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MINIMIZATION OF FUNCTIONAL WITH INTEGRAND EXPRESSED AS MINIMUM OF QUASICONVEX FUNCTIONS — GENERAL AND SPECIAL CASES

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Dedicated to the memory of Professor Zdzisław Naniewicz
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Abstract. We present Z. Naniewicz method of optimization a coercive integral functional $\mathcal{J}$ with integrand being a minimum of quasiconvex functions. This method is applied to the minimization of functional with integrand expressed as a minimum of two quadratic functions. This is done by approximating the original nonconvex problem by appropriate convex ones.

1. Introduction

1.1. Outline of the origin of the problem. Minimization of the integral functional of the form

$$\mathcal{I}(v) := \int_\Omega f(x, v(x), Dv(x))\, dx$$

lies at the heart of the calculus of variations. Here Ω is an open, bounded subset of $\mathbb{R}^n$ with sufficiently smooth boundary, v is an element of a suitable Sobolev space V of functions on Ω with values in $\mathbb{R}^m$ and integrand $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn} \to \mathbb{R} \cup \{+\infty\}$ is assumed to satisfy certain regularity and growth conditions. In particular $\mathcal{I}$ has to be bounded from below.

The problem has a history going back to C. F. Gauss, Lord Kelvin, G. L. Dirichlet, B. Riemann and their wrong argumentation that the existence of minimum of the prototype of $\mathcal{I}$, the Dirichlet integral, is guaranteed by its boundedness from below.

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Karl Weierstrass' counterexample and further effort of Arzela, Hilbert, Lebesgue, Tonelli, Poincaré (and many others; we can also mention Stanisław Zaremba, who contributed to the Dirichlet problem as well) has led to the method of minimizing $\mathcal{I}$ called 'the direct method'. In this method we consider a minimizing sequence for $\mathcal{I}$, that is, a sequence $(u_n) \subset V$ of functions belonging to the suitable function space V, such that $\lim_{n\to\infty} \mathcal{I}(u_n) = \inf\{\mathcal{I}(v) : v \in V\}$.

We have to prove that the sequence (u_n) is convergent (passing to a subsequence if necessary) to some $u_0 \in V$ with respect to an appropriate topology. Here coercivity of $\mathcal{I}$ is important. If we show next that $\mathcal{I}$ is sequentially lower semicontinuous with respect to this topology then we are done, for (up to a subsequence) we have

$$\inf\{\mathcal{I}(v) : v \in V\} = \lim_{n\to\infty} \mathcal{I}(u_n) \geq \mathcal{I}(u_0).$$

It follows that u_0 is a minimum of $\mathcal{I}$. Usually the most challenging task when carrying this procedure is proving lower semicontinuity of $\mathcal{I}$.

It is known that weak (sequential) lower semicontinuity of $\mathcal{I}$ is equivalent to the convexity of its integrand with respect to the third variable.

Variational problems outlined above are also met in applied sciences. In physics or engineering, especially in nonlinear elasticity, we have $m = n$, v is called the displacement of the elastic body Ω, f—the density of the internal energy, while $\mathcal{I}$—the energy functional.

However, it turns out that this convexity requirement is too restrictive for engineering applications. More precisely, mere convexity of the energy density with respect to its third variable is inconsistent with the principle of the material frame indifference. This raises need for generalizations of the notion of convexity. One of them is quasiconvexity in the Morrey sense.

Definition 1.1 ([Mor]). We say that a continuous function $h : \mathbb{R}^{mn} \to \mathbb{R}$ is *quasiconvex* (in the Morrey sense), if for every $\xi \in \mathbb{R}^{mn}$, for every bounded open subset $\omega \subset \mathbb{R}^n$ of Lebesgue measure $\operatorname{meas}(\omega)$ and for every function $z \in C_0^1(\omega, \mathbb{R}^m)$ the Jensen type inequality is satisfied:

$$\operatorname{meas}(\omega) h(\xi) \leq \int_\omega h(\xi + Dz(x))\, dx.$$

It is also known ([AF], see also [D]) that if $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn} \to \mathbb{R}$ fulfils the conditions:

(i) $\Omega \ni x \mapsto f(x, s, \xi)$ is measurable for all $(s, \xi) \in \mathbb{R}^m \times \mathbb{R}^{mn}$,
(ii) for a.e. $x \in \Omega$, $\mathbb{R}^m \times \mathbb{R}^{mn} \ni (s, \xi) \mapsto f(x, s, \xi)$ is continuous;
(iii) $0 \leq f(x, s, \xi) \leq A(x) + C(|s|^2 + |\xi|^2)$,

where $A(\cdot)$ is a nonnegative summable function in Ω and C is a nonnegative constant, then the functional $\mathcal{I}$ is sequentially weakly lower semicontinuous (*swlsc* for short) on the Sobolev space $H^1(\Omega; \mathbb{R}^m)$ iff for every $s \in \mathbb{R}^m$ and a.e. $x \in \Omega$ the function $\mathbb{R}^{mn} \ni \xi \mapsto f(x, s, \xi)$ is quasiconvex.

The situation becomes much more complicated if the integrand is not quasiconvex. In this case the minimized functional does not generally attain its infimum and any minimizing (sub)sequence for $\mathcal{I}$ does not converge strongly but only weakly. The elements of the minimizing sequence oscillate wildly around its weak limit. Basically, there are two ways to proceed in this case (see for instance references quoted in [NP]). The first one is to 'quasiconvexify' the original functional and to gather "nonconvexities" into its quasiconvex envelope. However, computing explicit form of the quasiconvex envelope is very difficult in practice. Further, carrying out this procedure (when possible) erases some important information concerning the behaviour of the minimizing sequences, because the weak limit of each minimizing sequence does not provide by itself all interesting aspects of the considered problem. Minimizers of quasiconvexifications themselves are not sufficient to characterize properly oscillatory phenomena of such problems (microstructural features describing fine mixtures of the phases in the phase transition problems, for instance).

Another way is to enlarge the space of admissible functions from Sobolev spaces to the space of parametrized Young measures ([Y]). In this approach the Young measures can be regarded as means of summarizing the spatial oscillatory properties of minimizing sequences, thus conserving some of that information. From the application point of view, the detailed structure of minimizing sequences including the behavioural characteristics of the phases involved appears to be as much important as the minimizers themselves. Unfortunately, it is a very difficult task to compute the parametrized Young measures associated with a minimizing sequence.

1.2. Problem description. Nonconvex minimization was one of the fields of vast mathematical interests of Professor Naniewicz. In this article we describe his original method of solving a particular class of non-(quasi)convex minimization problems, namely those involving integrands expressed as minimima of a finite number of (quasi)convex functions, called *phases*. (Without loss of generality we can consider minimum of only two such functions.) We do not present any new results here. The presentation consists of two parts. In the first part we discuss the general case; the presented results form the core of the paper [N]. They are further applied to the analysis of a variational problem involving double-well potential—this is second part of the article based on [NP] (see also [NP1]—the version of [NP] accepted by professor Naniewicz, unchanged due to the reviewers remarks).

From now on we will often write $\int_\Omega f(v)\,d\Omega$ or $\int_\Omega f(v, Dv)\,d\Omega$ instead of rather lenghty $\int_\Omega f(x, v(x), Dv(x))\,dx$.

