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WAVE FRONTS TO SOME MODIFICATIONS OF KORTEWEG–de VRIES AND BURGERS–KORTEWEG–de VRIES EQUATIONS

BOGDAN PRZERADZKI and KATARZYNA SZYMAŃSKA-DĘBOWSKA

Institute of Mathematics, Technical University of Łódź
90-924 Łódź, ul. Wólczańska 215, Poland
E-mail: przeradz@p.lodz.pl, katarzyna.szymanska-debowska@p.lodz.pl

Abstract. The existence of a traveling wave with special properties to modified KdV and BKdV equations is proved. Nonlinear terms in the equations are defined by means of a function f of an unknown u satisfying some conditions.

1. Introduction. Waves on shallow water can be described by nonlinear evolution equations such as Korteweg–de Vries equation

$$u_t + u_{xxx} - 6uu_x = 0.$$

One can extend possible applications if the dispersive term has a more general form $f(u)u_x$ with some appropriate function f. If we want to include dissipation in the model, the Burgers–Korteweg–de Vries equation will fit better:

$$u_t + u_{xxx} + \mu u_{xx} - 6uu_x = 0.$$

The difference between the two equations lies in the term μu_{xx} which has the effect that the second equation is similar to the diffusion equation (and also has similar properties). Again, we replace the dispersive term by a general one $f(u)u_x$. Equations of these type model many physical phenomena such as shallow-water waves with weakly non-linear restoring forces, ion-acoustic waves in a plasma, and acoustic waves on a crystal lattice. They first appeared in [8] but the history is long and complicated, see [1, 2].

Most nonlinear partial differential equations cannot be explicitly solved; one can study only special solutions such as steady-state ones ($u_t = 0$) or traveling waves

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$u(t,x) = z(x - vt)$, [9]. This last case is especially important since all considered equations are models of wave phenomena and solutions of this form vary strongly in time in the whole future: they do not converge as $t \to \infty$ to stationary solutions. The resulting equation for the function z is an ordinary differential equation which simplifies considerations: methods from the theory of dynamical systems can be used. On the other hand, ODEs have many solutions and we can put additional conditions on the behavior of the function z. If the limits $z_\pm := \lim_{\xi\to\pm\infty} z(\xi)$ exist and $z_+ = z_-$, then we have a solitary wave; if $z_+ \neq z_-$, we have found a wave front. Some authors distinguish wave front solutions (which have constant sign) from so called kick-profile waves (which change sign infinitely many times). The existence of traveling waves has been shown for many other equations [10, 12].

Sometimes, a nonlinear equation has such a form that the method of inverse scattering can be applied [3]. This method gives the exact solution although the formula for the solution is not explicit. Many authors search for exact solutions to a given equation which has a special form (sine or cosine functions, exponential functions) [5, 7, 11, 12] but this is impossible if the nonlinearity f is general. Here, we will show the existence of a traveling wave assuming only qualitative behavior of f.

2. Definitions. Let us consider the following modified KdV equation

$$u_{xxx} + u_t + f(u)u_x = 0, \tag{1}$$

and the modified BKdV equation

$$u_{xxx} + u_t + f(u)u_x + \mu u_{xx} = 0, \tag{2}$$

where $\mu > 0$.

Definition 1. By a *traveling wave* of equations (1) and (2) we mean any solution

$$u(x,t) = z(\xi),$$

where $z \in C^3(\mathbb{R})$, $(t,x) \in \mathbb{R}\times\mathbb{R}$, $\xi = x - vt$, $v \in \mathbb{R}\setminus\{0\}$ such that there exist finite limits

$$z_- := \lim_{\xi\to-\infty} z(\xi) \quad \text{and} \quad z_+ := \lim_{\xi\to+\infty} z(\xi).$$

Definition 2. We say that the traveling wave is a *wave front* if

$$z_- \neq z_+.$$

Definition 3. We say that a wave front is a *kick-profile wave solution* if $z(\xi)$ tends oscillating to z_- (alternatively z_+) when $\xi \to -\infty$ (alternatively $\xi \to +\infty$).

