi^{-1}(\log \frac{2}{r})\big)} \, dr \right\|_{Y(0,1)} \leq C_1 \|f\|_{X(0,1)} \tag{65}$$

holds for some constant C_1, and for every nonnegative $f \in X(0,1)$.

(ii) *The embedding*

$$V^m X(\mathbb{R}^n, \mu_{\Phi,n}) \to Y(\mathbb{R}^n, \mu_{\Phi,n}) \tag{66}$$

holds.

(iii) *The Poincaré inequality*

$$\|u\|_{Y(\mathbb{R}^n,\mu_{\Phi,n})} \leq C_2 \|\nabla^m u\|_{X(\mathbb{R}^n,\mu_{\Phi,n})} \tag{67}$$

holds for some constant C_2, and for every $u \in V^m_\perp X(\mathbb{R}^n, \mu_{\Phi,n})$.

When $m = 1$ and the measure $\mu_{\Phi,n}$ agrees with the Gauss measure γ_n, the result of Theorem 3.14 is by now standard (see e.g. [CP]).

The rearrangement-invariant function norm $\|\cdot\|_{X_{m,\Phi}(0,1)}$ which yields the optimal rearrangement-invariant target space $Y(\mathbb{R}^n, \mu_{\Phi,n})$ in embedding (66) is defined as follows. Let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm, and let $n, m \in \mathbb{N}$. Then $\|\cdot\|_{X_{m,\Phi}(0,1)}$ is the rearrangement-invariant function norm whose associate function norm is given by

$$\|f\|_{X'_{m,\Phi}(0,1)} = \left\| \int_0^r f^*(s) \, \frac{\big(\Phi^{-1}(\log \frac{2}{s}) - \Phi^{-1}(\log \frac{2}{r})\big)^{m-1}}{r\Phi'\big(\Phi^{-1}(\log \frac{2}{r})\big)} \, ds \right\|_{X'(0,1)} \tag{68}$$

for $f \in \mathcal{M}_+(0,1)$.

The reduction theorem takes a simpler form in the case of Gaussian measure.

THEOREM 3.15. *Let $X(\mathbb{R}^n, \gamma_n)$ and $Y(\mathbb{R}^n, \gamma_n)$ be r.i. spaces, and let $m \geq 1$. There exists a constant C_1 such that*

$$\|u\|_{Y(\mathbb{R}^n,\gamma_n)} \leq C_1 \|\nabla^m u\|_{X(\mathbb{R}^n,\gamma_n)}$$

for every $u \in V^m_\perp X(\mathbb{R}^n, \gamma_n)$ if and only if there exists a constant C_2 such that

$$\left\| \frac{1}{\big(1 + \log \frac{1}{s}\big)^{(m-1)/2}} \int_s^1 f(r) \, \frac{\big(\log \frac{r}{s}\big)^{m-1}}{r\big(1 + \log \frac{1}{r}\big)^{1/2}} \, dr \right\|_{Y(0,1)} \leq C_2 \|f\|_{X(0,1)}$$

for every $f \in X(0,1)$.

Given $n, m \in \mathbb{N}$, and a rearrangement-invariant function norm $\|\cdot\|_{X(0,1)}$, let us define $\|\cdot\|_{X_{m,G}(0,1)}$ as the rearrangement-invariant function norm whose associate function norm is given by

$$\|f\|_{X'_{m,G}(0,1)} = \left\| \frac{1}{r\big(\log \frac{2}{r}\big)^{1/2}} \int_0^r f^*(s) \, \frac{\big(\log \frac{r}{s}\big)^{m-1}}{\big(\log \frac{2}{s}\big)^{(m-1)/2}} \, ds \right\|_{X'(0,1)} \tag{69}$$

for $f \in \mathcal{M}_+(0,1)$.

Theorem 3.16. *Let $n \in \mathbb{N}$, $m \in \mathbb{N}$, and let $\|\cdot\|_{X(0,1)}$ be a rearrangement-invariant function norm. Then the functional $\|\cdot\|_{X'_{m,G}(0,1)}$, given by* (69)*, is a rearrangement-invariant function norm, whose associate norm $\|\cdot\|_{X_{m,G}(0,1)}$ satisfies*

$$V^m X(\mathbb{R}^n, \gamma_n) \to X_{m,G}(\mathbb{R}^n, \gamma_n) \tag{70}$$

with norm independent of n, and

$$\|u\|_{X_{m,G}(\mathbb{R}^n,\gamma_n)} \le C \|\nabla^m u\|_{X(\mathbb{R}^n,\gamma_n)} \tag{71}$$

for some constant C independent of n, for every $u \in V^m_\perp X(\mathbb{R}^n, \gamma_n)$. Moreover, the function norm $\|\cdot\|_{X_{m,G}(0,1)}$ is optimal in (70) *and* (71) *among all rearrangement-invariant norms.*

We finish with an application of our results to the particular case when $\mu_{\Phi,n}$ is a Boltzmann measure, and the norms are of Lorentz–Zygmund type.

Theorem 3.17. *Let $n, m \in \mathbb{N}$, let $\beta \in [1,2]$ and let $p, q \in [1,\infty]$ and $\alpha \in \mathbb{R}$ be such that one of the conditions in* (15) *is satisfied. Then*

$$V^m L^{p,q;\alpha}(\mathbb{R}^n, \gamma_{n,\beta}) \to \begin{cases} L^{p,q;\alpha+m(\beta-1)/\beta}(\mathbb{R}^n, \gamma_{n,\beta}) & \text{if} \quad p < \infty; \\ L^{\infty,q;\alpha-m/\beta}(\mathbb{R}^n, \gamma_{n,\beta}) & \text{if} \quad p = \infty. \end{cases} \tag{72}$$

Moreover, in both cases, the target space is optimal among all rearrangement-invariant spaces.

When $\beta = 2$, Theorem 3.17 yields the following sharp Sobolev type embeddings in Gauss space.

Theorem 3.18. *Let $n, m \in \mathbb{N}$, and let $p, q \in [1,\infty]$ and $\alpha \in \mathbb{R}$ be such that one of the conditions in* (15) *is satisfied. Then*

$$V^m L^{p,q;\alpha}(\mathbb{R}^n, \gamma_n) \to \begin{cases} L^{p,q;\alpha+m/2}(\mathbb{R}^n, \gamma_n) & \text{if } p < \infty; \\ L^{\infty,q;\alpha-m/2}(\mathbb{R}^n, \gamma_n) & \text{if } p = \infty. \end{cases}$$

Moreover, in both cases, the target space is optimal among all rearrangement-invariant spaces.

A further specialization of the indices p, q, α appearing in Theorem 3.18 leads to the following basic embeddings.

Corollary 3.19. *Let $n, m \in \mathbb{N}$.*

(i) *Assume that $p \in [1,\infty)$. Then*

$$V^m L^p(\mathbb{R}^n, \gamma_n) \to L^p(\log L)^{mp/2}(\mathbb{R}^n, \gamma_n),$$

and the target space is optimal among all rearrangement-invariant spaces.

(ii) *Assume that* $\gamma > 0$. *Then*

$$V^m \exp L^{\gamma}(\mathbb{R}^n, \gamma_n) \to \exp L^{2\gamma/(2+m\gamma)}(\mathbb{R}^n, \gamma_n),$$

and the target space is optimal among all rearrangement-invariant spaces.

(iii)
$$V^m L^{\infty}(\mathbb{R}^n, \gamma_n) \to \exp L^{2/m}(\mathbb{R}^n, \gamma_n),$$

and the target space is optimal among all rearrangement-invariant spaces.

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