1.2.1. *The first part.* In this part we present those results contained in [N] which are necessary to develop analysis of the double-well potential case. We consider the problem

$$\inf\Big\{\mathcal{J}(v) := \int_\Omega \min\{f_1(v, Dv), f_2(v, Dv)\}\,d\Omega : v \in H^1(\Omega; \mathbb{R}^m)\Big\} := \alpha, \tag{P}$$

under the hypothesis that the two integral functionals

$$v \mapsto \int_\Omega f_1(v, Dv)\,d\Omega \quad \text{and} \quad v \mapsto \int_\Omega f_2(v, Dv)\,d\Omega$$

are swlsc and coercive in $H^1(\Omega; \mathbb{R}^m)$. The functional $\mathcal{J}$ defined in (P) is not in general swlsc, so it does not attain its infimum over $H^1(\Omega; \mathbb{R}^m)$. If a function u is a solution of (P), then there exist functions χ_{Ω_1} and χ_{Ω_2} defined on Ω with values in $\{0, 1\}$ such that

$$\mathcal{J}(u) := \int_\Omega \min\{f_1(u, Du), f_2(u, Du)\}\,d\Omega = \int_\Omega \big[\chi_{\Omega_1} f_1(u, Du) + \chi_{\Omega_2} f_2(u, Du)\big]\,d\Omega = \alpha.$$

We want not only to find the weak limit of the minimizing subsequence, but to get the lacking necessary information concerning its spatial oscillatory properties. In [N] Z. Naniewicz proposed to search not only for $u \in H^1(\Omega;\mathbb{R}^m)$, but also for functions $\chi_1, \chi_2 : \Omega \to [0,1]$ (associated with phases f_1 and f_2 respectively) with $\chi_1 + \chi_2 \equiv 1$, together with a function $\mathcal{R}(u,\chi_1,\chi_2)$ (referred to as *the relaxation term*) such that

$$\int_\Omega \big[\chi_1 f_1(u,Du) + \chi_2 f_2(u,Du)\big]\, d\Omega - \mathcal{R}(u,\chi_1,\chi_2) = \alpha.$$

We require that for χ_1 and χ_2 taking only values 0 or 1 we have $\mathcal{R} = 0$. It is also important to find an explicit formula for $\mathcal{R}$.

We can now state the extended version of the optimization problem (P):

Find $u \in H^1(\Omega;\mathbb{R}^m)$, $\chi_1 : \Omega \to [0,1]$, $\chi_2 : \Omega \to [0,1]$ with $\chi_1 + \chi_2 \equiv 1$ and the function $\mathcal{R} = \mathcal{R}(u,\chi_1,\chi_2)$ such that

$$\int_\Omega \big[\chi_1 f_1(u,Du) + \chi_2 f_2(u,Du)\big]\, d\Omega - \mathcal{R}(u,\chi_1,\chi_2) = \alpha,$$

and

$$\big[\chi_1 f_1(u,Du) + \chi_2 f_2(u,Du) = \min\{f_1(u,Du), f_2(u,Du)\}\big] \Rightarrow \mathcal{R}(u,\chi_1,\chi_2) = 0.$$

In order to solve this extended problem we approximate the 'primal' functional $\mathcal{J}$ by suitably defined functionals $\mathcal{J}^k$. Despite not being quasiconvex they attain their infimum α^k over $H^1(\Omega;\mathbb{R}^m)$. For a sequence (u_k) suitably constructed we also have $\mathcal{J}^k(u_k) = \alpha^k$ and moreover, $\alpha^k \to \alpha$ as $k \to \infty$.

It is now possible to take the advantage of the fact that u_k is a minimizer of $\mathcal{J}^k$, by constructing two sequences (χ_1^k) and (χ_2^k) of functions on Ω with values in the set $\{0,1\}$. The elements of these sequences are connected with each other by the relation: $\chi_1^k + \chi_2^k \equiv 1$ for each $k \in \mathbb{N}$. This enables us to write values of the approximating functionals $\mathcal{J}^k$ on the elements of (u_k) as

$$\mathcal{J}^k(u_k) = \int_\Omega \big[\chi_1^k(u_k) f_1(u_k) + \chi_2^k(u_k) f_2(u_k)\big]\, d\Omega.$$

Further, it follows that (u_k) is a minimizing sequence for $\mathcal{J}$. As it is coercive by assumption, there exists a weakly convergent (in $H^1(\Omega;\mathbb{R}^m)$) subsequence of (u_k). Its limit u is a solution of (P), i.e.

$$\int_\Omega \min\big\{f_1(u,Du), f_2(u,Du)\big\}\, d\Omega = \alpha$$

and $\chi_i = \chi_{\Omega_i}$, $i = 1,2$. Now the relaxation term $\mathcal{R}$ can be determined.

1.2.2. *The second part.* In the second part of the presentation we specialize our considerations to the functional whose integrand is expressed as a minimum of two quadratic functions. We formulate and fully solve the nonconvex optimization problem, that is, we obtain an explicit formula for infimum.

The functional to be minimized is

$$\mathcal{J}(v) := \int_\Omega \min\big\{\tfrac{1}{2}a\,|\varepsilon(v) + C|^2, \tfrac{1}{2}b\,|\varepsilon(v) + D|^2\big\}\, dx,$$

where $\varepsilon(v) \in L^2(\Omega; \mathbb{R}^{n\times n}_{\mathrm{sym}})$ is the symmetrized gradient of the function $v \in H^1_0(\Omega; \mathbb{R}^n)$, i.e. $\varepsilon(v) := \frac{1}{2}(\nabla v + \nabla^T v)$. The minimization problem is of the form

$$\inf\big\{\mathcal{J}(v) : v \in H^1_0(\Omega; \mathbb{R}^n)\big\} := \alpha. \tag{Q}$$

The results of the previous section enable us to formulate convex optimization problems approximating (Q). Next we formulate their Fenchel duals with solutions (p^k). This makes it possible to write down the representations for the sequence (α^k) of approximate infima in (Q). The next step is a compensated compactness type result for the sequences (p^k) and (u^k)—a minimizing sequence for $\mathcal{J}$. Now we are ready to state the main results of this part. The first one is the explicit formula for α—the infimum of the functional with nonconvex integrand. This formula involves:

— the weak limit of the sequence (u^k);
— the weak limit of the sequence (p^k);
— the weak* limits of the sequences of characteristic functions (χ^k_a) and (χ^k_b) related to the phases $\frac{1}{2}a\,|\varepsilon(v) + C|^2$ and $\frac{1}{2}b\,|\varepsilon(v) + D|^2$, respectively.

The second main result is expressing the infimum α by the Young measures associated with the minimizing sequence and establishing some relations between the weak* limit of the sequence $(\psi^k) = (\chi^k_b - \chi^k_a)$ and the related parametrized measures.

2. General case—functional with integrand built of minimum of quasiconvex functions. We want to minimize the integral functional

$$\mathcal{J}(v) := \int_\Omega \min\big\{f_1(x, v(x), Dv(x)), f_2(x, v(x), Dv(x))\big\}\, dx$$

subject to $v \in H^1(\Omega; \mathbb{R}^m)$. This is our problem (P). We further assume that:

(i) Ω is a bounded domain in $\mathbb{R}^n$ with Lipschitz continuous boundary;

the functions f_i, $i = 1, 2$, satisfy Carathéodory conditions:

(ii) $\Omega \ni x \mapsto f_i(x, s, \xi)$ is measurable for every $(s, \xi) \in \mathbb{R}^m \times \mathbb{R}^{mn}$;
(iii) for almost all (a.e., with respect to the Lebesgue measure dx) $x \in \Omega$, the function $\mathbb{R}^m \times \mathbb{R}^{mn} \ni (s, \xi) \mapsto f_i(x, s, \xi)$ is continuous;

and growth condition

(iv) $a(x) + c\big(|s|^2 + |\xi|^2\big) \le f_i(x, s, \xi) \le A(x) + C\big(|s|^2 + |\xi|^2\big)$,

where $a(\cdot)$ and $A(\cdot)$ are summable functions in Ω, c and C are positive constants.

2.1. Approximation result. The first step in constructing a minimizing sequence for $\mathcal{J}$ is defining, for each $k \in \mathbb{N}$, an open partition $\Pi^k = \{\Omega^k_i\}^{l_k}_{i=1}$ of Ω. This partition has the following properties:

(a) $\forall i \ne j$, $1 \le i, j \le l_k$, we have $\Omega^k_i \cap \Omega^k_j = \emptyset$;
(b) $\bigcup^{l_k}_{i=1} \overline{\Omega}^k_i = \overline{\Omega}$;
(c) $\forall i \in \{1, \ldots, l_k\}$ we have $\Omega^k_i = \mathrm{interior}(\overline{\Omega}^k_i)$;
(d) $\forall \Omega^k_i \in \Pi^k\ \exists \mathcal{N}^k_i \subset \{1, \ldots, l_{k+1}\}$ such that $\overline{\Omega}^k_i = \bigcup_{j\in\mathcal{N}^k_i} \overline{\Omega}^{k+1}_j$, where $\Omega^{k+1}_j \in \Pi^{k+1}$;
(e) $\mathrm{diameter}(\Pi^k) := \max\big\{\mathrm{diameter}(\Omega^k_i) : 1 \le i \le l_k\big\} \to 0$ as $k \to \infty$.