3. The modified KdV equation. In this section, we shall study equation (1) under the following hypotheses on $f \in C^1([0,\infty))$:

(i) $f(0) = 0$,
(ii) there exists z_0 such that $f'(z) > 0$ for $z \in (0, z_0)$ and $f'(z) < 0$ for $z > z_0$,
(iii) there exists $R > 0$ such that $zf'(z)$ is decreasing for $z > R$.

First, we shall show properties of three functions defined through f which will be used later. The function $k : \mathbb{R}_+ \to \mathbb{R}$ given by

$$k(z) = zf(z) - \int_0^z f(s)\,ds. \tag{3}$$

Notice that

$$k'(z) = zf'(z) \quad \text{and} \quad k''(z) = zf''(z) + f'(z).$$

By assumptions (i) and (ii), k is positive and increasing for $z \in (0, z_0)$ and decreasing for $z > z_0$. By assumption (iii), $zf''(z) + f'(z) \le 0$, hence k is concave for z sufficiently large, so

$$\lim_{z\to\infty} k(z) = -\infty.$$

So, there exists only one point $z_k > z_0$ such that $k(z_k) = 0$. Moreover, observe that

$$k(z) < 0 \quad \text{for} \quad z > z_k. \tag{4}$$

Now, let us consider the function $g : \mathbb{R}_+ \to \mathbb{R}$ given by

$$g(z) = \tfrac{1}{2} z \int_0^z f(s)\, ds - \int_0^z s f(s)\, ds. \tag{5}$$

We have

$$g'(z) = -\tfrac{1}{2}\Big(zf(z) - \int_0^z f(s)\, ds \Big) = -\tfrac{1}{2} k(z)$$

and

$$g''(z) = -\tfrac{1}{2} z f'(z).$$

By assumptions (i) and (ii) we see that $g'(0) = 0$ and $g''(z) < 0$ for $z \in (0, z_0)$, hence $g'(z) < 0$ in $(0, z_0)$. Thus g is negative, decreasing and concave in $(0, z_0)$. Moreover, if $z > z_k$ then by (4) $g'(z) > 0$ and by (ii) $g''(z) > 0$. Therefore g becomes increasing and convex for $z > z_k$. Finally, there exists the unique $z_g > z_k$ such that $g(z_g) = 0$.

Define the third auxiliary function by the formula

$$h(z) = \frac{1}{z} \int_0^z f(s)\, ds. \tag{6}$$

Observe that $\lim_{z\to\infty} f(z) = -\infty$. Indeed, let $\bar{z} > \max(z_0, R)$, then, by (iii), we get $zf'(z) < \bar{z} f'(\bar{z})$, for $z \ge \bar{z}$. Hence

$$\int_{\bar{z}}^z f'(s)\, ds < \int_{\bar{z}}^z \frac{\bar{z} f'(\bar{z})}{s}\, ds,$$

and

$$f(z) < \bar{z} f'(\bar{z}) (\ln z - \ln \bar{z}) + f(\bar{z}).$$

Since $\bar{z} f'(\bar{z}) < 0$, we have $\lim_{z\to\infty} f(z) = -\infty$.

Now, by assumptions (i) and (ii), $\lim_{z\to 0^+} h(z) = 0$. Moreover, $\lim_{z\to\infty} h(z) = -\infty$. By (3) and (6), we get

$$h'(z) = \frac{zf(z) - \int_0^z f(s)\, ds}{z^2} = \frac{k(z)}{z^2}.$$

By (4), we know that h is increasing for $z \in (0, z_k)$ and then decreasing.

Now, assume that

(iv) $\int_0^{z_g} f(s)\, ds > 0$.

THEOREM 1. *Under assumptions* (i)–(iv), *there exists a velocity of wave* $v > 0$ *such that the equation* (1) *has at least one wave front solution.*

Proof. Looking for a traveling wave of (1) we get the differential equation

$$z''' - vz' + f(z)z' = 0, \tag{7}$$

which is equivalent to

$$\begin{cases} z' = x \\ x' = y \\ y' = vx - f(z)x. \end{cases} \tag{8}$$

We can write down equation (7) as

$$\Big(z'' - vz + \int_0^z f(s)\,ds\Big)' = 0.$$

Hence, we get

$$z'' = vz - \int_0^z f(s)\,ds + A, \quad A \in \mathbb{R}.$$

We have got a conservative system with the potential

$$U(z) = -\tfrac{1}{2}vz^2 - Az + \int_0^z (z-s)f(s)\,ds.$$

The existence of wave fronts of (1) is equivalent to existence of a heteroclinic orbit of the system (8) between points $(z_-, 0, 0)$ and $(z_+, 0, 0)$.