Now for $v \in H^1(\varOmega;\mathbb{R}^m)$ we define the approximation functional $\mathcal{J}^k$ as follows.

$$\mathcal{J}^k(v) := \sum_{i=1}^{l_k} \min\Big\{\int_{\varOmega_i^k} f_1(v)\,d\varOmega, \int_{\varOmega_i^k} f_2(v)\,d\varOmega\Big\}.$$

We also set

$$\alpha := \inf\Big\{\int_{\varOmega} \min\{f_1(v), f_2(v)\}\,d\varOmega : v \in H^1(\varOmega;\mathbb{R}^m)\Big\}.$$

By definition, the functional $\mathcal{J}^k$, $k \in \mathbb{N}$, is a global majorant of $\mathcal{J}$. Moreover, thanks to (d) above, the sequence $(\mathcal{J}^k(v))$ of real numbers, $v \in H^1(\varOmega;\mathbb{R}^m)$, is decreasing.

We can state approximating optimization problems for (P):

$$\inf\big\{\mathcal{J}^k(v) : v \in H^1(\varOmega;\mathbb{R}^m)\big\} := \alpha^k, \tag{P^k}$$

with (α^k)—the decreasing sequence of approximate solutions. Indeed, the following proposition is true.

Proposition 2.1. *Assume that f_1, f_2 are quasiconvex and satisfy* (ii), (iii), (iv) *and that* (a), (b), (c), (d), (e) *hold. Then the problem* (P^k) *has at least one solution, i.e. there exists $u_k \in H^1(\varOmega;\mathbb{R}^m)$ such that*

$$\alpha^k = \mathcal{J}^k(u_k). \tag{1}$$

Moreover,

$$\alpha^k \searrow \alpha \text{ as } k \to \infty. \tag{2}$$

Proof. We will prove (1) first. We will show for $k \in \mathbb{N}$ that although $\mathcal{J}^k$ is not in general quasiconvex, it attains its infimum. Let $(\widetilde{u}_j^k)$ be a minimizing sequence for $\mathcal{J}^k$. Its boundedness in $H^1(\varOmega;\mathbb{R}^m)$ is guaranteed by the coercivity condition, so (up to a subsequence) it is weakly convergent in $H^1(\varOmega;\mathbb{R}^m)$ to some u_k as $j \to \infty$.

By definition, Π^k contains finite number of subregions $\varOmega_i^k$, $i = 1, \dots, l_k$, so passing to an appropriate subsequence, if necessary, we may also suppose that for each $\varOmega_i^k \in \Pi^k$ and for all $j \in \mathbb{N}$, either

$$\int_{\varOmega_i^k} f_1(\widetilde{u}_j^k)\,d\varOmega < \int_{\varOmega_i^k} f_2(\widetilde{u}_j^k)\,d\varOmega,$$

or

$$\int_{\varOmega_i^k} f_2(\widetilde{u}_j^k)\,d\varOmega < \int_{\varOmega_i^k} f_1(\widetilde{u}_j^k)\,d\varOmega,$$

or

$$\int_{\varOmega_i^k} f_1(\widetilde{u}_j^k)\,d\varOmega = \int_{\varOmega_i^k} f_2(\widetilde{u}_j^k)\,d\varOmega.$$

Denote by $I_{f_1}^k$, $I_{f_2}^k$ and I_0^k all of those indices from $\{1, \dots, l_k\}$ for which respectively the first, the second and the third relation holds. Thus we have

$$\mathcal{J}^k(\widetilde{u}_j^k) = \int_{\bigcup_{i \in \mathcal{I}_{f_1}^k \cup I_0^k} \varOmega_i^k} f_1(\widetilde{u}_j^k)\,d\varOmega + \int_{\bigcup_{i \in \mathcal{I}_{f_2}^k} \varOmega_i^k} f_2(\widetilde{u}_j^k)\,d\varOmega, \quad j = 1, 2, \dots$$

By the weak lower semicontinuity of these functionals we are led to

$$\alpha^k = \lim_{j\to\infty} \mathcal{J}^k(\widetilde{u}_j^k) \geq \int_{\bigcup_{i\in\mathcal{I}_{f_1}^k \cup I_0^k} \Omega_i^k} f_1(u_k)\, d\Omega + \int_{\bigcup_{i\in\mathcal{I}_{f_2}^k} \Omega_i^k} f_2(u_k)\, d\Omega = \mathcal{J}^k(u_k).$$

Observe that if the above inequality were sharp, it would contradict the definition of α^k. Equation (1) now follows.

We now sketch the proof of (2). Choose and fix an arbitrary $w \in H^1(\Omega;\mathbb{R}^m)$. We claim that

$$\mathcal{J}^k(w) \searrow \mathcal{J}(w) \quad \text{as } k\to\infty. \tag{3}$$

Let us introduce:

— subsets of Ω:

$$\Omega_{f_1}(w) := \big\{x\in\Omega : f_1(x,w(x),Dw(x)) < f_2(x,w(x),Dw(x))\big\};$$
$$\Omega_{f_2}(w) := \big\{x\in\Omega : f_1(x,w(x),Dw(x)) > f_2(x,w(x),Dw(x))\big\};$$
$$\Omega_0(w) := \big\{x\in\Omega : f_1(x,w(x),Dw(x)) = f_2(x,w(x),Dw(x))\big\},$$

— sets of indices:

$$i\in I_{f_1}^k(w) \Leftrightarrow \int_{\Omega_i^k} f_1(w)\, d\Omega < \int_{\Omega_i^k} f_2(w)\, d\Omega;$$
$$i\in I_{f_2}^k(w) \Leftrightarrow \int_{\Omega_i^k} f_1(w)\, d\Omega > \int_{\Omega_i^k} f_2(w)\, d\Omega;$$
$$i\in I_0^k(w) \Leftrightarrow \int_{\Omega_i^k} f_1(w)\, d\Omega = \int_{\Omega_i^k} f_2(w)\, d\Omega$$

— and further subsets of Ω:

$$\Gamma_{f_1}^k(w) := \bigcup_{i\in I_{f_1}^k(w)} \Omega_i^k;\quad \Gamma_{f_2}^k(w) := \bigcup_{i\in I_{f_2}^k(w)} \Omega_i^k;\quad \Gamma_0^k(w) := \bigcup_{i\in I_0^k(w)} \Omega_i^k.$$

We know that f_1 and f_2 are measurable with respect to the first variable. By the Lusin theorem, for any $\varepsilon > 0$ we can find a sufficiently small $\omega\subset\Omega$ such that

— $\Omega\setminus\omega$ is closed;
— $f_1\big(\cdot,w(\cdot),Dw(\cdot)\big)$ and $f_2\big(\cdot,w(\cdot),Dw(\cdot)\big)$ are continuous on $\Omega\setminus\omega$.

We can also suppose that

— $\int_\omega \max\big(|f_1(w)|,|f_2(w)|\big)\, d\Omega \leq \varepsilon$.

By the definition of $\mathcal{J}^k$ and continuity of f_1 and f_2 on $\Omega\setminus\omega$ we can write

$$\begin{aligned}\lim_{k\to\infty} \mathcal{J}^k(w) &\leq \int_{\Omega_{f_1}(w)\setminus\omega} f_1(w)\, d\Omega + \int_{\Omega_{f_2}(w)\setminus\omega} f_2(w)\, d\Omega + \int_{\Omega_0(w)\setminus\omega} f_1(w)\, d\Omega + \varepsilon\\ &\leq \int_{\Omega_{f_1}(w)} f_1(w)\, d\Omega + \int_{\Omega_{f_2}(w)} f_2(w)\, d\Omega + \int_{\Omega_0(w)} f_1(w)\, d\Omega + 2\varepsilon = \mathcal{J}(w) + 2\varepsilon,\end{aligned}$$

and (3) follows. Let $(\widetilde{u}_j)$ be a minimizing sequence for $\mathcal{J}$ and an arbitrary $\varepsilon > 0$ be fixed. For every sufficiently large j there exists $K(j)\in\mathbb{N}$ with the property we have

$$\mathcal{J}^k(\widetilde{u}_j) \leq \mathcal{J}(\widetilde{u}_j) + \varepsilon \quad \text{for all } k \geq K(j)$$

and consequently

$$\alpha^k \le \alpha + 2\varepsilon$$

for sufficiently large k. Thus we have

$$\alpha \le \lim_{k\to\infty} \alpha^k \le \alpha + 2\varepsilon,$$

which implies (2) and finishes the proof of Proposition 2.1. ∎

We will now take advantage of the fact that u_k is the minimizer of $\mathcal{J}^k$. We have just proved that α^k is an infimum in the approximation problem (P^k). For any $w \in H^1(\Omega;\mathbb{R}^m)$ we define the functions $\chi^k_{f_i}(w) : \Omega \to \{0,1\}$, $i = 1,2$, by the following formulae:

$$\chi^k_{f_1}(w)(x) := \begin{cases} 1, & \text{if } x \in (\Gamma^k_{f_1}(w) \cup \Gamma^k_0(w)) \\ 0, & \text{otherwise;} \end{cases}$$

$$\chi^k_{f_2}(w)(x) := \begin{cases} 1, & \text{if } x \in \Gamma^k_{f_2}(w) \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, the sequences $(\chi^k_{f_i}(u_k))$, $i = 1,2$, are bounded in $L^\infty(\Omega)$, so (up to a subsequence) we have $\chi^k_{f_i}(u_k) \to \chi_{f_i}$ weakly* in $L^\infty(\Omega)$. The range of the limit functions is the interval $[0,1]$ and $\chi_{f_1} + \chi_{f_2} \equiv 1$. With the help of these functions we can write

$$\mathcal{J}^k(u_k) = \int_\Omega \big[\chi^k_{f_1}(u_k) f_1(u_k) + \chi^k_{f_2}(u_k) f_2(u_k)\big]\, d\Omega.$$

From the definition of α^k and the fact that u_k is a minimizer of $\mathcal{J}^k$ it also follows that for every $v, w \in H^1(\Omega;\mathbb{R}^m)$

$$\int_\Omega \big[\chi^k_{f_1}(u_k) f_1(u_k) + \chi^k_{f_2}(u_k) f_2(u_k)\big]\, d\Omega \le \int_\Omega \big[\chi^k_{f_1}(v) f_1(w) + \chi^k_{f_2}(v) f_2(w)\big]\, d\Omega.$$

The coercivity condition ensures that the sequence (u_k) is bounded in $H^1(\Omega;\mathbb{R}^m)$, so (up to a subsequence) we have $u_k \to u$ weakly in $H^1(\Omega;\mathbb{R}^m)$. Thus from Proposition 2.1 we obtain that

$$\mathcal{J}^k(u_k) = \int_\Omega \big[\chi^k_{f_1}(u_k) f_1(u_k) + \chi^k_{f_2}(u_k) f_2(u_k)\big]\, d\Omega \to \mathcal{J}(u) \quad \text{as } k \to \infty.$$

We can now sum up above considerations by stating the first of the main results of the article.

THEOREM 2.2. *Suppose that f_1 and f_2 are quasiconvex and satisfy* (ii), (iii), (iv). *Then there exist a sequence $(u_k) \subset H^1(\Omega;\mathbb{R}^m)$, sequences $(\chi^k_{f_i})$, $i = 1,2$, of functions from $H^1(\Omega;\mathbb{R}^m)$ to $\{0,1\}$ with $\chi^k_{f_1}(u_k) + \chi^k_{f_2}(u_k) \equiv 1$, such that*

- (i′) *$u_k \to u$ weakly in $H^1(\Omega;\mathbb{R}^m)$ as $k \to \infty$;*
- (ii′) *$\chi^k_{f_i}(u_k) \to \chi_{f_i}$ weakly* in $L^\infty(\Omega)$ as $k \to \infty$, $\chi_{f_i} : \Omega \to [0,1]$, $i = 1,2$, and $\chi_{f_1} + \chi_{f_2} \equiv 1$;*
- (iii′) $\lim_{k\to\infty} \int_\Omega \big[\chi^k_{f_1}(u_k) f_1(u_k) + \chi^k_{f_2}(u_k) f_2(u_k)\big]\, d\Omega = \alpha$;
- (iv′) *for all $v, w \in H^1(\Omega;\mathbb{R}^m)$*

$$\int_\Omega \big[\chi^k_{f_1}(u_k) f_1(u_k) + \chi^k_{f_2}(u_k) f_2(u_k)\big]\, d\Omega \le \int_\Omega \big[\chi^k_{f_1}(v) f_1(w) + \chi^k_{f_2}(v) f_2(w)\big]\, d\Omega.$$

REMARK 2.3. There is much more in the article [N] than it has been presented above:

(a) the sufficient condition for u to be a solution of (P) is stated and proved;
(b) there is a detailed analysis of the one-dimensional Dirichlet problem with the relaxation term $\mathcal{R}$ calculated explicitly;
(c) the optimization problem (P), where Ω is an elastic body in the three-dimensional Euclidean space and integrand of $\mathcal{J}$ is of the form
$$\min\{C_{ijkl}(v)\varepsilon_{ij}(v)\varepsilon_{kl}(v), D_{ijkl}(v)\varepsilon_{ij}(v)\varepsilon_{kl}(v)\},$$
with C and D being Hooke's tensors of elasticity and ε the strain tensor field with respect to the displacement vector field v, is analyzed.

3. Special case—quadratic double-well energy. This section is devoted to the analysis of the special case of the problem described above. Namely, we want to minimize the functional with integrand being a minimum of two quadratic functions:
$$\mathcal{J}(u) = \int_\Omega \min\{\tfrac{1}{2}a\,|\varepsilon(u)+C|^2\,, \tfrac{1}{2}b\,|\varepsilon(u)+D|^2\}\,dx,$$
where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is a competing vector-valued function from the Sobolev space $H_0^1(\Omega;\mathbb{R}^n)$, $\Omega \subset \mathbb{R}^n$—a bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$. Further, the symbol "$|\cdot|$" stands for the Euclidean norm in $\mathbb{R}^{n\times n}_{\mathrm{sym}}$, $\varepsilon(u) \in L^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ is the symmetrized gradient of $u \in H_0^1(\Omega;\mathbb{R}^n)$ (i.e. $\varepsilon(v) := \frac{1}{2}(\nabla v + \nabla^T v)$), $C, D \in L^\infty(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ and $a, b \in L^\infty(\Omega)$ are such that $a(x), b(x) \geq \delta > 0$ a.e. in Ω for a positive constant δ.

3.1. Statement of the problem and its approximation. Our problem (Q) has the form
$$\inf\{\mathcal{J}(u) : u \in H_0^1(\Omega;\mathbb{R}^n)\} := \alpha.$$
Theorem 2.2 ensures the existence of sequences $u^k \in H_0^1(\Omega;\mathbb{R}^n)$, $\chi_a^k \subset \{0,1\}$ and $\chi_b^k \subset \{0,1\}$, $\chi_a^k + \chi_b^k \equiv 1$, with the properties that

(a) $\{u_k\}$ is a minimizing sequence for (Q),
(b) $u^k \to u$ weakly in $H_0^1(\Omega;\mathbb{R}^n)$ as $k \to \infty$,
(c) $\chi_a^k \to \chi_a$, $\quad \chi_b^k \to \chi_b$ weak* in $L^\infty(\Omega)$ as $k \to \infty$, where $\chi_a : \Omega \to [0,1]$, $\chi_b : \Omega \to [0,1]$ with $\chi_a + \chi_b \equiv 1$,
(d) $\displaystyle\int_\Omega \big[\tfrac{1}{2}\chi_a^k a|\varepsilon(u^k)+C|^2 + \tfrac{1}{2}\chi_b^k b|\varepsilon(u^k)+D|^2\big]\,dx := \alpha^k \to \alpha$ as $k \to \infty$,
(e) $$\int_\Omega \big[\tfrac{1}{2}\chi_a^k a|\varepsilon(u^k)+C|^2 + \tfrac{1}{2}\chi_b^k b|\varepsilon(u^k)+D|^2\big]\,dx$$
$$\leq \int_\Omega \big[\tfrac{1}{2}\chi_a^k a|\varepsilon(w)+C|^2 + \tfrac{1}{2}\chi_b^k b|\varepsilon(w)+D|^2\big]\,dx \quad \forall w \in H_0^1(\Omega;\mathbb{R}^n).$$

Let us now introduce the function which in some sense describes the behaviour of the minimizing sequence $\{u_k\}$:
$$\psi^k = \chi_b^k - \chi_a^k \tag{4}$$
with the property
$$(\psi^k)^2 = 1. \tag{5}$$

Observe that we can write

$$\chi_a^k = \frac{1-\psi^k}{2} \quad \text{and} \quad \chi_b^k = \frac{1+\psi^k}{2}. \tag{6}$$

From (e) above it follows that u^k is a solution of the convex optimization problem

$$\inf\{\mathcal{J}^k(v) : v \in H_0^1(\Omega;\mathbb{R}^n)\} := \alpha^k, \tag{Q^k}$$

where

$$\mathcal{J}^k(v) = \int_\Omega \big[\tfrac{1}{2}\chi_a^k a|\varepsilon(v)+C|^2 + \tfrac{1}{2}\chi_b^k b|\varepsilon(v)+D|^2\big]\,dx, \quad v \in H_0^1(\Omega;\mathbb{R}^n),$$

i.e.

$$\mathcal{J}^k(u^k) = \alpha^k.$$

Convex problems (Q^k) are approximating problems of the nonconvex one (Q). According to Proposition 2.1 the sequence (α^k) converges to α as $k \to \infty$.