In our case, to get the heteroclinic orbit of (8) the potential U might have two maximum points: z_- and z_+ at which U has the same values and one minimum point z_1, where $z_- < z_1 < z_+$.

Let $z_- = 0$ and $U(0) = 0$, hence $A = 0$. We have

$$U(z) = -\tfrac{1}{2}vz^2 + \int_0^z (z-s)f(s)\,ds, \tag{9}$$

and

$$U'(z) = -vz + \int_0^z f(s)\,ds. \tag{10}$$

Notice that $U'(0) = 0$ and $U''(0) = -v + f(0) < 0$. Hence, the potential U has a maximum at 0.

Set

$$v := \frac{1}{z_g}\int_0^{z_g} f(s)\,ds, \tag{11}$$

where z_g is the zero of g. By (iv), we get $v > 0$. Now, we have

$$U(z_g) = g(z_g) = 0.$$

Observe that there exists a point z_1, $z_1 < z_k < z_g$ such that $h(z_1) = h(z_g)$. By (10), we have

$$U'(z_1) = U'(z_g) = 0.$$

Moreover,

$$U''(z_g) = -\frac{1}{z_g}\int_0^{z_g} f(s)\,ds + f(z_g) < 0.$$

Indeed, by (4), $k(z_g) < 0$, so $z_g f(z_g) - \int_0^{z_g} f(s)\,ds < 0$. Similarly, we get

$$U''(z_1) = -\frac{1}{z_1}\int_0^{z_1} f(t)\,dt + f(z_1) > 0.$$

Finally, we get that the potential U has a maximum equal to 0 at 0 and z_g and a minimum at z_1. Moreover, $U(z) \le 0$ for $z \in (0, z_g)$. Indeed, by (6), (10) and (11), $U(z)$ is increasing for $z \in (z_1, z_g)$ and decreasing in the remaining cases. Hence, there exists a heteroclinic orbit between 0 and z_g and the proof is complete. ∎

4. The modified BKdV equation. Here, we shall consider equation (2).

THEOREM 2. *Let $f \in C^2([0,\infty),\mathbb{R})$ satisfy the following assumptions*:

(i) $f(0) = 0$,
(ii) $f'(z) > 0$ *for* $z > 0$.

Then, for all $v \in (0, v_0)$, where $v_0 = \lim_{z\to\infty} f(z) \in (0, +\infty]$, the equation (2) *has at least one wave front solution* (*a kick-profile wave solution in a case*).

Proof. When we look for traveling wave solutions of (2) we get the ordinary differential equation

$$z''' - vz' + f(z)z' + \mu z'' = 0.$$

Hence

$$(z'' - vz + F(z) + \mu z')' = 0,$$

where $F(z) = \int_0^z f(s)\,ds$. By the above, we get

$$z'' = vz - F(z) - \mu z' + A,$$

which is equivalent to the system

$$\begin{cases} z' = y \\ y' = vz - F(z) - \mu y + A. \end{cases} \tag{12}$$

Due to the definition, the existence of wave fronts of (2) is equivalent to the existence of an orbit of system (12) connecting points $(z_-, 0)$ and $(z_+, 0)$.