Using the properties of the scalar product we get

$$\begin{aligned}
\chi_a^k aC + \chi_b^k bC &= \frac{aC+bD}{2} + \psi^k \frac{bD-aC}{2},\\
\frac{1}{2}\,\chi_a^k a|C|^2 + \frac{1}{2}\,\chi_b^k b|D|^2 &= \frac{1}{2}\bigg(\frac{a|C|^2+b|D|^2}{2} + \psi^k\,\frac{b|D|^2-a|C|^2}{2}\bigg),
\end{aligned}$$

so that $\mathcal{J}^k(\cdot)$ admits the representation

$$\mathcal{J}^k(v) = \int_\Omega \Big[\frac{1}{2}(\chi_a^k a + \chi_b^k b)|\varepsilon(v)|^2 + (\chi_a^k aC + \chi_b^k bD)\varepsilon(v) + \frac{1}{2}\,\chi_a^k a|C|^2 + \frac{1}{2}\,\chi_b^k b|D|^2\Big]\,dx.$$

It will be convenient to introduce the notation:

$$\begin{aligned}
m^k &:= \chi_a^k a + \chi_b^k b,\\
\mathcal{A}^+ &:= \frac{aC+bD}{2}, \quad \mathcal{A}^- := \frac{bD-aC}{2},\\
\mathcal{B}^k &:= \frac{a|C|^2+b|D|^2}{2} + \psi^k \frac{b|D|^2-a|C|^2}{2}.
\end{aligned} \tag{7}$$

With the help of the above and (6) we can write the approximation functional in the form

$$\mathcal{J}^k(v) = \int_\Omega \Big[\frac{1}{2} m^k|\varepsilon(v)|^2 + (\mathcal{A}^+ + \psi^k\mathcal{A}^-)\varepsilon(v) + \frac{1}{2}\mathcal{B}^k\Big]\,dx. \tag{R}$$

3.2. Fenchel dual of the approximating problems. Recall that if X is a Hilbert space and a function $f : X \to \mathbb{R}\cup\{+\infty\}$ has a nonempty domain, then its Fenchel conjugate f^c is defined by

$$X^* \ni p \mapsto f^c := \sup\{\langle p, x\rangle - f(x) : x \in X\} \in \mathbb{R}\cup\{+\infty\}.$$

Roughly speaking, the Fenchel idea is to associate with primal minimization problem S its Fenchel dual S^*. Their (finite) infima, s and s^* respectively, are linked by the equation $s + s^* = 0$. See [A] (especially Chapter 3) for details.

Let $m > 0$ be fixed constant and consider the function

$$\phi : \mathbb{R}^{n\times n}_{\text{sym}} \ni \xi \mapsto \phi(\xi) := \tfrac{1}{2}|\xi|^2 + E\cdot\xi \in \mathbb{R}\cup\{+\infty\}.$$

Calculating its Fenchel conjugate ϕ^c we get:

$$\forall p \in \mathbb{R}^{n\times n}_{\mathrm{sym}} \quad \phi^c(p) = \sup\{p\cdot\xi - \phi(\xi) : \xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}}\}$$
$$= p\cdot \tfrac{1}{m}(p-E) - \phi(\tfrac{1}{m}(p-E)) = \tfrac{1}{2m}|p-E|^2.$$

Define next a continuous linear operator L as follows:

$$L : H^1_0(\Omega;\mathbb{R}^n) \ni v \mapsto L(v) := \varepsilon(v) \in L^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}),$$

with the transpose $L^* : L^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}) \to H^{-1}(\Omega;\mathbb{R}^n)$ given by:

$$\forall p \in L^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})\ \forall v \in H^1_0(\Omega;\mathbb{R}^n) \quad \langle L^*p, v\rangle = \int_\Omega p\cdot\varepsilon(v)\,dx.$$

The kernel of L^* consists of all vectors $p \in L^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ satisfying the equation $\int_\Omega p\cdot\varepsilon(v)\,dx = 0$. Finally, define the integral operators $\mathcal{I}^k$, $k\in\mathbb{N}$:

$$\mathcal{I}^k(q) := \int_\Omega \Big[\frac{1}{2m^k}\,\big|q - (\mathcal{A}^+ + \psi^k\mathcal{A}^-)\big|^2 - \frac{1}{2}\,\mathcal{B}^k\Big]\,dx, \quad q \in L^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}}).$$

We can now state the Fenchel dual of (Q^k):

$$\inf\{\mathcal{I}^k(q) : q \in \operatorname{Ker} L^*\} := \beta^k. \tag{Q^{k*}}$$

The Fenchel duality theorem now yields for each $v \in H^1_0(\Omega;\mathbb{R}^n)$ and each $q \in \operatorname{Ker} L^*$

$$\begin{aligned}\mathcal{J}^k(v) &\ge \int_\Omega \Big[\frac{1}{2}m^k|\varepsilon(u^k)|^2 + (\mathcal{A}^+ + \psi^k\mathcal{A}^-)\cdot\varepsilon(u^k) + \frac{1}{2}\,\mathcal{B}^k\Big]\,dx\\ &= \mathcal{J}^k(u^k) = \alpha^k = -\beta^k = -\mathcal{I}^k(p^k)\\ &= -\int_\Omega \Big[\frac{1}{2m^k}\,\big|p^k - (\mathcal{A}^+ + \psi^k\mathcal{A}^-)\big|^2 - \frac{1}{2}\,\mathcal{B}^k\Big]\,dx\\ &\ge -\int_\Omega \Big[\frac{1}{2m^k}\,\big|q - (\mathcal{A}^+ + \psi^k\mathcal{A}^-)\big|^2 - \frac{1}{2}\,\mathcal{B}^k\Big]\,dx = -\mathcal{I}^k(q), \end{aligned}\tag{8}$$

where

$$p^k = m^k\varepsilon(u^k) + \mathcal{A}^+ + \psi^k\mathcal{A}^- \in \operatorname{Ker} L^* \tag{9}$$

is a solution of the dual problem (Q^{k*}). From the above formula we obtain

$$\varepsilon(u^k) = \frac{1}{m^k}(p^k - \mathcal{A}^+ - \psi^k\mathcal{A}^-). \tag{10}$$

To get various representations of approximate infima it would be convenient to introduce some notation. Consider the obvious equalities

$$a = \frac{a+b}{2} - \frac{b-a}{2} := \overline{m} - \underline{m} \quad \text{and} \quad b = \frac{a+b}{2} + \frac{b-a}{2} := \overline{m} + \underline{m}.$$

Observe that for any element of the set

$$\Omega_0 := \{x \in \Omega : a(x) = b(x)\} \tag{11}$$

we have $\underline{m} = 0$ and $\overline{m} = a$.

Recalling (6) and definition of m^k in (7) we can express m^k as

$$m^k = \frac{1-\psi^k}{2}(\overline{m} - \underline{m}) + \frac{1+\psi^k}{2}(\overline{m} + \underline{m}),$$

and (9) becomes

$$p^k = \Big[\frac{1-\psi^k}{2}\,(\overline{m}-\underline{m}) + \frac{1+\psi^k}{2}\,(\overline{m}+\underline{m})\Big]\varepsilon(u^k) + \mathcal{A}^+ + \psi^k\mathcal{A}^-, \tag{12}$$

which allows us to write

$$\psi^k\varepsilon(u^k) = \frac{p^k}{\underline{m}} - \frac{\overline{m}}{\underline{m}}\,\varepsilon(u^k) - \frac{1}{\underline{m}}\mathcal{A}^+ - \frac{1}{\underline{m}}\psi^k\mathcal{A}^-. \tag{13}$$

This makes it possible to obtain various forms of α^k.

3.3. Representations of the approximate infima. Since $p^k \in \operatorname{Ker} L^*$ we have

$$\int_\Omega p^k\cdot\varepsilon(u^k)\,dx = \int_\Omega \big[m^k|\varepsilon(u^k)|^2 + (\mathcal{A}^+ + \psi^k\mathcal{A}^-)\cdot\varepsilon(u^k)\big]\,dx = 0. \tag{14}$$

Taking into account that $\alpha^k = -\mathcal{I}^k(p^k)$ and equation (14) we get the following representations of α^k:

$$\alpha^k = \frac{1}{2}\int_\Omega \big[-m^k|\varepsilon(u^k)|^2 + \mathcal{B}^k\big]\,dx = \frac{1}{2}\int_\Omega \big[(\mathcal{A}^+ + \psi^k\mathcal{A}^-)\cdot\varepsilon(u^k) + \mathcal{B}^k\big]\,dx. \tag{15}$$

From (10), (14) and (15) it follows that

$$\int_\Omega \Big[\frac{1}{m^k}|p^k|^2 - \frac{1}{m^k}(\mathcal{A}^+ + \psi^k\mathcal{A}^-)\cdot p^k\Big]\,dx = 0$$

and further

$$\begin{aligned}\alpha^k &= -\frac{1}{2}\int_\Omega \Big[-\frac{1}{m^k}|p^k|^2 + \frac{1}{m^k}\left|\mathcal{A}^+ + \psi^k\mathcal{A}^-\right|^2 - \mathcal{B}^k\Big]\,dx\\ &= \frac{1}{2}\int_\Omega \Big[\frac{1}{m^k}(\mathcal{A}^+ + \psi^k\mathcal{A}^-)\cdot p^k - \frac{1}{m^k}\left|\mathcal{A}^+ + \psi^k\mathcal{A}^-\right|^2 + \mathcal{B}^k\Big]\,dx,\end{aligned} \tag{16}$$

which leads us to the equality

$$\int_\Omega \Big[m^k\left|\varepsilon(u^k)\right|^2 + \frac{1}{m^k}|p^k|^2\Big]\,dx = \int_\Omega \Big[\frac{1}{m^k}\left|\mathcal{A}^+ + \psi^k\mathcal{A}^-\right|^2\Big]\,dx. \tag{17}$$

After some algebraic manipulations we can write the right hand side of (17) as

$$\int_\Omega \Big(\frac{a+b}{2ab} + \psi^k\,\frac{a-b}{2ab}\Big)\big(|\mathcal{A}^+|^2 + 2\psi^k\mathcal{A}^+\mathcal{A}^- + |\mathcal{A}^-|^2\big)\,dx.$$

Next, using the facts that $\chi_a^k + \chi_b^k = 1$, $(\psi^k)^2 = 1$ and that $\{\psi^k\}$ is weakly* convergent in L^∞, it is possible to pass to the limit with k and get

$$\lim_{k\to\infty}\int_\Omega \Big[m^k\left|\varepsilon(u^k)\right|^2 + \frac{1}{m^k}|p^k|^2\Big]\,dx = \frac{1}{2}\int_\Omega \big(a|C|^2 + b|D|^2\big)\,dx + \frac{1}{2}\int_\Omega \psi\big(b|D|^2 - a|C|^2\big)\,dx.$$

Abbreviating

$$\mathcal{B} := \frac{a|C|^2 + b|D|^2}{2} + \psi\,\frac{b|D|^2 - a|C|^2}{2}$$

we can write

$$\lim_{k\to\infty}\int_\Omega \Big[m^k|\varepsilon(u^k)|^2 + \frac{1}{m^k}\,|p^k|^2\Big]\,dx = \int_\Omega \mathcal{B}\,dx. \tag{18}$$

Since in $\Omega \setminus \Omega_0$ (with Ω_0 given by (11)) the equation (13) is valid, we obtain from (15) the following representation of an approximate infimum:

$$\begin{aligned}\alpha^k = \frac{1}{2}\int_{\Omega\setminus\Omega_0} \Big[&\frac{ab(C-D)}{b-a}\cdot\varepsilon(u^k) + \frac{bD-aC}{b-a}\cdot p^k \\ &+ \frac{ab(|C|^2-|D|^2)}{2(b-a)} - \psi^k\,\frac{ab|C-D|^2}{2(b-a)}\Big]\,dx \\ + \frac{1}{2}\int_{\Omega_0}&\Big[-a|\varepsilon(u^k)|^2 + \frac{a(|C|^2+|D|^2)}{2} + \psi^k\cdot\frac{a(|D|^2-|C|^2)}{2}\Big]\,dx \\ &+ \frac{1}{2}\int_{\Omega_0} p^k\cdot\varepsilon(u^k)\,dx.\end{aligned} \tag{19}$$

If we observe that $\mathcal{B}^k = \frac{1}{a}(|\mathcal{A}^+|^2 + |\mathcal{A}^-|^2 + 2\psi^k\mathcal{A}^+\cdot\mathcal{A}^-)$ in Ω_0 and recall that $(\psi^k)^2 = 1$, then we can use equation (16) to derive from (9) yet another representation of α^k:

$$\begin{aligned}\alpha^k = \frac{1}{2}\int_{\Omega\setminus\Omega_0}\Big[&\frac{ab(C-D)}{b-a}\cdot\varepsilon(u^k) + \frac{bD-aC}{b-a}\cdot p^k + \frac{ab(|C|^2-|D|^2)}{2(b-a)} \\ &- \psi^k\,\frac{ab|C-D|^2}{2(b-a)}\Big]\,dx + \frac{1}{2}\int_{\Omega_0}\frac{1}{a}|p^k|^2\,dx - \tfrac{1}{2}\int_{\Omega_0} p^k\cdot\varepsilon(u^k)\,dx.\end{aligned} \tag{20}$$

We will further use formulas (19) and (20) to express the infimum α in terms of weak limits u, p and ψ.

3.4. Explicit formulas for infimum. In this section we will present the explicit formulas for infimum in (Q) – the second of the main results of the article. The following lemma will be of use.

LEMMA 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain in $\mathbb{R}^n$ with Lipschitz continuous boundary $\partial\Omega$. Then*

$$p^k\cdot\varepsilon(u^k) \to p\cdot\varepsilon(u) \quad \text{weakly in } L^1(\Omega).$$

This can be proved with the help of the Rellich compactness theorem, Chacon's biting lemma, characterization of weak convergence in L^1 via biting convergence (see Lemma 6.9 in [P]) and Vitali covering theorem (see [NP] for details). The weak lower semicontinuity of convex functionals, the upper semicontinuity of concave functionals and Lemma 3.1 yield

$$\liminf_{k\to\infty}\frac{1}{2}\int_{\Omega_0}\frac{1}{a}|p^k|^2\,dx \geq \frac{1}{2}\int_{\Omega_0}\frac{1}{a}|p|^2\,dx,$$

$$\limsup_{k\to\infty}\frac{1}{2}\int_{\Omega_0}\big[-a|\varepsilon(u^k)|^2 + \mathcal{B}^k\big]\,dx \leq \frac{1}{2}\int_{\Omega_0}\big[-a|\varepsilon(u)|^2 + \mathcal{B}\big]\,dx,$$

$$\lim_{k\to\infty}\frac{1}{2}\int_{\Omega_0} p^k\cdot\varepsilon(u^k)\,dx = \frac{1}{2}\int_{\Omega_0} p\cdot\varepsilon(u)\,dx,$$

where

$$\mathcal{B} = \frac{a(|C|^2+|D|^2)}{2} + \psi\,\frac{a(|D|^2-|C|^2)}{2} \quad \text{in } \Omega_0,$$

and $\psi = \chi_b - \chi_a$.