Set $A = 0$. The stationary points of (12) sit on z axis, where $vz = F(z)$. Since F is convex and $F(0) = 0$, we have at most two such points and exactly two (notice that F is defined only for $z \ge 0$) iff $v \in (0, v_0)$. The Jacobi matrix of the vector field defined by the right-hand side of (12) equals at these stationary points

$$J(z_\pm, 0) = \begin{bmatrix} 0 & 1 \\ v - f(z_\pm) & -\mu \end{bmatrix}$$

and the characteristic polynomial is

$$P_\pm(\lambda) = \lambda^2 + \mu\lambda + f(z_\pm) - v.$$

Since $0 = f(z_-) < v$, the eigenvalues at z_- have opposite signs and this stationary point is a saddle. Similarly $f(z_+) > v$ and the eigenvalues

$$\lambda_{1,2} = \left(-\frac{1}{2}\mu \pm \sqrt{\frac{1}{4}\mu^2 + v - f(z_+)}\right)$$

are both real negative if $f(z_+) - v \le \frac{1}{4}\mu^2$—in this case the second stationary point is a stable node, or complex conjugate with negative real parts if $f(z_+) - v > \frac{1}{4}\mu^2$—it is a stable focus.

It remains to show that the trajectory tending to $(z_-, 0)$ as $t \to -\infty$ is the sought heteroclinic orbit. Let us consider the function of energy $E(z, y) = \frac{1}{2}y^2 + \int_0^z (F(t) - vt)\,dt$. The directional derivative of E in direction given by the vector field (12) $E' = -y^2$, hence E is decreasing along all trajectories. On the other hand, E tends to ∞ if $\|(z, y)\| \to \infty$, that gives all trajectories are bounded as $t \to +\infty$. Moreover, $E(z_-, 0) = 0$ and $E(0, y) = \frac{1}{2}y^2 \ge 0$, hence the trajectory outgoing from $(z_-, 0)$ cannot escape from the half-plane $z > 0$.

On the other hand, the divergence of the vector field equals -1, thus, by the Bendixson criterion, there are no periodic orbits neither homoclinic ones. Therefore the trajectory outgoing from $(z_-, 0)$ will tend to the second stationary point due to the Poincaré–Bendixson Theorem. This ends the proof; the kick-profile wave is obtained for the case $f(z_+) - v > \frac{1}{4}$. ■

Remark. We have set arbitrarily $A = 0$. For $A \ne 0$, at least one of two stationary points of (12) is lost: if $A > 0$ it remains only z_+, if $A < 0$ we can even lose both points for sufficiently small v.

References

[1] L. Debnath, *Nonlinear Partial Differential Equations for Scientists and Engineers*, Birkhäuser, Boston, 1997.

[2] P. G. Drazin, R. S. Johnson, *Solitons: an Introduction*, Cambridge Texts Appl. Math., Cambridge Univ. Press, Cambridge, 1989.

[3] C. S. Gardner, J. M. Greene, M. D. Kruskal, R. M. Miura, *Method for solving the Korteweg–de Vries equation*, Phys. Rev. Lett. 19 (1967), 1095–1097.

[4] S. Haq, Siraj-Ul-Islam, M. Uddin, *A mesh-free method for the numerical solution of the KdV–Burgers equation*, Appl. Math. Model. 33 (2009), 3442–3449.

[5] M. M. Hassan, *Exact solitary wave solutions for a generalized KdV–Burgers equation*, Chaos Solitons Fractals 19 (2004), 1201–1206.

[6] M. A. Helal, *Soliton solution of some nonlinear partial differential equations and its applications in fluid mechanics*, Chaos Solitons Fractals 13 (2002), 1917–1929.

[7] M. A. Helal, M. S. Mehanna, *A comparison between two different methods for solving KdV–Burgers equation*, Chaos Solitons Fractals 28 (2006), 320–326.

[8] D. J. Korteweg, G. de Vries, *On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary wave*, Philos. Mag. (5) 39 (1895), 422–443.

[9] J. D. Logan, *An Introduction to Nonlinear Partial Differential Equations*, Pure Appl. Math. (N.Y.), Wiley-Interscience, New York, 1994.

[10] B. Przeradzki, *Travelling waves for reaction-diffusion equations with time depending nonlinearities*, J. Math. Anal. Appl. 281 (2003), 164–170.

[11] S. Tanaka, *Korteweg–de Vries equation: construction of solutions in terms of scattering data*, Osaka J. Math. 11 (1974), 49–59.

[12] L. P. Wu, S. F. Chen, C. P. Pang, *Traveling wave solutions for generalized Drinfeld–Sokolov equations*, Appl. Math. Model. 33 (2009), 4126–4130.