Consider now the first integral on the right hand side in (19). It can be shown (see [NP]) that although the functions $\frac{ab(C-D)}{b-a}$ and $\frac{bD-aC}{b-a}$ are not assumed to belong to $L^2(\Omega\setminus\Omega_0;\mathbb{R}^{n\times n}_{\mathrm{sym}})$, it is true that

$$\begin{aligned}\lim_{k\to\infty}\frac{1}{2}\int_{\Omega\setminus\Omega_0}\Big[&\frac{ab(C-D)}{b-a}\cdot\varepsilon(u^k)+\frac{bD-aC}{b-a}\cdot p^k\\&+\frac{ab(|C|^2-|D|^2)}{2(b-a)}-\psi^k\,\frac{ab|C-D|^2}{2(b-a)}\Big]\,dx\\=\frac{1}{2}\int_{\Omega\setminus\Omega_0}\Big[&\frac{ab(C-D)}{b-a}\cdot\varepsilon(u)+\frac{bD-aC}{b-a}\cdot p\\&+\frac{ab(|C|^2-|D|^2)}{2(b-a)}-\psi\,\frac{ab|C-D|^2}{2(b-a)}\Big]\,dx.\end{aligned}\tag{21}$$

Now, for $v\in H^1_0(\Omega;\mathbb{R}^n)$ and $q\in\operatorname{Ker}L^*$ let us set

$$\mathcal{I}(v,q):=\int_{\Omega\setminus\Omega_0}\Big[\frac{ab(C-D)}{b-a}\cdot\varepsilon(v)+\frac{bD-aC}{b-a}\cdot q+\frac{ab(|C|^2-|D|^2)}{2(b-a)}-\psi\,\frac{ab|C-D|^2}{2(b-a)}\Big]\,dx.\tag{22}$$

Using the fact that for all $k\in\mathbb{N}$ we have $p^k\in\operatorname{Ker}L^*$, (21) and passing to the limit as $k\to\infty$ in (19) and (20) we obtain the system of inequalities

$$\begin{aligned}\frac{1}{2}\int_{\Omega_0}\Big[-a|\varepsilon(u)|^2\Big]\,dx&+\int_{\Omega_0}p\cdot\varepsilon(u)\,dx\\&\ge\alpha+\frac{1}{2}\int_{\Omega_0}p\cdot\varepsilon(u)\,dx-\mathcal{I}(u,p)-\int_{\Omega_0}\mathcal{B}\,dx\\&\ge\frac{1}{2}\int_{\Omega_0}\Big[\frac{1}{a}|p|^2-\frac{1}{a}\big(|\mathcal{A}^+|^2+|\mathcal{A}^-|^2+2\psi\mathcal{A}^+\cdot\mathcal{A}^-\big)\Big]\,dx.\end{aligned}\tag{23}$$

Since $p=a\varepsilon(u)+\mathcal{A}^++\psi\mathcal{A}^-$ and $\mathcal{A}^-=a\frac{D-C}{2}$ in Ω_0, $\psi^2-1=-4\chi_a\chi_b$ it follows from (23) that

$$\begin{aligned}0\ge\alpha-\int_{\Omega_0}\big[(\mathcal{A}^++\psi\mathcal{A}^-)\cdot\varepsilon(u)+\mathcal{B}\big]\,dx-\frac{1}{2}\int_{\Omega_0}p\cdot\varepsilon(u)\,dx-\mathcal{I}(u,p)\\\ge-\frac{1}{2}\int_{\Omega_0}\chi_a\chi_b\,a|C-D|^2\,dx.\end{aligned}$$

Thus we are allowed to conclude that there exists a $\theta\in[0,1]$ such that

$$\begin{aligned}\alpha=&\int_{\Omega_0}\big[(\mathcal{A}^++\psi\mathcal{A}^-)\cdot\varepsilon(u)+\mathcal{B}\big]\,dx+\frac{1}{2}\int_{\Omega_0}p\cdot\varepsilon(u)\,dx\\&+\int_{\Omega\setminus\Omega_0}\Big[\frac{ab(C-D)}{b-a}\cdot\varepsilon(u)+\frac{bD-aC}{b-a}\cdot p+\frac{ab(|C|^2-|D|^2)}{2(b-a)}-\psi\,\frac{ab|C-D|^2}{2(b-a)}\Big]\,dx\\&-\frac{\theta}{2}\int_{\Omega_0}\chi_a\chi_b\,a|C-D|^2\,dx.\end{aligned}$$

We can at last formulate the theorem being the main application of Theorem 2.2.

THEOREM 3.2. *Let $u \in H_0^1(\Omega;\mathbb{R}^n)$ be the weak limit of $\{u^k\}$—the minimizing sequence for $\mathcal{J}$ in* (Q). *Let $p \in L^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ the weak limit of $\{p^k\}$—the sequence of solutions of the dual problems (Q^{k*}).*

Then there exists $\theta \in [0,1]$ such that the infimum α in the minimization problem (Q) *can be expressed as*

$$\begin{aligned}\alpha = \int_{\Omega_0} \Big[\Big(\frac{a(C+D)}{2} + \psi\,\frac{a(D-C)}{2}\Big)\cdot\varepsilon(u) + \frac{a(|C|^2+|D|^2)}{2} \\ + \psi\,\frac{a(|D|^2-|C|^2)}{2}\Big]\,dx + \frac{1}{2}\int_{\Omega_0} p\cdot\varepsilon(u)\,dx \\ + \int_{\Omega\setminus\Omega_0}\Big[\frac{ab(C-D)}{b-a}\cdot\varepsilon(u) + \frac{bD-aC}{b-a}\cdot p + \frac{ab(|C|^2-|D|^2)}{2(b-a)} \\ - \psi\,\frac{ab|C-D|^2}{2(b-a)}\Big]\,dx - \frac{\theta}{2}\int_{\Omega_0}\chi_a\chi_b\,a|C-D|^2\,dx.\end{aligned} \tag{24}$$

Moreover, the above formula allows us to express α in the following ways:

$$\begin{aligned}\alpha = \int_{\Omega_0}\Big[-a|\varepsilon(u)|^2 + \frac{a(|C|^2+|D|^2)}{2} + \psi\,\frac{a(|D|^2-|C|^2)}{2}\Big]\,dx + \frac{3}{2}\int_{\Omega_0} p\cdot\varepsilon(u)\,dx \\ + \int_{\Omega\setminus\Omega_0}\Big[\frac{ab(C-D)}{b-a}\cdot\varepsilon(u) + \frac{bD-aC}{b-a}\cdot p + \frac{ab(|C|^2-|D|^2)}{2(b-a)} \\ - \psi\,\frac{ab|C-D|^2}{2(b-a)}\Big]\,dx - \frac{\theta}{2}\int_{\Omega_0}\chi_a\chi_b\,a|C-D|^2\,dx,\end{aligned} \tag{25}$$

and

$$\begin{aligned}\alpha = \int_{\Omega_0}\frac{1}{a}|p|^2\,dx - \frac{1}{2}\int_{\Omega_0} p\cdot\varepsilon(u)\,dx \\ + \int_{\Omega\setminus\Omega_0}\Big[\frac{ab(C-D)}{b-a}\cdot\varepsilon(u) + \frac{bD-aC}{b-a}\cdot p + \frac{ab(|C|^2-|D|^2)}{2(b-a)} \\ - \psi\,\frac{ab|C-D|^2}{2(b-a)}\Big]\,dx + \frac{2-\theta}{2}\int_{\Omega_0}\chi_a\chi_b\,a|C-D|^2\,dx,\end{aligned} \tag{26}$$

and finally

$$\begin{aligned}\alpha = \frac{1}{2}\int_{\Omega_0}\Big[\frac{1}{a}\,|p|^2 - a|\varepsilon(u)|^2 + \frac{a(|C|^2+|D|^2)}{2} + \psi\,\frac{a(|D|^2-|C|^2)}{2}\Big]\,dx \\ + \int_{\Omega\setminus\Omega_0}\Big[\frac{ab(C-D)}{b-a}\cdot\varepsilon(u) + \frac{bD-aC}{b-a}\cdot p + \frac{ab(|C|^2-|D|^2)}{2(b-a)} \\ - \psi\,\frac{ab|C-D|^2}{2(b-a)}\Big]\,dx + \frac{1-\theta}{2}\int_{\Omega_0}\chi_a\chi_b\,a|C-D|^2\,dx + \frac{1}{2}\int_{\Omega_0} p\cdot\varepsilon(u)\,dx.\end{aligned} \tag{27}$$

3.5. Young measure representations for infimum. The last part of the article is devoted to expressing the infimum in (Q) by the Young measure associated with the minimizing for $\mathcal{J}$ sequence (u^k). This is possible because of the structure of this sequence. We will also see that sometimes it is possible to calculate an explicit form of the (nonhomogeneous) Young measure. We will need some notation.

Denote by ω_0^+ and ω_0^- such subsets of Ω that $\psi^k \to 1$ weakly in $L^1(\omega_0^+)$ and $\psi^k \to -1$ weakly in $L^1(\omega_0^-)$. Let $\omega_0 := \omega_0^+ \cup \omega_0^-$ and

$$h(x,\lambda) := \min\{\tfrac{1}{2}a(x)|\lambda + C(x)|^2,\ \tfrac{1}{2}b(x)|\lambda + D(x)|^2\}, \quad \lambda \in \mathbb{R}^{n\times n}_{\mathrm{sym}},\ x \in \Omega.$$

Theorem 3.3. *Let $\nu = \{\nu_x\}_{x\in\Omega}$ be the parametrized Young measure associated with the minimizing sequence $\{u^k\}$. Then*

$$\begin{aligned}\alpha = \int_\Omega \int_{\mathbb{R}^n} h(x,\lambda)\,d\nu_x(\lambda)\,dx &= \int_{\Omega\setminus\Omega_0} \Big[\frac{ab(C-D)}{b-a}\cdot\varepsilon(u) + \frac{bD-aC}{b-a}\cdot p \\ &+ \frac{ab(|C|^2-|D|^2)}{2(b-a)} - \psi\,\frac{ab|C-D|^2}{2(b-a)}\Big]\,dx + \int_{\Omega_0}\Big[-\int_{\mathbb{R}^{n\times n}} a\,|\lambda|^2\,d\nu_x(\lambda) \\ &+ \frac{a(|C|^2+|D|^2)}{2} + \psi\,\frac{a(|D|^2-|C|^2)}{2}\Big]\,dx + \frac{3}{2}\int_{\Omega_0} p\cdot\varepsilon(u)\,dx.\end{aligned} \tag{28}$$

Moreover, we have

$$\nu_x = \delta_{\varepsilon(u(x))} \quad \textit{a.e. in } \omega_0. \tag{29}$$

Sketch of the proof. Let us put for $k \in \mathbb{N}$ $h_1^k := a(x)\left|\varepsilon(u^k(x))\right|^2$, $x \in \Omega$. The equiintegrability of the sequence $(p^k \cdot \varepsilon(u^k))$ together with (9) multiplied by $\varepsilon(u^k)$ yields the equiintegrability and weak convergence (by passing to a subsequence, if necessary) in $L^1(\Omega)$, of (h_1^k). Its weak limit, according to Theorem 6.2 in [P], equals $\int_{\mathbb{R}^{n\times n}} a|\lambda|^2\,d\nu_x(\lambda)$. Now from the inequality

$$h\big(x,\varepsilon(u^k(x))\big) \le \tfrac{1}{2}m^k\left|\varepsilon(u^k)\right|^2 + (\mathcal{A}^+ + \psi^k\mathcal{A}^-)\cdot\varepsilon(u^k) + \tfrac{1}{2}\mathcal{B}^k$$

and Theorem 6.2 in [P] again we can conclude that the weak limit of the sequence $\{h\big(x,\varepsilon(u^k(x))\big)\}$ is the same as in the right hand side of (28).

Strong convergence of the given sequence of measurable functions is the necessary and sufficient condition for the Dirac measure to be the Young measure associated with this sequence, as Proposition 6.12 in [P] states. We thus have to prove strong convergence of $(\varepsilon(u^k))$ in $L^2(\omega_0;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ to establish (29). The fact that the upper Kuratowski limit of the sequence $\{\psi^k\} = \{\chi_b^k - \chi_a^k\}$ is the set $\{-1, 1\}$ allows us to use the Balder theorem (see [V], Theorem 4) and to conclude that $\psi^k \to 1$ a.e. in ω_0^+ ($\psi^k \to -1$ a.e. in ω_0^-), by passing to a subsequence if necessary. Further, the equiintegrability of $(|\varepsilon(u^k)|^2)$ and Lemma 3.1 yield

$$\int_{\omega_0^\pm} b\left|\varepsilon(u^k)\right|^2\,dx \to \int_{\omega_0^\pm} b\left|\varepsilon(u)\right|^2\,dx.$$

Since we also have $\varepsilon(u^k) \rightharpoonup \varepsilon(u)$ in $L^2(\omega_0^\pm;\mathbb{R}^{n\times n}_{\mathrm{sym}})$, we can deduce the strong convergence of $(\varepsilon(u^k))$ in $L^2(\omega_0;\mathbb{R}^{n\times n}_{\mathrm{sym}})$. ■

Corollary 3.4. *From* (25) *and* (28) *it follows that*

$$\begin{aligned}\lim_{k\to\infty}\int_{\Omega_0} a|\varepsilon(u^k)|^2\,dx = \int_{\Omega_0}\int_{\mathbb{R}^n} a\,|\lambda|^2\,d\nu_x(\lambda)\,dx = \int_{\Omega_0} a|\varepsilon(u)|^2\,dx \\ + \frac{\theta}{2}\int_{\Omega_0}\chi_a\chi_b\,a|C-D|^2\,dx,\end{aligned}$$

giving rise to the formula that allows us to calculate $\theta \in [0,1]$. *Namely, if we let*

$$d := \lim_{k\to\infty} \int_{\Omega_0} a\left|\varepsilon(u^k)\right|^2\,dx - \int_{\Omega_0} a\left|\varepsilon(u)\right|^2\,dx, \tag{30}$$

then from the equation

$$d = \frac{\theta}{2}\int_{\Omega_0} \chi_a\chi_b\, a|C-D|^2\,dx$$

we obtain

$$\theta = \begin{cases} \dfrac{2d}{\int_{\Omega_0}\chi_a\chi_b\,a|C-D|^2\,dx} & \text{if } \int_{\Omega_0}\chi_a\chi_b\,a|C-D|^2\,dx > 0\\ 0 & \text{otherwise,}\end{cases} \tag{31}$$

or equivalently

$$\theta = \begin{cases} \dfrac{2\int_{\Omega_0}\int_{\mathbb{R}^n} a\,|\lambda|^2\,d\nu_x(\lambda)\,dx - \int_{\Omega_0} a|\varepsilon(u)|^2\,dx}{\int_{\Omega_0}\chi_a\chi_b\,a|C-D|^2\,dx} & \text{if } \int_{\Omega_0}\chi_a\chi_b\,a|C-D|^2\,dx > 0\\ 0 & \text{otherwise.}\end{cases} \tag{32}$$

Remark 3.5. It is worth to point out that the formulas (25), (26), (27) make it possible to express the infimum of (Q) via (31) in terms of the limits u, p, χ_a, χ_b, d only. On the other hand, the formula (28) expresses it in terms of the parametrized Young measures $\{\nu_x(\cdot)\}$ which, in practice, are much more difficult to derive.

Example 3.6. Let $\omega_0 = \{x\in\Omega : \chi_a(x)\chi_b(x) = 0\}$. Without loss of generality one can suppose that $\psi = 1$ a.e. in ω_0. Let $\omega_{0k}^- = \{x\in\omega_0 : \psi^k(x) = -1\}$. Since $\psi^k \to 1$ weak* in $L^\infty(\omega_0)$, we have

$$2|\omega_{0k}^-| = \int_{\omega_0} (1-\psi_k)\,dx \to 0.$$

Thus $\psi^k \to 1$ strongly in $L^1(\omega_0)$ (in fact, in $L^p(\omega_0)$ for any $p\ge 1$). By the above theorem this means that $\nu_x = \delta_{\varepsilon(u(x))}$ a.e. in ω_0.

Remark. It should be stressed that the main ideas and techniques presented in this article are essentially due to Professor Zdzisław Naniewicz, who prematurely died in March 2012.

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